(E1.4) (Kazakhstan MO 2008)

First, $II_b \perp B_1B_2$.

Thus, $IB_2 \perp B_1I_b \iff I$ is orthocenter of $\Delta B_1B_2I_b$.

Let M and L be the midpoint of AC and the midpoint of arc $\widehat{AC}$ that does not contains B.

It is known that L is the center of a circle Γ that passes through A, C, I, I_b.

Also, $\angle LMB_2 = \angle LBB_2 = 90^\circ \Rightarrow \odot(L, M, B, B_2) \Rightarrow$

$$\Rightarrow B_1B \cdot B_1B_2 = B_1M \cdot B_1L \Rightarrow$$

$$\Rightarrow B_1B \cdot (B_1B + BB_2) = B_1M \cdot B_1L \Rightarrow$$

$$\Rightarrow B_1B^2 - B_1M \cdot B_1L = BB_1 \cdot BB_2 \Rightarrow$$

$$\Rightarrow BB_1 \cdot BB_2 = (2R\,\mathrm{sen}(\alpha-\gamma))^2 - \left(R \cdot \mathrm{sen}\beta \cdot \frac{\cos\beta}{\mathrm{sen}\beta}\right)(2R)$$

$$\boxed{BB_1 \cdot BB_2 = 4R^2\left(\mathrm{sen}^2(\alpha-\gamma) - \cos^2\beta\right)}$$

$$BI \cdot BI_b = BL^2 - r_\Gamma^2 = (2R\cos(\alpha-\gamma))^2 - (2R\,\mathrm{sen}\beta)^2$$

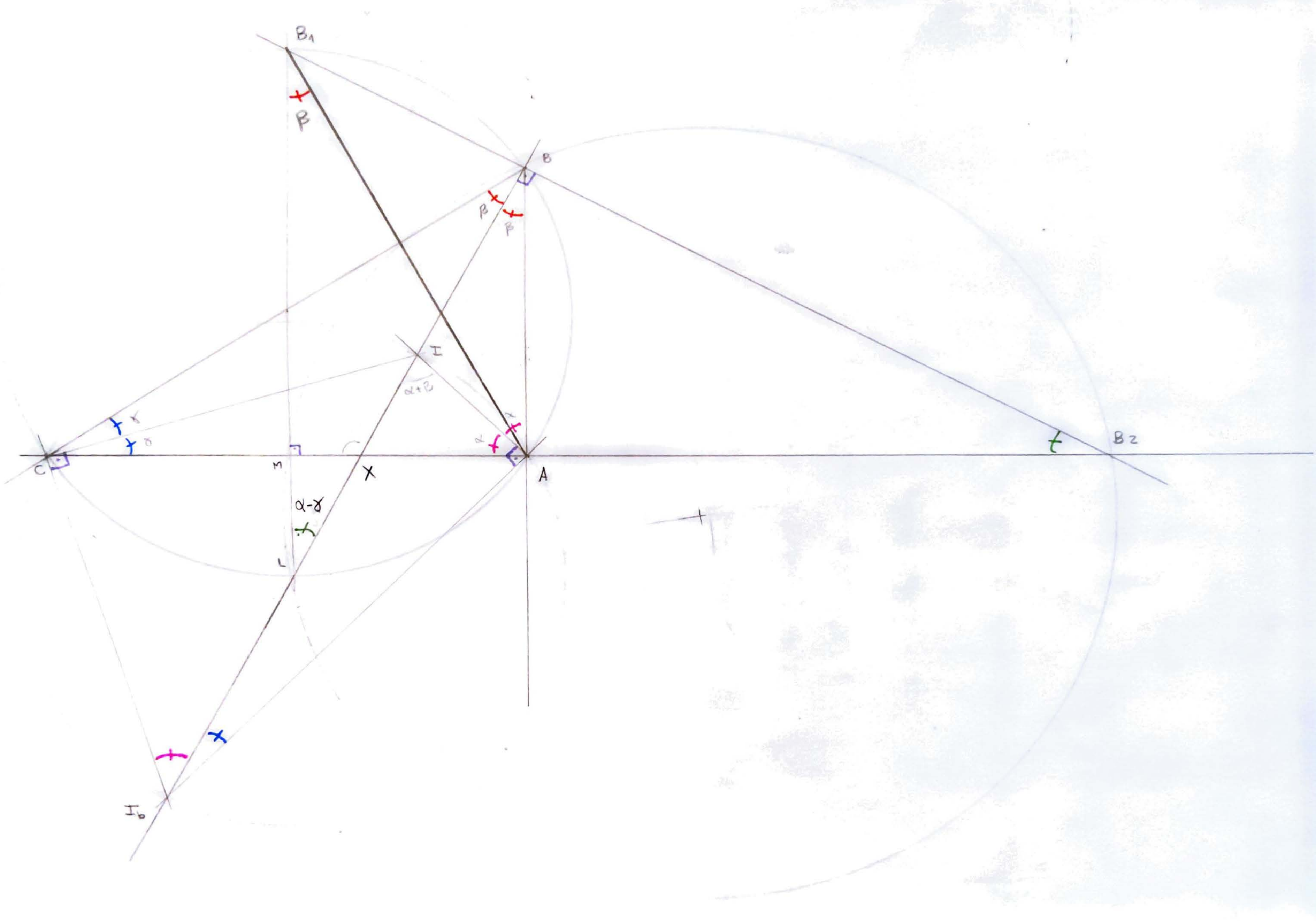
$$= 4R^2\left(\cos^2(\alpha-\gamma) - \mathrm{sen}^2\beta\right)$$

$$= 4R^2\left(\cos^2\beta - \mathrm{sen}^2(\alpha-\gamma)\right)$$

$$= -BB_1 \cdot BB_2.$$

$\Rightarrow I$ is the orthocenter of $\Delta B_1B_2I_b$.

$\square$



B_1
β
B
β
β
I
$\alpha+\beta$
γ
γ
α
α
C
M
X
A
$\alpha-\gamma$
L
I_b
B_2