PROBLEMA 1
There are $n \geq 3$ distinct positive real numbers. Show that there are at most $n-2$ different integer power of three that can be written as the sum of three distinct elements from these n numbers.

PROBLEMA 2
$ABCD$ is a fixed rhombus. Segment PQ is tangent to the inscribed circle of $ABCD$, where P is on side AB, Q is on side AD. Show that, when segment PQ is moving, the area of ΔCPQ is a constant.

PROBLEMA 3
There are finite many coins in David's purse. The values of these coins are pair wisely distinct positive integers. Is that possible to make such a purse, such that David has exactly 2020 different ways to select the coins in his purse and the sum of these selected coins is 2020?

PROBLEMA 4
$S = \{1, 4, 8, 9, 16, ...\}$is the set of perfect integer power. ($S = \{n^k | n, k \in Z, k \geq 2\}$.)We arrange the elements in S into an increasing sequence $\{a_i\}$. Show that there are infinite many n, such that $9999 | a_{n+1} - a_n$

PROBLEMA 5
Simple graph G has 19998 vertices. For any subgraph $\bar{G}$ of G with 9999 vertices, $\bar{G}$ has at least 9999 edges. Find the minimum number of edges in G

PROBLEMA 1
Points A, B, C are on a plane such that $AB = BC = CA = 6$. At any step, you may choose any three existing points and draw that triangle's circumcentre. Prove that you can draw a point such that its distance from an previously drawn point is:(a) greater than $7(b)$ greater than 2019

PROBLEMA 2
Let a, b be positive integers such that $a + b^3$ is divisible by $a^2 + 3ab + 3b^2 - 1$. Prove that $a^2 + 3ab + 3b^2 - 1$ is divisible by the cube of an integer greater than 1.

PROBLEMA 3
You have a $2m$ by $2n$ grid of squares coloured in the same way as a standard checkerboard. Find the total number of ways to place mn counters on white squares so that each square contains at most one counter and no two counters are in diagonally adjacent white squares.

PROBLEMA 4
Prove that for $n > 1$ and real numbers $a_0, a_1, \ldots, a_n, k$ with $a_1 = a_{n-1} = 0$,

$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

PROBLEMA 5
A 2-player game is played on $n \geq 3$ points, where no 3 points are collinear. Each move consists of selecting 2 of the points and drawing a new line segment connecting them. The first player to draw a line segment that creates an odd cycle loses. (An odd cycle must have all its vertices among the n points from the start, so the vertices of the cycle cannot be the intersections of the lines drawn.) Find all n such that the player to move first wins.

PROBLEMA 1

Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: select a pair of tokens at points A and B and move both of them to the midpoint of A and B.

We say that an arrangement of n tokens is collapsible if it is possible to end up with all n tokens at the same point after a finite number of moves. Prove that every arrangement of n tokens is collapsible if and only if n is a power of 2.

PROBLEMA 2

Let five points on a circle be labelled A, B, C, D, and E in clockwise order. Assume $AE = DE$ and let P be the intersection of AC and BD. Let Q be the point on the line through A and B such that A is between B and Q and $AQ = DP$ Similarly, let R be the point on the line through C and D such that D is between C and R and $DR = AP$. Prove that PE is perpendicular to QR.

PROBLEMA 3

Two positive integers a and b are prime-related if $a = pb$ or $b = pa$ for some prime p. Find all positive integers n, such that n has at least three divisors, and all the divisors can be arranged without repetition in a circle so that any two adjacent divisors are prime-related.

Note that 1 and n are included as divisors.

PROBLEMA 4

Find all polynomials $p(x)$ with real coefficients that have the following property: there exists a polynomial $q(x)$ with real coefficients such that

$$p(1) + p(2) + p(3) + \cdots + p(n) = p(n)q(n)$$

for all positive integers n.

PROBLEMA 5

Let k be a given even positive integer. Sarah first picks a positive integer N greater than 1 and proceeds to alter it as follows: every minute, she chooses a prime divisor p of the current value of N, and multiplies the current N by $p^k - p^{-1}$ to produce the next value of N. Prove that there are infinitely many even positive integers k such that, no matter what choices Sarah makes, her number N will at some point be divisible by 2018.

