Olimpíada Búlgara de Matemática

Todo e qualquer feedback, especialmente sobre erros neste livreto (mesmo erros tipográficos pequenos), é apreciado. Você também pode contribuir enviando suas soluções (de preferência, formatadas em TEX).
Você pode enviar comentários e soluções para zeusdanmou+tex@gmail.com.
A versão mais atualizada desse arquivo (provavelmente) está disponível clicando aqui. Última atualização: 23 de dezembro de 2021.

Primeiro Dia

PROBLEMA 1
A city has 4 horizontal and $n \geq 3$ vertical boulevards which intersect at $4n$ crossroads. The crossroads divide every horizontal boulevard into $n-1$ streets and every vertical boulevard into 3 streets. The mayor of the city decides to close the minimum possible number of crossroads so that the city doesn't have a closed path, i.e., this means that starting from any street and going only through open crossroads without turning back you can't return to the same street.

(a) Prove that exactly n crossroads are closed.

(b) Prove that if from any street you can go to any other street and none of the 4 corner crossroads are closed then exactly 3 crossroads on the border are closed. (A crossroad is on the border if it lies either on the first or fourth horizontal boulevard, or on the first or the n-th vertical boulevard.)

PROBLEMA 2
A point T is given on the altitude through point C in the acute triangle ABC with circumcenter O, such that $\measuredangle TBA = \measuredangle ACB$. If the line CO intersects side AB at point K, prove that the perpendicular bisector of AB, the altitude through A and the segment KT are concurrent.

PROBLEMA 3
Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f(f(x)+y)f(x) = f(xy+1)$$

for all $x, y \in \mathbb{R}_{>0}$.

Segundo Dia

PROBLEMA 4
Let $a_1 < a_2 < a_3 < \cdots$ and $b_1 < b_2 < b_3 < \cdots$ be two infinite arithmetic sequences with positive integers. It is known that there are infinitely many pairs of positive integers (i, j) for which $i \leq j \leq i + 2021$ and a_i divides b_j. Prove that for every positive integer i there exists a positive integer j such that a_i divides b_j.

PROBLEMA 5
Does there exist a set S of 100 points in a plane such that the center of mass of any 10 points in S is also a point in S?

PROBLEMA 6
Point S is the midpoint of arc ACB of the circumscribed circle k around triangle ABC with $AC > BC$. Let I be the incenter of triangle ABC. Line SI intersects k again at point T. Let D be the reflection of I across T and M be the midpoint of side AB. Line IM intersects the line through D, parallel to AB, at point E. Prove that $AE = BD$.

Primeiro Dia

PROBLEMA 1

On the sides of $\triangle ABC$ points $P, Q \in AB$ (P is between A and Q) and $R \in BC$ are chosen. The points M and N are defined as the intersection point of AR with the segments CP and CQ, respectively. If $BC = BQ$, $CP = AP$, $CR = CN$ and $\angle BPC = \angle CRA$, prove that $MP + NQ = BR$.

PROBLEMA 2

Let $b_1, \ldots, b_n$ be nonnegative integers with sum 2 and $a_0, a_1, \ldots, a_n$ be real numbers such that $a_0 = a_n = 0$ and $|a_i - a_{i-1}| \leq b_i$ for each $i = 1, \ldots, n$. Prove that

$$\sum_{i=1}^{n}(a_i + a_{i-1})b_i \leq 2$$

PROBLEMA 3

Let $a_1 \in \mathbb{Z}$, $a_2 = a_1^2 - a_1 - 1, \ldots, a_{n+1} = a_n^2 - a_n - 1$. Prove that a_{n+1} and $2n+1$ are coprime.

Segundo Dia

PROBLEMA 4

Are there positive integers $m > 4$ and n, such that a) $\binom{m}{3} = n^2$ b) $\binom{m}{4} = n^2 + 9$

PROBLEMA 5

There are n points in the plane, some of which are connected by segments. Some of the segments are colored in white, while the others are colored black in such a way that there exist a completely white as well as a completely black closed broken line of segments, each of them passing through every one of the n points exactly once. It is known that the segments AB and BC are white. Prove that it is possible to recolor the segments in red and blue in such a way that AB and BC are recolored as red, , and that there exist a completely red as well as a completely blue closed broken line of segments, each of them passing through every one of the n points exactly once.

PROBLEMA 6

Let $f(x)$ be a nonconstant real polynomial. The sequence $\{a_i\}_{i=1}^{\infty}$ of real numbers is strictly increasing and unbounded, as

$$a_{i+1} < a_i + 2020.$$

The integers $\lfloor |f(a_1)| \rfloor$, $\lfloor |f(a_2)| \rfloor$, $\lfloor |f(a_3)| \rfloor$, ... are written consecutively in such a way that their digits form an infinite sequence of digits $\{s_k\}_{k=1}^{\infty}$ (here $s_k \in \{0, 1, \ldots, 9\}$). If $n \in \mathbb{N}$, prove that among the numbers $\overline{s_{n(k-1)+1}s_{n(k-1)+2}\cdots s_{nk}}$, where $k \in \mathbb{N}$, all n-digit numbers appear.

Primeiro Dia

PROBLEMA 1
Let $f(x) = x^2 + bx + 1$, where b is a real number. Find the number of integer solutions to the inequality $f(f(x)+x) < 0$.

PROBLEMA 2
Let ABC be an acute triangle with orthocenter H and circumcenter O. Let the intersection points of the perpendicular bisector of CH with AC and BC be X and Y respectively. Lines XO and YO cut AB at P and Q respectively. If $XP + YQ = AB + XY$, determine $\measuredangle OHC$.

PROBLEMA 3
Find all real numbers a, which satisfy the following condition:

For every sequence $a_1, a_2, a_3, \ldots$ of pairwise different positive integers, for which the inequality $a_n \leq an$ holds for every positive integer n, there exist infinitely many numbers in the sequence with sum of their digits in base 4038, which is not divisible by 2019.

Segundo Dia

PROBLEMA 4
Determine all positive integers d, such that there exists an integer $k \geq 3$, such that One can arrange the numbers $d, 2d, \ldots, kd$ in a row, such that the sum of every two consecutive of them is a perfect square.

PROBLEMA 5
Let P be a 2019$-$gon, such that no three of its diagonals concur at an internal point. We will call each internal intersection point of diagonals of P a knot. What is the greatest number of knots one can choose, such that there doesn't exist a cycle of chosen knots? (Every two adjacent knots in a cycle must be on the same diagonal and on every diagonal there are at most two knots from a cycle.)

PROBLEMA 6
Let $ABCDEF$ be an inscribed hexagon with

$$AB.CD.EF = BC.DE.FA$$

Let B_1 be the reflection point of B with respect to AC and D_1 be the reflection point of D with respect to CE, and finally let F_1 be the reflection point of F with respect to AE. Prove that $\triangle B_1D_1F_1 \sim BDF$.

Primeiro Dia

PROBLEMA 1

Let n be an odd positive integer.Let M be a set of n positive integers ,which are 2x2 different. A set $T \in M$ is called "good"if the product of its elements is divisible by the sum of the elements in M, but is not divisible by the square of the same sum. Given that M is "good",how many "good"subsets of M can there be?

PROBLEMA 2

Let $ABCD$ be a cyclic quadrilateral. Let H_1 be the orthocentre of triangle ABC. Point A_1 is the image of A after reflection about BH_1. Point B_1 is the image of of B after reflection about AH_1. Let O_1 be the circumcentre of $(A_1B_1H_1)$. Let H_2 be the orthocentre of triangle ABD. Point A_2 is the image of A after reflection about BH_2. Point B_2 is the image of of B after reflection about AH_2. Let O_2 be the circumcentre of $(A_2B_2H_2)$. Lets denote by ℓ_{AB} be the line through O_1 and O_2. ℓ_{AD} ,ℓ_{BC} ,ℓ_{CD} are defined analogously. Let $M = \ell_{AB} \cap \ell_{BC}$, $N = \ell_{BC} \cap \ell_{CD}$, $P = \ell_{CD} \cap \ell_{AD}$,$Q = \ell_{AD} \cap \ell_{AB}$. Prove that $MNPQ$ is cyclic.

PROBLEMA 3

Prove that

$$\left(\frac{6}{5}\right)^{\sqrt{3}} > \left(\frac{5}{4}\right)^{\sqrt{2}}.$$

Segundo Dia

PROBLEMA 4

Let $ABCD$ be a quadrilateral ,circumscribed about a circle. Let M be a point on the side AB. Let I_1,I_2 and I_3 be the incentres of triangles AMD, CMD and BMC respectively. Prove that $I_1I_2I_3M$ is circumscribed.

PROBLEMA 5

Given a polynomial $P(x) = a_dx^d + \ldots + a_2x^2 + a_0$ with positive integers for coefficients and degree $d \geq 2$. Consider the sequence defined by

$$b_1 = a_0, b_{n+1} = P(b_n)$$

for $n \geq 1$. Prove that for all $n \geq 2$ there exists a prime p such that p divides b_n but does not divide $b_1b_2 \ldots b_{n-1}$.

PROBLEMA 6

On a planet there are M countries and N cities. There are two-way roads between some of the cities. It is given that:

(1) In each county there are at least three cities;

(2) For each country and each city in the country is connected by roads with at least half of the other cities in the countries;

(3) Each city is connceted with exactly one other city ,that is not in its country;

(4) There are at most two roads between cities from cities in two different countries;

(5) If two countries contain less than $2M$ cities in total then there is a road between them.

Prove that there is cycle of lenght at least $M + \frac{N}{2}$.

Primeiro Dia

PROBLEMA 1

An convex qudrilateral $ABCD$ is given. O is the intersection point of the diagonals AC and BD. The points A_1, B_1, C_1, D_1 lie respectively on AO, BO, CO, DO such that $AA_1 = CC_1, BB_1 = DD_1$. The circumcircles of $\triangle AOB$ and $\triangle COD$ meet at second time at M and the the circumcircles of $\triangle AOD$ and $\triangle BOC$ - at N. The circumcircles of $\triangle A_1OB_1$ and $\triangle C_1OD_1$ meet at second time at P and the the circumcircles of $\triangle A_1OD_1$ and $\triangle B_1OC_1$ - at Q. Prove that the quadrilateral $MNPQ$ is cyclic.

PROBLEMA 2

Let $m > 1$ be a natural number and $N = m^{2017} + 1$. On a blackboard, left to right, are written the following numbers:

$$N, N - m, N - 2m, \ldots, 2m + 1, m + 1, 1.$$

On each move, we erase the most left number, written on the board, and all its divisors (if any). This procces continues till all numbers are deleted. Which numbers will be deleted on the last move.

PROBLEMA 3

Let M be a set of 2017 positive integers. For any subset A of M, let $f(A)$ be the set of all $x \in M$ such that $|\{y \in A : x \text{ is a multiple of } y\}|$ is odd. Find the minimal natural number k, satisfying that, for any M, we can color all the subsets of M with k colors, such that whenever $A \neq f(A)$, A and $f(A)$ are colored with different colors.

Segundo Dia

PROBLEMA 4

Find all triples (p, a, m) such that p is a prime number, a and m are nonnegative integers, $a \leq 5p^2$ and $(p-1)!+a = p^m$.

PROBLEMA 5

Let n be a natural number and $f(x)$ be a polynomial with real coefficients having n different positive real roots. Is it possible the polynomial:

$$x(x+1)(x+2)(x+4)f(x) + a$$

to be presented as the k-th power of a polynomial with real coefficients, for some natural $k \geq 2$ and real a?

PROBLEMA 6

An acute non-isosceles $\triangle ABC$ is given. CD, AE, BF are its altitudes. The points E', F' are symetrical of E, F with respect accordingly to A and B. The point C_1 lies on $\overrightarrow{CD}$, such that $DC_1 = 3CD$. Prove that $\angle E'C_1F' = \angle ACB$

Primeiro Dia

PROBLEMA 1
Find all positive integers m and n such that $(2^{2^n}+1)(2^{2^m}+1)$ is divisible by $m \cdot n$.

PROBLEMA 2
At a mathematical competition n students work on 6 problems each one with three possible answers. After the competition, the Jury found that for every two students the number of the problems, for which these students have the same answers, is 0 or 2. Find the maximum possible value of n.

PROBLEMA 3
For $a, b, c, d > 0$ prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{4} \leq \sqrt[4]{a.\frac{a+b}{2}.\frac{a+b+c}{3}.\frac{a+b+c+d}{4}}$$

Segundo Dia

PROBLEMA 4
Determine whether there exist a positive integer $n < 10^9$, such that n can be expressed as a sum of three squares of positive integers by more than 1000 distinct ways?

PROBLEMA 5
Let $\triangle ABC$ be isosceles triangle with $AC = BC$. The point D lies on the extension of AC beyond C and is that $AC > CD$. The angular bisector of $\angle BCD$ intersects BD at point N and let M be the midpoint of BD. The tangent at M to the circumcircle of triangle AMD intersects the side BC at point P. Prove that points A, P, M and N lie on a circle.

PROBLEMA 6
Let n be positive integer.A square A of side length n is divided by n^2 unit squares. All unit squares are painted in n distinct colors such that each color appears exactly n times. Prove that there exists a positive integer N , such that for any $n > N$ the following is true: There exists a square B of side length $\sqrt{n}$ and side parallel to the sides of A such that B contains completely cells of 4 distinct colors.

