Answer Form

Contestant Code	BRA 1	Problem Number	4	Page Number	1 /9

Write your solution only on this side of this form.

$\mathbb{N} = \{1, 2, \dots\}$.

A is sum-free $\Leftrightarrow (x, y \in A \Rightarrow x+y \notin A)$

We wish to find all surjective $f : \mathbb{N} \to \mathbb{N}$ such that, if A is sum-free, then $f(A)$ is sum-free.

In other words,

if $f(A)$ is not sum-free, then A cannot be sum-free.

Let x_i be a number such that $f(x_i) = i$.

$\{i, 2i\}$ is not sum-free $\Rightarrow \{f(x_i), f(x_{2i})\}$ is not sum free $\Rightarrow$

$\Rightarrow \{x_i, x_{2i}\}$ is not sum-free $\Rightarrow$

$\Rightarrow$ There exists, $x, y \in \{x_i, x_{2i}\}$ such that $x+y \in \{x_i, x_{2i}\}$.

$\Rightarrow \{x, y, x+y\}$ has less than 3 elements $\Rightarrow$

$x = y$ or $x = x+y$ ⊗ or $y = x+y$ ⊗ $\Rightarrow x = y. \Rightarrow \left.\begin{matrix} x_i = 2x_{2i} \\ \text{or} \\ 2x_i = x_{2i} \end{matrix}\right\}$ (Lemma 1)

In this solution, # means contradiction.

Answer Form

Contestant Code: BRA 1 | Problem Number: 4 | Page Number: 2 / 9

Suppose that

$x_i = 2x_{2i}$, for some i.

Thus, $i \to 2i$ (L. 1), it cannot happen, $2x_{2i} = x_{4i}$, else

$x_i = x_{4i}$

$\Downarrow$

$i = 4i$ Abs!

$\Rightarrow$ it is true that $x_{2i} = 2x_{4i}$

The argument can be repeated such that

$x_i = 2x_{2i} = 4x_{4i} = \cdots = 2^n \cdot x_{ni}$, $\forall n$.

$\Rightarrow$ $x_i = 2^n x_{2^n i} \geq 2^n \Rightarrow x_i \geq 2^n$, $\forall n$. Absurd!

Thus: <u>(Lemma 2)</u>:

$2x_i = x_{2i}$

Answer Form

Contestant Code: BGA 1 | Problem Number: 4 | Page Number: 3/9

Write your solution only on this side of this form.

RASC

Let us prove by induction that:
strong.

$n x_i = x_{ni}$

Bases: $n=1$ and $n=2$ are OK!

We know that $k x_i = x_{ki}$, $\forall k < n$.

$\{i, (n-1)i, ni\}$ is not sum-free $\Rightarrow$

$\Rightarrow \{f(x_i), f(x_{(n-1)i}), f(x_{ni})\}$ is not sum-free $\Rightarrow$

$\Rightarrow \{x_i, x_{(n-1)i}, x_{ni}\}$ is not sum-free $\Rightarrow$

$\Rightarrow \{x_i, (n-1)x_i, x_{ni}\}$ is not sum-free. $\Rightarrow$

$x_i + x_i = x_{ni} \Rightarrow x_{2i} = x_{ni} \Rightarrow 2i = ni \Rightarrow 2=n$ ⊗

or

$x_i + (n-1)x_i = x_{ni} \Rightarrow n x_i = x_{ni}$ (That's what we want)

$(n-i)x_i + (n-1)x_i = x_{ni} \Rightarrow 2(n-1$

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Answer Form

Contestant Code	BRA1	Problem Number	4	Page Number	4/9

Write your solution only on this side of this form.

(RASC)

All the cases are:

$x_i + x_i = x_i \Rightarrow x_i = 0 \Rightarrow$ Nope!

$x_i + x_i = (n-1)x_i \Rightarrow (n-3)x_i = 0 \Rightarrow \underline{n=3}$ maybe! (need more bases for induction!)

not that easy!

