Problema 4 (Álgebra / Shine)

(APMO 2009) Sejam a_1, a_2, a_3, a_4, a_5 números reais t.q.

$$\frac{a_1}{k^2+1}+\frac{a_2}{k^2+2}+\frac{a_3}{k^2+3}+\frac{a_4}{k^2+4}+\frac{a_5}{k^2+5}=\frac{1}{k^2} \quad (\text{para } k=1,2,3,4,5)$$

Calcule: $\frac{a_1}{37}+\frac{a_2}{38}+\frac{a_3}{39}+\frac{a_4}{40}+\frac{a_5}{41}$.

Considere o polinômio

$$P(k)=k^2\cdot(k^2+1)\cdots(k^2+5)\left(\frac{a_1}{k^2+1}+\cdots+\frac{a_5}{k^2+5}\right)-(k^2+1)\cdots(k^2+5) \quad (*)$$

Note que P tem grau 10 e raízes $1, 2, 3, 4, 5, -1, -2, -3, -4, -5$.

Logo: $P(k)=A\cdot(k+5)\cdots(k+1)(k-1)\cdots(k-5)$. $\quad (+)$

$P(0)=-5!$, por $(*)$, mas $P(0)=-A\cdot(5!)^2$, por $(+)$

$\Rightarrow A=1/5!$.

Logo: $P(6)=\frac{1}{\cancel{5!}}\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot\cancel{5\cdot 4\cdot 3\cdot 2\cdot 1}=\frac{11!}{6!}$, por $(+)$.

$\Rightarrow$ Por $(*)$: $\frac{11!}{6!}=6^2\cdot(6^2+1)\cdots(6^2+5)\left(\frac{a_1}{6^2+1}+\cdots+\frac{a_5}{6^2+5}\right)-(6^2+1)\cdots(6^2+5)$

$$\Rightarrow \frac{a_1}{6^2+1}+\cdots+\frac{a_5}{6^2+5}=\frac{11!}{6!\cdot 6^2\cdots(6^2+5)}+\frac{1}{6^2} \Rightarrow$$

$$\Rightarrow \frac{a_1}{37}+\frac{a_2}{38}+\frac{a_3}{39}+\frac{a_4}{40}+\frac{a_5}{41}=\frac{1}{36}+\frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{36\cdot 37\cdot 38\cdot 39\cdot 40\cdot 41}$$

$\square$