Answer Form

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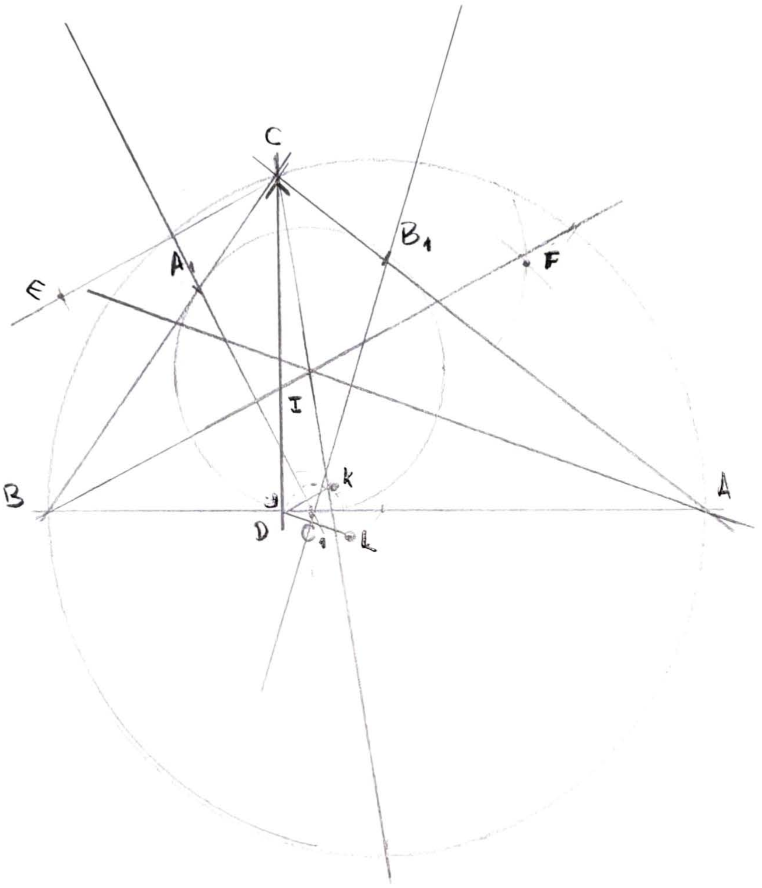

Figure 1:

Fact 1: (E,C,F) has center C_1.

Fact 2: (K,D,L) has center C_1.

Fact 3: $\# CA_1IB_1$ is a square.

Conjecture 4: C, I, K, L are collinear

Fact 5: B, I, F are collinear
A, I, E are collinear

Define
$P := (A_1EI) \cap (B_1FI)$.

We wish to prove that:

C_1, K, L, P is cyclic.

