**$19^{\text{th}}$ Olympic Revenge**
**$23^{\text{rd}}$ Olympic Week – Natal, RN**
**January 30 and 31, 2020**

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- Don't write more than one question per sheet.
- Write your name in every sheet of paper you use.

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▶ **PROBLEM 1**
Let $n$ be a positive integer and $a_1, a_2, \ldots, a_n$ non-zero real numbers. What is the least number of non-zero coeficients that the polynomial $P(x) = (x - a_1)(x - a_2)\cdots(x - a_n)$ can have?

▶ **PROBLEM 2**
For a positive integer $n$, we say an *$n$-shuffling* is a bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that there exist exactly two elements $i$ of $\{1, 2, \ldots, n\}$ such that $\sigma(i) \neq i$.
Fix some three pairwise distinct $n$-shufflings $\sigma_1, \sigma_2, \sigma_3$. Let $q$ be any prime, and let $\mathbb{F}_q$ be the integers modulo $q$. Consider all functions $f : (\mathbb{F}_q^n)^n \to \mathbb{F}_q$ that satisfy, for all integers $i$ with $1 \leq i \leq n$ and all $x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_n, y, z \in \mathbb{F}_q^n$,

$$f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, y + z, x_{i+1}, \ldots, x_n),$$

and that satisfy, for all $x_1, \ldots, x_n \in \mathbb{F}_q^n$ and all $\sigma \in \{\sigma_1, \sigma_2, \sigma_3\}$,

$$f(x_1, \ldots, x_n) = -f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

For a given tuple $(x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^n$, let $g(x_1, \ldots, x_n)$ be the number of different values of $f(x_1, \ldots, x_n)$ over all possible functions $f$ satisfying the above conditions.
Pick $(x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^n$ uniformly at random, and let $\varepsilon(q, \sigma_1, \sigma_2, \sigma_3)$ be the expected value of $g(x_1, \ldots, x_n)$. Finally, let

$$\kappa(\sigma_1, \sigma_2, \sigma_3) = -\lim_{q \to \infty} \log_q \left( -\ln \left( \frac{\varepsilon(q, \sigma_1, \sigma_2, \sigma_3) - 1}{q - 1} \right) \right).$$

Pick three pairwise distinct $n$-shufflings $\sigma_1, \sigma_2, \sigma_3$ uniformly at random from the set of all $n$-shufflings. Let $\pi(n)$ denote the expected value of $\kappa(\sigma_1, \sigma_4, \sigma_3)$. Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients such that $q(-3) \neq 0$ and such that $\pi(n) = \frac{p(n)}{q(n)}$ for infinitely many positive integers $n$. Compute $\frac{p(-3)}{q(-3)}$.

▶ **PROBLEM 3**
Let $ABC$ be a triangle and $\omega$ its circumcircle. Let $D$ and $E$ be the foot of the angle bissectors relative to $B$ and $C$, respectively. The line $DE$ meets $\omega$ at $F$ and $G$. Prove that the tangents to $\omega$ through $F$ and $G$ are tangents to the excircle of $\triangle ABC$ opposite to $A$.

▶ **PROBLEM 4**
Let $n$ be a positive integer and $A$ a set of integers such that the set $\{x = a + b \mid a, b \in A\}$ contains $\{1^2, 2^2, \ldots, n^2\}$. Prove that there is a positive integer $N$ such that if $n \geq N$, then $|A| > n^{0.666}$.

▶ **PROBLEM 5**
Let $n$ be a positive integer. Given $n$ points in the plane, prove that it is possible to draw an angle with measure $\frac{2\pi}{n}$ with vertex as each one of the given points, such that any point in the plane is covered by at least one of the angles.

*Linguagem: English*

*Duration: 5 hours.*
*Each problem is worth 7 points.*