conjunto em que pode ter gente repetida.

Sejam $\{a_1, \ldots, a_n\}$ e $\{b_1, \ldots, b_n\}$ dois multiconjuntos distintos, cada um deles formado por inteiros positivos. Se a igualdade dos seguintes multiconjuntos é verdadeira

$$\{a_i + a_j ; 1 \leqslant i < j \leqslant n\} = \{b_i + b_j ; 1 \leqslant i < j \leqslant n\},$$

SOMA DE PARES

prove que n é uma potência de 2.

$\underline{n=3}$ $\{a_1 \le a_2 \le a_3\}$, $\{b_1 \le b_2 \le b_3\}$

$$\{a_1+a_2,\ a_1+a_3,\ \underline{a_2+a_3}\} = \{b_1+b_2,\ b_1+b_3,\ \underline{b_2+b_3}\}$$

- $2(a_1+a_2+a_3) = 2(b_1+b_2+b_3)$
- $\min(1°) = a_1+a_2 = \min(2°) = b_1+b_2$

$\Rightarrow a_3 = b_3 \Rightarrow \{a_1, a_2, a_3\} = \{b_1, b_2, b_3\}$ $\underline{\text{OK!}}$

$\underline{n=5}$:

$$\begin{array}{c|cccc}
 & a_1 & a_2 & a_3 & a_4 \\
a_2 & \boxed{a_1+a_2} & & & \\
a_3 & \boxed{a_1+a_3} & a_2+a_3 & & \\
a_4 & a_1+a_4 & a_2+a_4 & a_3+a_4 & \\
a_5 & a_1+a_5 & a_2+a_5 & \boxed{a_3+a_5} & \boxed{a_4+a_5}
\end{array}
\qquad
\begin{array}{c|cccc}
 & b_1 & b_2 & b_3 & b_4 \\
b_2 & \boxed{b_1+b_2} & & & \\
b_3 & \boxed{b_1+b_3} & b_2+b_3 & & \\
b_4 & b_1+b_4 & b_2+b_4 & b_3+b_4 & \\
b_5 & b_1+b_5 & b_2+b_5 & \boxed{b_3+b_5} & \boxed{b_4+b_5}
\end{array}$$

$a_1+a_2 = b_1+b_2$ $\qquad$ $a_4+a_5 = b_4+b_5$

$a_1+a_3 = b_1+b_3$ $\qquad$ $a_3+a_5 = b_3+b_5$

$\underline{n=4}$

$$\begin{array}{c|ccc}
 & a_1 & a_2 & a_3 \\
a_2 & a_1+a_2 & & \\
a_3 & a_1+a_3 & a_2+a_3 & \\
a_4 & a_1+a_4 & a_2+a_4 & a_3+a_4
\end{array}
\qquad
\begin{array}{c|ccc}
 & b_1 & b_2 & b_3 \\
b_2 & b_1+b_2 & & \\
b_3 & b_1+b_3 & b_2+b_3 & \\
b_4 & b_1+b_4 & b_2+b_4 & b_3+b_4
\end{array}$$

- $\overset{1}{a_1} + \overset{2}{a_2} = \overset{0}{b_1} + \overset{3}{b_2}$ $\qquad$ - $\overset{3}{a_3} + \overset{6}{a_4} = \overset{4}{b_3} + \overset{5}{b_4}$
- $\overset{1}{a_1} + \overset{3}{a_3} = \overset{0}{b_1} + \overset{4}{b_3}$ $\qquad$ - $\overset{2}{a_2} + \overset{6}{a_4} = \overset{3}{b_2} + \overset{5}{b_4}$

4 7

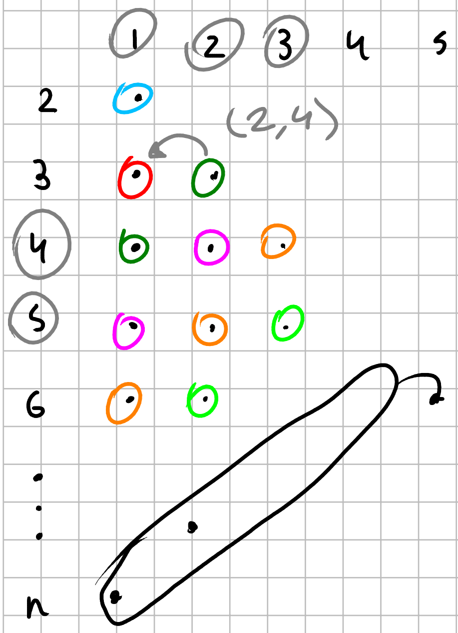
$A = \{1, 2, 3, 6\}$, $\qquad$ $B = \{0, 3, 4, 5\}$

