

## Geometria 2

Guilherme Zeus Moura  
zeusdanmou@gmail.com

**Problema 1 (OBM 2016, 1)** Seja  $ABC$  um triângulo. As retas  $r$  e  $s$  são bissetrizes internas de  $\angle ABC$  e  $\angle BCA$ , respectivamente. Os pontos  $E$  sobre  $r$  e  $D$  sobre  $s$  são tais que  $AD \parallel BE$  e  $AE \parallel CD$ . As retas  $BD$  e  $CE$  se cortam em  $F$ . Seja  $I$  o incentro do triângulo  $ABC$ . Mostre que se os pontos  $A$ ,  $F$  e  $I$  são colineares então  $AB = AC$ .

**Problema 2 (OBM 2015, 1)** Seja  $ABC$  um triângulo escaleno e acutângulo e  $N$  o centro do círculo que passa pelos pés das três alturas do triângulo. Seja  $D$  a interseção das retas tangentes ao circuncírculo de  $ABC$  e que passam por  $B$  e  $C$ . Prove que  $A$ ,  $D$  e  $N$  são colineares se, e somente se,  $\angle BAC = 45^\circ$ .

**Problema 3 (OBM 2014, 1)** Seja  $ABCD$  um quadrilátero convexo e seja  $P$  a interseção das diagonais  $AC$  e  $BD$ . Os raios dos círculos inscritos nos triângulos  $ABP$ ,  $BPC$ ,  $CPD$  e  $DAP$  são iguais. Prove que  $ABCD$  é um losango.

**Problema 4 (OBM 2010, 4)** Seja  $ABCD$  um quadrilátero convexo e  $M$  e  $N$  os pontos médios dos lados  $CD$  e  $AD$ , respectivamente. As retas perpendiculares a  $AB$  passando por  $M$  e a  $BC$  passando por  $N$  cortam-se no ponto  $P$ . Prove que  $P$  pertence à diagonal  $BD$  se, e somente se, as diagonais  $AC$  e  $BD$  são perpendiculares.

**Problema 5 (OBM 2008, 4)** Seja  $ABCD$  um quadrilátero cíclico e  $r$  e  $s$  as retas simétricas à reta  $AB$  em relação às bissetrizes internas dos ângulos  $\angle CAD$  e  $\angle CBD$ , respectivamente. Sendo  $P$  a interseção de  $r$  e  $s$  e  $O$  o centro do círculo circunscrito a  $ABCD$ , prove que  $OP$  é perpendicular a  $CD$ .

**Problema 6 (Banco IMO 2018, G1)** Seja  $\Gamma$  o circuncírculo do triângulo acutângulo  $ABC$ . Os pontos  $D$  e  $E$  estão sobre os segmentos  $AB$  e  $AC$ , respectivamente, de modo que  $AD = AE$ . As mediatrizes de  $BD$  e  $CE$  intersectam os arcos menores  $AB$  e  $AC$  de  $\Gamma$  nos pontos  $F$  e  $G$ , respectivamente. Prove que as retas  $DE$  e  $FG$  são paralelas (ou são a mesma reta).

**Problema 7 (Banco IMO 2017, G1)** Let  $ABCDE$  be a convex pentagon such that  $AB = BC = CD$ ,  $\angle EAB = \angle BCD$ , and  $\angle EDC = \angle CBA$ . Prove that the perpendicular line from  $E$  to  $BC$  and the line segments  $AC$  and  $BD$  are concurrent.

**Problema 8 (Banco IMO 2016, G1)** Triangle  $BCF$  has a right angle at  $B$ . Let  $A$  be the point on line  $CF$  such that  $FA = FB$  and  $F$  lies between  $A$  and  $C$ . Point  $D$  is chosen so that  $DA = DC$  and  $AC$  is the bisector of  $\angle DAB$ . Point  $E$  is chosen so that  $EA = ED$  and  $AD$  is the bisector of  $\angle EAC$ . Let  $M$  be the midpoint of  $CF$ . Let  $X$  be the point such that  $AMXE$  is a parallelogram. Prove that  $BD$ ,  $FX$  and  $ME$  are concurrent.

**Problema 9 (Banco IMO 2015, G1)** Let  $ABC$  be an acute triangle with orthocenter  $H$ . Let  $G$  be the point such that the quadrilateral  $ABGH$  is a parallelogram. Let  $I$  be the point on the line  $GH$  such that  $AC$  bisects  $HI$ . Suppose that the line  $AC$  intersects the circumcircle of the triangle  $GCI$  at  $C$  and  $J$ . Prove that  $IJ = AH$ .