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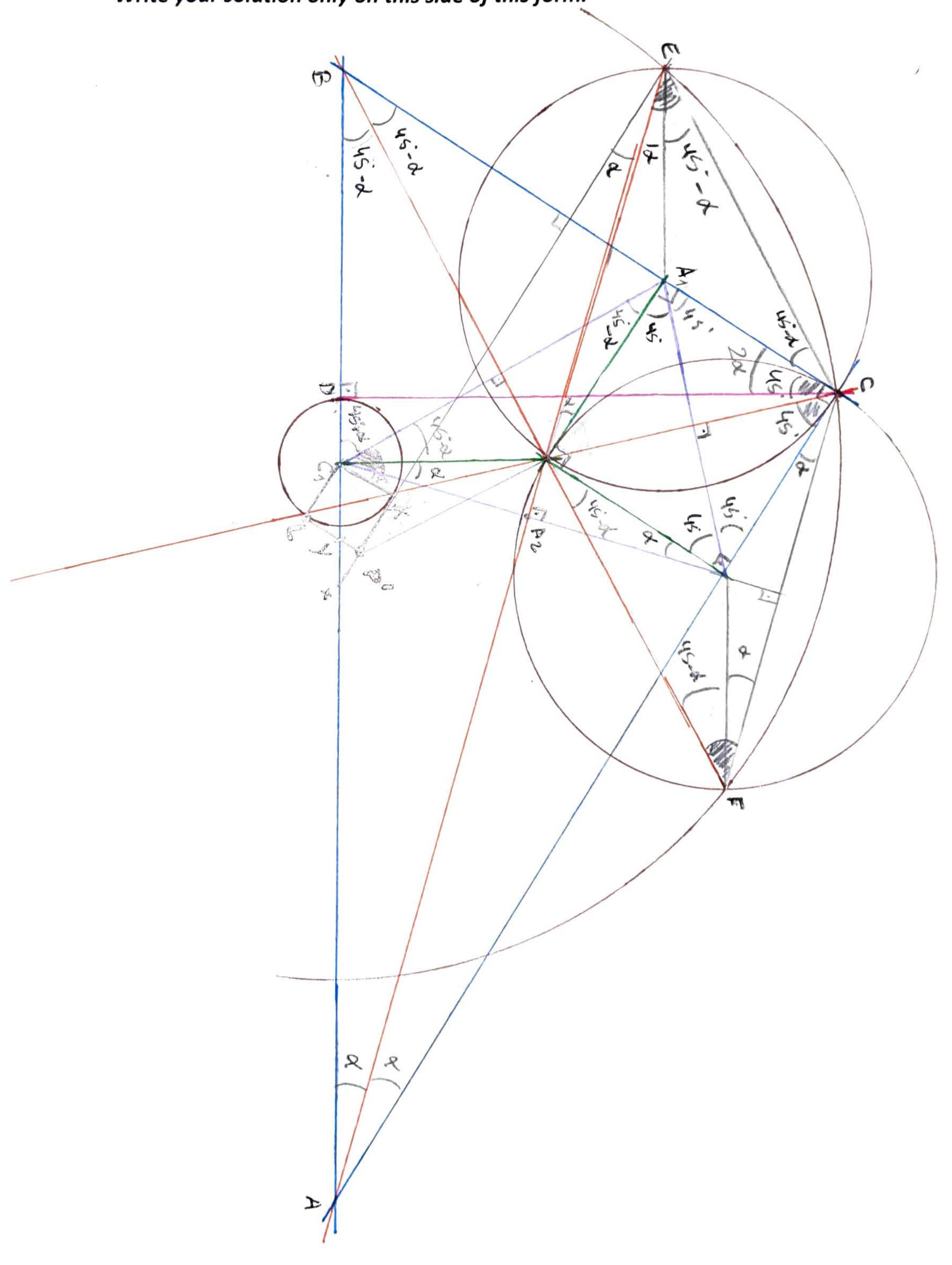

Figure 2.

Proof of Fact 5: B, I, F are collinear.

- $C_1B_1 \perp CF$ (because F is the reflection of C in B_1C_1)
- $C_1B_1 \perp IA$ (because A is the intersection of the tangents of ω by C_1 and by B_1.)

$\Rightarrow CF \parallel IA \Rightarrow \angle FCA = \angle FAI = \alpha$.

$\Rightarrow \angle FCI = 45^\circ + \alpha$. As $\angle CFI = \frac{\angle CB_1I}{2} = 45^\circ$

$\Rightarrow \angle CFI = 90^\circ - \alpha$.

But, $\angle CIB = 90^\circ + \alpha \Rightarrow \angle BIF = \angle BIC + \angle CIF = 180^\circ$

$\Rightarrow B, I, F$ are collinear.

Also, A, I, E are collinear.

Fact 6: $EA_1 \parallel BA \parallel FB_1$.

Proof: $\angle A_1EA = \alpha = \angle EAB. \Rightarrow EA_1 \parallel AB$

$\angle B_1FB = 45^\circ - \alpha = \angle FBA \Rightarrow FB_1 \parallel BA$

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By reflections in A_1C_1 and B_1C_1:

$$90^\circ = \angle CDC_1 = \angle EKC_1 = \angle FLC_1$$

Define $A_2 := AI \cap B_1C_1$; $B_2 = BI \cap A_1C_1$; $C_2 = CI \cap A_1B_1$

$\angle EA_2C_1 = 90^\circ \Rightarrow \# EA_2KC_1$ is cyclic.

Also, $\# FB_1LC_1$ is cyclic.

