Treinamento para Provas de Velocidade em Equipe, #3

Equipe Bizueiros

Round 1

1. Let right triangle ABC have hypotenuse AC and $AB = 8$. Let Dbe the foot of the altitude from B to AC. If ABC has area 60, then the length of BD can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime, positive integers. Find $a + b$.

2. Zack climbs stairs at 2 meters per minute, Kevin climbs stairs at 7 meters per minute, and Rahul climbs stairs at 8 meters per minute. The Shanghai Tower is 632 meters tall. Zack begins climbing at noon, Kevin begins x minutes after him, and Rahul begins y minutes after Kevin. If they all arrive at the top at exactly the same time, compute $x + y$.

3. Milan colors two squares, chosen uniformly at random without replacement, on a 3×3 grid. The expected number of vertices shared by the two squares is expressible as a/b where a, b are coprime positive integers. Find $a + b$.

4. A cylinder has radius 6 and height 12. An equilateral triangle is inscribed in both bases of the cylinder, then a triangular prism is removed from the cylinder by drilling perpendicular to the bases through the image of the equilateral triangles. If the surface area of the new figure can be expressed as $a\pi + b\sqrt{c}$ for integers a, b, and c (where c is divisible by no perfect squares) then find $a + b + c$.

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Round 2

5. Let f be the cubic polynomial that passes through the points $(1, 30)$, $(2, 15)$, $(3, 10)$, and $(5, 6)$. Compute the product of the roots of f.

6. In the top left corner of a grid with 100 rows and 100 columns is a ball. On each move the ball moves down one unit with probability 1/3 or right one unit with probability 2/3. After 99 moves, the ball will be in the nth column from the left, where $n = 1$ in the leftmost column. Find the expected value of n.

7. Let σ be a permutation of $\{1, 2, 3, 4\}$. If

$$\sigma(a) \cdot \sigma^2(a) \cdot \sigma^3(a) \cdot \sigma^4(a) \equiv -1 \pmod 5$$

for all $a \in \{1, 2, 3, 4\}$, compute the number of possible σ.

8. The base of a square pyramid is divided into a 2-by-2 grid. In how many ways can the 4 congruent square sections of the base and the 4 triangular faces be painted either orange or black, up to rotational symmetry?

Round 3

9. Charlie's graduating class has 10 homerooms each of size 16, into which students are randomly assigned. At graduation, the homerooms are randomly numbered and the students in the first homeroom are called alphabetically, followed by the second homeroom, etc. If Charlie is 54th overall in her class alphabetically, find the expected number of students to get called before her.

10. Let S be the set of all quadratic polynomials of the form $P(x) = x^2 + bx + 2$ where b is a positive integer and P has real roots. For any polynomial P, let $s(P)$ be the sum of the squares of P's roots. If $\sum_{P \in S} \frac{1}{s(P)} = \frac{m}{n}$, where m and n are coprime positve integers, find $m + n$.

11. A 4-digit number is *what-it-is* if, when it is written as $ABCD$, the two-digit integers AB and CD both divide $ABCD$. (Note: a two-digit integer may not have leading zeroes.) What is the sum of all what-it-is numbers?

12. The polynomials $P_1(x) = 4x^5 - 311x^4 - 704x^3 - 1255x^2 - 1964x - 2880$ and $P_2(x) = 4x^5 - 279x^4 - 632x^3 - 1127x^2 - 1764x - 2592$ have four roots in common. If $P_1(s) = P_2(t) = 0$ and $P_1(t)$ and $P_2(s)$ are both nonzero, find $s + t$.

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Round 4

13. Let a, b, and c be real numbers that satisfy the following equations: $a + b + c = 0$ and $abc = 2019$. Compute $a^3 + b^3 + c^3$.

14. Consider a triangle where the sum of the three side lengths is equal to the product of the three side lengths. If the circumcircle has 25 times the area of the incircle, the distance between the incenter and the circumcenter can be expressed in the form $\frac{\sqrt{x}}{y}$, for integers x and y, with x square-free. Find $x + y$.

15. Find the number of different possible values of the number of parts in which we can cut a circle with 2018 distinct lines (assuming that all the lines cut the circle in its interior).

16. Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a non-constant polynomial. Furthermore, P satisfies the following conditions:

- The coefficients $a_0, \dots, a_n$ are all integers.
- If x is a nonnegative integer, then $P(x)$ is positive and composite.

Find the minimum possible value of the sum of the squares of P's coefficients.

Round 5

17. What is the sum of all positive integers n with at most three digits that satisfy $n = (a + b) \cdot (b + c)$ when n is written in base 10 as $10^2 a + 10b + c$?

Note: The integer n can have leading zeroes.

18. Let k be the number of nonintersecting paths a King can take on a $6 \cdot 6$ square board from one corner to the opposite corner such that the number of steps the King is from its starting point never decreases. Compute $k \mod 23$. (Note that a King can move to any square that shares an edge or a vertex with its current square.)