PROBLEMA 1
For pairwise distinct nonnegative reals a, b, c, prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2} > 2$$

.

PROBLEMA 2
Define a function $f(n)$ from the positive integers to the positive integers such that $f(f(n))$ is the number of positive integer divisors of n. Prove that if p is a prime, then $f(p)$ is prime.

PROBLEMA 3
Define S_n as the set $1, 2, \cdots, n$. A non-empty subset T_n of S_n is called *balanced* if the average of the elements of T_n is equal to the median of T_n. Prove that, for all n, the number of balanced subsets T_n is odd.

PROBLEMA 4
Let $ABCD$ be a parallelogram. Points P and Q lie inside $ABCD$ such that $\triangle ABP$ and $\triangle BCQ$ are equilateral. Prove that the intersection of the line through P perpendicular to PD and the line through Q perpendicular to DQ lies on the altitude from B in $\triangle ABC$.

PROBLEMA 5
There are 100 circles of radius one in the plane. A triangle formed by the centres of any three given circles has area at most 2017. Prove that there is a line intersecting at least three of the circles.

PROBLEMA 1

The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

(a) Prove that there is a sequence of replacements that will make the final number equal to 2.

(b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

PROBLEMA 2

Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

PROBLEMA 3

Find all polynomials $P(x)$ with integer coefficients such that $P(P(n) + n)$ is a prime number for infinitely many integers n.

PROBLEMA 4

Let A, B, and F be positive integers, and assume $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m+1, m+2, \ldots, m+A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

(i) any two distinct intervals are disjoint and not adjacent;

(ii) there are at least F positive integers with no lava between any two intervals; and

(iii) no lava is placed at any integer less than F.

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is $F = (n-1)A + B$, where n is the positive integer such that $\frac{A}{n+1} \leq B - A < \frac{A}{n}$.

PROBLEMA 5

Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

PROBLEMA 1
Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer n.

PROBLEMA 2
Let ABC be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that

$$\frac{AB \cdot AC + BC \cdot CA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2.$$

PROBLEMA 3
On a $(4n+2) \times (4n+2)$ square grid, a turtle can move between squares sharing a side.The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?

PROBLEMA 4
Let ABC be an acute-angled triangle with circumcenter O. Let I be a circle with center on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that

$$BP \cdot CQ = AP \cdot AQ.$$

Prove that I is tangent to the circumcircle of triangle BOC.

PROBLEMA 5
Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

PROBLEMA 1

Let $a_1, a_2, \dots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

is greater than or equal to $\frac{2^n-1}{2^n}$.

PROBLEMA 2

Let m and n be odd positive integers. Each square of an m by n board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of m and n.

PROBLEMA 3

Let p be a fixed odd prime. A p-tuple $(a_1, a_2, a_3, \dots, a_p)$ of integers is said to be good if(i) $0 \le a_i \le p-1$ for all i, and(ii) $a_1 + a_2 + a_3 + \cdots + a_p$ is not divisible by p, and(iii) $a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_pa_1$ is divisible by p.

Determine the number of good p-tuples.

PROBLEMA 4

The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.

PROBLEMA 5

Fix positive integers n and $k \ge 2$. A list of n integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after a finite number of steps, you can reach a state where at least $n-k+2$ of the numbers on the blackboard are all simultaneously divisible by k.

PROBLEMA 1
Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

PROBLEMA 2
The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that the $n+1$ numbers $0, a_1, a_1+a_2, a_1+a_2+a_3, \ldots, a_1+a_2+\cdots+a_n$ all have different remainders when divided by $n+1$?

PROBLEMA 3
Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90°$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

PROBLEMA 4
Let n be a positive integer. For any positive integer j and positive real number r, define $f_j(r)$ and $g_j(r)$ by

$$f_j(r) = \min(jr, n) + \min\left(\frac{j}{r}, n\right), \text{ and } g_j(r) = \min(\lceil jr \rceil, n) + \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Prove that

$$\sum_{j=1}^{n} f_j(r) \leq n^2 + n \leq \sum_{j=1}^{n} g_j(r)$$

for all positive real numbers r.