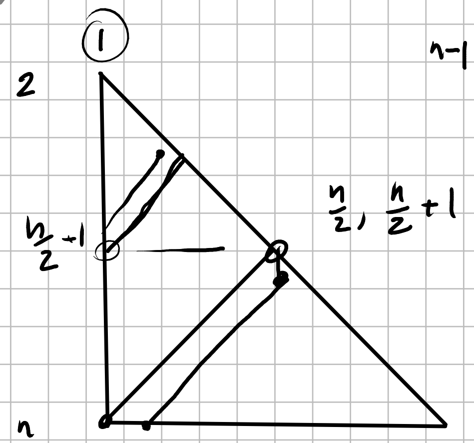
3 5 9 8 $\qquad$ 7 9 $\qquad$ 3 4 5 8

$n=6$

- $a_1 + a_2 = b_1 + b_2 \leftarrow a_3 + a_4 = b_3 + b_4 \rightarrow$ • $a_5 + a_6 = b_5 + b_6$
- $a_1 + a_3 = b_1 + b_3 \leftarrow a_2 + a_5 = b_2 + b_5 \rightarrow$ • $a_4 + a_6 = b_4 + b_6$

$(1,2,4)$





$P \neq Q$

$$P(x) = \sum_{i=1}^{n} x^{a_i} \qquad Q(x) = \sum_{i=1}^{n} x^{b_j}$$

$$P(x)P(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} x^{c_i + c_j} \qquad Q(x)Q(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} x^{b_i + b_j}$$

$$= \sum_{i=1}^{n} x^{2c_i} + 2\underbrace{\sum_{i<j} x^{a_i + a_j}} \qquad = \sum_{i=1}^{n} x^{2b_i} + 2\underbrace{\sum_{i<j} x^{b_i + b_j}}$$

$$\Rightarrow P(x)^2 - Q(x)^2 = P(x^2) - Q(x^2)$$

$$(P(x) + Q(x))(P(x) - Q(x)) = P(x^2) - Q(x^2)$$

$$P(x) + Q(x) = \frac{(P-Q)(x^2)}{(P-Q)(x)} = \frac{(P-Q)^{(n)}(x^2)\cdot 2^n x}{(P-Q)^{(n)}(x)} = 2^n$$

$2n$

$\lim \quad x \to 1$

xoornel ++

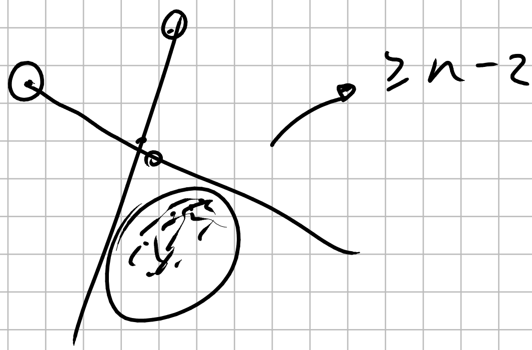
$$\lim_{x\to 1} \underbrace{f(x)}_{\text{polinomio}} \cdot x = \left[\lim_{x\to 1} f(x)\right] \cdot \underbrace{\left[\lim_{x\to 1} x\right]}$$

Um plano tem um ponto especial O chamado origem. Seja P um conjunto de ~~2021~~ n pontos no plano tal que

- não existem três pontos em P que sejam colineares.
- não existem dois pontos que estejam numa reta que passe pela origem.

Um triângulo com vértices em P é chamado de *gordo* se O está estritamente interno ao triângulo. Encontre o maior número de triângulos gordos.

CONSTRUÇÕES:



$\approx n/3$ $\approx n/3$

$$\left(\frac{n}{3}\right)^3 = \frac{n^3}{27}$$

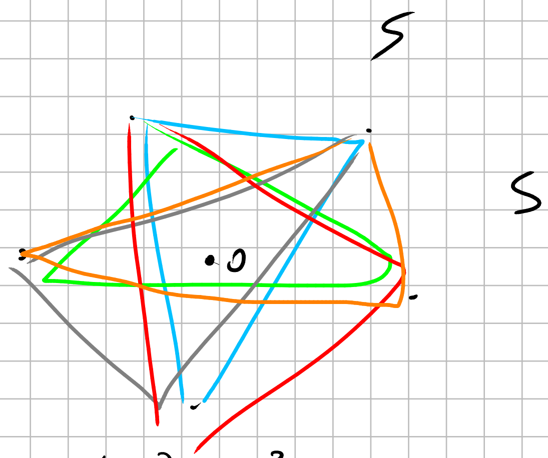
0

$\approx n/3$

2

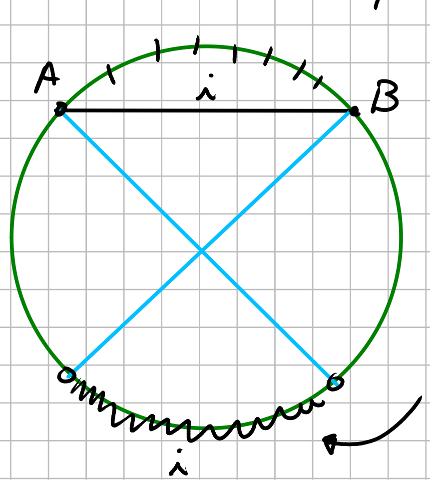
?

