Tutoria para RMM

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1. (RMM 2017, 1) (a) Prove that every positive integer n can be written uniquely in the form

$$n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j},$$

where $k \geq 0$ and $0 \leq m_1 < m_2 \cdots < m_{2k+1}$ are integers.

This number k is called *weight* of n.

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

2. (RMM 2018, 2) Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

3. (RMM 2018, 3) Ann and Bob play a game on an infinite checkered plane making moves in turn. Ann makes the first move. A move consists in orienting any unit grid-segment that has not been oriented before. If at some stage some oriented segments form an oriented cycle, Bob wins. Does Bob have a strategy that guarantees him to win?

4. (RMM 2020, 4) Let $\mathbb{N}$ be the set of all positive integers. A subset A of $\mathbb{N}$ is sum-free if, whenever x and y are (not necessarily distinct) members of A, their sum $x+y$ does not belong to A. Determine all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that, for each sum-free subset A of $\mathbb{N}$, the image $\{f(a) : a \in A\}$ is also sum-free.Note: a function $f : \mathbb{N} \to \mathbb{N}$ is surjective if, for every positive integer n, there exists a positive integer m such that $f(m) = n$.