PROBLEMA 5
Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

PROBLEMA 1
Let x, y and z be positive real numbers. Show that $x^2 + xy^2 + xyz^2 \geq 4xyz - 4$.

PROBLEMA 2
For any positive integers n and k, let $L(n, k)$ be the least common multiple of the k consecutive integers $n, n+1, \ldots, n+k-1$. Show that for any integer b, there exist integers n and k such that $L(n, k) > bL(n+1, k)$.

PROBLEMA 3
Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$ and $\angle APB$ meet at a common point.

PROBLEMA 4
A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either passable or impassable. All edges on the boundary of the grid are impassable. You can give any of the commands up, down, left, or right.

All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of up, down, left, or right, then another, for as long as you want. Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that all of the robots end up on the same square at the same time.

PROBLEMA 5
A bookshelf contains n volumes, labelled 1 to n, in some order. The librarian wishes to put them in the correct order as follows. The librarian selects a volume that is too far to the right, say the volume with label k, takes it out, and inserts it in the k-th position. For example, if the bookshelf contains the volumes $1, 3, 2, 4$ in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the correct order $1, 2, 3, 4$.

(a) Show that if this process is repeated, then, however the librarian makes the selections, all the volumes will eventually be in the correct order.

(b) What is the largest number of steps that this process can take?

PROBLEMA 1
Consider 70-digit numbers with the property that each of the digits $1, 2, 3, ..., 7$ appear 10 times in the decimal expansion of n (and $8, 9, 0$ do not appear). Show that no number of this form can divide another number of this form.

PROBLEMA 2
Let $ABCD$ be a cyclic quadrilateral with opposite sides not parallel. Let X and Y be the intersections of AB, CD and AD, BC respectively. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively, and let the angle bisectors of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EFGH$ is a parallelogram.

PROBLEMA 3
Amy has divided a square into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of white equals the total area of red, determine the minimum of x.

PROBLEMA 4
Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a, which is divisible by 2011.

Note. A contiguous substring of an integer a is an integer with a decimal expansion equivalent to a sequence of consecutive digits taken from the decimal expansion of a.

PROBLEMA 5
Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence of n integers $\epsilon_1, \epsilon_2, ..., \epsilon_n$, where $\epsilon_i = \pm 1$ (not necessarily dependent on each other) for all integers $1 \leq i \leq n$, such that $S = \sum_{i=1}^{n} \epsilon_i (1 + id)^2$.

PROBLEMA 1
For all natural n, an n-staircase is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically.

Let $f(n)$ denote the minimum number of square tiles requires to tile the n-staircase, where the side lengths of the square tiles can be any natural number. e.g. $f(2) = 3$ and $f(4) = 7$.

(a) Find all n such that $f(n) = n$.

(b) Find all n such that $f(n) = n + 1$.

PROBLEMA 2
Let A, B, P be three points on a circle. Prove that if a, b are the distances from P to the tangents at A, B respectively, and c is the distance from P to the chord AB, then $c^2 = ab$.

PROBLEMA 3
Three speed skaters have a friendly "race"on a skating oval. They all start from the same point and skate in the same direction, but with different speeds that they maintain throughout the race. The slowest skater does 1 lap per minute, the fastest one does 3.14 laps per minute, and the middle one does L laps a minute for some $1 < L < 3.14$. The race ends at the moment when all three skaters again come together to the same point on the oval (which may differ from the starting point.) Determine the number of different choices for L such that exactly 117 passings occur before the end of the race.

Note: A passing is defined as when one skater passes another one. The beginning and the end of the race when all three skaters are together are not counted as passings.

PROBLEMA 4
Each vertex of a finite graph can be coloured either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the colour of P and all of its neighbours. Is it possible to change the colour of every vertex from black to white by a

sequence of operations of this type?

Note: A finite graph consists of a finite set of vertices and a finite set of edges between vertices. If there is an edge between vertex A and vertex B, then A and B are neighbours of each other.

PROBLEMA 5
Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $\frac{P(a_n)}{Q(a_n)}$ is an integer for every n, then $\frac{P(n)}{Q(n)}$ is an integer for every integer n such that $Q(n) \neq 0$.

