Terceiro Teste de Seleção

LX Olímpíada Internacional de Matemática e XXXIV Olimpíada Iberoamericana

Problema 1 Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

Problema 2 Dado um triângulo ABC, sejam A_1, B_1 e C_1 pontos sobre os lados BC, CA e AB, respectivamente, tais que o trângulo $A_1B_1C_1$ é equilátero. Sejam I_1 e ω_1 o incentro e o incírculo de AB_1C_1. Defina I_2, ω_2 e I_3, ω_3 similarmente com respeito aos triângulos BA_1C_1 e CA_1B_1, respectivamente. Seja ℓ_1 a reta tangente externamente a ω_2 e ω_3 diferente da reta BC. Defina ℓ_2 e ℓ_3 similarmente com respeito aos pares ω_1, ω_3 e ω_1, ω_2. Sabendo que $A_1I_2 = A_1I_3$, mostre que ℓ_1, ℓ_2 e ℓ_3 são concorrentes.

Problema 3 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

Problema 4 Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m+n) | f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.