Primeiro Dia

PROBLEMA 1
The hexagon $ABLCDK$ is inscribed and the line LK intersects the segments AD, BC, AC and BD in points M, N, P and Q, respectively. Prove that $NL \cdot KP \cdot MQ = KM \cdot PN \cdot LQ$.

PROBLEMA 2
One hundred and one of the squares of an $n \times n$ table are colored blue. It is known that there exists a unique way to cut the table to rectangles along boundaries of its squares with the following property: every rectangle contains exactly one blue square. Find the smallest possible n.

PROBLEMA 3
The sequence $a_1, a_2, ...$ is defined by the equalities $a_1 = 2, a_2 = 12$ and $a_{n+1} = 6a_n - a_{n-1}$ for every positive integer $n \geq 2$. Prove that no member of this sequence is equal to a perfect power (greater than one) of a positive integer.

Segundo Dia

PROBLEMA 4
Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in R^+$ the followings hold: $i)$ $f(x+y) \geq f(x) + y$ $ii)$ $f(f(x)) \leq x$

PROBLEMA 5
In a triangle $\triangle ABC$ points L, P and Q lie on the segments AB, AC and BC, respectively, and are such that $PCQL$ is a parallelogram. The circle with center the midpoint M of the segment AB and radius CM and the circle of diameter CL intersect for the second time at the point T. Prove that the lines AQ, BP and LT intersect in a point.

PROBLEMA 6
In a mathematical olympiad students received marks for any of the four areas: algebra, geometry, number theory and combinatorics. Any two of the students have distinct marks for all four areas. A group of students is called *nice* if all students in the group can be ordered in increasing order simultaneously of at least two of the four areas. Find the least positive integer N, such that among any N students there exist a *nice* group of ten students.

Primeiro Dia

PROBLEMA 1
Let k be a given circle and A is a fixed point outside k. BC is a diameter of k. Find the locus of the orthocentre of $\triangle ABC$ when BC varies.

PROBLEMA 2
Every cell of a $m \times n$ chess board, $m \geq 2, n \geq 2$, is colored with one of four possible colors, e.g white, green, red, blue. We call such coloring good if the four cells of any 2×2 square of the chessboard are colored with pairwise different colors. Determine the number of all good colorings of the chess board.

PROBLEMA 3
A real number $f(X) \neq 0$ is assigned to each point X in the space. It is known that for any tetrahedron $ABCD$ with O the center of the inscribed sphere, we have :

$$f(O) = f(A)f(B)f(C)f(D).$$

Prove that $f(X) = 1$ for all points X.

Segundo Dia

PROBLEMA 4
Find all pairs of prime numbers $p\,,q$ for which:

$$p^2 \mid q^3 + 1 \text{ and } q^2 \mid p^6 - 1$$

PROBLEMA 5
Find all functions $f : \mathbb{Q}^+ \to \mathbb{R}^+$ with the property:

$$f(xy) = f(x+y)(f(x) + f(y))\,, \forall x, y \in \mathbb{Q}^+$$

PROBLEMA 6
Let $ABCD$ be a quadrilateral inscribed in a circle k. AC and BD meet at E. The rays $\overrightarrow{CB}, \overrightarrow{DA}$ meet at F. Prove that the line through the incenters of $\triangle ABE\,, \triangle ABF$ and the line through the incenters of $\triangle CDE\,, \triangle CDF$ meet at a point lying on the circle k.

Primeiro Dia

PROBLEMA 1
Find all prime numbers p, q, for which $p^{q+1} + q^{p+1}$ is a perfect square.

PROBLEMA 2
Find all $f : \mathbb{R} \to \mathbb{R}$, bounded in $(0, 1)$ and satisfying:
$x^2 f(x) - y^2 f(y) = (x^2 - y^2) f(x + y) - xyf(x - y)$
for all $x, y \in \mathbb{R}$

PROBLEMA 3
The integer lattice in the plane is colored with 3 colors. Find the least positive real S with the property: for any such coloring it is possible to find a monochromatic lattice points A, B, C with $S_{\triangle ABC} = S$.
EDIT: It was the problem 3 (not 2), corrected the source title.

Segundo Dia

PROBLEMA 4
Suppose $\alpha, \beta, \gamma \in [0.\pi/2)$ and $\tan\alpha + \tan\beta + \tan\gamma \leq 3$. Prove that:

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma \geq 0$$

PROBLEMA 5
Consider acute $\triangle ABC$ with altitudes AA_1, BB_1 and CC_1 ($A_1 \in BC, B_1 \in AC, C_1 \in AB$). A point C' on the extension of B_1A_1 beyond A_1 is such that $A_1C' = B_1C_1$. Analogously, a point B' on the extension of A_1C_1 beyond C_1 is such that $C_1B' = A_1B_1$ and a point A' on the extension of C_1B_1 beyond B_1 is such that $B_1A' = C_1A_1$. Denote by A'', B'', C'' the symmetric points of A', B', C' with respect to BC, CA and AB respectively. Prove that if R, R' and R" are circumradiii of $\triangle ABC, \triangle A'B'C'$ and $\triangle A''B''C''$, then R, R' and R'' are sidelengths of a triangle with area equals one half of the area of $\triangle ABC$.

PROBLEMA 6
Given $m \in \mathbb{N}$ and a prime number p, $p > m$, let

$$M = \{n \in \mathbb{N} \mid m^2 + n^2 + p^2 - 2mn - 2mp - 2np \text{ is a perfect square}\}$$

Prove that $|M|$ does not depend on p.

Primeiro Dia

PROBLEMA 1
The sequence $a_1, a_2, a_3 \ldots$, consisting of natural numbers, is defined by the rule:

$$a_{n+1} = a_n + 2t(n)$$

for every natural number n, where $t(n)$ is the number of the different divisors of n (including 1 and n). Is it possible that two consecutive members of the sequence are squares of natural numbers?

PROBLEMA 2
Prove that the natural numbers can be divided into two groups in a way that both conditions are fulfilled: 1) For every prime number p and every natural number n, the numbers p^n, p^{n+1} and p^{n+2} do not have the same colour. 2) There does not exist an infinite geometric sequence of natural numbers of the same colour.

PROBLEMA 3
We are given a real number a, not equal to 0 or 1. Sacho and Deni play the following game. First is Sasho and then Deni and so on (they take turns). On each turn, a player changes one of the "*" symbols in the equation:

$$*x^4 + *x^3 + *x^2 + *x^1 + * = 0$$

with a number of the type a^n, where n is a whole number. Sasho wins if at the end the equation has no real roots, Deni wins otherwise. Determine (in term of a) who has a winning strategy

---

Segundo Dia

PROBLEMA 4
Let n be an even natural number and let A be the set of all non-zero sequences of length n, consisting of numbers 0 and 1 (length n binary sequences, except the zero sequence $(0, 0, \ldots, 0)$). Prove that A can be partitioned into groups of three elements, so that for every triad $\{(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n)\}$, and for every $i = 1, 2, \ldots, n$, exactly zero or two of the numbers a_i, b_i, c_i are equal to 1.

PROBLEMA 5
Let $Q(x)$ be a quadratic trinomial. Given that the function $P(x) = x^2Q(x)$ is increasing in the interval $(0, \infty)$, prove that:

$$P(x) + P(y) + P(z) > 0$$

for all real numbers x, y, z such that $x + y + z > 0$ and $xyz > 0$.

PROBLEMA 6
We are given an acute-angled triangle ABC and a random point X in its interior, different from the centre of the circumcircle k of the triangle. The lines AX, BX and CX intersect k for a second time in the points A_1, B_1 and C_1 respectively. Let A_2, B_2 and C_2 be the points that are symmetric of A_1, B_1 and C_1 in respect to BC, AC and AB respectively. Prove that the circumcircle of the triangle A_2, B_2 and C_2 passes through a constant point that does not depend on the choice of X.

Primeiro Dia

PROBLEMA 1

Prove whether or not there exist natural numbers n, k where $1 \leq k \leq n-2$ such that

$$\binom{n}{k}^2 + \binom{n}{k+1}^2 = \binom{n}{k+2}^4$$

PROBLEMA 2

Let $f_1(x)$ be a polynomial of degree 2 with the leading coefficient positive and $f_{n+1}(x) = f_1(f_n(x))$ for $n \geq 1$. Prove that if the equation $f_2(x) = 0$ has four different non-positive real roots, then for arbitrary n then $f_n(x)$ has 2^n different real roots.

PROBLEMA 3

Triangle ABC and a function $f : \mathbb{R}^+ \to \mathbb{R}$ have the following property: for every line segment DE from the interior of the triangle with midpoint M, the inequality $f(d(D)) + f(d(E)) \leq 2f(d(M))$, where $d(X)$ is the distance from point X to the nearest side of the triangle (X is in the interior of $\triangle ABC$). Prove that for each line segment PQ and each point interior point N the inequality $|QN|f(d(P)) + |PN|f(d(Q)) \leq |PQ|f(d(N))$ holds.

Segundo Dia

PROBLEMA 4

Point O is inside $\triangle ABC$. The feet of perpendicular from O to BC, CA, AB are D, E, F. Perpendiculars from A and B respectively to EF and FD meet at P. Let H be the foot of perpendicular from P to AB. Prove that D, E, F, H are concyclic.

PROBLEMA 5

For each natural number a we denote $\tau(a)$ and $\phi(a)$ the number of natural numbers dividing a and the number of natural numbers less than a that are relatively prime to a. Find all natural numbers n for which n has exactly two different prime divisors and n satisfies $\tau(\phi(n)) = \phi(\tau(n))$.

PROBLEMA 6

In the interior of the convex 2011-gon are 2011 points, such that no three among the given 4022 points (the interior points and the vertices) are collinear. The points are coloured one of two different colours and a colouring is called "good"if some of the points can be joined in such a way that the following conditions are satisfied: 1) Each segment joins two points of the same colour. 2) None of the line segments intersect. 3) For any two points of the same colour there exists a path of segments connecting them. Find the number of "good"colourings.

Primeiro Dia

PROBLEMA 1
A table 2×2010 is divided to unit cells. Ivan and Peter are playing the following game. Ivan starts, and puts horizontal 2×1 domino that covers exactly two unit table cells. Then Peter puts vertical 1×2 domino that covers exactly two unit table cells. Then Ivan puts horizontal domino. Then Peter puts vertical domino, etc. The person who cannot put his domino will lose the game. Find who have winning strategy.

PROBLEMA 2
Each of two different lines parallel to the the axis Ox have exactly two common points on the graph of the function $f(x) = x^3 + ax^2 + bx + c$. Let ℓ_1 and ℓ_2 be two lines parallel to Ox axis which meet the graph of f in points K_1, K_2 and K_3, K_4, respectively. Prove that the quadrilateral formed by K_1, K_2, K_3 and K_4 is a rhombus if and only if its area is equal to 6 units.

PROBLEMA 3
Let $a_0, a_1, \ldots, a_9$ and $b_1, b_2, \ldots, b_9$ be positive integers such that $a_9 < b_9$ and $a_k \neq b_k, 1 \leq k \leq 8$. In a cash dispenser/automated teller machine/ATM there are $n \geq a_9$ levs (Bulgarian national currency) and for each $1 \leq i \leq 9$ we can take a_i levs from the ATM (if in the bank there are at least a_i levs). Immediately after that action the bank puts b_i levs in the ATM or we take a_0 levs. If we take a_0 levs from the ATM the bank doesn't put any money in the ATM. Find all possible positive integer values of n such that after finite number of takings money from the ATM there will be no money in it.

Segundo Dia

PROBLEMA 4
Does there exist a number $n = \overline{a_1a_2a_3a_4a_5a_6}$ such that $\overline{a_1a_2a_3} + 4 = \overline{a_4a_5a_6}$ (all bases are 10) and $n = a^k$ for some positive integers a, k with $k \geq 3$?

PROBLEMA 5
Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(1) = 1$ and

$$f(n) = n - f(f(n-1)), \quad \forall n \geq 2.$$

Prove that $f(n + f(n)) = n$ for each positive integer n.

PROBLEMA 6
Let k be the circumference of the triangle ABC. The point D is an arbitrary point on the segment AB. Let I and J be the centers of the circles which are tangent to the side AB, the segment CD and the circle k. We know that the points A, B, I and J are concyclic. The excircle of the triangle ABC is tangent to the side AB in the point M. Prove that $M \equiv D$.

Primeiro Dia

PROBLEMA 1

The natural numbers a and b satisfy the inequalities $a > b > 1$. It is also known that the equation $\frac{a^x-1}{a-1} = \frac{b^y-1}{b-1}$ has at least two solutions in natural numbers, when $x > 1$ and $y > 1$. Prove that the numbers a and b are coprime (their greatest common divisor is 1).

PROBLEMA 2

In the triangle ABC its incircle with center I touches its sides BC, CA and AB in the points A_1, B_1, C_1 respectively. Through I is drawn a line ℓ. The points A', B' and C' are reflections of A_1, B_1, C_1 with respect to the line ℓ. Prove that the lines AA', BB' and CC' intersects at a common point.