PROBLEMA 1
Given an $m \times n$ grid with unit squares coloured either black or white, a black square in the grid is stranded if there is some square to its left in the same row that is white and there is some square above it in the same column that is white.

Find a closed formula for the number of $2 \times n$ grids with no stranded black square.

Note that n is any natural number and the formula must be in terms of n with no other variables.

PROBLEMA 2
Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, so that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

PROBLEMA 3
Define $f(x, y, z) = \frac{(xy+yz+zx)(x+y+z)}{(x+y)(y+z)(z+x)}$.

Determine the set of real numbers r for which there exists a triplet of positive real numbers satisfying $f(x, y, z) = r$.

PROBLEMA 4
Find all ordered pairs of integers (a, b) such that $3^a + 7^b$ is a perfect square.

PROBLEMA 5
A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.

PROBLEMA 1
$ABCD$ is a convex quadrilateral for which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

PROBLEMA 2
Determine all functions f defined on the set of rational numbers that take rational values for which

$$f(2f(x)+f(y)) = 2x + y,$$

for each x and y.

PROBLEMA 3
Let a, b, c be positive real numbers for which $a+b+c=1$. Prove that

$$\frac{a-bc}{a+bc} + \frac{b-ca}{b+ca} + \frac{c-ab}{c+ab} \leq \frac{3}{2}.$$

PROBLEMA 4
Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

PROBLEMA 5
A self-avoiding rook walk on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, i.e., the rook's path is non-self-intersecting.

Let $R(m,n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m,1) = 1$ for all natural numbers m; $R(2,2) = 2$; $R(3,2) = 4$; $R(3,3) = 11$. Find a formula for $R(3,n)$ for each natural number n.

PROBLEMA 1
What is the maximum number of non-overlapping 2×1 dominoes that can be placed on a 8×9 checkerboard if six of them are placed as shown? Each domino must be placed horizontally or vertically so as to cover two adjacent squares of the board.

PROBLEMA 2
You are given a pair of triangles for which two sides of one triangle are equal in length to two sides of the second triangle, and the triangles are similar, but not necessarily congruent. Prove that the ratio of the sides that correspond under the similarity is a number between $\frac{1}{2}(\sqrt{5}-1)$ and $\frac{1}{2}(\sqrt{5}+1)$.

PROBLEMA 3
Suppose that f is a real-valued function for which

$$f(xy) + f(y-x) \geq f(y+x)$$

for all real numbers x and y.

a) Give a non-constant polynomial that satisfies the condition.

b) Prove that $f(x) \geq 0$ for all real x.

PROBLEMA 4
For two real numbers a, b, with $ab \neq 1$, define the $*$ operation by

$$a * b = \frac{a + b - 2ab}{1 - ab}.$$

Start with a list of $n \geq 2$ real numbers whose entries x all satisfy $0 < x < 1$. Select any two numbers a and b in the list; remove them and put the number $a * b$ at the end of the list, thereby reducing its length by one. Repeat this procedure until a single number remains.a. Prove that this single number is the same regardless of the choice of pair at each stage.b. Suppose that the condition on the numbers x is weakened to $0 < x \leq 1$. What happens if the list contains exactly one 1?

PROBLEMA 5
Let the incircle of triangle ABC touch sides BC, CA and AB at D, E and F, respectively. Let ω, ω_1, ω_2 and ω_3 denote the circumcircles of triangle ABC, AEF, BDF and CDE respectively.

Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and $R.a$. Prove that ω_1, ω_2 and ω_3 intersect in a common point.b. Show that PD, QE and RF are concurrent.

PROBLEMA 1
Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most 2 candies. For example $f(3,7) = 0, f(3,6) = 1, f(3,4) = 6$. Determine the value of $f(2006,1) + f(2006,4) + \ldots + f(2006,1000) + f(2006,1003) + \ldots + f(2006,4012)$.

PROBLEMA 2
Let ABC be acute triangle. Inscribe a rectangle $DEFG$ in this triangle such that $D \in AB, E \in AC, F \in BC, G \in BC$. Describe the locus of (i.e., the curve occupied by) the intersections of the diagonals of all possible rectangles $DEFG$.

