Generating Functions and the Residue Theorem
Treinamento IMO

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1. (HMMT 2007) Let S denote the set of all triples (i, j, k) of positive integers where $i + j + k = 17$. Compute
$$\sum_{(i,j,k)\in S} ijk$$

2. Is it possible to partition the set of positive integers into a finite number of (more than one) distinct arithmetic progressions?

3. (IMO Shortlist 1998) Let $a_0, a_1, a_2, \dots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

4. For $n \geq 0$, compute
$$\sum_{k\geq 0} \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k$$

5. (Putnam 2003) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

6. (IMO 1995) Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \dots, 2p\}$ are there such that the sums of its elements are divisible by p?

7. (PUMaC 2015). Let p be an odd prime. Prove that $p^2 | 2^p - 2$ if and only if
$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(p-2)(p-1)} \equiv 0 \pmod{p}.$$

8. Let $b(n)$ be the n^{th} Bell number, i.e., $b(n)$ is the number of ways of partitioning the set $\{1, 2, \dots, n\}$ into disjoint subsets. Find its exponential generating function and use it to find a recurrence formula for the Bell numbers.

9. (IMO Shortlist 2014 N6) Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geq n + 2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \dots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

10. (Putnam 2018) Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most
$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$