P2

Ache todos $f: \mathbb{R} \to \mathbb{R}$ tais que

$$f(x + y f(x+y)) = y^2 + f(x) f(y), \qquad \forall x, y \in \mathbb{R}.$$

$P(x,y)$

$P(x,0)$: $f(x) = f(x) \cdot f(0)$. $\Rightarrow$ $f(x) = 0, \forall x$ } Teste e vê que não funciona.

ou

$f(0) = 1$

$\Rightarrow f(0) = 1$.

$f(x) \equiv 1 + x$ funciona.

$f(x) = 1 - x$ funciona.

$P(x,-x)$: $1 = x^2 + f(x) f(-x)$

$f(x) f(-x) = 1 - x^2 = (1+x)(1-x)$

$x \to 1$: $0 = f(1) \cdot f(-1)$ $\Rightarrow$ $f(1) = 0$ ou $f(-1) = 0$

Seja $a \in \{1, -1\}$ tq $f(a) = 0$.

• Se $f(x_0) = 0$:

$P(0, x_0)$: $f(x_0 f(x_0)) = x_0^2 + f(x_0)$

$1 = f(0) = x_0^2 \Rightarrow x_0 = \pm 1$.

$P(x,1)$: $f(x + f(x+1)) = 1$

• Se $f(x_1) = 1$.

$P(0, x_1)$: $f(x_1 f(x_1)) = x_1^2 + f(0) f(x_1)$

$1 = x_1^2 + 1$

$x_1 = 0$

$\boxed{\text{Se } f(x_1) = 1 \Rightarrow x_1 = 0}$

$P(x,a)$: $f(x + a f(x+a)) = 1$ $\Rightarrow$

$\Rightarrow x + a f(x+a) = 0$

$f(x+1) = -x \Rightarrow f(x) = 1 - x$.

$x = f(x-1) \Rightarrow f(x) = x + 1$.

A B

P3

$p=3 \qquad 2p-2=4.$

$\{1,2,3,4\}$

$2\cdot 1+1 \equiv_3 0.$

Ana ganha.

A nunca perde. A tem estratégia não-perdedora: Nunca pegar p.

$p=5: \qquad 2p-2=8$

1 2 3 4 5
6 7 8

foram

$(p-1)!+1 \equiv 0 \pmod p$

CABO!

Ana ganha

• $n=1$.

• $n=2$. a_1, a_2.

$$(1,1) \longrightarrow (1,2) \longrightarrow (2,1) \longrightarrow \cdots$$

DIMINUIR O MAIOR

$$\sum \frac{1}{a_i} \leq \frac{1}{2} \Rightarrow \frac{1}{a_i} \leq \frac{1}{2} \Rightarrow a_i \geq 2 \qquad \underline{\text{OK!}}$$

• $n=3$. a_1, a_2, a_3

$$(1,1,1) \to (2,2,1) \to (3,1,2) \to (1,2,3) \to (2,3,1) \to \cdots$$

$$\sum \frac{1}{a_i} \leq \frac{1}{2} \Rightarrow \frac{1}{a_i} < \frac{1}{2} \Rightarrow a_i > 2 \Rightarrow a_i \geq 3 \qquad \underline{\text{OK!}}$$

• $\underline{n=4}$: a_1, a_2, a_3, a_4

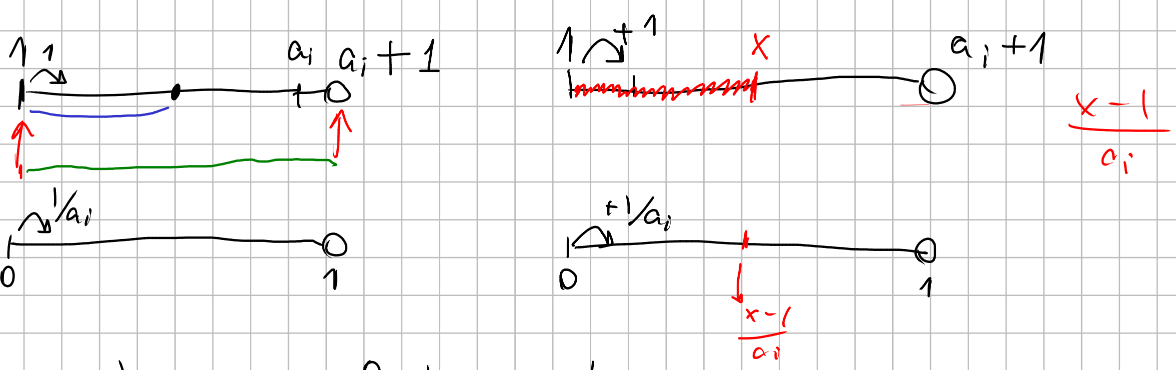
$a_1 = 3 \qquad a_2 = a_3 = a_4 = 18$

DIMINUIR O MAIS PERTO DO TETO.