PROBLEMA 3
In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

PROBLEMA 4
Consider a round-robin tournament with $2n+1$ teams, where each team plays each other team exactly one. We say that three teams X, Y and Z, form a cycle triplet if X beats Y, Y beats Z and Z beats X. There are no ties.

a)Determine the minimum number of cycle triplets possible.

b)Determine the maximum number of cycle triplets possible.

PROBLEMA 5
The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc BC and arc AC are supplementary. To each of three arcs, we draw a tangent such that its point of tangency is the mid point of that portion of the tangent intercepted by the extended lines AB, AC. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where tangent at D intersects the extended lines AB, AC. Similarly for E on arc AC and F on arc AB. Prove that triangle DEF is equilateral.

PROBLEMA 1
An equilateral triangle of side length n is divided into unit triangles. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in a path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example is shown on the picture for $n = 5$. Determine the value of $f(2005)$.

PROBLEMA 2
Let (a, b, c) be a Pythagorean triple, i.e. a triplet of positive integers with $a^2+b^2 = c^2$.$a)$ Prove that $\left(\frac{c}{a} + \frac{c}{b}\right)^2 > 8$.$b)$ Prove that there are no integer n and Pythagorean triple (a, b, c) satisfying $\left(\frac{c}{a} + \frac{c}{b}\right)^2 = n$.

PROBLEMA 3
Let S be a set of $n \geq 3$ points in the interior of a circle.$a)$ Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S.$b)$ Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

PROBLEMA 4
Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of: $\frac{KP}{R^3}$.

PROBLEMA 5
Let's say that an ordered triple of positive integers (a, b, c) is n-powerful if $a \leq b \leq c, \gcd(a, b, c) = 1$ and $a^n+b^n+c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful.$a)$ Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$.$b)$ Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

PROBLEMA 1

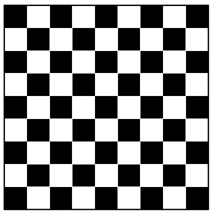
Find all ordered triples (x, y, z) of real numbers which satisfy the following system of equations:

$$\begin{cases} xy &= z - x - y \\ xz &= y - x - z \\ yz &= x - y - z \end{cases}$$

PROBLEMA 2

How many ways can 8 mutually non-attacking rooks be placed on the 9×9 chessboard (shown here) so that all 8 rooks are on squares of the same color?

(Two rooks are said to be attacking each other if they are placed in the same row or column of the board.)



PROBLEMA 3

Let A, B, C, D be four points on a circle (occurring in clockwise order), with $AB < AD$ and $BC > CD$. The bisectors of angles BAD and BCD meet the circle at X and Y, respectively. Consider the hexagon formed by these six points on the circle. If four of the six sides of the hexagon have equal length, prove that BD must be a diameter of the circle.

PROBLEMA 4

Let p be an odd prime. Prove that:

$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}$$

PROBLEMA 5

Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements of a subset S of T such that no element in S divides any other element in S?

PROBLEMA 1
Consider a standard twelve-hour clock whose hour and minute hands move continuously. Let m be an integer, with $1 \leq m \leq 720$. At precisely m minutes after 12:00, the angle made by the hour hand and minute hand is exactly 1°.
Determine all possible values of m.

PROBLEMA 2
Find the last three digits of the number $2003^{2002^{2001}}$.

PROBLEMA 3
Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$
$$x^2 + y^2 + z^2 = xyz.$$

PROBLEMA 4
Prove that when three circles share the same chord AB, every line through A different from AB determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the first circle while Y and Z are the points where AX intersects the other two circles (labeled so that Y is between X and Z).

PROBLEMA 5
Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart.
Prove there is a subset T of S with at least $\frac{n}{7}$ points such that any two points of T are at least $\sqrt{3}$ units apart.

PROBLEMA 1

Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

What is the maximum number of elements that S can contain?