**Problema 10 (Banco IMO 2014, G1)** Let  $P$  and  $Q$  be on segment  $BC$  of an acute triangle  $ABC$  such that  $\angle PAB = \angle BCA$  and  $\angle CAQ = \angle ABC$ . Let  $M$  and  $N$  be the points on  $AP$  and  $AQ$ , respectively, such that  $P$  is the midpoint of  $AM$  and  $Q$  is the midpoint of  $AN$ . Prove that the intersection of  $BM$  and  $CN$  is on the circumference of triangle  $ABC$ .

**Problema 11 (Banco IMO 2013, G1)** Let  $ABC$  be an acute triangle with orthocenter  $H$ , and let  $W$  be a point on the side  $BC$ , lying strictly between  $B$  and  $C$ . The points  $M$  and  $N$  are the feet of the altitudes from  $B$  and  $C$ , respectively. Denote by  $\omega_1$  is the circumcircle of  $BWN$ , and let  $X$  be the point on  $\omega_1$  such that  $WX$  is a diameter of  $\omega_1$ . Analogously, denote by  $\omega_2$  the circumcircle of triangle  $CWM$ , and let  $Y$  be the point such that  $WY$  is a diameter of  $\omega_2$ . Prove that  $X$ ,  $Y$  and  $H$  are collinear.

**Problema 12 (Banco IMO 2012, G1)** Given triangle  $ABC$  the point  $J$  is the centre of the excircle opposite the vertex  $A$ . This excircle is tangent to the side  $BC$  at  $M$ , and to the lines  $AB$  and  $AC$  at  $K$  and  $L$ , respectively. The lines  $LM$  and  $BJ$  meet at  $F$ , and the lines  $KM$  and  $CJ$  meet at  $G$ . Let  $S$  be the point of intersection of the lines  $AF$  and  $BC$ , and let  $T$  be the point of intersection of the lines  $AG$  and  $BC$ . Prove that  $M$  is the midpoint of  $ST$ .

**Problema 13 (Banco IMO 2018, G2)** Seja  $ABC$  um triângulo com  $AB = AC$ , e seja  $M$  o ponto médio de  $BC$ . Seja  $P$  um ponto tal que  $PB < PC$  e  $PA$  paralelo a  $BC$ . Sejam  $X$  e  $Y$  pontos nas retas  $PB$  e  $PC$ , respectivamente, tal que  $B$  cai no segmento  $PX$ ,  $C$  cai no segmento  $PY$ , e  $\angle PXM = \angle PYM$ . Prove que o quadrilátero  $APXY$  é cíclico.

**Problema 14 (Banco IMO 2017, G2)** Let  $R$  and  $S$  be different points on a circle  $\Omega$  such that  $RS$  is not a diameter. Let  $\ell$  be the tangent line to  $\Omega$  at  $R$ . Point  $T$  is such that  $S$  is the midpoint of the line segment  $RT$ . Point  $J$  is chosen on the shorter arc  $RS$  of  $\Omega$  so that the circumcircle  $\Gamma$  of triangle  $JST$  intersects  $\ell$  at two distinct points. Let  $A$  be the common point of  $\Gamma$  and  $\ell$  that is closer to  $R$ . Line  $AJ$  meets  $\Omega$  again at  $K$ . Prove that the line  $KT$  is tangent to  $\Gamma$ .

**Problema 15 (Banco IMO 2016, G2)** Let  $ABC$  be a triangle with circumcircle  $\Gamma$  and incenter  $I$  and let  $M$  be the midpoint of  $\overline{BC}$ . The points  $D, E, F$  are selected on sides  $\overline{BC}, \overline{CA}, \overline{AB}$  such that  $\overline{ID} \perp \overline{BC}, \overline{IE} \perp \overline{AI}$ , and  $\overline{IF} \perp \overline{AI}$ . Suppose that the circumcircle of  $\triangle AEF$  intersects  $\Gamma$  at a point  $X$  other than  $A$ . Prove that lines  $XD$  and  $AM$  meet on  $\Gamma$ .

**Problema 16 (Banco IMO 2015, G2)** Triangle  $ABC$  has circumcircle  $\Omega$  and circumcenter  $O$ . A circle  $\Gamma$  with center  $A$  intersects the segment  $BC$  at points  $D$  and  $E$ , such that  $B, D, E$ , and  $C$  are all different and lie on line  $BC$  in this order. Let  $F$  and  $G$  be the points of intersection of  $\Gamma$  and  $\Omega$ , such that  $A, F, B, C$ , and  $G$  lie on  $\Omega$  in this order. Let  $K$  be the second point of intersection of the circumcircle of triangle  $BDF$  and the segment  $AB$ . Let  $L$  be the second point of intersection of the circumcircle of triangle  $CGE$  and the segment  $CA$ .

Suppose that the lines  $FK$  and  $GL$  are different and intersect at the point  $X$ . Prove that  $X$  lies on the line  $AO$ .