$\angle CBD = 90^\circ - 2\alpha \Rightarrow \angle BCD = 2\alpha \Rightarrow \angle ECD = 45^\circ + \alpha \Rightarrow$

$\Rightarrow \angle CEK = 45^\circ + \alpha \Rightarrow \angle A_2EK = \alpha \Rightarrow \angle A_2C_1K = \alpha \Rightarrow \angle IC_1K = 2\alpha$.

But, $\angle BCD = 2\alpha$. As $CD \parallel IC_1$, $\Rightarrow$ $\underline{CB \parallel C_1K}$.

$\hookrightarrow$ Fact 7.

Also, $CA \parallel C_1L$.

$\angle KEA = \alpha = \angle EAC \Rightarrow EK \parallel CA$.

Let $X := EK \cap BA$. $\Rightarrow \Delta BCA \sim \Delta C_1KX$.

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~~Let $K' := EK \cap CI$.~~

~~$\angle CEK = 45^\circ + \alpha$ $= \angle CEK'$; $\angle ECI = 90^\circ - \alpha$ $= \angle ECK'$~~

~~$\Rightarrow \angle EK'C = 45^\circ$.~~

$C_1K \parallel BC$ and $C_1L \parallel CA$ $\Rightarrow$ $\angle KC_1L = \angle BCA = 90^\circ$

Thus, $\# C_1KPL$ is cyclic $\Leftrightarrow$ $\angle KPL = 90^\circ$

Suppose Conjecture 4 is true.

Define P' such that KC_1LP' is a square. (thus, cyclic)

$$2\alpha = \angle IC_1K \overset{\text{Reflect on } KL}{=} \angle IP'K = \angle IP'E.$$

But $\angle IAE = 180^\circ - 2\alpha$ $\Rightarrow$ $\# IAEP'$ is cyclic.

Also, $\# IBFP'$ is cyclic $\Rightarrow$ $P' \equiv P$.

$\Rightarrow$ KC_1LP is cyclic.

Now, it is enough to prove Conjecture 4.

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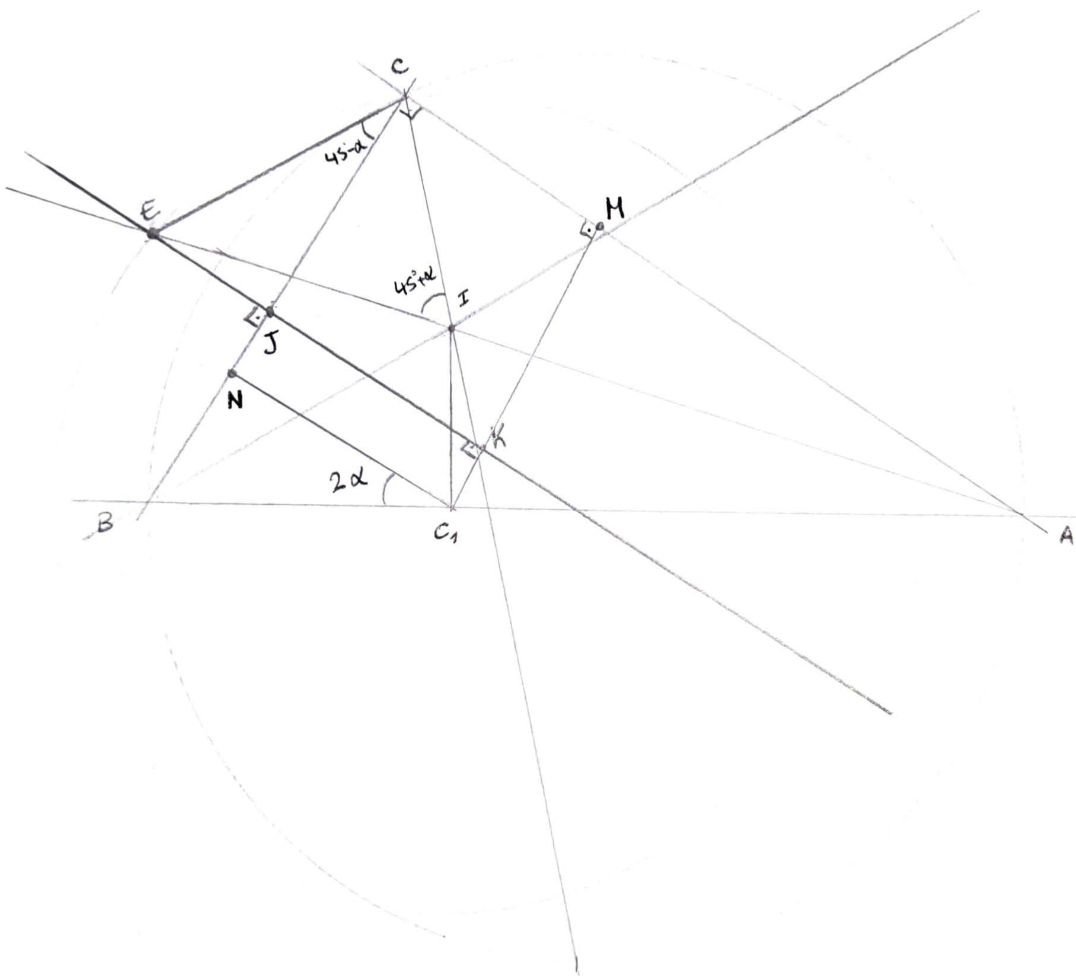