PROBLEMA 2

Call a positive integer n practical if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

For example, the divisors of 6 are 1, 2, 3, and 6. Since

$$1=\mathbf{1}, \quad 2=\mathbf{2}, \quad 3=\mathbf{3}, \quad 4=\mathbf{1}+\mathbf{3}, \quad 5=\mathbf{2}+\mathbf{3}, \quad 6=\mathbf{6},$$

we see that 6 is practical.

Prove that the product of two practical numbers is also practical.

PROBLEMA 3

Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

and determine when equality occurs.

PROBLEMA 4

Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let the line CP meet Γ again at Q.

Prove that $PQ = r$.

PROBLEMA 5

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all x and y in $\mathbb{N}$.

PROBLEMA 1

Randy: "Hi Rachel, that's an interesting quadratic equation you have written down. What are its roots?"Rachel: "The roots are two positive integers. One of the roots is my age, and the other root is the age of my younger brother, Jimmy."Randy: "That is very neat! Let me see if I can figure out how old you and Jimmy are. That shouldn't be too difficult since all of your coefficients are integers. By the way, I notice that the sum of the three coefficients is a prime number."Rachel: "Interesting. Now figure out how old I am."Randy: "Instead, I will guess your age and substitute it for x in your quadratic equation ... darn, that gives me -55, and not 0."Rachel: "Oh, leave me alone!"

(1) Prove that Jimmy is two years old.

(2) Determine Rachel's age.

PROBLEMA 2

There is a board numbered -10 to 10. Each square is coloured either red or white, and the sum of the numbers on the red squares is n. Maureen starts with a token on the square labeled 0. She then tosses a fair coin ten times. Every time she flips heads, she moves the token one square to the right. Every time she flips tails, she moves the token one square to the left. At the end of the ten flips, the probability that the token finishes on a red square is a rational number of the form $\frac{a}{b}$. Given that $a + b = 2001$, determine the largest possible value for n.

PROBLEMA 3

Let ABC be a triangle with $AC > AB$. Let P be the intersection point of the perpendicular bisector of BC and the internal angle bisector of $\angle A$. Construct points X on AB (extended) and Y on AC such that PX is perpendicular to AB and PY is perpendicular to AC. Let Z be the intersection point of XY and BC.

Determine the value of $\frac{BZ}{ZC}$.

PROBLEMA 4

Let n be a positive integer. Nancy is given a rectangular table in which each entry is a positive integer. She is permitted to make either of the following two moves:

(1) select a row and multiply each entry in this row by n;

(2) select a column and subtract n from each entry in this column.

Find all possible values of n for which the following statement is true:

Given any rectangular table, it is possible for Nancy to perform a finite sequence of moves to create a table in which each entry is 0.

PROBLEMA 5

Let P_0, P_1, P_2 be three points on the circumference of a circle with radius 1, where $P_1P_2 = t < 2$. For each $i \geq 3$, define P_i to be the centre of the circumcircle of $\triangle P_{i-1}P_{i-2}P_{i-3}$.

(1) Prove that the points $P_1, P_5, P_9, P_{13}, \cdots$ are collinear.

(2) Let x be the distance from P_1 to P_{1001}, and let y be the distance from P_{1001} to P_{2001}. Determine all values of t for which $\sqrt[500]{\frac{x}{y}}$ is an integer.

PROBLEMA 1
At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length 300 meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least 100 meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners.)

PROBLEMA 2
A permutation of the integers $1901, 1902, \cdots, 2000$ is a sequence $a_1, a_2, \cdots, a_{100}$ in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

$$s_1 = a_1,\ \ s_2 = a_1 + a_2,\ \ s_3 = a_1 + a_2 + a_3,\ \ldots\ ,\ s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

How many of these permutations will have no terms of the sequence $s_1, \ldots, s_{100}$ divisible by three?

PROBLEMA 3
Let $A = (a_1, a_2, \cdots, a_{2000})$ be a sequence of integers each lying in the interval $[-1000, 1000]$. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

PROBLEMA 4
Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$ and $AB = CB$.
Prove that $AD = CD$.

