Geometria 2

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Problema 1. (OBM 2016) Seja ABC um triângulo. As retas r e s são bissetrizes internas de $\angle ABC$ e $\angle BCA$, respectivamente. Os pontos E sobre r e D sobre s são tais que $AD \parallel BE$ e $AE \parallel CD$. As retas BD e CE se cortam em F. Seja I o incentro do triângulo ABC. Mostre que se os pontos A, F e I são colineares então $AB = AC$.

Problema 2. (OBM 2015) Seja ABC um triângulo escaleno e acutângulo e N o centro do círculo que passa pelos pés das três alturas do triângulo. Seja D a interseção das retas tangentes ao circuncírculo de ABC e que passam por B e C. Prove que A, D e N são colineares se, e somente se, $\angle BAC = 45°$.

Problema 3. (OBM 2014) Seja $ABCD$ um quadrilátero convexo e seja P a interseção das diagonais AC e BD. Os raios dos círculos inscritos nos triângulos ABP, BCP, CDP e DAP são iguais. Prove que $ABCD$ é um losango.

Problema 4. (OBM 2010) Seja $ABCD$ um quadrilátero convexo e M e N os pontos médios dos lados CD e AD, respectivamente. As retas perpendiculares a AB passando por M e a BC passando por N cortam-se no ponto P. Prove que P pertence à diagonal BD se, e somente se, as diagonais AC e BD são perpendiculares.

Problema 5. (OBM 2008) Seja $ABCD$ um quadrilátero cíclico e r e s as retas simétricas à reta AB em relação às bissetrizes internas dos ângulos $\angle CAD$ e $\angle CBD$, respectivamente. Sendo P a interseção de r e s e O o centro do círculo circunscrito a $ABCD$, prove que OP é perpendicular a CD.

Problema 6. (Banco IMO 2018) Seja Γ o circuncírculo do triângulo acutângulo ABC. Os pontos D e E estão sobre os segmentos AB e AC, respectivamente, de modo que $AD = AE$. As mediatrizes de BD e CE intersectam os arcos menores AB e AC de Γ nos pontos F e G, respectivamente. Prove que as retas DE e FG são paralelas (ou são a mesma reta).

Problema 7. (Banco IMO 2017) Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

Problema 8. (Banco IMO 2016) Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

Problema 9. (Banco IMO 2015) Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

Problema 10. (Banco IMO 2014) Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC.

Problema 11. (Banco IMO 2013) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problema 12. (Banco IMO 2012) Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

Problema 13. (Banco IMO 2018) Seja ABC um triângulo com $AB = AC$, e seja M o ponto médio de BC. Seja P um ponto tal que $PB < PC$ e PA paralelo a BC. Sejam X e Y pontos nas retas PB e PC, respectivamente, tal que B cai no segmento PX, C cai no segmento PY, e $\angle PXM = \angle PYM$. Prove que o quadrilátero $APXY$ é cíclico.

Problema 14. (Banco IMO 2017) Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problema 15. (Banco IMO 2016) Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Problema 16. (Banco IMO 2015) Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problema 17. (Banco IMO 2014) Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

Problema 18. (Banco IMO 2013) Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

Problema 19. (Banco IMO 2012) Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Problema 20. (OBM 2012) Dado um triângulo ABC, o exincentro relativo ao vértice A é o ponto de interseção das bissetrizes externas de DB e DC. Sejam I_A, I_B e I_C os exincentros do triângulo escaleno ABC relativos a A, B e C, respectivamente, e X, Y e Z os pontos médios de I_BI_C, I_CI_A e I_AI_B, respectivamente. O incírculo do triângulo ABC toca os lados BC, CA e AB nos pontos D, E e F, respectivamente. Prove que as retas DX, EY e FZ têm um ponto em comum pertencente à reta IO, sendo I e O o incentro e o circuncentro do triângulo ABC, respectivamente.

Problema 21. (OBM 2011) Seja ABC um triângulo acutângulo e H seu ortocentro. As retas BH e CH cortam AC e AB em D e E, respectivamente. O circuncírculo de ADE corta o circuncírculo de ABC em $F \neq A$. Provar que as bissetrizes internas de $\angle BFC$ e $\angle BHC$ se cortam em um ponto sobre o segmento BC.

Problema 22. (OBM 2007) Seja $ABCD$ um quadrilátero convexo, P a interseção das retas AB e CD, Q a interseção das retas AD e BC e O a interseção das diagonais AC e BD. Prove que se $\angle POQ$ é um ângulo reto então PO é bissetriz de $\angle AOD$ e QO é bissetriz de $\angle AOB$.

Problema 23. (OBM 2017) No triângulo ABC, seja r_A a reta que passa pelo ponto médio de BC e é perpendicular à bissetriz interna de $\angle BAC$. Defina r_B e r_C da mesma forma. Sejam H e I o ortocentro e o incentro de ABC, respectivamente. Suponha que as três retas r_A, r_B, r_C definem um triângulo. Prove que o circuncentro desse triângulo é o ponto médio de HI.

Problema 24. (OBM 2017) Um quadrilátero $ABCD$ tem um círculo inscrito ω e é tal que as semirretas AB e DC se cortam no ponto P e as semirretas AD e BC se cortam no ponto Q. As retas AC e PQ se cortam no ponto R. Seja T o ponto de ω mais próximo da reta PQ. Prove que a reta RT passa pelo incentro do triângulo PQC.