Problemas para Universitários

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- Caso você esteja vendo esse arquivo antes de 06 de Agosto de 2021, às 14:45, horário de Brasília, recomendo que pense primeiro nos problemas com numeração maior.
- Caso contrário, pense nos problemas na ordem em que eles estão propostos.
- Se não souber a definição de um termo, pergunte para seus colegas.

1. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

2. Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is unique real $n \times n$ matrix X that satisfies the equation $X + AX + XA^2 = A$.

(b) Express X in terms of A

3. Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

4. For a prime number p, let $GL_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\phi : GL_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

5. We say that a positive real number d is *good* if there exists an infinite squence $a_1, a_2, a_3, ... \in (0, d)$ such that for each positive integer n, the points $a_1, a_2, ..., a_n$ partition the interval $[0, d]$ into segments of length at most $\frac{1}{n}$ each. Find $\sup\{d : d \text{ is good}\}$.