19. The sequence a_n satisfies $a_1 = 1$, $a_2 = 3$ and, for $n \geq 2$, $a_n = \frac{1}{n+2}\left(\sum_{j=1}^{n+1} a_j - 1\right)$. Compute $\left\lceil \log_2 \frac{a_{2020}}{a_2 018} \right\rceil$

20. A positive integer $N > 1$ is *unhappy* if there is no permutation σ of the integers $\{1, 2, \ldots, N\}$ for which

$$\frac{\sigma(1)}{2^{n-1}} + \frac{1}{2}\sum_{k=2}^{n} \frac{\sigma(k)}{2^{n-k}}$$

is an integer for each $2 \leq n \leq N$. Find the sum of all the unhappy numbers.

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Round 6

21. Kapil the Kingly, Casimir the Conjurer, and Zack the Zombie all stand in clockwise order on the perimeter of a circular hockey rink with radius 10 meters. The central angle between Kapil and Casimir is $30°$ and the central angle between Casimir and Zack is $130°$. Kapil smacks a hockey puck in a certain direction, and the puck bounces against the edge of the rink without losing velocity. Kapil aims his shot so that it takes the minimal number of bounces to reach both Casimir and Zack. If the total number of meters traveled by the puck when it has reached the final person is $a \sin(b°)$, where a and b are positive integers and $0 \leq b \leq 90$, find $a + b$.

(Note: the puck bounces off of the rink according to the rule *angle of incidence equals angle of reflection* with respect to the tangent line.)

22. There exists a (possibly not unique) $n \in \mathbb{N}$ such that the equation

$$2x^2 + (yz)^5 + 22xyz + n = 0$$

has a maximal number of solutions $(x, y, z) \in \mathbb{Z}^3$, where $yz > 0$. Find this maximal number of solutions.

23. An ordered pair (m, n) of positive integers is called *gutsy* it satisfies the following properties:

- $m, n \leq 2017$
- $n \neq 1$
- There exists a real sequence $\{x_i\}_{i=1}^n$ for which $|x_i| = 1$ for every i and

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = m.$$

How many gutsy ordered pairs are there?

24. Points E and F lie on sides AC and AB (respectively) of $\triangle ABC$ so that $EF//BC$. Suppose there is a point P on the circumcircle of $\triangle ABC$ such that $\angle BPE = \angle CPF = 90°$. Given that $AB = 13$, $BC = 14$, and $CA = 15$, the distance from P to line EF can be written in the form $\frac{p}{q}$, where p and q are coprime positive integers. Find $p + q$.

Round 7

25. Find the sum of all positive integers k such that there exists a positive integer a such that $7k^2 = a^3 + a! + 2767$.

26. If a, b, c are positive reals such that $abc = 64$ and $3a^2 + 2b^3 + c^6 = 384$, compute maximum value of $a + b + c$.

27. Three fair twenty-sided dice are rolled, and then arranged in decreasing order. The expected value of the largest die can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.

28. Call distinct nonzero real numbers x and y disparate if they, as well as $\frac{1}{x}$ and $\frac{1}{y}$, differ by an integer. Call the ordered pair (x, y) separated if x and y are disparate. Find the number of separated pairs that are no further than $\sqrt{\frac{2016}{2017}}$ units away from the origin.

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Round 8

29. A (b, k)-*palindrome* code is a sequence of k integers between 0 and $b - 1$, inclusive, that reads the same forwards as backwards. Note that a (b, k)-palindrome code can be interpreted as a base-b integer, if one ignores the initial zeros. A positive integer n is reliable if there exist at least two distinct pairs of positive integers b and k, both greater than 1, such that the average of all (b, k)-palindrome codes (interpreted as base-b integers) is equal to n. Find the sum of the three smallest reliable numbers.

30. Define $S, T \subset 1, ..., 2016$ such that S consists of all such integers that are divisible by 3 and T consists of the rest. Compute

$$\sum_{s \in S} \binom{2019}{s} - \frac{1}{2} \sum_{t \in T} \binom{2019}{t}.$$

31. In $\triangle ABC$, let $\angle CAB = 45°$, and $|AB| = \sqrt{2}$, $|AC| = 6$. Let M be the midpoint of side BC. The line AM intersects the circumcircle of $\triangle ABC$ at P. The circle centered at M with radius MP intersects the circumcircle of ABC again at $Q \neq P$. Suppose the tangent to the circumcircle of $\triangle ABC$ at B intersects AQ at T. Find TC^2.

32. At Bizueiros Airlines, tickets do not come with assigned seats, but the passengers are given a fixed boarding order. The plane has 40 rows and an aisle down the middle, so each person chooses a row with an available seat and then goes to the left or the right, sliding all the way down. (e.g., the first person to board must choose a window seat.) If each row has three seats to the left and three seats to the right, and the flight is sold out, then there are N different ways that the plane can be boarded. Find the greatest integer k such that 2^k divides N.