PROBLEMA 3

Through the points with integer coordinates in the right-angled coordinate system $Oxyz$ are constructed planes, parallel to the coordinate planes and in this way the space is divided to unit cubes. Find all triples (a, b, c) consisting of natural numbers $(a \le b \le c)$ for which the cubes formed can be coloured in abc colors in such a way that every palellepiped with dimensions $a \times b \times c$, having vertices with integer coordinates and sides parallel to the coordinate axis doesn't contain unit cubes in the same color.

Segundo Dia

PROBLEMA 4

Let $n \ge 3$ be a natural number. Find all nonconstant polynomials with real coeficcietns $f_1(x), f_2(x), \dots, f_n(x)$, for which

$$f_k(x) f_{k+1}(x) = f_{k+1}(f_{k+2}(x)), \quad 1 \le k \le n,$$

for every real x (with $f_{n+1}(x) \equiv f_1(x)$ and $f_{n+2}(x) \equiv f_2(x)$).

PROBLEMA 5

We divide a convex 2009-gon in triangles using non-intersecting diagonals. One of these diagonals is colored green. It is allowed the following operation: for two triangles ABC and BCD from the dividing/separating with a common side BC if the replaced diagonal was green it loses its color and the replacing diagonal becomes green colored. Prove that if we choose any diagonal in advance it can be colored in green after applying the operation described finite number of times.

PROBLEMA 6

Prove that if $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ are arbitrary taken real numbers and $c_1, c_2, \dots, c_n$ are positive real numbers, than $\left(\sum_{i,j=1}^n \frac{a_i a_j}{c_i + c_j}\right)\left(\sum_{i,j=1}^n \frac{b_i b_j}{c_i + c_j}\right) \ge \left(\sum_{i,j=1}^n \frac{a_i b_j}{c_i + c_j}\right)^2$.

Primeiro Dia

PROBLEMA 1
Let ABC be an acute triangle and CL be the angle bisector of $\angle ACB$. The point P lies on the segment CL such that $\angle APB = \pi - \frac{1}{2}\angle ACB$. Let k_1 and k_2 be the circumcircles of the triangles APC and BPC. $BP \cap k_1 = Q, AP \cap k_2 = R$. The tangents to k_1 at Q and k_2 at B intersect at S and the tangents to k_1 at A and k_2 at R intersect at T. Prove that $AS = BT$.

PROBLEMA 2
Is it possible to find 2008 infinite arithmetical progressions such that there exist finitely many positive integers not in any of these progressions, no two progressions intersect and each progression contains a prime number bigger than 2008?

PROBLEMA 3
Let $n \in \mathbb{N}$ and $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq \pi$ and $b_1, b_2, \ldots, b_n$ are real numbers for which the following inequality is satisfied :

$$\left|\sum_{i=1}^{n} b_i \cos(ka_i)\right| < \frac{1}{k}$$

for all $k \in \mathbb{N}$. Prove that $b_1 = b_2 = \ldots = b_n = 0$.

Segundo Dia

PROBLEMA 4
Find the smallest natural number k for which there exists natural numbers m and n such that $1324 + 279m + 5^n$ is k-th power of some natural number.

PROBLEMA 5
Let n be a fixed natural number. Find all natural numbers m for which

$$\frac{1}{a^n} + \frac{1}{b^n} \geq a^m + b^m$$

is satisfied for every two positive numbers a and b with sum equal to 2.

PROBLEMA 6
Let M be the set of the integer numbers from the range $[-n, n]$. The subset P of M is called a *base subset* if every number from M can be expressed as a sum of some different numbers from P. Find the smallest natural number k such that every k numbers that belongs to M form a base subset.

Primeiro Dia

PROBLEMA 1
The quadrilateral $ABCD$, where $\angle BAD + \angle ADC > \pi$, is inscribed a circle with centre I. A line through I intersects AB and CD in points X and Y respectively such that $IX = IY$. Prove that $AX \cdot DY = BX \cdot CY$.

PROBLEMA 2
Find the greatest positive integer n such that we can choose 2007 different positive integers from $[2 \cdot 10^{n-1}, 10^n)$ such that for each two $1 \le i < j \le n$ there exists a positive integer $\overline{a_1 a_2 \dots a_n}$ from the chosen integers for which $a_j \ge a_i + 2$.

A. Ivanov, E. Kolev

PROBLEMA 3
Find the least positive integer n such that $\cos \frac{\pi}{n}$ cannot be written in the form $p + \sqrt{q} + \sqrt[3]{r}$ with $p, q, r \in \mathbb{Q}$.

O. Mushkarov, N. Nikolov

Click to reveal hidden text

Segundo Dia

PROBLEMA 4
Let $k > 1$ be a given positive integer. A set S of positive integers is called *good* if we can colour the set of positive integers in k colours such that each integer of S cannot be represented as sum of two positive integers of the same colour. Find the greatest t such that the set $S = \{a+1, a+2, \dots, a+t\}$ is *good* for all positive integers a.

A. Ivanov, E. Kolev

PROBLEMA 5
Find the least real number m such that with all 5 equilaterial triangles with sum of areas m we can cover an equilaterial triangle with side 1.

O. Mushkarov, N. Nikolov

PROBLEMA 6
Let $P(x) \in \mathbb{Z}[x]$ be a monic polynomial with even degree. Prove that, if for infinitely many integers x, the number $P(x)$ is a square of a positive integer, then there exists a polynomial $Q(x) \in \mathbb{Z}[x]$ such that $P(x) = Q(x)^2$.

Primeiro Dia

PROBLEMA 1
Consider the set $A = \{1, 2, 3 \ldots, 2^n\}, n \geq 2$. Find the number of subsets B of A such that for any two elements of A whose sum is a power of 2 exactly one of them is in B.
Aleksandar Ivanov

PROBLEMA 2
Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function that satisfies for all $x > y > 0$

$$f(x + y) - f(x - y) = 4\sqrt{f(x)f(y)}$$

a) Prove that $f(2x) = 4f(x)$ for all $x > 0$; b) Find all such functions.
Nikolai Nikolov, Oleg Mushkarov

PROBLEMA 3
The natural numbers are written in sequence, in increasing order, and by this we get an infinite sequence of digits. Find the least natural k, for which among the first k digits of this sequence, any two nonzero digits have been written a different number of times.
Aleksandar Ivanov, Emil Kolev

Segundo Dia

PROBLEMA 4
Let p be a prime such that p^2 divides $2^{p-1} - 1$. Prove that for all positive integers n the number $(p - 1)(p! + 2^n)$ has at least 3 different prime divisors.
Aleksandar Ivanov

PROBLEMA 5
The triangle ABC is such that $\angle BAC = 30°, \angle ABC = 45°$. Prove that if X lies on the ray AC, Y lies on the ray BC and $OX = BY$, where O is the circumcentre of triangle ABC, then S_{XY} passes through a fixed point.
Emil Kolev

PROBLEMA 6
Consider a point O in the plane. Find all sets S of at least two points in the plane such that if $A \in S$ ad $A \neq O$, then the circle with diameter OA is in S.
Nikolai Nikolov, Slavomir Dinev

Primeiro Dia

PROBLEMA 1
Determine all triples (x, y, z) of positive integers for which the number

$$\sqrt{\frac{2005}{x+y}}+\sqrt{\frac{2005}{y+z}}+\sqrt{\frac{2005}{z+x}}$$

is an integer .

PROBLEMA 2
Consider two circles k_1, k_2 touching externally at point T. a line touches k_2 at point X and intersects k_1 at points A and B. Let S be the second intersection point of k_1 with the line XT . On the arc $\widehat{TS}$ not containing A and B is chosen a point C . Let CY be the tangent line to k_2 with $Y \in k_2$, such that the segment CY does not intersect the segment ST . If $I = XY \cap SC$. Prove that :

(a) the points C, T, Y, I are concyclic.

(b) I is the excenter of triangle ABC with respect to the side BC.

PROBLEMA 3
Let $M = (0, 1) \cap \mathbb{Q}$. Determine, with proof, whether there exists a subset $A \subset M$ with the property that every number in M can be uniquely written as the sum of finitely many distinct elements of A.

Segundo Dia

PROBLEMA 4
Let ABC be a triangle with $AC \neq BC$, and let $A'B'C$ be a triangle obtained from ABC after some rotation centered at C. Let M, E, F be the midpoints of the segments BA', AC and CB' respectively. If $EM = FM$, find $\widehat{EMF}$.

PROBLEMA 5
For positive integers t, a, b,a (t, a, b)-*game* is a two player game defined by the following rules. Initially, the number t is written on a blackboard. At his first move, the 1st player replaces t with either $t - a$ or $t - b$. Then, the 2nd player subtracts either a or b from this number, and writes the result on the blackboard, erasing the old number. After this, the first player once again erases either a or b from the number written on the blackboard, and so on. The player who first reaches a negative number loses the game. Prove that there exist infinitely many values of t for which the first player has a winning strategy for all pairs (a, b) with $a + b = 2005$.

PROBLEMA 6
Let a, b and c be positive integers such that ab divides $c(c^2 - c + 1)$ and $a + b$ is divisible by $c^2 + 1$. Prove that the sets $\{a, b\}$ and $\{c, c^2 - c + 1\}$ coincide.

Primeiro Dia

PROBLEMA 1
Let I be the incenter of triangle ABC, and let A_1, B_1, C_1 be arbitrary points on the segments (AI), (BI), (CI), respectively. The perpendicular bisectors of AA_1, BB_1, CC_1 intersect each other at A_2, B_2, and C_2. Prove that the circumcenter of the triangle $A_2B_2C_2$ coincides with the circumcenter of the triangle ABC if and only if I is the orthocenter of triangle $A_1B_1C_1$.

PROBLEMA 2
For any positive integer n the sum $1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}$ is written in the form $\dfrac{P(n)}{Q(n)}$, where $P(n)$ and $Q(n)$ are relatively prime.

a) Prove that $P(67)$ is not divisible by 3;

b) Find all possible n, for which $P(n)$ is divisible by 3.

PROBLEMA 3
A group consist of n tourists. Among every 3 of them there are 2 which are not familiar. For every partition of the tourists in 2 buses you can find 2 tourists that are in the same bus and are familiar with each other. Prove that is a tourist familiar to at most $\dfrac{2}{5}n$ tourists.

Segundo Dia

PROBLEMA 4
In a word formed with the letters a, b we can change some blocks: aba in b and back, bba in a and backwards. If the initial word is $aaa\ldots ab$ where a appears 2003 times can we reach the word $baaa\ldots a$, where a appears 2003 times.

PROBLEMA 5
Let a, b, c, d be positive integers such that the number of pairs $(x, y) \in (0,1)^2$ such that both $ax+by$ and $cx+dy$ are integers is equal with 2004. If $\gcd(a, c) = 6$ find $\gcd(b, d)$.

PROBLEMA 6
Let p be a prime number and let $0 \le a_1 < a_2 < \cdots < a_m < p$ and $0 \le b_1 < b_2 < \cdots < b_n < p$ be arbitrary integers. Let k be the number of distinct residues modulo p that $a_i + b_j$ give when i runs from 1 to m, and j from 1 to n. Prove that

a) if $m + n > p$ then $k = p$;

b) if $m + n \le p$ then $k \ge m + n - 1$.

Primeiro Dia

PROBLEMA 1
Let $x_1, x_2 \ldots, x_5$ be real numbers. Find the least positive integer n with the following property: if some n distinct sums of the form $x_p + x_q + x_r$ (with $1 \leq p < q < r \leq 5$) are equal to 0, then $x_1 = x_2 = \cdots = x_5 = 0$.

PROBLEMA 2
Let H be an arbitrary point on the altitude CP of the acute triangle ABC. The lines AH and BH intersect BC and AC in M and N, respectively.

(a) Prove that $\angle NPC = \angle MPC$.

(b) Let O be the common point of MN and CP. An arbitrary line through O meets the sides of quadrilateral $CNHM$ in D and E. Prove that $\angle EPC = \angle DPC$.

PROBLEMA 3
Given the sequence $\{y_n\}_{n=1}^{\infty}$ defined by $y_1 = y_2 = 1$ and

$$y_{n+2} = (4k - 5)y_{n+1} - y_n + 4 - 2k, \qquad n \geq 1$$

find all integers k such that every term of the sequence is a perfect square.

Segundo Dia

PROBLEMA 4
A set A of positive integers is called *uniform* if, after any of its elements removed, the remaining ones can be partitioned into two subsets with equal sum of their elements. Find the least positive integer $n > 1$ such that there exist a uniform set A with n elements.

PROBLEMA 5
Let a, b, c be rational numbers such that $a + b + c$ and $a^2 + b^2 + c^2$ are *equal* integers. Prove that the number abc can be written as the ratio of a perfect cube and a perfect square which are relatively prime.

PROBLEMA 6
Determine all polynomials $P(x)$ with integer coefficients such that, for any positive integer n, the equation $P(x) = 2^n$ has an integer root.

Primeiro Dia

PROBLEMA 1
Let $a_1, a_2...$ be an infinite sequence of real numbers such that $a_{n+1} = \sqrt{{a_n}^2 + a_n - 1}$. Prove that $a_1 \notin (-2, 1)$

PROBLEMA 2
Consider the orthogonal projections of the vertices A, B and C of triangle ABC on external bisectors of $\angle ACB$, $\angle BAC$ and $\angle ABC$, respectively. Prove that if d is the diameter of the circumcircle of the triangle, which is formed by the feet of projections, while r and p are the inradius and the semiperimeter of triangle ABC, prove that $r^2 + p^2 = d^2$

PROBLEMA 3
Given are n^2 points in the plane, such that no three of them are collinear, where $n \geq 4$ is the positive integer of the form $3k + 1$. What is the minimal number of connecting segments among the points, such that for each n-plet of points we can find four points, which are all connected to each other?

