AIME 2020

Rodada 1, até 16 horas.

PROBLEMA 1
In $\triangle ABC$ with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that $AE = ED = DB = BC$. The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

PROBLEMA 2
There is a unique positive real number x such that the three numbers $\log_8(2x), \log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

PROBLEMA 3
A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

PROBLEMA 4
Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. For example, $42,020$ is in S because 4 is a divisor of $42,020$. Find the sum of all the digits of all the numbers in S. For example, the number $42,020$ contributes $4 + 2 + 0 + 2 + 0 = 8$ to this total.

PROBLEMA 5
Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.

Rodada 2, até 17 horas.

PROBLEMA 6

A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

PROBLEMA 7

A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

PROBLEMA 8

A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

PROBLEMA 9

Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random from the set S and labeled a_1, a_2, and a_3 in the order they are chosen. The probability that both a_1 divides a_2 and a_2 divides a_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

PROBLEMA 10

Let m and n be positive integers satisfying the conditions

$\gcd(m+n, 210) = 1, m^m$ is a multiple of n^n, and$m$ is not a multiple of n.

Find the least possible value of $m+n$.

Rodada 3, até 18 horas.

PROBLEMA 11
For integers a, b, c, and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

PROBLEMA 12
Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive divisors of n.

PROBLEMA 13
Point D lies on side BC of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

PROBLEMA 14
Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

PROBLEMA 15
Let ABC be an acute triangle with circumcircle ω and orthocenter H. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3$, $HX = 2$, $HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.