Figure 4.

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(Look at figure 4)

$$CJ = EC \cdot \cos(45^\circ - \alpha) = 2A_1C \cdot \operatorname{sen}(45^\circ + \alpha)\cos(45^\circ - \alpha)$$
$$= 2(p-c) \cdot \operatorname{sen}^2(45^\circ + \alpha)$$
$$= 2(p-c) \cdot \left(\frac{2}{\sqrt{2}}\cos\alpha + \frac{2}{\sqrt{2}}\operatorname{sen}\alpha\right)^2 = (p-c)(\cos\alpha + \operatorname{sen}\alpha)^2$$
$$= (p-c)(1 + \operatorname{sen} 2\alpha)$$

$$JK = NC_1 = BC_1 \cdot \cos(2\alpha)$$
$$= (p-b)\cos(2\alpha)$$

Thus: $CJ = JK \iff (p-c)(1+\operatorname{sen} 2\alpha) = (p-b)\cos(2\alpha)$

$\iff (a+b-c)(1+\operatorname{sen} 2\alpha) = (a-b+c)\cos(2\alpha)$

$\iff (\operatorname{sen} 2\alpha + \cos 2\alpha - 1)(1+\operatorname{sen} 2\alpha) = (\operatorname{sen} 2\alpha - \cos 2\alpha + 1)\cos(2\alpha)$

$\iff \cancel{\operatorname{sen} 2\alpha} + \cancel{\cos 2\alpha} - 1 + \operatorname{sen}^2 2\alpha + \cancel{\cos 2\alpha \operatorname{sen} 2\alpha} - \cancel{\operatorname{sen} 2\alpha}$
$=$
$\cancel{\operatorname{sen} 2\alpha \cos 2\alpha} - \cos^2 2\alpha + \cancel{\cos 2\alpha}$

$\iff \operatorname{sen}^2 2\alpha + \cos^2 2\alpha = 1$, which is true.

Thus, $CJ = JK \Rightarrow JK = KH \Rightarrow K$ is in the angle bisector of $\angle C$

$\Rightarrow C, I, K$ are collinear.

Also, C, I, L are collinear. $\Rightarrow$ Conjecture 4 is true, and the problem is solved. $\square$

Romanian Master of Mathematics
2020

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RASC

Define $Y := LF \cap BA$.

Proof of Conjecture 4:

$\Delta BCA \sim \Delta C_1KX \sim \Delta YLC_1$.

Let $O_K := CK \cap BA$ (the center of the homothety $T_K : \Delta BCA \to \Delta C_1KX$)

$O_L := CL \cap BA$ (the center of the homothety $T_L : \Delta BCA \to \Delta YLC_1$)

$O := KL \cap BA$ (the center of the homothety $T : \Delta C_1KX \to \Delta YLC_1$)

As $\angle KC_1L = 90° \Rightarrow \angle C_1KL = \angle C_1LK = 45°$.

$\Rightarrow O$ is the foot of the angle bisector of $\angle C_1KX$ and $\angle YLC_1$.