**Problema 17 (Banco IMO 2014, G2)** Let  $ABC$  be a triangle. The points  $K, L$ , and  $M$  lie on the segments  $BC, CA$ , and  $AB$ , respectively, such that the lines  $AK, BL$ , and  $CM$  intersect in a common point. Prove that it is possible to choose two of the triangles  $ALM, BMK$ , and  $CKL$  whose inradii sum up to at least the inradius of the triangle  $ABC$ .

**Problema 18 (Banco IMO 2013, G2)** Let  $\omega$  be the circumcircle of a triangle  $ABC$ . Denote by  $M$  and  $N$  the midpoints of the sides  $AB$  and  $AC$ , respectively, and denote by  $T$  the midpoint of the arc  $BC$  of  $\omega$  not containing  $A$ . The circumcircles of the triangles  $AMT$  and  $ANT$  intersect the perpendicular bisectors of  $AC$  and  $AB$  at points  $X$  and  $Y$ , respectively; assume that  $X$  and  $Y$  lie inside the triangle  $ABC$ . The lines  $MN$  and  $XY$  intersect at  $K$ . Prove that  $KA = KT$ .

**Problema 19 (Banco IMO 2012, G2)** Let  $ABCD$  be a cyclic quadrilateral whose diagonals  $AC$  and  $BD$  meet at  $E$ . The extensions of the sides  $AD$  and  $BC$  beyond  $A$  and  $B$  meet at  $F$ . Let  $G$  be the point such that  $ECGD$  is a parallelogram, and let  $H$  be the image of  $E$  under reflection in  $AD$ . Prove that  $D, H, F, G$  are concyclic.

**Problema 20 (OBM 2012, 2)** Dado um triângulo  $ABC$ , o exincentro relativo ao vértice  $A$  é o ponto de interseção das bissetrizes externas de  $DB$  e  $DC$ . Sejam  $I_A, I_B$  e  $I_C$  os exincentros do triângulo escaleno  $ABC$  relativos a  $A, B$  e  $C$ , respectivamente, e  $X, Y$  e  $Z$  os pontos médios de  $I_B I_C, I_C I_A$  e  $I_A I_B$ , respectivamente. O incírculo do triângulo  $ABC$  toca os lados  $BC, CA$  e  $AB$  nos pontos  $D, E$  e  $F$ , respectivamente. Prove que as retas  $DX, EY$  e  $FZ$  têm um ponto em comum pertencente à reta  $IO$ , sendo  $I$  e  $O$  o incentro e o circuncentro do triângulo  $ABC$ , respectivamente.

**Problema 21 (OBM 2011, 5)** Seja  $ABC$  um triângulo acutângulo e  $H$  seu ortocentro. As retas  $BH$  e  $CH$  cortam  $AC$  e  $AB$  em  $D$  e  $E$ , respectivamente. O circuncírculo de  $ADE$  corta o circuncírculo de  $ABC$  em  $F \neq A$ . Provar que as bissetrizes internas de  $\angle BFC$  e  $\angle BHC$  se cortam em um ponto sobre o segmento  $BC$ .

**Problema 22 (OBM 2007, 5)** Seja  $ABCD$  um quadrilátero convexo,  $P$  a interseção das retas  $AB$  e  $CD$ ,  $Q$  a interseção das retas  $AD$  e  $BC$  e  $O$  a interseção das diagonais  $AC$  e  $BD$ . Prove que se  $\angle POQ$  é um ângulo reto então  $PO$  é bissetriz de  $\angle AOD$  e  $QO$  é bissetriz de  $\angle AOB$ .

**Problema 23 (OBM 2017, 5)** No triângulo  $ABC$ , seja  $r_A$  a reta que passa pelo ponto médio de  $BC$  e é perpendicular à bissetriz interna de  $\angle BAC$ . Defina  $r_B$  e  $r_C$  da mesma forma. Sejam  $H$  e  $I$  o ortocentro e o incentro de  $ABC$ , respectivamente. Suponha que as três retas  $r_A, r_B, r_C$  definem um triângulo. Prove que o circuncentro desse triângulo é o ponto médio de  $HI$ .

**Problema 24 (OBM 2017, 3)** Um quadrilátero  $ABCD$  tem um círculo inscrito  $\omega$  e é tal que as semirretas  $AB$  e  $DC$  se cortam no ponto  $P$  e as semirretas  $AD$  e  $BC$  se cortam no ponto  $Q$ . As retas  $AC$  e  $PQ$  se cortam no ponto  $R$ . Seja  $T$  o ponto de  $\omega$  mais próximo da reta  $PQ$ . Prove que a reta  $RT$  passa pelo incentro do triângulo  $PQC$ .