Problemas Sortidos

Guilherme Zeus Dantas e Moura
zeusdanmou@gmail.com

1. A positive integer is called *square-free* if it is not a multiple of any square other than 1. George and his n friends sit around a table. George thinks of a positive integer $k > 1$ and writes it on the blackboard. The person to his left then divides the number on the blackboard by a square-free number to obtain another positive integer $k_1 < k$, and replaces k with k_1 on the blackboard. The process repeats with each person in succession, going clockwise around the table, generating positive integers $k_1 > k_2 > k_3 > \cdots$ and so on. The first person to write 1 on the blackboard wins. Prove that for any value of n, George can always think of a positive integer k such that he is guaranteed to win.

2. Let $\mathbb{Q}$ denote the set of rational numbers. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all rational a and b,

$$f(a)f(b) = f(a + b).$$

3. Find all positive integers a and b such that $a^2 + 2b^2$ is a power of 2.

4. Do there exist points A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, and Z in the Euclidean plane, not all the same, such that $ABCD$, $EFGH$, $IJKL$, $MNOP$, $QRST$, $UVWX$, $YZAB$, $CDEF$, $GHIJ$, $KLMN$, $OPQR$, $STUV$, and $WXYZ$ are all squares? (Note that the vertices of a square do not necessarily have to be in order, so that if $ABCD$ is a square then so is $ACBD$.)

5. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(xf(x) + f(y)) = xf(x + y).$$

6. Let n be a given positive integer. Find the minimum m such that for all real sequences x_1, $x_2, \ldots, x_n$ there exists a real number y such that

$$\langle y - x_1\rangle + \langle y - x_2\rangle + \cdots + \langle y - x_n\rangle \le m,$$

where $\langle x\rangle = x - \lfloor x\rfloor$ is the difference between x and the greatest integer less than or equal to x.

7. Let ABC be a triangle and denote by M the midpoint of BC. Suppose X is the point on the perimeter of ABC such that MX bisects the perimeter of ABC. Show that MX is parallel to the internal angle bisector of $\angle BAC$.

8. Let n and k be given positive integers. Find the number of k-tuples $(S_1, S_2, \ldots, S_k)$ of sets S_i such that $S_i \subseteq \{1, 2, \ldots, n\}$ and $S_1 \subseteq S_2 \supseteq S_3 \subseteq S_4 \supseteq S_5 \subseteq \cdots S_k$.

9. Sharky has a collection of 2^n strips of $n \times 1$ strips of paper, with each strip divided into n unit squares. Each square on a strip is coloured black or white such that every strip is unique. Find the smallest m such that for any m strips, Sharky can choose n of these strips and arrange them (without flipping any of the strips) into a $n \times n$ square grid with the property that a main diagonal is monochromatic.

10. Let Γ be the circumcircle of $\triangle ABC$. O lies on the internal angle bisector of $\angle BAC$ such that a circle centred at O is tangent to the segment BC at P and the arc BC of Γ without A at Q. Prove that $\angle PAO = \angle QAO$.

11. Let ABC be a triangle with circumcentre O, and let P be a point on BC distinct from B and C. Construct X and Y on AB and AC respectively such that $XB = XP$ and $YP = YC$. Prove that $AXOY$ is cyclic.

12. Prove that for all Pythagorean triples A and B there exists a finite sequence of Pythagorean triples starting with A and ending with B such that any two consecutive triples share at least one number.

(A *Pythagorean triple* is a triple of positive integers (a, b, c) such that a, b, and c are the side lengths of a right-angled triangle.)