1	1	1	…	1	2	3 →
1	2	3	…	16 →	1	2
1	2	3	…	16	17 →	1
1	2	3	…	16	17	18 →

1 →	1	2 →	1	2 →	1	2 →	1
1	2 →	1	2	3	4	5	6
1	2	3	4 →	1	2	3	4
1	2	3	4	5	6 →	1	2

DIMINUIR O MAIS PERTO DO TETO QUE NÃO ESTÁ NO CHÃO.



$$(b_1, \ldots, b_n) \qquad b_i = \frac{0}{a_i}, \frac{1}{a_i}, \ldots, 1 - \frac{1}{a_i}.$$

0 →	0	1/3 →	0	1/3 →	0	1/3 →	0
0	1/18 →	0	1/18	2/18	3/18	4/18	5/18
0	1/18	2/18	3/18 →	0	1/18	2/18	3/18
0	1/18	2/18	3/18	4/18	5/18 →	0	1/18
	↓	↓				↓	
	3/18	10/18				12/18	

$a_1 = 3$

$a_2 = a_3 = a_4 = 18$. ✓

$$\frac{12}{18} = \frac{2}{3} = 1 - \frac{1}{n-2}$$

$c_1 \to c_1 + 1/a_1$
$c_2 \to c_2 + 1/a_2$
$c_3 \to$
$c_4 \to$
$\vdots \to c_{n-1} + 1/c_{n-1}$
$c_n \to \cancel{c_n + 1/c_n}$ 0

0 →	0 →	0	...	0	1/3 →	0	1/3 →	0	1/3 →	0
0	1/18	2/18	...	6/18 →	0	1/18	2/18	3/18	4/18	5/18
0	1/18	2/18		6/18	7/18	8/18 →	0	1/18	2/18	3/18
0	1/18	2/18		6/18	7/18	8/18	9/18	10/18 →	0	1/18

$c_1 \geq 1 - 0/a_1$

Na jogada anterior,

$c_1 \geq 1 - 1/a_1 \quad c_2 \geq 1 - 1/a_2$

Na jogada anterior

$c_i \geq 1 - 2/a_i \, , \quad \forall\, i \in \{1, \dots, 3\}$

Na jogada anterior

$c_i \geq 1 - 3/a_i \, , \quad \forall\, i \in \{1, \dots, 4\}$

$\vdots$

$c_i \geq 1 - \frac{n-1}{a_i} \, , \quad \forall\, i \in \{1, \dots, n\}$

$\Rightarrow \sum c_i \geq n - (n-1) \cdot \sum a_i \geq \frac{n+1}{2}$

$\leq (n-1)/a_1$
$\leq (n-1)/a_2$
$\vdots$
$\leq (n-1)/a_3$

$\Leftrightarrow \quad \dots$

$\leq 2/a_1$
$\leq 2/a_2$
$\leq 2/a_3$

$\Leftrightarrow$

$\leq 1/a_1$
$\leq 1/a_2$

$\Leftrightarrow$

$0/a_2$

$2/a_1 \to 1$
$2/a_2 \to 1$
$3/a_2$

$c_i \geq 1 - \frac{n-1}{a_i}$

$\sum c_i \geq n - (n-1) \sum \frac{1}{a_i} \geq n - \frac{n-1}{2} = \frac{n+1}{2}$

ESTRATÉGIA INTELIGENTE
&
PERDEU NA RODADA $X+N-1$

$\Rightarrow$ NA RODADA X, $\sum c_i \geq \frac{n+1}{2}$.

se $\exists$ estratégia inteligente que garante $\sum a_i < \frac{n+1}{2}$. ACABOU.

$c_1 = (b_1 - 1)/a_1 \longrightarrow \cancel{b_1/a_1} \to 0$ ($a_i - b_i$ mínimo, desde que $b_i \neq 0$)

$c_2 = (b_2 - 1)/a_2 \longrightarrow b_2/a_2$

$\vdots$

$c_n = (b_n - 1)/c_2 \longrightarrow b_n/a_n$

RODADA é x-suicida Se existem x índices $i_1, \dots, i_x$ t.q

$a_{i_0} - b_{i_0} \leq x$ 1.

1-suicida é A terminal.

$c_1 = 3 \quad a_2 = a_3 = c_4 = 18$

FALTA

$1 \to 3 \to 2 \to 1 \to \emptyset$

$2 \to \cancel{1}$

$3 \to 2 \to \cancel{1}$

$4 \to 3 \to 2 \to \cancel{1}$

$AB \cap CD = P$

$AD \cap BC = Q$

(A, B, C, D)