$$2016 = 2^5 \cdot 3^2 \cdot 7 \mid (x_a - x_b)^2 + (y_a - y_b)^2$$

$$\Leftrightarrow$$

$$(\ast)\begin{cases} 2^5 \mid (x_a - x_b)^2 + (y_a - y_b)^2 & (\ast) \\ \text{e} \\ 3^2 \mid (x_a - x_b)^2 + (y_a - y_b)^2 & (\ast) \\ \text{e} \\ 7 \mid (x_a - x_b)^2 + (y_a - y_b)^2 \end{cases}$$

$(\ast)$ $3^2 \mid (x_a - x_b)^2 + (y_a - y_b)^2$

$\Rightarrow$ $x^2 + y^2 \equiv 0 \pmod 3$

Se $y \not\equiv 0$:

$\Rightarrow$ $(x \cdot y^{-1})^2 \equiv -1 \pmod 3$ Absurdo!

Logo, $y \equiv 0$ e $x \equiv 0 \pmod 3$.

$y_a \equiv y_b$ e $x_a \equiv x_b \pmod 3$

Lema: $3^2 \mid (x_a - x_b)^2 + (y_a - y_b)^2 \Leftrightarrow x_a \equiv x_b$ e $y_a \equiv y_b \pmod 3$

Lema 2: $7 \mid (x_a - x_b)^2 + (y_a - y_b)^2 \Leftrightarrow x_a \equiv x_b$ e $y_a \equiv y_b \pmod 7$

Lema 3: $2^5 \mid (x_a - x_b)^2 + (y_a - y_b) \Leftrightarrow x_a \equiv x_b$ e $y_a \equiv y_b \pmod 4$ e $x_a - x_b \equiv y_a - y_b \pmod 8$

$(\ast) \iff \begin{matrix} x_a \equiv x_b \\ y_a \equiv y_b \end{matrix} \pmod{\underbrace{3\cdot 7\cdot 4}_{n}}$ e $\boxed{\underbrace{x_a - x_b}_{\equiv 0\,ou\,4} \equiv y_a - y_b}$ (8)

$(x_a, y_a) \mapsto \underbrace{(x_a \pmod{3\cdot 7\cdot 4},\ y_a \pmod{3\cdot 7\cdot 4})}_{(3\cdot 7\cdot 4)^2}$

$\to$ Se $n > 2\cdot(3\cdot 7\cdot 4)^2$: . .

P.C.P: $\exists\, a, b, c \in S$ t.q.

$x_a \equiv x_b \equiv x_c$ e $y_a \equiv y_b \equiv y_c \pmod{3\cdot 7\cdot 4}$

Será que vale algum desses?

- $x_a - x_b \equiv y_a - y_b + 4 \quad (8)$
- $x_b - x_c \equiv y_b - y_c + 4 \quad (8)$
- $x_c - x_a \equiv y_c - y_a + 4 \quad (8)$

$\oplus$ Sim

$0 \equiv 4$ Abs. (8)

$\to$ Construir exemplo p/ $n = 2\cdot(3\cdot 7\cdot 4)^2$

$S = \{(x, y) : 0 \le x < 2\cdot 3\cdot 7\cdot 4,\ 0 \le y < 3\cdot 7\cdot 4\}$

Os possíveis pares são

(x, y), $(x + 3\cdot 7\cdot 4, y)$, mas $\begin{matrix} y - y \equiv 0 \quad (8) \\ (3\cdot 7\cdot 4 + x) - x \equiv 4 \quad (8). \end{matrix}$

Resposta: $2\cdot(3\cdot 4\cdot 7)^2 + 1 = 14113$.

$$\begin{cases} a_0 = m \\ a_i = \dfrac{P(a_{i-1})}{2016} \end{cases}$$

Quando que (a_i) possui infinitos negativos?

