2017 Russian Mathematical Olympiad

9th grade

9.1. In country some cities are connected with one-directional direct flights (between any two cities, there is at most one flight). City A called *available* for city B, if there is a chain of fights that start from B and ends at A (perhaps with zero, one or more flights). It is known, that for every 2 cities P and Q exist city R, such that P and Q are available from R. Prove, that exist city A, such that every city is available for A.

9.2. $ABCD$ is an isosceles trapezoid with $BC||AD$. A circle ω passing through B and C intersects the side AB and the diagonal BD at points X and Y respectively. Tangent to ω at C intersects the line AD at Z. Prove that the points X, Y, and Z are collinear.

9.3. There are 100 dwarfes with weight $1, 2, ..., 100$. They sit on the left riverside. They can not swim, but they have one boat with capacity 100. River has strong river flow, so every dwarf has power only for one passage from right side to left as oarsman. On every passage can be only one oarsman. Can all dwarfes get to right riverside?

9.4. Are there infinite increasing sequence of natural numbers, such that sum of every 2 different numbers are relatively prime with sum of every 3 different numbers?

9.5. There are $n > 3$ different natural numbers, less than $(n-1)!$. For every pair of numbers Sergey divides bigest on lowest and write integer quotient (for example, 100 divides $7 = 14$) and write result on the paper. Prove, that not all numbers on paper are different.

9.6. Determine whether for every three distinct positive integers a, b, c there exists a quadratic polynomial $P(x) = kx^2 + lx + m$, with k, l, m integers, $k > 0$, such that for some integer points this polynomial takes the values a^3, b^3, c^3?

9.7. In the scalene triangle ABC, $\angle ACB = 60^\circ$ and Ω is its cirumcirle. On the bisectors of the angles BAC and CBA points A', B' are chosen respectively such that $AB' \parallel BC$ and $BA' \parallel AC$. $A'B'$ intersects with Ω at D, E. Prove that triangle CDE is isosceles.

9.8. Every cell of 100×100 table is colored black or white. Every cell on table border is black. It is known, that in every 2×2 square there are cells of two colors. Prove, that exist 2×2 square that is colored in chess order.

10th grade

10.1. In the Cartesian plane, two graphs Γ_1 and Γ_2 of monic quadratic trinomials and two non-parallel lines ℓ_1 and ℓ_2 are drawn. Assume that Γ_1 abd Γ_2 cut out segments of equal lenghts on ℓ_1, and cut out segments of equal lenghts on ℓ_2. Prove that Γ_1 and Γ_2 coincide.

10.2. Let ABC be an acute angled isosceles triangle with $AB = AC$ and circumcentre O. Lines BO and CO intersect AC, AB respectively at B', C'. A straight line l is drawn through C' parallel to AC. Prove that the line l is tangent to the circumcircle of $\triangle B'OC$.

10.3. There are 3 heaps with $100, 101, 102$ stones. Ilya and Kostya play following game. Every step they take one stone from some heap, but not from same, that was on previous step. They make his steps in turn, Ilya make first step. Player loses if can not make step. Who has winning strategy?

10.4. Positive numbers $a_1, a_2, \dots, a_n$ are written on the board in a row. For every $i = 1, 2, \dots, n$, Vasya wishes to write a number $b_i \geq a_i$, so that for every integer $i, j \in \{1, 2, \dots, n\}$, at least one of the ratios b_i/b_j or b_j/b_i is an integer. Prove that Vasya can reach this goal so that $b_1 b_2 \cdots b_n \leq 2^{(n-1)/2} a_1 a_2 \dots a_n$.

10.5. Assume that n is a composite positive integer. For each its proper divisor d, we write down the number $d+1$ on the board. Find all values of n for which the written numbers appear to be exactly all the proper divisors of some positive integer m.

(A proper divisor of a positive integer $a > 1$ is any of its positive divisors distinct from 1 and a.)

10.6. Let $P(x)$ be a polynomial of degree $n \geq 2$ with non-negative coefficients. Let a, b, c be the sides of an acute-angled triangle. Prove that the numbers $\sqrt[n]{P(a)}$, $\sqrt[n]{P(b)}$, $\sqrt[n]{P(c)}$ are also the sides of some acute-angled triangle.

10.7. Every cell of 100×100 table is colored black or white. Every cell on table border is black. It is known, that in every 2×2 square there are cells of two colors. Prove, that exist 2×2 square that is colored in chess order.

10.8. In a non-isosceles triangle ABC, O and I are circumcenter and incenter, respectively. B' is reflection of B with respect to OI and lies inside the angle ABI.Prove that the tangents to circumcirle of $\triangle BB'I$ at B' and I intersect on AC.

11th grade

11.1. Um real x é escolhido tal que cada uma das somas $S = \text{sen}64x + \text{sen}65x$ e $C = \cos 64x + \cos 65x$ é racional. Prove que, em uma dessas somas, ambas as parcelas são racionais.

11.2. Let ABC be an acute angled isosceles triangle with $AB = AC$ and circumcentre O. Lines BO and CO intersect AC, AB respectively at B', C'. A straight line l is drawn through C' parallel to AC. Prove that the line l is tangent to the circumcircle of $\triangle B'OC$.

11.3. Positive numbers $a_1, a_2, \dots, a_n$ are written on the board in a row. For every $i = 1, 2, \dots, n$, Vasya wishes to write a number $b_i \geq a_i$, so that for every integer $i, j \in \{1, 2, \dots, n\}$, at least one of the ratios b_i/b_j or b_j/b_i is an integer. Prove that Vasya can reach this goal so that $b_1 b_2 \cdots b_n \leq 2^{(n-1)/2} a_1 a_2 \dots a_n$.

11.4. A prestidigitator and his assistant have a deck of cards; back sides of all the cards look identical, the front side of each card is painted in one of 2017 colors (there are 1000000 cards of each color in the deck). They want to perform the following trick. The prestidigitator goes out of the room. The spectators put n cards facing front in a row onto the table. The assistant looks at them and turns all the cards except one facing back side (he does not change the order of the cards). Finally, the prestidigitator comes in, stares at the table, and guesses the color of one of the cards facing back. Find the least value of n for which the prestidigitator and the assistant can agree on their actions in advance so that they will do the trick for sure.

11.5. Let $P(x)$ be a polynomial of degree $n \geq 2$ with non-negative coefficients. Let a, b, c be the sides of an acute-angled triangle. Prove that the numbers $\sqrt[n]{P(a)}$, $\sqrt[n]{P(b)}$, $\sqrt[n]{P(c)}$ are also the sides of some acute-angled triangle.

11.6. Some cells of a 200×200 checkered square contain red or blue tokens - one per cell; the others are empty. We say that a token *sees* another if they are situated either in one row or in one column. Assume that each red token sees exactly five blue tokens (and, perhaps, some red tokens), and that each blue token sees exactly five red tokens (and, perhaps, some blue tokens). Determine the maximal possible total amount of tokens on the board.

11.7. Initially, a positive integer N is written on the board. At each moment, Misha may choose a number $a > 1$ on the board, remove it, and write all its positive divisors except a. After some time on the board there are N^2 numbers. Detemine all values of N for which this can happen.

11.8. Let $ABCD$ be a convex quadrilateral. Let I_A, I_B, I_C and I_D be the incenters of triangles DAB, ABC, BCD and CDA, respectively. Given that $\angle BI_AA + \angle I_CI_AI_D = 180^\circ$, prove that $\angle BI_BA + \angle I_CI_BI_D = 180^\circ$.