Problemas Sortidos VI

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Problema 1

How many positive integers N satisfy all of the following three conditions?

(i) N is divisible by 2020.
(ii) N has at most 2020 decimal digits.
(iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Problema 2

For a positive integer n, define $d(n)$ to be the sum of the digits of n when written in binary (for example, $d(13) = 1 + 1 + 0 + 1 = 3$). Let

$$S = \sum_{k=1}^{2020} (-1)^{d(k)} k^3.$$

Determine S modulo 2020.

Problema 3

Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}.$$

Problema 4

Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

Problema 5

Let a_n be the number of sets S of positive integers for which

$$\sum_{k \in S} F_k = n,$$

where the Fibonacci sequence $(F_k)_{k \geq 1}$ satisfies $F_{k+2} = F_{k+1} + F_k$ and begins $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$. Find the largest number n such that $a_n = 2020$.

Problema 6

Consider a horizontal strip of $N + 2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbors with equal probability, and color tis neighboring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbors. Let $w(N)$ be the expected number of white squares remaining. Find

$$\lim_{N \to \infty} \frac{w(N)}{N}.$$