Segundo Dia

PROBLEMA 4
Let I be the incenter of a non-equilateral triangle ABC and T_1, T_2, and T_3 be the tangency points of the incircle with the sides BC, CA and AB, respectively. Prove that the orthocenter of triangle $T_1T_2T_3$ lies on the line OI, where O is the circumcenter of triangle ABC.

PROBLEMA 5
Find all pairs (b, c) of positive integers, such that the sequence defined by $a_1 = b$, $a_2 = c$ and $a_{n+2} = |3a_{n+1} - 2a_n|$ for $n \geq 1$ has only finite number of composite terms.

PROBLEMA 6
Find the smallest number k, such that $\frac{l_a + l_b}{a + b} < k$ for all triangles with sides a and b and bisectors l_a and l_b to them, respectively.

Primeiro Dia

PROBLEMA 1
Consider the sequence $\{a_n\}$ such that $a_0 = 4$, $a_1 = 22$, and $a_n - 6a_{n-1} + a_{n-2} = 0$ for $n \geq 2$. Prove that there exist sequences $\{x_n\}$ and $\{y_n\}$ of positive integers such that

$$a_n = \frac{y_n^2 + 7}{x_n - y_n}$$

for any $n \geq 0$.

PROBLEMA 2
Suppose that $ABCD$ is a parallelogram such that $DAB > 90$. Let the point H to be on AD such that BH is perpendicular to AD. Let the point M to be the midpoint of AB. Let the point K to be the intersecting point of the line DM with the circumcircle of ADB. Prove that $HKCD$ is concyclic.

PROBLEMA 3
Given a permutation $(a_1, a_1, \ldots, a_n)$ of the numbers $1, 2, \ldots, n$ one may interchange any two consecutive "blocks"' – that is, one may transform

$$(a_1, a_2, \ldots, a_i, \underbrace{a_{i+1}, \ldots a_{i+p}}_{A}, \underbrace{a_{i+p+1}, \ldots, a_{i+q}}_{B}, \ldots, a_n)$$

into

$$(a_1, a_2, \ldots, a_i, \underbrace{a_{i+p+1}, \ldots, a_{i+q}}_{B}, \underbrace{a_{i+1}, \ldots a_{i+p}}_{A}, \ldots, a_n)$$

by interchanging the "blocks" A and B. Find the least number of such changes which are needed to transform $(n, n-1, \ldots, 1)$ into $(1, 2, \ldots, n)$.

Segundo Dia

PROBLEMA 4
Let $n \geq 2$ be a given integer. At any point (i, j) with $i, j \in \mathbb{Z}$ we write the remainder of $i + j$ modulo n. Find all pairs (a, b) of positive integers such that the rectangle with vertices $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$ has the following properties: *(i)* the remainders $0, 1, \ldots, n-1$ written at its interior points appear the same number of times; *(ii)* the remainders $0, 1, \ldots, n-1$ written at its boundary points appear the same number of times.

PROBLEMA 5
Find all real values t for which there exist real numbers x, y, z satisfying : $3x^2 + 3xz + z^2 = 1$, $3y^2 + 3yz + z^2 = 4$, $x^2 - xy + y^2 = t$.

PROBLEMA 6
Let p be a prime number congruent to 3 modulo 4, and consider the equation $(p+2)x^2 - (p+1)y^2 + px + (p+2)y = 1$. Prove that this equation has infinitely many solutions in positive integers, and show that if $(x, y) = (x_0, y_0)$ is a solution of the equation in positive integers, then $p|x_0$.

Primeiro Dia

PROBLEMA 1
In the coordinate plane, a set of 2000 points $\{(x_1, y_1), (x_2, y_2), ..., (x_{2000}, y_{2000})\}$ is called *good* if $0 \leq x_i \leq 83$, $0 \leq y_i \leq 83$ for $i = 1, 2, \ldots, 2000$ and $x_i \neq x_j$ when $i \neq j$. Find the largest positive integer n such that, for any good set, the interior and boundary of some unit square contains exactly n of the points in the set on its interior or its boundary.

PROBLEMA 2
Let be given an acute triangle ABC. Show that there exist unique points $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$ such that each of these three points is the midpoint of the segment whose endpoints are the orthogonal projections of the other two points on the corresponding side. Prove that the triangle $A_1B_1C_1$ is similar to the triangle whose side lengths are the medians of $\triangle ABC$.

PROBLEMA 3
Let p be a prime number and let $a_1, a_2, \ldots, a_{p-2}$ be positive integers such that p doesn't a_k or ${a_k}^k - 1$ for any k. Prove that the product of some of the a_i's is congruent to 2 modulo p.

Segundo Dia

PROBLEMA 4
Find all polynomials $P(x)$ with real coefficients such that

$$P(x)P(x+1) = P(x^2), \quad \forall x \in \mathbb{R}.$$

PROBLEMA 5
Let D be the midpoint of the base AB of the isosceles acute triangle ABC. Choose point E on segment AB, and let O be the circumcenter of triangle ACE. Prove that the line through D perpendicular to DO, the line through E perpendicular to BC, and the line through B parallel to AC are concurrent.

PROBLEMA 6
Let A be the set of all binary sequences of length n and denote $o = (0, 0, \ldots, 0) \in A$. Define the addition on A as $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (c_1, \ldots, c_n)$, where $c_i = 0$ when $a_i = b_i$ and $c_i = 1$ otherwise. Suppose that $f\colon A \to A$ is a function such that $f(0) = 0$, and for each $a, b \in A$, the sequences $f(a)$ and $f(b)$ differ in exactly as many places as a and b do. Prove that if a , b, $c \in A$ satisfy $a + b + c = 0$, then $f(a) + f(b) + f(c) = 0$.

Primeiro Dia

PROBLEMA 1
The faces of a box with integer edge lengths are painted green. The box is partitioned into unit cubes. Find the dimensions of the box if the number of unit cubes with no green face is one third of the total number of cubes.

PROBLEMA 2
Let $\{a_n\}$ be a sequence of integers satisfying $(n-1)a_{n+1} = (n+1)a_n - 2(n-1) \forall n \geq 1$. If $2000|a_{1999}$, find the smallest $n \geq 2$ such that $2000|a_n$.

PROBLEMA 3
The vertices of a triangle have integer coordinates and one of its sides is of length $\sqrt{n}$, where n is a square-free natural number. Prove that the ratio of the circumradius and the inradius is an irrational number.

Segundo Dia

PROBLEMA 4
Find the number of all integers n with $4 \leq n \leq 1023$ which contain no three consecutive equal digits in their binary representations.

PROBLEMA 5
The vertices A, B, C of an acute-angled triangle ABC lie on the sides B_1C_1, C_1A_1, A_1B_1, respectively, of a triangle $A_1B_1C_1$ similar to the triangle ABC, i.e., $\angle A = \angle A_1$, $\angle B = \angle B_1$, and $\angle C = \angle C_1$. Prove that the orthocenters of triangles ABC and $A_1B_1C_1$ are equidistant from the circumcenter of ABC.

PROBLEMA 6
Prove that $x^3 + y^3 + z^3 + t^3 = 1999$ has infinitely many solutions over $\mathbb{Z}$.

Primeiro Dia

PROBLEMA 1
Let n be a natural number. Find the least natural number k for which there exist k sequences of 0 and 1 of length $2n+2$ with the following property: any sequence of 0 and 1 of length $2n+2$ coincides with some of these k sequences in at least $n+2$ positions.

PROBLEMA 2
The polynomials $P_n(x,y), n=1,2,...$ are defined by

$$P_1(x,y)=1, P_{n+1}(x,y)=(x+y-1)(y+1)P_n(x,y+2)+(y-y^2)P_n(x,y)$$

Prove that $P_n(x,y)=P_n(y,x)$ for all $x,y\in\mathbb{R}$ and n.

PROBLEMA 3
On the sides of a non-obtuse triangle ABC a square, a regular n-gon and a regular m-gon (m,$n>5$) are constructed externally, so that their centers are vertices of a regular triangle. Prove that $m=n=6$ and find the angles of $\triangle ABC$.

Segundo Dia

PROBLEMA 4
Let $a_1,a_2,\cdots,a_n$ be real numbers, not all zero. Prove that the equation:

$$\sqrt{1+a_1x}+\sqrt{1+a_2x}+\cdots+\sqrt{1+a_nx}=n$$

has at most one real nonzero root.

PROBLEMA 5
let m and n be natural numbers such that: $3m|(m+3)^n+1$ Prove that $\frac{(m+3)^n+1}{3m}$ is odd

PROBLEMA 6
The sides and diagonals of a regular n-gon R are colored in k colors so that: (i) For each color a and any two vertices A,B of R , the segment AB is of color a or there is a vertex C such that AC and BC are of color a. (ii) The sides of any triangle with vertices at vertices of R are colored in at most two colors. Prove that $k\leq 2$.

Primeiro Dia

PROBLEMA 1

Consider the polynomial $P_n(x) = \binom{n}{2} + \binom{n}{5}x + \binom{n}{8}x^2 + \cdots + \binom{n}{3k+2}x^{3k}$ where $n \geq 2$ is a natural number and $k = \left\lfloor \frac{n-2}{3} \right\rfloor$

(a) Prove that $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$ *(b)* Find all integer numbers a such that $P_n(a^3)$ is divisible by $3^{\lfloor \frac{n-1}{2} \rfloor}$ for all $n \geq 2$

PROBLEMA 2

Let M be the centroid of ΔABC Prove the inequality $\sin\angle CAM + \sin\angle CBM \leq \frac{2}{\sqrt{3}}$

(a) if the circumscribed circle of ΔAMC is tangent to the line AB (b) for any ΔABC

PROBLEMA 3

Let n and m be natural numbers such that $m + i = a_i b_i^2$ for $i = 1, 2, \cdots n$ where a_i and b_i are natural numbers and a_i is not divisible by a square of a prime number. Find all n for which there exists an m such that $\sum_{i=1}^{n} a_i = 12$

Segundo Dia

PROBLEMA 4

Let a, b, c be positive real numbers such that $abc = 1$. Prove that $\frac{1}{1+b+c} + \frac{1}{1+c+a} + \frac{1}{1+a+b} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}$.

PROBLEMA 5

Given a triangle ABC. Let M and N be the points where the angle bisectors of the angles ABC and BCA intersect the sides CA and AB, respectively. Let D be the point where the ray MN intersects the circumcircle of triangle ABC.

Prove that $\frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}$.

PROBLEMA 6

Let X be a set of $n + 1$ elements, $n \geq 2$. Ordered n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ formed from distinct elements of X are called *disjoint* if there exist distinct indices $1 \leq i \neq j \leq n$ such that $a_i = b_j$. Find the maximal number of pairwise disjoint n-tuples.

Primeiro Dia

PROBLEMA 1
Find all prime numbers p, q for which pq divides $(5^p - 2^p)(5^q - 2^q)$.

PROBLEMA 2
Find the side length of the smallest equilateral triangle in which three discs of radii $2, 3, 4$ can be placed without overlap.

PROBLEMA 3
The quadratic polynomials f and g with real coefficients are such that if $g(x)$ is an integer for some $x > 0$, then so is $f(x)$. Prove that there exist integers m, n such that $f(x) = mg(x) + n$ for all x.

Segundo Dia

PROBLEMA 4
Sequence $\{a_n\}$ it define $a_1 = 1$ and

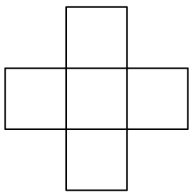
$$a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}$$

for all $n \geq 1$ Prove that $\lfloor a_n^2 \rfloor = n$ for all $n \geq 4$.

PROBLEMA 5
The quadrilateral $ABCD$ is inscribed in a circle. The lines AB and CD meet each other in the point E, while the diagonals AC and BD in the point F. The circumcircles of the triangles AFD and BFC have a second common point, which is denoted by H. Prove that $\angle EHF = 90°$.

PROBLEMA 6
A square table of size 7×7 with the four corner squares deleted is given.

- What is the smallest number of squares which need to be colored black so that a $5-$square entirely uncolored Greek cross (see Figure) cannot be found on the table?
- Prove that it is possible to write integers in each square in a way that the sum of the integers in each Greek cross is negative while the sum of all integers in the square table is positive.