Qual o valor mínimo de $P(x)$?

$$-3024 = P(-3/2)$$

Qual é o valor mínimo de $P(x)$, dado que

$$x \geqslant \frac{-3024}{2016} = \frac{-3}{2} \; ? \qquad -3024.$$

Quando $\dfrac{P(x)}{2016} > x$?

$\iff$

$$P(x) - 2016x > 0$$

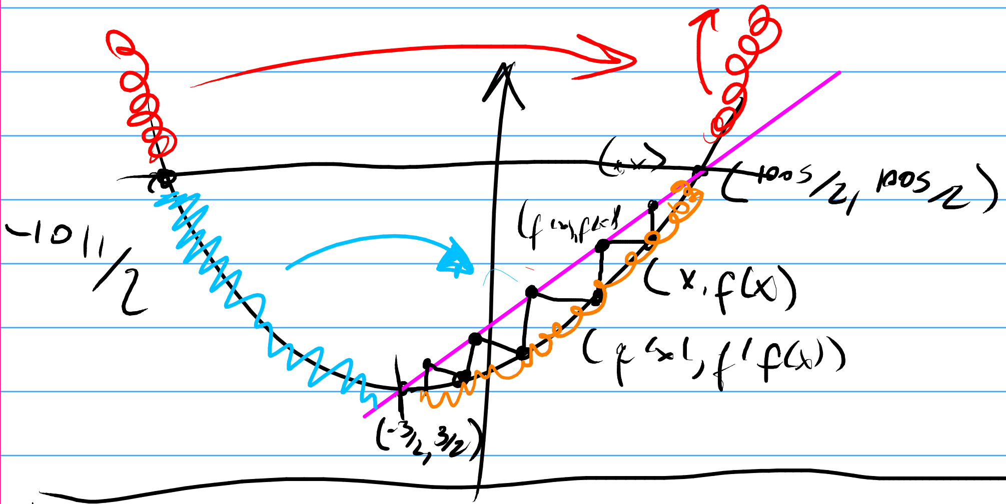
$\iff$

$$4x^2 - 2004x - 3015 > 0$$

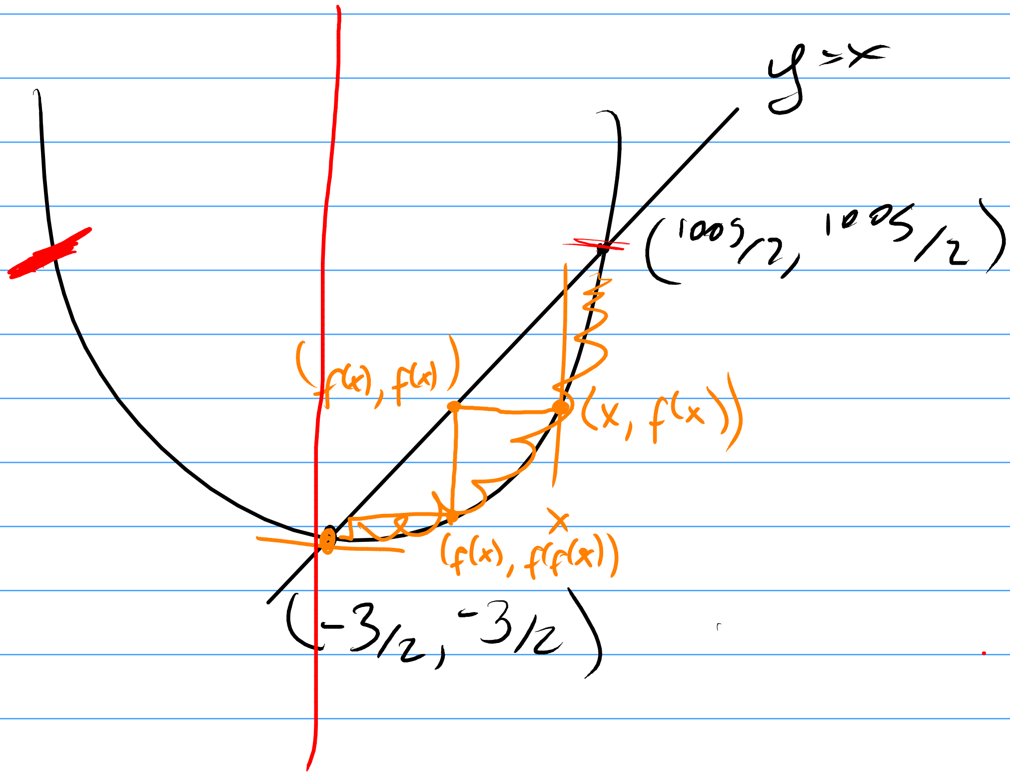
$\iff$ $x < -3/2$ ou $x > 1005/2$

$$f(x) = \frac{P(x)}{2016} = \frac{4x^2 + 12x - 3015}{2016}$$

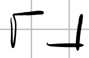
$\rightarrow$ Se $x > 1005/2 \Rightarrow f(x) > x$

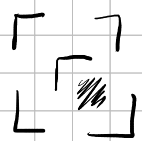


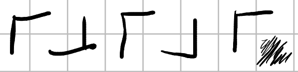