From Lemma 2, we have:

$$2^n x_i = x_{2^n i}.$$

In special, we have $4x_i = x_{4i}$

$\Rightarrow \{i, 3i, 4i\}$ is not sum-free, $\Rightarrow$

$\Rightarrow \{f(x_i), f(x_{3i}), f(x_{4i})\}$ is not sum-free $\Rightarrow$

$\Rightarrow \{x_i, x_{3i}, 4x_i\}$ is not sum free.

All the possible sums are: (it never can happen $x+y=z$, when $x=z$ or $y=z$)

~~$x_i + x_i = x_i \Rightarrow 2=1$ #~~

$x_i + x_i = x_{3i} \Rightarrow x_{2i} = x_{3i} \Rightarrow 3i = 2i$ #

$x_i + x_i = 4x_i \Rightarrow 2=4$ #

$x_i + x_{3i} = 4x_i \Rightarrow x_{3i} = 3x_i$ ✓

$x_i + 4x_i = x_{3i} \Rightarrow x_{3i} = 5x_i$ ✓

$x_{3i} + 4x_i = x_i \Rightarrow x_{3i} = -3x_i$ #

$x_{3i} + x_{3i} = x_i \Rightarrow 2x_{3i} = x_i \Rightarrow x_{6i} = x_i \Rightarrow 6i = i$ #.

$x_{3i} + x_{3i} = 4x_i \Rightarrow x_{6i} = x_{4i} \Rightarrow 6i = 4i$ #

$x_{4i} + x_{4i} = x_i \Rightarrow 8i = i$ #

$x_{4i} + x_{4i} = x_{3i} \Rightarrow 8i = 3i$ #

$\Rightarrow$ Then:

$$x_{3i} = 3x_i$$

or

$$x_{3i} = 5x_i$$

Suppose that, for some i,

$$x_{3i} = 5x_i \quad .$$

Thus, $\{2i, 3i, 5i\}$ is not sum-free

$\Rightarrow \{x_{2i}, x_{3i}, x_{5i}\}$ is not sum-free

$\Rightarrow$

$2x_{2i} = x_{3i} \Rightarrow 4i = 3i$ #

$2x_{2i} = x_{5i} \Rightarrow 4i = 5i$ #

$2x_{3i} = x_{5i} \Rightarrow 6i = 5i$ #

$2x_{3i} = x_{2i} \Rightarrow 6i = 2i$ #

$x_{2i} + x_{3i} = x_{5i} \Rightarrow 7x_i = x_{5i}$

$x_{2i} + x_{5i} = x_{3i} \Rightarrow x_{5i} = 3x_i$

$x_{3i} + x_{5i} = x_{2i} \Rightarrow x_{5i} = -3x_i$ #

$\Rightarrow$ $x_{5i} = 7x_i$ (A) or $x_{5i} = 3x_i$ (B)

If (B): $\{2i, 5i, 7i\}$ not sum-free $\Rightarrow \{x_{2i}, x_{5i}, x_{7i}\}$ not sum-free

$\Rightarrow$ $x_{7i} = x_{2i} + x_{5i} = 5x_i = x_{3i}$ or $x_{7i} = x_{5i} - x_{2i} = x_i = x_i$ $\Rightarrow$ $7i = 3i$ or $7i = i$ #

If (A): $\{2i, 5i, 7i\}$ not sum-free $\Rightarrow \{x_{2i}, x_{5i}, x_{7i}\}$ not sum-free

$\Rightarrow$ $x_{7i} = x_{2i} + x_{5i} = 9x_i$ or $x_{7i} = x_{5i} - x_{2i} = 5x_i = x_{3i} \Rightarrow 7i = 3i$ # $\Rightarrow$ $\boxed{x_{7i} = 9i}$

Answer Form

Contestant Code	BRA1	Problem Number	4	Page Number	7/9

Write your solution only on this side of this form.