Primeiro Dia

PROBLEMA 1

Through a random point C_1 from the edge DC of the regular tetrahedron $ABCD$ is drawn a plane, parallel to the plane ABC. The plane constructed intersects the edges DA and DB at the points A_1, B_1 respectively. Let the point H is the midpoint of the altitude through the vertex D of the tetrahedron $DA_1B_1C_1$ and M is the center of gravity (barycenter) of the triangle ABC_1. Prove that the measure of the angle HMC doesn't depend on the position of the point C_1. *(Ivan Tonov)*

PROBLEMA 2

Prove that there exists 1904-element subset of the set $\{1, 2, \ldots, 1992\}$, which doesn't contain an arithmetic progression consisting of 41 terms. *(Ivan Tonov)*

PROBLEMA 3

Let m and n are fixed natural numbers and Oxy is a coordinate system in the plane. Find the total count of all possible situations of $n+m-1$ points $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_{n+m-1}(x_{n+m-1}, y_{n+m-1})$ in the plane for which the following conditions are satisfied:

(i) The numbers x_i and y_i $(i = 1, 2, \ldots, n+m-1)$ are integers and $1 \le x_i \le n, 1 \le y_i \le m$. (ii) Every one of the numbers $1, 2, \ldots, n$ can be found in the sequence $x_1, x_2, \ldots, x_{n+m-1}$ and every one of the numbers $1, 2, \ldots, m$ can be found in the sequence $y_1, y_2, \ldots, y_{n+m-1}$. (iii) For every $i = 1, 2, \ldots, n+m-2$ the line P_iP_{i+1} is parallel to one of the coordinate axes. *(Ivan Gochev, Hristo Minchev)*

Segundo Dia

PROBLEMA 4

Let p be a prime number in the form $p = 4k+3$. Prove that if the numbers x_0, y_0, z_0, t_0 are solutions of the equation $x^{2p} + y^{2p} + z^{2p} = t^{2p}$, then at least one of them is divisible by p. *(Plamen Koshlukov)*

PROBLEMA 5

Points D, E, F are midpoints of the sides AB, BC, CA of triangle ABC. Angle bisectors of the angles BDC and ADC intersect the lines BC and AC respectively at the points M and N, and the line MN intersects the line CD at the point O. Let the lines EO and FO intersect respectively the lines AC and BC at the points P and Q. Prove that $CD = PQ$. *(Plamen Koshlukov)*

PROBLEMA 6

There are given one black box and n white boxes (n is a random natural number). White boxes are numbered with the numbers $1, 2, \ldots, n$. In them are put n balls. It is allowed the following rearrangement of the balls: if in the box with number k there are exactly k balls, that box is made empty - one of the balls is put in the black box and the other $k-1$ balls are put in the boxes with numbers: $1, 2, \ldots, k-1$. *(Ivan Tonov)*

Primeiro Dia

PROBLEMA 1
Let M be a point on the altitude CD of an acute-angled triangle ABC, and K and L the orthogonal projections of M on AC and BC. Suppose that the incenter and circumcenter of the triangle lie on the segment KL.

(a) Prove that $CD = R+r$, where R and r are the circumradius and inradius, respectively. (b) Find the minimum value of the ratio $CM : CD$.

PROBLEMA 2
Let K be a cube with edge n, where $n > 2$ is an even integer. Cube K is divided into n^3 unit cubes. We call any set of n^2 unit cubes lying on the same horizontal or vertical level a layer. We dispose of $\frac{n^3}{4}$ colors, in each of which we paint exactly 4 unit cubes. Prove that we can always select n unit cubes of distinct colors, no two of which lie on the same layer.

PROBLEMA 3
Prove that for every prime number $p \geq 5$,

(a) p^3 divides $\binom{2p}{p} - 2$; (b) p^3 divides $\binom{kp}{p} - k$ for every natural number k.

Segundo Dia

PROBLEMA 4
Let $f(x)$ be a polynomial of degree n with real coefficients, having n (not necessarily distinct) real roots. Prove that for all real x,

$$f(x)f''(x) \leq f'(x)^2.$$

PROBLEMA 5
On a unit circle with center O, AB is an arc with the central angle $\alpha < 90°$. Point H is the foot of the perpendicular from A to OB, T is a point on arc AB, and l is the tangent to the circle at T. The line l and the angle AHB form a triangle Δ.

(a) Prove that the area of Δ is minimal when T is the midpoint of arc AB. (b) Prove that if S_α is the minimal area of Δ then the function $\frac{S_\alpha}{\alpha}$ has a limit when $\alpha \to 0$ and find this limit.

PROBLEMA 6
White and black checkers are put on the squares of an $n \times n$ chessboard ($n \geq 2$) according to the following rule. Initially, a black checker is put on an arbitrary square. In every consequent step, a white checker is put on a free square, whereby all checkers on the squares neighboring by side are replaced by checkers of the opposite colors. This process is continued until there is a checker on every square. Prove that in the final configuration there is at least one black checker.

Primeiro Dia

PROBLEMA 1
Consider the number obtained by writing the numbers $1, 2, \ldots, 1990$ one after another. In this number every digit on an even position is omitted; in the so obtained number, every digit on an odd position is omitted; then in the new number every digit on an even position is omitted, and so on. What will be the last remaining digit?

PROBLEMA 2
Let be given a real number $\alpha \neq 0$. Show that there is a unique point P in the coordinate plane, such that for every line through P which intersects the parabola $y = \alpha x^2$ in two distinct points A and B, segments OA and OB are perpendicular (where O is the origin).

PROBLEMA 3
Let $n = p_1 p_2 \cdots p_s$, where $p_1, \ldots, p_s$ are distinct odd prime numbers. (a) Prove that the expression

$$F_n(x) = \prod \left(x^{\frac{n}{p_{i_1} \cdots p_{i_k}}} - 1 \right)^{(-1)^k},$$

where the product goes over all subsets $\{p_{i_1}, \ldots, p_{i_k}\}$ or $\{p_1, \ldots, p_s\}$ (including itself and the empty set), can be written as a polynomial in x with integer coefficients. (b) Prove that if p is a prime divisor of $F_n(2)$, then either $p \mid n$ or $n \mid p - 1$.

Segundo Dia

PROBLEMA 4
Suppose M is an infinite set of natural numbers such that, whenever the sum of two natural numbers is in M, one of these two numbers is in M as well. Prove that the elements of any finite set of natural numbers not belonging to M have a common divisor greater than 1.

PROBLEMA 5
Given a circular arc, find a triangle of the smallest possible area which covers the arc so that the endpoints of the arc lie on the same side of the triangle.

PROBLEMA 6
The base ABC of a tetrahedron $MABC$ is an equilateral triangle, and the lateral edges MA, MB, MC are sides of a triangle of the area S. If R is the circumradius and V the volume of the tetrahedron, prove that $RS \geq 2V$. When does equality hold?

Primeiro Dia

PROBLEMA 1
In triangle ABC, point O is the center of the excircle touching the side BC, while the other two excircles touch the sides AB and AC at points M and N respectively. A line through O perpendicular to MN intersects the line BC at P. Determine the ratio AB/AC, given that the ratio of the area of $\triangle ABC$ to the area of $\triangle MNP$ is $2R/r$, where R is the circumradius and r the inradius of $\triangle ABC$.

PROBLEMA 2
Prove that the sequence (a_n), where

$$a_n = \sum_{k=1}^{n} \left\{ \frac{\left\lfloor 2^{k-\frac{1}{2}} \right\rfloor}{2} \right\} 2^{1-k},$$

converges, and determine its limit as $n \to \infty$.

PROBLEMA 3
Let p be a real number and $f(x) = x^p - x + p$. Prove that:

(a) Every root α of $f(x)$ satisfies $|\alpha| < p^{\frac{1}{p-1}}$; (b) If p is a prime number, then $f(x)$ cannot be written as the product of two non-constant polynomials with integer coefficients.

Segundo Dia

PROBLEMA 4
At each of the given n points on a circle, either $+1$ or -1 is written. The following operation is performed: between any two consecutive numbers on the circle their product is written, and the initial n numbers are deleted. Suppose that, for any initial arrangement of $+1$ and -1 on the circle, after finitely many operations all the numbers on the circle will be equal to $+1$. Prove that n is a power of two.

PROBLEMA 5
Prove that the perpendiculars, drawn from the midpoints of the edges of the base of a given tetrahedron to the opposite lateral edges, have a common point if and only if the circumcenter of the tetrahedron, the centroid of the base, and the top vertex of the tetrahedron are collinear.

PROBLEMA 6
Let x, y, z be pairwise coprime positive integers and $p \geq 5$ and q be prime numbers which satisfy the following conditions:

(i) $6p$ does not divide $q - 1$; (ii) q divides $x^2 + xy + y^2$; (iii) q does not divide $x + y - z$.

Prove that $x^p + y^p \neq z^p$.

Primeiro Dia

PROBLEMA 1
Find all real parameters q for which there is a $p \in [0, 1]$ such that the equation

$$x^4 + 2px^3 + (2p^2 - p)x^2 + (p - 1)p^2x + q = 0$$

has four real roots.

PROBLEMA 2
Let n and k be natural numbers and p a prime number. Prove that if k is the exact exponent of p in $2^{2^n} + 1$ (i.e. p^k divides $2^{2^n} + 1$, but p^{k+1} does not), then k is also the exact exponent of p in $2^{p-1} - 1$.

PROBLEMA 3
Let M be an arbitrary interior point of a tetrahedron $ABCD$, and let S_A, S_B, S_C, S_D be the areas of the faces BCD, ACD, ABD, ABC, respectively. Prove that

$$S_A \cdot MA + S_B \cdot MB + S_C \cdot MC + S_D \cdot MD \geq 9V,$$

where V is the volume of $ABCD$. When does equality hold?

Segundo Dia

PROBLEMA 4
Let A, B, C be non-collinear points. For each point D of the ray AC, we denote by E and F the points of tangency of the incircle of $\triangle ABD$ with AB and AD, respectively. Prove that, as point D moves along the ray AC, the line EF passes through a fixed point.

PROBLEMA 5
The points of space are painted in two colors. Prove that there is a tetrahedron such that all its vertices and its centroid are of the same color.

PROBLEMA 6
Find all polynomials $p(x)$ satisfying $p(x^3 + 1) = p(x + 1)^3$ for all x.

Primeiro Dia

PROBLEMA 1
Let $f(x) = x^n + a_1x^{n-1} + \ldots + a_n$ $(n \geq 3)$ be a polynomial with real coefficients and n real roots, such that $\frac{a_{n-1}}{a_n} > n+1$. Prove that if $a_{n-2} = 0$, then at least one root of $f(x)$ lies in the open interval $\left(-\frac{1}{2}, \frac{1}{n+1}\right)$.

PROBLEMA 2
Let there be given a polygon P which is mapped onto itself by two rotations: ρ_1 with center O_1 and angle ω_1, and ρ_2 with center O_2 and angle ω_2 $(0 < \omega_i < 2\pi)$. Show that the ratio $\frac{\omega_1}{\omega_2}$ is rational.

PROBLEMA 3
Let $MABCD$ be a pyramid with the square $ABCD$ as the base, in which $MA = MD$, $MA^2 + AB^2 = MB^2$ and the area of $\triangle ADM$ is equal to 1. Determine the radius of the largest ball that is contained in the given pyramid.

Segundo Dia

PROBLEMA 4
The sequence $(x_n)_{n\in\mathbb{N}}$ is defined by $x_1 = x_2 = 1$, $x_{n+2} = 14x_{n+1} - x_n - 4$ for each $n \in \mathbb{N}$. Prove that all terms of this sequence are perfect squares.

PROBLEMA 5
Let E be a point on the median AD of a triangle ABC, and F be the projection of E onto BC. From a point M on EF the perpendiculars MN to AC and MP to AB are drawn. Prove that if the points N, E, P lie on a line, then M lies on the bisector of $\angle BAC$.

PROBLEMA 6
Let Δ be the set of all triangles inscribed in a given circle, with angles whose measures are integer numbers of degrees different than $45°, 90°$ and $135°$. For each triangle $T \in \Delta$, $f(T)$ denotes the triangle with vertices at the second intersection points of the altitudes of T with the circle.

(a) Let $f^0(T) = T$ and $f^k(T) = f(f^{k-1}(T))$. Prove that there exists a natural number n such that for every triangle $T \in \Delta$, among the triangles $T, f(T), \ldots, f^n(T)$ at least two are equal.

(b) Find the smallest n with the property above.

Primeiro Dia

PROBLEMA 1
Find the smallest natural number n for which the number $n^2 - n + 11$ has exactly four prime factors (not necessarily distinct).

PROBLEMA 2
Let $f(x)$ be a quadratic polynomial with two real roots in the interval $[-1, 1]$. Prove that if the maximum value of $|f(x)|$ in the interval $[-1, 1]$ is equal to 1, then the maximum value of $|f'(x)|$ in the interval $[-1, 1]$ is not less than 1.

PROBLEMA 3
A regular tetrahedron of unit edge is given. Find the volume of the maximal cube contained in the tetrahedron, whose one vertex lies in the feet of an altitude of the tetrahedron.

Segundo Dia

PROBLEMA 4
Find the smallest integer $n \geq 3$ for which there exists an n-gon and a point within it such that, if a light bulb is placed at that point, on each side of the polygon there will be a point that is not lightened. Show that for this smallest value of n there always exist two points within the n-gon such that the bulbs placed at these points will lighten up the whole perimeter of the n-gon.

PROBLEMA 5
Let A be a fixed point on a circle k. Let B be any point on k and M be a point such that $AM : AB = m$ and $\angle BAM = \alpha$, where m and α are given. Find the locus of point M when B describes the circle k.

PROBLEMA 6
Let $0 < k < 1$ be a given real number and let $(a_n)_{n \geq 1}$ be an infinite sequence of real numbers which satisfies $a_{n+1} \leq \left(1 + \frac{k}{n}\right) a_n - 1$. Prove that there is an index t such that $a_t < 0$.