1 4



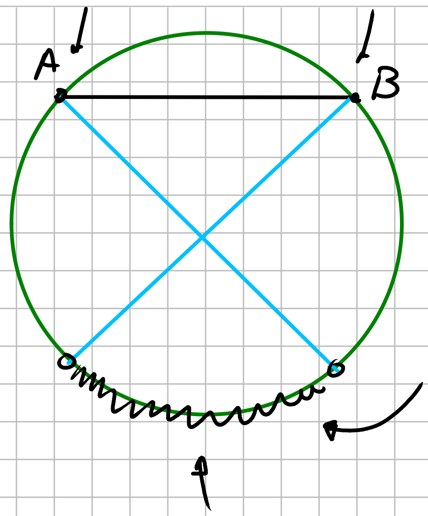
$$5\cdot\left(\frac{n}{5}\right)^3 = \frac{n^3}{25}.$$

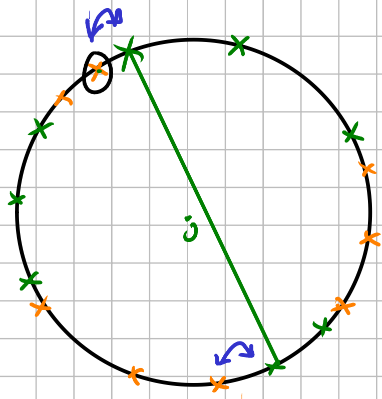
n ímpar

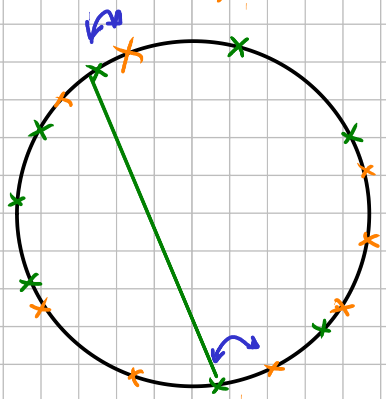


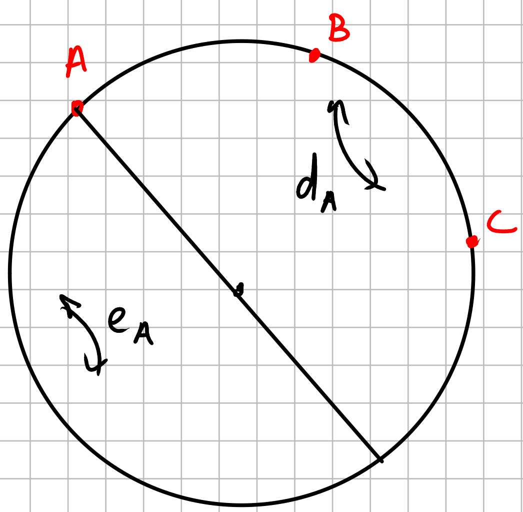
$$\sum_{i=1}^{\frac{n-1}{2}} n\cdot i = n\cdot\frac{\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)}{2}$$

$$\#\Delta = \frac{n\cdot(n-1)(n+1)}{24} = \frac{1}{4}\binom{n+1}{3}$$









Vamos contar (A,B,C) t.q.
$B \in$ arco menor $\widehat{AC}$.

- contando por AC, é $2 \cdot \#$ triângulos magros

- contando por A

é $\displaystyle\sum_A \binom{d_A}{2} + \binom{e_A}{2} \geq \sum_A 2\binom{\frac{n-1}{2}}{2}$

$$\geq n \cdot \left(\frac{n-1}{2}\right) \cdot \left(\frac{n-3}{2}\right)$$

$$\#\text{triângulos magros} \geq \frac{n(n-1)(n-3)}{8}$$

$$\#\text{triângulos gordos} \leq \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)(n-3)}{8}$$

$$\leq \frac{n(n-1)}{2}\left(\frac{n-2}{3} - \frac{n-3}{4}\right)$$

$$\leq \frac{n(n-1)}{2} \cdot \frac{(n+1)}{12}.$$

ok!