~~But, $\{i, 6i, 7i\}$ not sum-free $\Rightarrow \{x_i, x_{6i}, x_{7i}\}$ not sum-free~~

$$\cancel{\Rightarrow \quad x_{7i} = x_{6i} + x_i = 2x_{3i} + x_i = 11x_i}$$

~~or~~

$$\cancel{x_{7i} = x_{6i} - x_i = 2x_{3i} - x_i = 9x_i}$$

But, $\{i, 8i, 9i\}$ not sum-free, $\Rightarrow \{x_i, x_{8i}, x_{9i}\}$ not sum-free

$$\Rightarrow \quad x_{9i} = x_{8i} + x_i = 9x_i = x_{7i} \quad \Rightarrow \quad 9i = 7i$$

or

$$x_{9i} = x_{8i} - x_i = 7x_i = x_{5i} \quad\quad 9i = 5i \quad \#.$$

Thus: $x_{3i} \neq 5x_i \Rightarrow \boxed{x_{3i} = 3x_i}$

Maybe induction will work!

Answer Form

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Contestant Code	BRA1	Problem Number	4	Page Number	8/9

Write your solution only on this side of this form.

Repeating the steps from page 3:

Bases: $n = 1, 2, 3$ ok!

If: $n \geq 4$: Suppose $k x_i = x_{ki}$, $\forall k < n$.

$\Rightarrow \{i, (n-1)i, ni\}$ is not sum-free

$\Rightarrow \{x_i, (n-1)x_i, x_{ni}\}$ is not sum-free

$\Rightarrow$ $2x_i = x_{(n-1)i} \Rightarrow 2i = (n-1)i \Rightarrow n = 3$ #

$2x_i = x_{ni} \Rightarrow 2i = ni \Rightarrow n = 2$ #

$2x_{(n-1)i} = x_i \Rightarrow 2(n-1)i = i \Rightarrow n = 3/2$ #

$2x_{(n-1)i} = x_{ni} \Rightarrow 2(n-1)i = ni \Rightarrow n = 1$ #

$2x_{ni} = x_i \Rightarrow 2ni = i \Rightarrow n = 1/2$ #

$2x_{ni} = x_{(n-1)i} \Rightarrow 2ni = (n-1)i \Rightarrow n = -1$ #

$x_i + (n-1)x_i = x_{ni} \Rightarrow x_{ni} = nx_i$ ✓ (That's what we want!)

$x_i + x_{ni} = (n-1)x_i \Rightarrow x_{ni} = (n-2)x_i = x_{(n-2)i} \Rightarrow ni = (n-2)i$ #.

$(n-1)x_i + x_{ni} = x_i \Rightarrow x_{ni} = (2-n)x_i < 0$ #.

Thus, $x_{ni} = nx_i$.

By induction, we proved that $x_{ni} = nx_i$, $\forall n, i$.

In especial, $i=1$: $c := x_1 \Rightarrow f(c)=1$.

$$x_n = n x_1 = n \cdot c.$$

$$\Rightarrow f(x_n) = f(n \cdot c)$$

$$\Rightarrow n = f(n \cdot c), \ \forall n.$$

Let us prove that f is injective.

Suppose $f(x_i) = f(y_i) = i$ and $f(x_{2i}) = 2i$

$\{i, 2i\}$ is not sum-free $\Rightarrow$ $\{x_i, x_{2i}\}$ and $\{y_i, x_{2i}\}$ are not sum-free.

By Lemma 1: $y_i = x_i$ or $y_i = 4x_i \Rightarrow y_i = x_{4i} \Rightarrow f(y_i) = 4i$ #

$\Rightarrow y_i = x_i \Rightarrow f$ is injective.

$\Rightarrow f(1) = t = f(t \cdot c) \Rightarrow 1 = t \cdot c \Rightarrow t = c = 1.$

$\Rightarrow f(n) = n, \ \forall n$ is the only solution, (which works because $f(A) = A$)