Primeiro Dia

PROBLEMA 1
Let $f(x)$ be a non-constant polynomial with integer coefficients and n, k be natural numbers. Show that there exist n consecutive natural numbers $a, a+1, \ldots, a+n-1$ such that the numbers $f(a), f(a+1), \ldots, f(a+n-1)$ all have at least k prime factors. (We say that the number $p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ has $\alpha_1 + \ldots + \alpha_s$ prime factors.)

PROBLEMA 2
Find all real parameters a for which all the roots of the equation

$$x^6 + 3x^5 + (6-a)x^4 + (7-2a)x^3 + (6-a)x^2 + 3x + 1$$

are real.

PROBLEMA 3
A pyramid $MABCD$ with the top-vertex M is circumscribed about a sphere with center O so that O lies on the altitude of the pyramid. Each of the planes ACM, BDM, ABO divides the lateral surface of the pyramid into two parts of equal areas. The areas of the sections of the planes ACM and ABO inside the pyramid are in ratio $(\sqrt{2}+2) : 4$. Determine the angle δ between the planes ACM and ABO, and the dihedral angle of the pyramid at the edge AB.

Segundo Dia

PROBLEMA 4
Seven points are given in space, no four of which are on a plane. Each of the segments with the endpoints in these points is painted black or red. Prove that there are two monochromatic triangles (not necessarily both of the same color) with no common edge. Does the statement hold for six points?

PROBLEMA 5
Let P be a point on the median CM of a triangle ABC with $AC \neq BC$ and the acute angle γ at C, such that the bisectors of $\angle PAC$ and $\angle PBC$ intersect at a point Q on the median CM. Determine $\angle APB$ and $\angle AQB$.

PROBLEMA 6
Let α_a denote the greatest odd divisor of a natural number a, and let $S_b = \sum_{a=1}^{b} \frac{\alpha_a}{a}$ Prove that the sequence S_b/b has a finite limit when $b \to \infty$, and find this limit.

Primeiro Dia

PROBLEMA 1
Solve the equation $5^x 7^y + 4 = 3^z$ in nonnegative integers.

PROBLEMA 2
The diagonals of a trapezoid $ABCD$ with bases AB and CD intersect in a point O, and $AB/CD = k > 1$. The bisectors of the angles AOB, BOC, COD, DOA intersect AB, BC, CD, DA respectively at K, L, M, N. The lines KL and MN meet at P, and the lines KN and LM meet at Q. If the areas of $ABCD$ and OPQ are equal, find the value of k.

PROBLEMA 3
Points $P_1, P_2, \ldots, P_n, Q$ are given in space ($n \geq 4$), no four of which are in a plane. Prove that if for any three distinct points $P_\alpha, P_\beta, P_\gamma$ there is a point P_δ such that the tetrahedron $P_\alpha P_\beta P_\gamma P_\delta$ contains the point Q, then n is an even number.

Segundo Dia

PROBLEMA 4
Let $a, b, a_2, \ldots, a_{n-2}$ be real numbers with $ab \neq 0$ such that all the roots of the equation

$$ax^n - ax^{n-1} + a_2 x^{n-2} + \ldots + a_{n-2} x^2 - n^2 bx + b = 0$$

are positive and real. Prove that these roots are all equal.

PROBLEMA 5
Let $0 < x_i < 1$ and $x_i + y_i = 1$ for $i = 1, 2, \ldots, n$. Prove that

$$(1 - x_1 x_2 \cdots x_n)^m + (1 - y_1^m)(1 - y_2^m) \cdots (1 - y_n^m) > 1$$

for any natural numbers m and n.

PROBLEMA 6
Let there be given a pyramid $SABCD$ whose base $ABCD$ is a parallelogram. Let N be the midpoint of BC. A plane λ intersects the lines SC, SA, AB at points P, Q, R respectively such that $\overline{CP}/\overline{CS} = \overline{SQ}/\overline{SA} = \overline{AR}/\overline{AB}$. A point M on the line SD is such that the line MN is parallel to λ. Show that the locus of points M, when λ takes all possible positions, is a segment of the length $\frac{\sqrt{5}}{2} SD$.

Primeiro Dia

PROBLEMA 1
Determine all natural numbers n for which there exists a permutation $(a_1, a_2, \ldots, a_n)$ of the numbers $0, 1, \ldots, n-1$ such that, if b_i is the remainder of $a_1 a_2 \cdots a_i$ upon division by n for $i = 1, \ldots, n$, then $(b_1, b_2, \ldots, b_n)$ is also a permutation of $0, 1, \ldots, n-1$.

PROBLEMA 2
Let $b_1 \geq b_2 \geq \ldots \geq b_n$ be nonnegative numbers, and $(a_1, a_2, \ldots, a_n)$ be an arbitrary permutation of these numbers. Prove that for every $t \geq 0$,

$$(a_1 a_2 + t)(a_3 a_4 + t) \cdots (a_{2n-1} a_{2n} + t) \leq (b_1 b_2 + t)(b_3 b_4 + t) \cdots (b_{2n-1} b_{2n} + t).$$

PROBLEMA 3
A regular triangular pyramid $ABCD$ with the base side $AB = a$ and the lateral edge $AD = b$ is given. Let M and N be the midpoints of AB and CD respectively. A line α through MN intersects the edges AD and BC at P and Q, respectively.

(a) Prove that $AP/AD = BQ/BC$. (b) Find the ratio AP/AD which minimizes the area of $MQNP$.

Segundo Dia

PROBLEMA 4
Find the smallest possible side of a square in which five circles of radius 1 can be placed, so that no two of them have a common interior point.

PROBLEMA 5
Can the polynomials $x^5 - x - 1$ and $x^2 + ax + b$, where $a, b \in Q$, have common complex roots?

PROBLEMA 6
Let $a, b, c > 0$ satisfy for all integers n, we have

$$\lfloor an \rfloor + \lfloor bn \rfloor = \lfloor cn \rfloor$$

Prove that at least one of a, b, c is an integer.

Primeiro Dia

PROBLEMA 1
Find all pairs of natural numbers (n, k) for which
$(n+1)^k - 1 = n!$.

PROBLEMA 2
Let n unit circles be given on a plane. Prove that on one of the circles there is an arc of length at least $\frac{2\pi}{n}$ not intersecting any other circle.

PROBLEMA 3
In a regular $2n$-gonal prism, bases $A_1A_2 \cdots A_{2n}$ and $B_1B_2 \cdots B_{2n}$ have circumradii equal to R. If the length of the lateral edge A_1B_1 varies, the angle between the line A_1B_{n+1} and the plane $A_1A_3B_{n+2}$ is maximal for $A_1B_1 = 2R\cos\frac{\pi}{2n}$.

Segundo Dia

PROBLEMA 4
If $x_1, x_2, \ldots, x_n$ are arbitrary numbers from the interval $[0, 2]$, prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j| \leq n^2$$

When is the equality attained?

PROBLEMA 5
Find all values of parameters a, b for which the polynomial

$$x^4 + (2a+1)x^3 + (a-1)^2x^2 + bx + 4$$

can be written as a product of two monic quadratic polynomials $\Phi(x)$ and $\Psi(x)$, such that the equation $\Psi(x) = 0$ has two distinct roots α, β which satisfy $\Phi(\alpha) = \beta$ and $\Phi(\beta) = \alpha$.

PROBLEMA 6
Find the locus of centroids of equilateral triangles whose vertices lie on sides of a given square $ABCD$.

Primeiro Dia

PROBLEMA 1
Five points are given in space, no four of which are coplanar. Each of the segments connecting two of them is painted in white, green or red, so that all the colors are used and no three segments of the same color form a triangle. Prove that among these five points there is one at which segments of all the three colors meet.

PROBLEMA 2
Let ABC be a triangle such that the altitude CH and the sides CA, CB are respectively equal to a side and two distinct diagonals of a regular heptagon. Prove that $\angle ACB < 120°$.

PROBLEMA 3
A quadrilateral pyramid is cut by a plane parallel to the base. Suppose that a sphere S is circumscribed and a sphere Σ inscribed in the obtained solid, and moreover that the line through the centers of these two spheres is perpendicular to the base of the pyramid. Show that the pyramid is regular.

Segundo Dia

PROBLEMA 4
Let n be an odd positive integer. Prove that if the equation $\frac{1}{x} + \frac{1}{y} = \frac{4}{n}$ has a solution in positive integers x, y, then n has at least one divisor of the form $4k - 1$, $k \in \mathbb{N}$.

PROBLEMA 5
Find all positive values of a, for which there is a number b such that the parabola $y = ax^2 - b$ intersects the unit circle at four distinct points. Also prove that for every such a there exists b such that the parabola $y = ax^2 - b$ intersects the unit circle at four distinct points whose x-coordinates form an arithmetic progression.

PROBLEMA 6
Planes $\alpha, \beta, \gamma, \delta$ are tangent to the circumsphere of a tetrahedron $ABCD$ at points A, B, C, D, respectively. Line p is the intersection of α and β, and line q is the intersection of γ and δ. Prove that if lines p and CD meet, then lines q and AB lie on a plane.

Primeiro Dia

PROBLEMA 1
Show that there exists a unique sequence of decimal digits $p_0 = 5, p_1, p_2, \ldots$ such that, for any k, the square of any positive integer ending with $\overline{p_k p_{k-1} \cdots p_0}$ ends with the same digits.

PROBLEMA 2
(a) Prove that the area of a given convex quadrilateral is at least twice the area of an arbitrary convex quadrilateral inscribed in it whose sides are parallel to the diagonals of the original one. (b) A tetrahedron with surface area S is intersected by a plane perpendicular to two opposite edges. If the area of the cross-section is Q, prove that $S > 4Q$.

PROBLEMA 3
Each diagonal of the base and each lateral edge of a 9-gonal pyramid is colored either green or red. Show that there must exist a triangle with the vertices at vertices of the pyramid having all three sides of the same color.

Segundo Dia

PROBLEMA 4
If a, b, c are arbitrary nonnegative real numbers, prove the inequality

$$a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4}(a+b+c)^3$$

with equality if and only if two of the numbers are equal and the third one equals zero.

PROBLEMA 5
Prove that the number of ways of choosing 6 among the first 49 positive integers, at least two of which are consecutive, is equal to $\binom{49}{6} - \binom{44}{6}$.

PROBLEMA 6
Show that if all lateral edges of a pentagonal pyramid are of equal length and all the angles between neighboring lateral faces are equal, then the pyramid is regular.

Primeiro Dia

PROBLEMA 1
Show that there are no integers x and y satisfying $x^2 + 5 = y^3$.
Daniel Harrer

PROBLEMA 2
Points P, Q, R, S are taken on respective edges AC, AB, BD, and CD of a tetrahedron $ABCD$ so that PR and QS intersect at point N and PS and QR intersect at point M. The line MN meets the plane ABC at point L. Prove that the lines AL, BP, and CQ are concurrent.

PROBLEMA 3
Each side of a triangle ABC has been divided into $n+1$ equal parts. Find the number of triangles with the vertices at the division points having no side parallel to or lying at a side of $\triangle ABC$.

Segundo Dia

PROBLEMA 4
For each real number k, denote by $f(k)$ the larger of the two roots of the quadratic equation

$$(k^2+1)x^2 + 10kx - 6(9k^2+1) = 0.$$

Show that the function $f(k)$ attains a minimum and maximum and evaluate these two values.

PROBLEMA 5
A convex pentagon $ABCDE$ satisfies $AB = BC = CA$ and $CD = DE = EC$. Let S be the center of the equilateral triangle ABC and M and N be the midpoints of BD and AE, respectively. Prove that the triangles SME and SND are similar.

PROBLEMA 6
The set $M = \{1, 2, \ldots, 2n\}$ $(n \geq 2)$ is partitioned into k nonintersecting subsets $M_1, M_2, \ldots, M_k$, where $k^3 + 1 \leq n$. Prove that there exist $k+1$ even numbers $2j_1, 2j_2, \ldots, 2j_{k+1}$ in M that are in one and the same subset M_j $(1 \leq j \leq k)$ such that the numbers $2j_1 - 1, 2j_2 - 1, \ldots, 2j_{k+1} - 1$ are also in one and the same subset M_r $(1 \leq r \leq k)$.

Primeiro Dia

PROBLEMA 1

We are given the sequence $a_1, a_2, a_3, \ldots$, for which:

$$a_n = \frac{a_{n-1}^2 + c}{a_{n-2}} \text{ for all } n > 2.$$

Prove that the numbers a_1, a_2 and $\frac{a_1^2+a_2^2+c}{a_1 a_2}$ are whole numbers.

PROBLEMA 2

k_1 denotes one of the arcs formed by intersection of the circumference k and the chord AB. C is the middle point of k_1. On the half line (ray) PC is drawn the segment PM. Find the locus formed from the point M when P is moving on k_1.

G. Ganchev

PROBLEMA 3

On the name day of a man there are 5 people. The men observed that of any 3 people there are 2 that knows each other. Prove that the man may order his guests around circular table in such way that every man have on its both side people that he knows.

N. Nenov, N. Hazhiivanov

Segundo Dia

PROBLEMA 4

Find the greatest possible real value of S and smallest possible value of T such that for every triangle with sides a, b, c $(a \le b \le c)$ to be true the inequalities:

$$S \le \frac{(a+b+c)^2}{bc} \le T.$$

PROBLEMA 5

Prove that for every convex polygon can be found such three sequential vertices for which a circle that they lie on covers the polygon.