$$\boxed{-3/2 \text{ é mínimo do } f \text{ e } -3/2 \text{ é raiz de } f(x)-x}$$



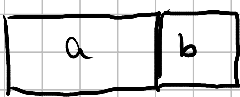
2x2  $\equiv$ 1

2x4
N x x N
N x x N
e 

4x4  ✓

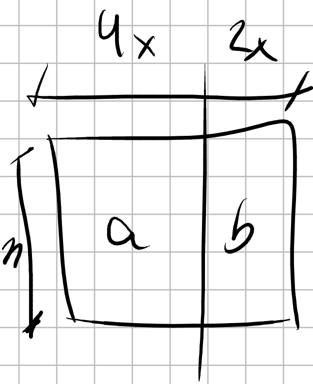
2x8  ✓

Se $3 | m$, $\Sigma \equiv 0 \ (3)$

2x6 = 

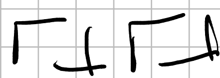
$8a + 4b \equiv 0 \ (3)$

$a \equiv b \ (3)$ ✓



$4nx(2a+b) \equiv 0 \ (3)$

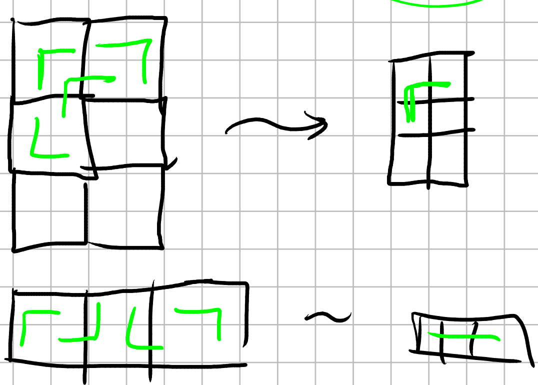
$\boxed{nx \equiv 0 \ (3)}$ ou $a \equiv b \ (3)$ ✓

divide em 2x6 e 

Soma 1 no a ou no b

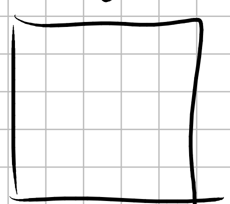
somo 3 no a ou no b

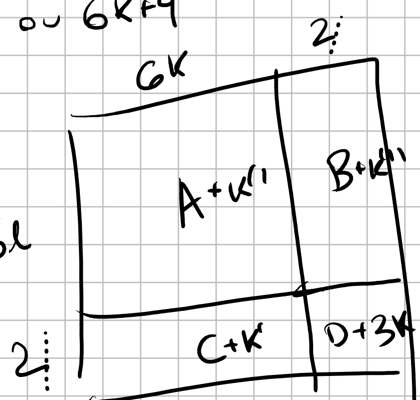
2nx2m ⟵ mxn ADAPTADO



• $3 \nmid mn$.

