$(a+b)^3 - 2(a^3+b^3) =$

$= (a+b)(6ab - (a+b)^3)$ é pot. de 2.

$\Rightarrow$ $a+b = 2^i$ ⊛
$6ab - (a+b)^2 = 2^j$

Como $a+b = 2^i$ e $a,b>0$ $\Rightarrow$

$$v_2(a) = v_2(b) = \alpha < i$$

⊛ $\Rightarrow$ $6ab = 2^j + 2^{2i}$

$v_2(6ab) = 2\alpha + 1$.

$$v_2(2^j + 2^{2i}) = \begin{cases} \min(j, 2i), & \text{se } j \neq 2i \\ j+1, & \text{se } j = 2i \end{cases}$$

Mas $v_2(6ab) = v_2(2^j - 2^{2i})$ $\Rightarrow$

$$2\alpha + 1 = \min(j, 2i), \text{ se } j \neq 2i.$$

Se $\min(j, 2i) = 2i$. Abs (mod 2)!

Logo, sobram:

$$2\alpha + 1 = j \,, \qquad \text{se } j < 2i$$

ou $$2\alpha + 1 = j + 1 \,, \qquad \text{se } j = 2i$$

(*) Caso II: $2\alpha = j = 2i$

$\Rightarrow \alpha = i$

$\Rightarrow a + b = 2^{i}$, com $v_2(a) = v_2(b) = i$.

Absurdo!

Logo, sobra:

$$2\alpha + 1 = j \quad \text{e} \quad j < 2i.$$

Seja $a = 2^{\alpha}A$; $b = 2^{\alpha}B$. $\Rightarrow A + B = 2^{i-\alpha}$

$$6 \cdot (2^{\alpha}A)(2^{\alpha}B) = 2^{2\alpha+1} + 2^{2i}$$

$$3 \cdot 2^{2\alpha+1} \cdot A \cdot B = 2^{2\alpha+1} + 2^{2i}$$

$$3 \cdot A \cdot B = 1 + 2^{2i-2\alpha-1}$$

Seja $t = i - \alpha$

$$A + B = 2^t .$$

$$A \cdot B = \frac{1 + 2^{2t-1}}{3} .$$

A e B são raízes de

$$3x^2 - 3 \cdot 2^t x + (1 + 2^{2t-1}) = 0 .$$

$$\Delta = 9 \cdot 2^{2t} - 4 \cdot 3 \cdot (1 + 2^{2t-1})$$
$$= 9 \cdot 2^{2t} - 12 - 6 \cdot 2^{2t}$$
$$= 3 \cdot 2^{2t} - 12 = 3 \cdot 4 \left(2^{2t-2} - 1\right) .$$

$\Delta = (6p)^2$

Seja $q = 2^{t-1}$.

$$2^{2t-2} - 1 = 3 \cdot p^2$$

$$q^2 - 1 = 3p^2$$

$$q^2 - 3p^2 = 1$$

A solução minimal de $q^2 - 3p^2 = 1$

é $(q_1, p_1) = (2, 1)$

Logo: $q_n + \sqrt{3}\, p_n = (q_1 + \sqrt{3}\, p_1)^n$

$$= (2 + \sqrt{3})^n$$

$$q_n = \frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2}.$$

(q_n) segue uma recorrência com eq. característica

$$P(x) = (x - (2+\sqrt{3}))(x - (2-\sqrt{3}))$$
$$= x^2 - 4x + 1.$$

Logo: $q_n - 4q_{n-1} + q_{n+2} = 0$, $q_0 = 1$,
$q_1 = 2$

$$\Rightarrow \quad q_n \equiv q_{n-2} \pmod{4}$$

$\Rightarrow \quad qn \not\equiv 0 \pmod 4.$

$\Rightarrow \quad q = 1 \quad \text{ou} \quad q = 2$

$\Rightarrow \quad 2^{t-1} = 1 \quad \text{ou} \quad 2^{t-1} = 2$

$\Rightarrow \quad t = 1 \quad \text{ou} \quad t = 2.$

- $\underline{t = 1}$

$$\begin{matrix} A + B = 2 \\ A \cdot B = 1 \end{matrix} \Rightarrow A = B = 1 \Rightarrow a = b = 2^{\alpha}$$

- $\underline{t = 2}$

$$\begin{matrix} A + B = 4 \\ A \cdot B = 3 \end{matrix} \Rightarrow \{A, B\} = \{1, 3\} \Rightarrow$$

$$\Rightarrow \{a, b\} = \{2^{\alpha}, 3 \cdot 2^{\alpha}\}.$$