Jordan Tabov

PROBLEMA 6

The base of the pyramid with vertex S is a pentagon $ABCDE$ for which $BC > DE$ and $AB > CD$. If AS is the longest edge of the pyramid prove that $BS > CS$.

Jordan Tabov

Primeiro Dia

PROBLEMA 1
For natural number n and real numbers α and x satisfy the inequalities $\alpha^{n+1} \leq x \leq 1$ and $0 < \alpha < 1$. Prove that

$$\prod_{k=1}^{n} \left| \frac{x - \alpha^k}{x + \alpha^k} \right| \leq \prod_{k=1}^{n} \left| \frac{1 - \alpha^k}{1 + \alpha^k} \right|.$$

Borislav Boyanov

PROBLEMA 2
In the space are given n points and no four of them belongs to a common plane. Some of the points are connected with segments. It is known that four of the given points are vertices of tetrahedron which edges belong to the segments given. It is also known that common number of the segments, passing through vertices of tetrahedron is $2n$. Prove that there exists at least two tetrahedrons every one of which have a common face with the first (initial) tetrahedron.

N. Nenov, N. Hadzhiivanov

PROBLEMA 3
A given truncated pyramid has triangular bases. The areas of the bases are B_1 and B_2 and the area of the surface is S. Prove that if there exists a plane parallel to the bases whose intersection divides the pyramid to two truncated pyramids in which may be inscribed by spheres then

$$S = (\sqrt{B_1} + \sqrt{B_2})(\sqrt[4]{B_1} + \sqrt[4]{B_2})^2$$

G. Gantchev

Segundo Dia

PROBLEMA 4
Vertices A and C of the quadrilateral $ABCD$ are fixed points of the circle k and each of the vertices B and D is moving to one of the arcs of k with ends A and C in such a way that $BC = CD$. Let M be the intersection point of AC and BD and F is the center of the circumscribed circle around $\triangle ABM$. Prove that the locus of F is an arc of a circle.

J. Tabov

PROBLEMA 5
Let $Q(x)$ be a non-zero polynomial and k be a natural number. Prove that the polynomial $P(x) = (x-1)^k Q(x)$ has at least $k+1$ non-zero coefficients.

PROBLEMA 6
A Pythagorean triangle is any right-angled triangle for which the lengths of two legs and the length of the hypotenuse are integers. We are observing all Pythagorean triangles in which may be inscribed a quadrangle with sidelength integer number, two of which sides lie on the cathets and one of the vertices of which lies on the hypotenuse of the triangle given. Find the side lengths of the triangle with minimal surface from the observed triangles.

St. Doduneko

Primeiro Dia

PROBLEMA 1
In a circle with a radius of 1 is an inscribed hexagon (convex). Prove that if the multiple of all diagonals that connects vertices of neighboring sides is equal to 27 then all angles of hexagon are equals.

V. Petkov, I. Tonov

PROBLEMA 2
Find all polynomials $p(x)$ satisfying the condition:

$$p(x^2 - 2x) = p(x-2)^2.$$

PROBLEMA 3
In the space is given a tetrahedron with length of the edge 2. Prove that distances from some point M to all of the vertices of the tetrahedron are integer numbers if and only if M is a vertex of tetrahedron.

J. Tabov

Segundo Dia

PROBLEMA 4
Let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \ldots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq \frac{x_2}{x_1} + \frac{x_3}{x_2} + \ldots + \frac{x_n}{x_{n-1}} + \frac{x_1}{x_n}$$

I. Tonov

PROBLEMA 5
It is given a tetrahedron $ABCD$ and a plane α intersecting the three edges passing through D. Prove that α divides the surface of the tetrahedron into two parts proportional to the volumes of the bodies formed if and only if α is passing through the center of the inscribed tetrahedron sphere.

PROBLEMA 6
It is given a plane with a coordinate system with a beginning at the point O. $A(n)$, when n is a natural number is a count of the points with whole coordinates which distances to O are less than or equal to n.

(a) Find

$$\ell = \lim_{n\to\infty} \frac{A(n)}{n^2}.$$

(b) For which β $(1 < \beta < 2)$ does the following limit exist?

$$\lim_{n\to\infty} \frac{A(n) - \pi n^2}{n^\beta}$$

Primeiro Dia

PROBLEMA 1
Find all pairs of natural numbers (m, n) bigger than 1 for which $2^m + 3^n$ is the square of whole number.
I. Tonov

PROBLEMA 2
Let F be a polygon the boundary of which is a broken line with vertices in the knots (units) of a given in advance regular square network. If k is the count of knots of the network situated over the boundary of F, and ℓ is the count of the knots of the network lying inside F, prove that if the surface of every square from the network is 1, then the surface S of F is calculated with the formulae:

$$S = \frac{k}{2} + \ell - 1$$

V. Chukanov

PROBLEMA 3
Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ be a polynomial with real coefficients ($a_0 \neq 0$) such that $|f(x)| \leq 1$ for every $x \in [-1, 1]$. Prove that
(a) there exist a constant c (one and the same for all polynomials with the given property), for which (b) $|a_0| \leq 4$.
V. Petkov

Segundo Dia

PROBLEMA 4
In the plane are given a circle k with radii R and the points $A_1, A_2, \ldots, A_n$, lying on k or outside k. Prove that there exist infinitely many points X from the given circumference for which

$$\sum_{i=1}^{n} A_iX^2 \geq 2nR^2.$$

Does there exist a pair of points on different sides of some diameter, X and Y from k, such that

$$\sum_{i=1}^{n} A_iX^2 \geq 2nR^2 \text{ and } \sum_{i=1}^{n} A_iY^2 \geq 2nR^2?$$

H. Lesov

PROBLEMA 5
Let the *subbishop* (a bishop is the figure moving only by a diagonal) be a figure moving only by diagonal but only in the next cells (squares) of the chessboard. Find the maximal count of subbishops over a chessboard $n \times n$, no two of which are not attacking.
V. Chukanov

PROBLEMA 6
Some of the faces of a convex polyhedron M are painted in blue, others are painted in white and there are no two walls with a common edge. Prove that if the sum of surfaces of the blue walls is bigger than half surface of M then it may be inscribed a sphere in the polyhedron given (M).
(H. Lesov)

Primeiro Dia

PROBLEMA 1
Find all natural numbers n with the following property: there exists a permutation $(i_1, i_2, \ldots, i_n)$ of the numbers $1, 2, \ldots, n$ such that, if on the circular table there are n people seated and for all $k = 1, 2, \ldots, n$ the k-th person is moving i_n places in the right, all people will sit on different places.
V. Drenski

PROBLEMA 2
Let $f(x)$ and $g(x)$ be non-constant polynomials with integer positive coefficients, m and n are given natural numbers. Prove that there exists infinitely many natural numbers k for which the numbers

$$f(m^n) + g(0), f(m^n) + g(1), \ldots, f(m^n) + g(k)$$

are composite.
I. Tonov

PROBLEMA 3
(a) Find all real numbers p for which the inequality

$$x_1^2 + x_2^2 + x_3^2 \geq p(x_1x_2 + x_2x_3)$$

is true for all real numbers x_1, x_2, x_3. (b) Find all real numbers q for which the inequality

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq q(x_1x_2 + x_2x_3 + x_3x_4)$$

is true for all real numbers x_1, x_2, x_3, x_4.
I. Tonov

Segundo Dia

PROBLEMA 4
Find the maximal count of shapes that can be placed over a chessboard with size 8×8 in such a way that no three shapes are not on two squares, lying next to each other by diagonal parallel $A1 - H8$ ($A1$ is the lowest-bottom left corner of the chessboard, $H8$ is the highest-upper right corner of the chessboard).
V. Chukanov

PROBLEMA 5
Find all point M lying into given acute-angled triangle ABC and such that the area of the triangle with vertices on the feet of the perpendiculars drawn from M to the lines BC, CA, AB is maximal.
H. Lesov

PROBLEMA 6
In triangle pyramid $MABC$ at least two of the plane angles next to the edge M are not equal to each other. Prove that if the bisectors of these angles form the same angle with the angle bisector of the third plane angle, the following inequality is true

$$8a_1b_1c_1 \leq a^2a_1 + b^2b_1 + c^2c_1$$

where a, b, c are sides of triangle ABC and a_1, b_1, c_1 are edges crossed respectively with a, b, c.
V. Petkov

Primeiro Dia

PROBLEMA 1
Let the sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by the conditions: $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ $(n = 1, 2, \ldots)$. Prove that:

(a) a_m and a_n are relatively prime numbers when $m \neq n$. (b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{a_k} = 1$

I. Tonov

PROBLEMA 2
Let the numbers a_1, a_2, a_3, a_4 form an arithmetic progression with difference $d \neq 0$. Prove that there are no exists geometric progressions b_1, b_2, b_3, b_4 and c_1, c_2, c_3, c_4 such that:

$$a_1 = b_1 + c_1, a_2 = b_2 + c_2, a_3 = b_3 + c_3, a_4 = b_4 + c_4.$$

PROBLEMA 3
Let $a_1, a_2, \ldots, a_n$ are different integer numbers in the range $[100, 200]$ for which: $a_1 + a_2 + \ldots + a_n \geq 11100$. Prove that it can be found at least number from the given in the representation of decimal system on which there are at least two equal digits.

L. Davidov

Segundo Dia

PROBLEMA 4
Find all functions $f(x)$ defined in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that are differentiable at 0 and satisfy

$$f(x) = \frac{1}{2}\left(1 + \frac{1}{\cos x}\right) f\left(\frac{x}{2}\right)$$

for every x in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

L. Davidov

PROBLEMA 5
Let the line ℓ intersects the sides AC, BC of the triangle ABC respectively at the points E and F. Prove that the line ℓ is passing through the incenter of the triangle ABC if and only if the following equality is true:

$$BC \cdot \frac{AE}{CE} + AC \cdot \frac{BF}{CF} = AB.$$

H. Lesov

PROBLEMA 6
In the tetrahedron $ABCD$, E and F are the midpoints of BC and AD, G is the midpoint of the segment EF. Construct a plane through G intersecting the segments AB, AC, AD in the points M, N, P respectively in such a way that the sum of the volumes of the tetrahedrons $BMNP$, $CMNP$ and $DMNP$ to be minimal.

H. Lesov

Primeiro Dia

PROBLEMA 1
Prove that there are don't exist integers a, b, c such that for every integer x the number $A = (x+a)(x+b)(x+c) - x^3 - 1$ is divisible by 9.

I. Tonov

PROBLEMA 2
Solve the system of equations:

$$\begin{cases} \sqrt{\frac{y(t-y)}{t-x} - \frac{4}{x}} + \sqrt{\frac{z(t-z)}{t-x} - \frac{4}{x}} = \sqrt{x} \\ \sqrt{\frac{z(t-z)}{t-y} - \frac{4}{y}} + \sqrt{\frac{x(t-x)}{t-y} - \frac{4}{y}} = \sqrt{y} \\ \sqrt{\frac{x(t-x)}{t-z} - \frac{4}{z}} + \sqrt{\frac{y(t-y)}{t-z} - \frac{4}{z}} = \sqrt{z} \\ x + y + z = 2t \end{cases}$$

if the following conditions are satisfied: $0 < x < t$, $0 < y < t$, $0 < z < t$.

H. Lesov

PROBLEMA 3
Prove the equality:

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2n}} = n^2$$

where n is a natural number.

H. Lesov

Segundo Dia

PROBLEMA 4
Find maximal possible number of points lying on or inside a circle with radius R in such a way that the distance between every two points is greater than $R\sqrt{2}$.

H. Lesov

PROBLEMA 5
In a circle with radius R, there is inscribed a quadrilateral with perpendicular diagonals. From the intersection point of the diagonals, there are perpendiculars drawn to the sides of the quadrilateral.

(a) Prove that the feet of these perpendiculars P_1, P_2, P_3, P_4 are vertices of the quadrilateral that is inscribed and circumscribed. (b) Prove the inequalities $2r_1 \leq \sqrt{2}R_1 \leq R$ where R_1 and r_1 are radii respectively of the circumcircle and inscircle to the quadrilateral $P_1P_2P_3P_4$. When does equality hold?

H. Lesov

PROBLEMA 6
It is given a tetrahedron $ABCD$ for which two points of opposite edges are mutually perpendicular. Prove that:

(a) the four altitudes of $ABCD$ intersects at a common point H; (b) $AH + BH + CH + DH < p + 2R$, where p is the sum of the lengths of all edges of $ABCD$ and R is the radii of the sphere circumscribed around $ABCD$.

H. Lesov

Primeiro Dia

PROBLEMA 1
A natural number is called *triangular* if it may be presented in the form $\frac{n(n+1)}{2}$. Find all values of a $(1 \leq a \leq 9)$ for which there exist a triangular number all digit of which are equal to a.

PROBLEMA 2
Prove that the equation

$$\sqrt{2-x^2} + \sqrt[3]{3-x^3} = 0$$

has no real solutions.

PROBLEMA 3
There are given 20 points in the plane, no three of which lie on a single line. Prove that there exist at least 969 quadrilaterals with vertices from the given points.

Segundo Dia

PROBLEMA 4
It is given a triangle ABC. Let R be the radius of the circumcircle of the triangle and O_1, O_2, O_3 be the centers of excircles of the triangle ABC and q is the perimeter of the triangle $O_1O_2O_3$. Prove that $q \leq 6R\sqrt{3}$. When does equality hold?

