RMM 2015 - P1

Yes! I am going to present an example:

Let $p_1 < p_2 < \cdots$ be the prime numbers.

We define a_n as

→ $(p_1 \cdot p_3 \cdot \cdots \cdot p_{2n-5}) \cdot (p_{2n-1} \cdot p_{2n})$, if n is odd

→ $(p_2 \cdot p_4 \cdot \cdots \cdot p_{2n-4}) \cdot (p_{2n-1} \cdot p_{2n})$, if n is even

i.e.,

$a_{2k} = (p_2 \cdot p_4 \cdot \cdots \cdot p_{4k-4}) \cdot (p_{4k-1} \cdot p_{4k})$

$a_{2k+1} = (p_1 \cdot p_2 \cdots \cdot p_{4k-3}) \cdot (p_{4k+1} \cdot p_{4k+2})$

Let us show that it works.

<u>Lemma 1</u>: if $|m-n| = 1 \Rightarrow a_m$ and a_n are coprimes.

<u>Proof</u>:
W.L.O.G., $m = n+1$.

$a_{2k} = p_2 \cdots p_{4k-4} \cdot (p_{4k-1} \cdot p_{4k})$ } coprimes! ($n=2k$ ok!)

coprimes! ($n=2k+1$ ok!) { $a_{2k+1} = p_1 \cdots p_{4k-3} \cdot (p_{4k+1} \, p_{4k+2})$

$a_{2k+2} = (p_2 \quad p_{4k}) \cdot (p_{4k+3} \, p_{4k+4})$

□

Lemma 2: if $|m-n|>1 \Rightarrow a_m$ and a_n are not coprimes

Proof: W.L.O.G. $n \leq m \Rightarrow n \leq m-2$.

We claim that either p_{2n-1} or p_{2n} divide both a_m and a_n, which proves that they are not coprimes.

By definition, $p_{2n-1} \mid a_n$ and $p_{2n} \mid a_n$

- $m=2k \Rightarrow a_m = (p_2 \cdots p_{4k-4}) \cdot (p_{4k-1}\, p_{4k})$

$$n \leq m-2 \Rightarrow 2n \leq 4k-4 \Rightarrow p_{2n} \mid (p_2 \cdots p_{4k-4})$$

$$\Rightarrow p_{2n} \mid a_m.$$

- $m=2k+1 \rightsquigarrow$ ANALOGOUS...

$\square$

Thus, using Lemmas 1 and 2, de facto, the sequence has the desired property.

$\square$