$2n = 6K + 2$ ou $6K+4$





Bases 2x ~

4x ~

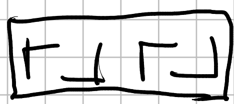
Base: 2×2, 2×4, 4×4.

OK!

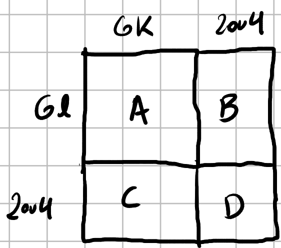
Suponha que a hipótese é verdadeira p/ todo tabuleiro $2n' \times 2m'$ onde $n' \cdot m' < n \cdot m$.

$\rightarrow$ Se $3 \nmid mn$ $\quad 2m = 6K+2$ ou $6K+4$

$2n = 6l+2$ ou $6l+4$



Somar 1 no 2×6 OK!



$A \mapsto A+1$

$B \mapsto B+1$

$C \mapsto C+1$

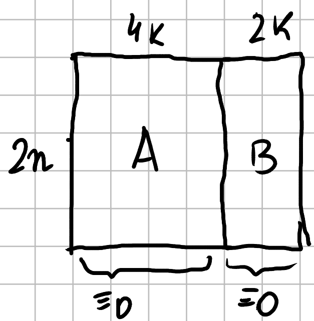
$D \mapsto D+3$

$\Rightarrow A = B = C = D$

OK!

$\rightarrow 3 \mid mn$. S.P.G $3 \mid m$. $\quad (\Sigma \equiv 0\ (3))$

$2m = 6K$



$4K \cdot 2n \cdot A \equiv 0 \quad (3)$

$2K \cdot 2n \cdot B \equiv 0 \quad (3)$

$K \equiv 0\ (3)$ ou $n \equiv 0\ (3)$ ou $A \equiv B\ (3)$

$K \equiv 0\ (3) \rightarrow$ Somar 1 no A/B ✓

$n \equiv 0\ (3) \rightarrow$ Somar 1 no A/B ✓

$A \equiv B\ (3)$: Somar 3 no A/B ✓

$x = 0,a_1 a_2 \dots$ é irracional

$\Leftrightarrow$

$a_1, a_2, \dots$ não é periódica a partir de nenhum ponto

$\underline{K=1}$: Supunha $p(1) \leq 1$

0, AAAAAA

$p(1) = p(2) = 2$.

$\underline{K=2}$

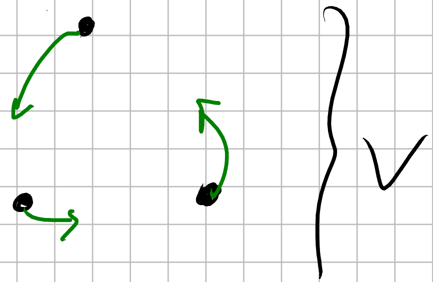
A A —

B B —

$\underline{K=3}$:

$p(2) = p(3) = 3$

Todas as possíveis seq de tamanho



$s_{1,1}\, s_{1,2} \longrightarrow s_{1,1}\, s_{1,2}\, s_{1,3}$

$s_{2,1}\, s_{2,2} \longrightarrow s_{2,1}\, s_{2,2}\, s_{2,3}$

$s_{3,1}\, s_{3,2} \longrightarrow s_{3,1}\, s_{3,2}\, s_{3,3}$

$x = 0, \underbrace{a_1 a_2}\, a_3 \dots$

$V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \dots \longrightarrow \dots \longrightarrow$

Prova por indução: $p(k) \geq k+1$.

$k = 1$ ∅

$p(k) \geq k+1 \to$ ok!