Let Z be the foot of the angle bisector of $\angle BCA$.

$$\underbrace{Z \xrightarrow{T_K} O \xrightarrow{(T_L)^{-1}} Z}_{(T_K)\circ(T_L)^{-1} =}$$

Z is a fixed point of $T_K \circ T_L^{-1} \Rightarrow \begin{cases} Z \text{ is the center of homothety } T_K \circ T_L^{-1} \\ \text{or} \\ T_K = T_L \Rightarrow K = L \Rightarrow \text{"circumcircle of } C_1KL\text{" makes no sense. Absurd!} \end{cases}$

$\Rightarrow Z$ is the center of $T_K \circ T_L^{-1}$.

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RASC

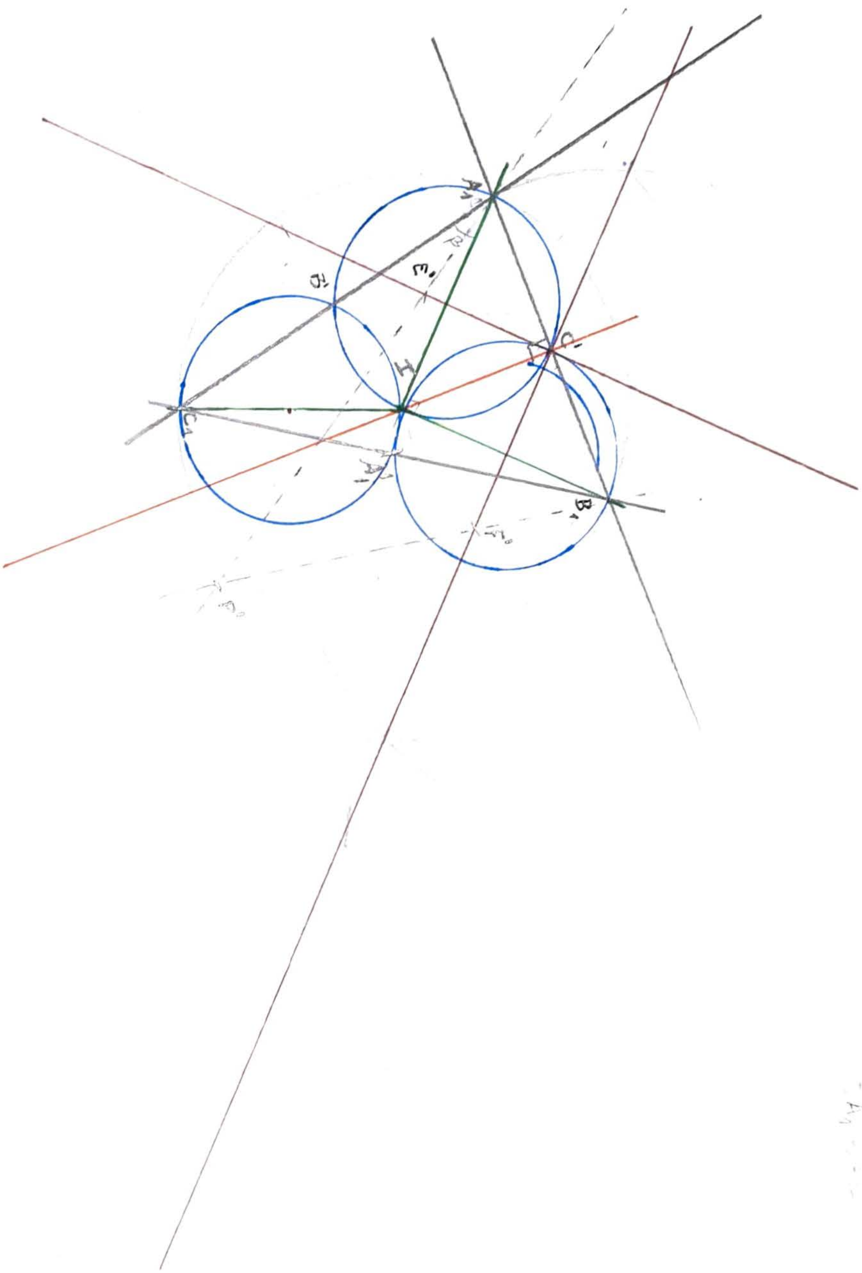

Figure 3. Inversion on the incircle.