Mexican Quarantine Mathematical Olympiad

2020

PROBLEMA 1

Let a, b and c be real numbers such that

$$\lceil a \rceil + \lceil b \rceil + \lceil c \rceil + \lfloor a+b \rfloor + \lfloor b+c \rfloor + \lfloor c+a \rfloor = 2020$$

Prove that

$$\lfloor a \rfloor + \lfloor b \rfloor + \lfloor c \rfloor + \lceil a+b+c \rceil \geq 1346$$

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x, and $\lceil x \rceil$ is the smallest integer greater than or equal to x. That is, $\lfloor x \rfloor$ is the unique integer satisfying $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, and $\lceil x \rceil$ is the unique integer satisfying $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

PROBLEMA 2

Let n be an integer greater than 1. A certain school has $1 + 2 + \cdots + n$ students and n classrooms, with capacities for $1, 2, \ldots, n$ people, respectively. The kids play a game in k rounds as follows: in each round, when the bell rings, the students distribute themselves among the classrooms in such a way that they don't exceed the room capacities, and if two students shared a classroom in a previous round, they cannot do it anymore in the current round. For each n, determine the greatest possible value of k.

PROBLEMA 3

Let Γ_1 and Γ_2 be circles intersecting at points A and B. A line through A intersects Γ_1 and Γ_2 at C and D respectively. Let P be the intersection of the lines tangent to Γ_1 at A and C, and let Q be the intersection of the lines tangent to Γ_2 at A and D. Let X be the second intersection point of the circumcircles of BCP and BDQ, and let Y be the intersection of lines AB and PQ. Prove that C, D, X and Y are concyclic.

PROBLEMA 4

Let ABC be an acute triangle with orthocenter H. Let A_1, B_1 and C_1 be the feet of the altitudes of triangle ABC opposite to vertices A, B, and C respectively. Let B_2 and C_2 be the midpoints of BB_1 and CC_1, respectively. Let O be the intersection of lines BC_2 and CB_2. Prove that O is the circumcenter of triangle ABC if and only if H is the midpoint of AA_1.

PROBLEMA 5

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$, such that for all positive integers n and prime numbers p:

$$p \mid f(n)f(p-1)! + n^{f(p)}.$$

PROBLEMA 6

Oriol has a finite collection of cards, each one with a positive integer written on it. We say the collection is *n-complete* if for any integer k from 1 to n (inclusive), he can choose some cards such that the sum of the numbers on them is exactly k. Suppose that Oriol's collection is n-complete, but it stops being n-complete if any card is removed from it. What is the maximum possible sum of the numbers on all the cards?