PROBLEMA 5
Let $A_1, A_2, \ldots, A_{2n}$ are the vertices of a regular $2n$-gon and P is a point from the incircle of the polygon. If $\alpha_i = \angle A_i P A_{i+n}$, $i = 1, 2, \ldots, n$. Prove the equality

$$\sum_{i=1}^{n} \tan^2 \alpha_i = 2n \frac{\cos^2 \frac{\pi}{2n}}{\sin^4 \frac{\pi}{2n}}.$$

PROBLEMA 6
In a triangular pyramid $SABC$ one of the plane angles with vertex S is a right angle and the orthogonal projection of S on the base plane ABC coincides with the orthocenter of the triangle ABC. Let $SA = m$, $SB = n$, $SC = p$, r is the inradius of ABC. H is the height of the pyramid and r_1, r_2, r_3 are radii of the incircles of the intersections of the pyramid with the plane passing through SA, SB, SC and the height of the pyramid. Prove that

(a) $m^2 + n^2 + p^2 \geq 18r^2$; (b) $\frac{r_1}{H}, \frac{r_2}{H}, \frac{r_3}{H}$ are in the range $(0.4, 0.5)$.

Primeiro Dia

PROBLEMA 1
Find all natural numbers $a > 1$, with the property that every prime divisor of $a^6 - 1$ divides also at least one of the numbers $a^3 - 1$, $a^2 - 1$.
K. Dochev

PROBLEMA 2
Two bicyclists traveled the distance from A to B, which is 100 km, with speed 30 km/h and it is known that the first started 30 minutes before the second. 20 minutes after the start of the first bicyclist from A, there is a control car started whose speed is 90 km/h and it is known that the car is reached the first bicyclist and is driving together with him for 10 minutes, went back to the second and was driving for 10 minutes with him and after that the car is started again to the first bicyclist with speed 90 km/h and etc. to the end of the distance. How many times will the car drive together with the first bicyclist?
K. Dochev

PROBLEMA 3
On a chessboard (with 64 squares) there are situated 32 white and 32 black pools. We say that two pools form a mixed pair when they are with different colors and they lie on the same row or column. Find the maximum and the minimum of the mixed pairs for all possible situations of the pools.
K. Dochev

Segundo Dia

PROBLEMA 4
Let $\delta_0 = \triangle A_0B_0C_0$ be a triangle. On each of the sides B_0C_0, C_0A_0, A_0B_0, there are constructed squares in the halfplane, not containing the respective vertex A_0, B_0, C_0 and A_1, B_1, C_1 are the centers of the constructed squares. If we use the triangle $\delta_1 = \triangle A_1B_1C_1$ in the same way we may construct the triangle $\delta_2 = \triangle A_2B_2C_2$; from $\delta_2 = \triangle A_2B_2C_2$ we may construct $\delta_3 = \triangle A_3B_3C_3$ and etc. Prove that:

(a) segments A_0A_1, B_0B_1, C_0C_1 are respectively equal and perpendicular to B_1C_1, C_1A_1, A_1B_1;

(b) vertices A_1, B_1, C_1 of the triangle δ_1 lies respectively over the segments A_0A_3, B_0B_3, C_0C_3 (defined by the vertices of δ_0 and δ_1) and divide them in ratio $2 : 1$.

PROBLEMA 5
Prove that for $n \geq 5$ the side of regular inscribable n-gon is bigger than the side of regular $n+1$-gon circumscribed around the same circle and if $n \leq 4$ the opposite statement is true.

PROBLEMA 6
In space, we are given the points A, B, C and a sphere with center O and radius 1. Find the point X from the sphere for which the sum $f(X) = |XA|^2 + |XB|^2 + |XC|^2$ attains its maximal and minimal value. Prove that if the segments OA, OB, OC are pairwise perpendicular and d is the distance from the center O to the centroid of the triangle ABC then:
(a) the maximum of $f(X)$ is equal to $9d^2 + 3 + 6d$; (b) the minimum of $f(X)$ is equal to $9d^2 + 3 - 6d$.
K. Dochev and I. Dimovski

Primeiro Dia

PROBLEMA 1
Prove that if the sum of x^5, y^5 and z^5, where x, y and z are integer numbers, is divisible by 25 then the sum of some two of them is divisible by 25.

PROBLEMA 2
Prove that

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} < 2$$

for every $n \in \mathbb{N}$.

PROBLEMA 3
Some of the points in the plane are white and some are blue (every point of the plane is either white or blue). Prove that for every positive number r:

(a) there are at least two points with different color such that the distance between them is equal to r; (b) there are at least two points with the same color and the distance between them is equal to r; (c) will the statements above be true if the plane is replaced with the real line?

Segundo Dia

PROBLEMA 4
Find the sides of a triangle if it is known that the inscribed circle meets one of its medians in two points and these points divide the median into three equal segments and the area of the triangle is equal to $6\sqrt{14}$ cm^2.

PROBLEMA 5
Prove the equality

$$\prod_{k=1}^{2m} \cos \frac{k\pi}{2m+1} = \frac{(-1)^m}{4m}.$$

PROBLEMA 6
It is given that $r = \left(3\left(\sqrt{6}-1\right) - 4\left(\sqrt{3}+1\right) + 5\sqrt{2}\right) R$ where r and R are the radii of the inscribed and circumscribed spheres in a regular n-angled pyramid. If it is known that the centers of the spheres given coincide,

(a) find n; (b) if $n = 3$ and the lengths of all edges are equal to a find the volumes of the parts from the pyramid after drawing a plane μ, which intersects two of the edges passing through point A respectively in the points E and F in such a way that $|AE| = p$ and $|AF| = q$ $(p < a, q < a)$, intersects the extension of the third edge behind opposite of the vertex A wall in the point G in such a way that $|AG| = t$ $(t > a)$.

Primeiro Dia

PROBLEMA 1
Find all natural values of k for which the system

$$\begin{cases} x_1 + x_2 + \ldots + x_k = 9 \\ \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_k} = 1 \end{cases}$$

has solutions in positive numbers. Find these solutions.
I. Dimovski

PROBLEMA 2
Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the inequality

$$xf(y) + yf(x) = (x + y)f(x)f(y)$$

for all reals x, y. Prove that exactly two of them are continuous.
I. Dimovski

PROBLEMA 3
Prove that a binomial coefficient $\binom{n}{k}$ is odd if and only if all digits 1 of k, when k is written in binary, are on the same positions when n is written in binary.
I. Dimovski

Segundo Dia

PROBLEMA 4
On the line g we are given the segment AB and a point C not on AB. Prove that on g, there exists at least one pair of points P, Q symmetrical with respect to C, which divide the segment AB internally and externally in the same ratios, i.e

$$\frac{PA}{PB} = \frac{QA}{QB} \qquad (1)$$

If A, B, P, Q are such points from the line g satisfying (1), prove that the midpoint C of the segment PQ is the external point for the segment AB.
K. Petrov

PROBLEMA 5
The point M is inside the tetrahedron $ABCD$ and the intersection points of the lines AM, BM, CM and DM with the opposite walls are denoted with A_1, B_1, C_1, D_1 respectively. It is given also that the ratios $\frac{MA}{MA_1}$, $\frac{MB}{MB_1}$, $\frac{MC}{MC_1}$, and $\frac{MD}{MD_1}$ are equal to the same number k. Find all possible values of k.
K. Petrov

PROBLEMA 6
Find the kind of a triangle if

$$\frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{a\sin\alpha + b\sin\beta + c\sin\gamma} = \frac{2p}{9R}.$$

(α, β, γ are the measures of the angles, a, b, c are the respective lengths of the sides, p the semiperimeter, R is the circumradius)
K. Petrov

PROBLEMA 1
The numbers $12, 14, 37, 65$ are one of the solutions of the equation $xy - xz + yt = 182$. What number corresponds to which letter?

PROBLEMA 2
Prove that: (a) if $y < \frac{1}{2}$ and $n \geq 3$ is a natural number then $(y+1)^n \geq y^n + (1+2y)^{\frac{n}{2}}$; (b) if x, y, z and $n \geq 3$ are natural numbers for which $x^2 - 1 \leq 2y$ then $x^n + y^n \neq z^n$.

PROBLEMA 3
It is given a right-angled triangle ABC and its circumcircle k. (a) prove that the radii of the circle k_1 tangent to the cathets of the triangle and to the circle k is equal to the diameter of the incircle of the triangle ABC. (b) on the circle k there may be found a point M for which the sum $MA + MB + MC$ is as large as possible.

PROBLEMA 4
Outside of the plane of the triangle ABC is given point D. (a) prove that if the segment DA is perpendicular to the plane ABC then orthogonal projection of the orthocenter of the triangle ABC on the plane BCD coincides with the orthocenter of the triangle BCD. (b) for all tetrahedrons $ABCD$ with base, the triangle ABC with smallest of the four heights that from the vertex D, find the locus of the foot of that height.

PROBLEMA 1
Prove that the equation
$$3x(x-3y)=y^2+z^2$$
doesn't have any integer solutions except $x=0, y=0, z=0$.

PROBLEMA 2
Prove that for every four positive numbers a, b, c, d the following inequality is true:
$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt[3]{\frac{abc+abd+acd+bcd}{4}}.$$

PROBLEMA 3
(a) In the plane of the triangle ABC, find a point with the following property: its symmetrical points with respect to the midpoints of the sides of the triangle lie on the circumscribed circle. (b) Construct the triangle ABC if it is known the positions of the orthocenter H, midpoint of the side AB and the midpoint of the segment joining the feet of the heights through vertices A and B.

PROBLEMA 4
It is given a tetrahedron with vertices A, B, C, D.

(a) Prove that there exists a vertex of the tetrahedron with the following property: the three edges of that tetrahedron through that vertex can form a triangle. (b) On the edges DA, DB and DC there are given the points M, N and P for which:
$$DM=\frac{DA}{n},\ DN=\frac{DB}{n+1}\ DP=\frac{DC}{n+2}$$
where n is a natural number. The plane defined by the points M, N and P is α_n. Prove that all planes α_n, $(n=1,2,3,\ldots)$ pass through a single straight line.

PROBLEMA 1

The numbers $2, 3, 7$ have the property that the product of any two of them increased by 1 is divisible by the third number. Prove that this triple of integer numbers greater than 1 is the only triple with the given property.

PROBLEMA 2

Prove the inequality:

$$(1+\sin^2\alpha)^n + (1+\cos^2\alpha)^n \geq 2\left(\frac{3}{2}\right)^n$$

is true for every natural number n. When does equality hold?

PROBLEMA 3

In the triangle ABC, angle bisector CD intersects the circumcircle of ABC at the point K.

(a) Prove the equalities:

$$\frac{1}{ID} - \frac{1}{IK} = \frac{1}{CI}, \ \frac{CI}{ID} - \frac{ID}{DK} = 1$$

where I is the center of the inscribed circle of triangle ABC. (b) On the segment CK some point P is chosen whose projections on AC, BC, AB respectively are P_1, P_2, P_3. The lines PP_3 and P_1P_2 intersect at a point M. Find the locus of M when P moves around segment CK.

PROBLEMA 4

In the space there are given crossed lines s and t such that $\angle(s,t) = 60°$ and a segment AB perpendicular to them. On AB it is chosen a point C for which $AC : CB = 2 : 1$ and the points M and N are moving on the lines s and t in such a way that $AM = 2BN$. The angle between vectors $\overrightarrow{AM}$ and $\overrightarrow{BM}$ is $60°$. Prove that:

(a) the segment MN is perpendicular to t; (b) the plane α, perpendicular to AB in point C, intersects the plane CMN on fixed line ℓ with given direction in respect to s; (c) all planes passing by *ell* and perpendicular to AB intersect the lines s and t respectively at points M and N for which $AM = 2BN$ and $MN \perp t$.

PROBLEMA 1

Find all three-digit numbers whose remainders after division by 11 give quotient, equal to the sum of the squares of its digits.

PROBLEMA 2

It is given the equation $x^2 + px + 1 = 0$, with roots x_1 and x_2;

(a) find a second-degree equation with roots y_1, y_2 satisfying the conditions $y_1 = x_1(1 - x_1)$, $y_2 = x_2(1 - x_2)$; (b) find all possible values of the real parameter p such that the roots of the new equation lies between -2 and 1.

PROBLEMA 3

In the trapezium $ABCD$, a point M is chosen on the non-base segment AB. Through the points M, A, D and M, B, C are drawn circles k_1 and k_2 with centers O_1 and O_2. Prove that:

(a) the second intersection point N of k_1 and k_2 lies on the other non-base segment CD or on its continuation; (b) the length of the line O_1O_2 doesn't depend on the location of M on AB; (c) the triangles O_1MO_2 and DMC are similar. Find such a position of M on AB that makes k_1 and k_2 have the same radius.

PROBLEMA 4

In the tetrahedron $ABCD$ three of the faces are right-angled triangles and the other is not an obtuse triangle. Prove that:

(a) the fourth wall of the tetrahedron is a right-angled triangle if and only if exactly two of the plane angles having common vertex with the some of vertices of the tetrahedron are equal. (b) its volume is equal to $\frac{1}{6}$ multiplied by the multiple of two shortest edges and an edge not lying on the same wall.