(k−1) $p(k-1) \geq k$

$p(k-1) = p(k) = k$

$X = \#$ bolas pretas c/ Bob, que é uma variável aleatória.

$$E[X] = \sum_{n=0}^{5} n \cdot P(X=n).$$

$$X(b,p) = \#$$

$$E[X(10,9)] = ?$$

$$E[X(b,p)] = y(b,p)$$

$$E[X(b,p)] = \frac{p}{b+p}\left(1+E[X(b,p-2)]\right) + \frac{b}{b+p}\left(E[X(b-1,p-1)]\right)$$

$$= \frac{p}{b+p} + \frac{p\,E[X(b,p-2)] + b\cdot E[X(b-1,p-1)]}{p+b}$$

$$(p+b)\,y(b,p) = p + p\cdot y(b,p-2) + b\cdot y(b-1,p-1)$$

$$y(x,0) = 0$$

$$y(x,1) = 1/_{x+1}$$

$$y(x,2) = \frac{2}{x+2} + \frac{1}{x+2} = 3/_{x+2}$$

$$y(x,3) = \frac{3}{x+3} + \frac{3\cdot(1/x+1) + x\cdot(3/x+1)}{x+3}$$

$$= 6/_{x+3}$$

$$Y(x,4) = \frac{4}{x+4} + \frac{4\cdot\left(3/x+2\right) + x\cdot\left(6\cdot/x+2\right)}{x+4}$$

$$= \frac{4}{x+4} + \frac{12 + 6x}{(x+4)(x+2)} = \frac{10}{x+4}$$

Conj: $Y(b,p) = \dfrac{p(p+1)}{2(b+p)}$

$$Y(b,p) = \frac{p}{b+p} + \frac{p\,Y(b,p-2) + b\cdot Y(b-1,p-1)}{p+b}$$

$$= \frac{1}{b+p}\left(p + p\cdot\frac{(p-2)(p-1)}{2(p+b-2)} + b\cdot\frac{(p-1)p}{2(p+b-2)}\right)$$

$$= \frac{p}{2(b+p)}\left(2 + \frac{(p-1)\cancel{(b+p-2)}}{\cancel{p+b-2}}\right)$$

$$= \frac{p(p+1)}{2(b+p)}$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
+	-	+	+	+	-	-	-	+	-	+	+	-	+	+	+	+	-	+	+
		+				-				+				+				+	

21	22	23	24
+	-	-	-
		+	

→ provar que $\forall n \leq 4k$: $x_1 + \dots + x_n \geq 0$.

↪ indução em k.

$$x_{4k} = -x_{2k}$$

$$x_{4k-1} = -x_{2k}$$

$$x_{4k-2} = -x_{2k-1}$$

$$x_{4k-3} = x_{2k-1}$$

$$S_k = \sum_{i=1}^{k} a_i$$

$$S_{4k} = 2S_k, \quad S_{4k+2} = 2S_k.$$

$$\begin{aligned} S_{4k+1} &= S_{4k} + x_{4k+1} \\ &= S_{4k} + x_{2k+1} \\ &= 2S_k + x_{2k+1} \\ &= 2S_k + (-1)^k x_{k+1} \end{aligned}$$

$S_k + S_{k+1} \geq 0$

$S_k + S_{k-1} \geq 0$

$$S_{4k+3} = S_{4k+2} + x_{4k+3}$$

$$= 2S_k + x_{k+1} = S_k + S_{k+1}.$$

$$\left.\begin{array}{l} \bullet\ P(1) \text{ verdade} \\ \bullet\ P(n-1) \to P(n) \text{ verdade} \end{array}\right\}$$

$\Rightarrow$ $P(n)$ verdade, $\forall n \in \mathbb{N}$.

$$\left.\begin{array}{l} \bullet\ P(1) \text{ verdade} \\ \bullet\ P(1), P(2), \ldots, P(n-1) \to P(n) \text{ verdade} \end{array}\right\}$$

$\Rightarrow$ $P(n)$ verdade.