PROBLEMA 5
Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$\begin{aligned} 0 \leq a_{100} \leq a_{99} \leq \cdots \leq a_2 &\leq a_1, \\ a_1 + a_2 &\leq 100 \\ a_3 + a_4 + \cdots + a_{100} &\leq 100. \end{aligned}$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

PROBLEMA 1
Find all real solutions to the equation $4x^2 - 40\lfloor x\rfloor + 51 = 0$.

PROBLEMA 2
Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and center on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

PROBLEMA 3
Determine all positive integers n with the property that $n = (d(n))^2$. Here $d(n)$ denotes the number of positive divisors of n.

PROBLEMA 4
Suppose $a_1, a_2, \cdots, a_8$ are eight distinct integers from $\{1, 2, \cdots, 16, 17\}$. Show that there is an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions.

Also, find a specific set of 7 distinct integers from $\{1, 2, \ldots, 16, 17\}$ such that the equation $a_i - a_j = k$ does not have three distinct solutions for any $k > 0$.

PROBLEMA 5
Let x, y, and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

$$x^2y + y^2z + z^2x \leq \frac{4}{27}$$

and find when equality occurs.

PROBLEMA 1
Determine the number of real solutions a to the equation:

$$\left[\frac{1}{2}\,a\right]+\left[\frac{1}{3}\,a\right]+\left[\frac{1}{5}\,a\right]=a.$$

Here, if x is a real number, then $[\,x\,]$ denotes the greatest integer that is less than or equal to x.

PROBLEMA 2
Find all real numbers x such that:

$$x=\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}$$

PROBLEMA 3
Let n be a natural number such that $n \geq 2$. Show that

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right).$$

PROBLEMA 4
Let ABC be a triangle with $\angle BAC = 40^\circ$ and $\angle ABC = 60^\circ$. Let D and E be the points lying on the sides AC and AB, respectively, such that $\angle CBD = 40^\circ$ and $\angle BCE = 70^\circ$. Let F be the point of intersection of the lines BD and CE. Show that the line AF is perpendicular to the line BC.

PROBLEMA 5
Let m be a positive integer. Define the sequence $a_0, a_1, a_2, \cdots$ by $a_0 = 0$, $a_1 = m$, and $a_{n+1} = m^2 a_n - a_{n-1}$ for $n = 1, 2, 3, \cdots$.

Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2+b^2}{ab+1}=m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

PROBLEMA 1
Determine the number of pairs of positive integers x, y such that $x \leq y$, $\gcd(x, y) = 5!$ and $\operatorname{lcm}(x, y) = 50!$.

PROBLEMA 2
The closed interval $A = [0, 50]$ is the union of a finite number of closed intervals, each of length 1. Prove that some of the intervals can be removed so that those remaining are mutually disjoint and have total length greater than 25.

Note: For reals $a \leq b$, the closed interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ has length $b - a$; disjoint intervals have empty intersection.

PROBLEMA 3
Prove that $\frac{1}{1999} < \prod_{i=1}^{999} \frac{2i-1}{2i} < \frac{1}{44}$.

PROBLEMA 4
The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$. Prove that $\angle OBC = \angle ODC$.

PROBLEMA 5
Write the sum $\sum_{i=0}^{n} \frac{(-1)^i \cdot \binom{n}{i}}{i^3+9i^2+26i+24}$ as the ratio of two explicitly defined polynomials with integer coefficients.

PROBLEMA 1
If α, β, and γ are the roots of $x^3 - x - 1 = 0$, compute $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$.

PROBLEMA 2
Find all real solutions to the following system of equations. Carefully justify your answer.

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y \\ \\ \dfrac{4y^2}{1+4y^2} = z \\ \\ \dfrac{4z^2}{1+4z^2} = x \end{cases}$$

PROBLEMA 3
We denote an arbitrary permutation of the integers 1, 2, …, n by a_1, a_2, …, a_n. Let $f(n)$ denote the number of these permutations such that:

(1) $a_1 = 1$;

(2):$|a_i - a_{i+1}| \leq 2$, $i = 1, \ldots, n-1$.

Determine whether $f(1996)$ is divisible by 3.

PROBLEMA 4
Let triangle ABC be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of its angle $\angle B$ meets the side AC at a point D and that $BC = BD + AD$.

Determine $\angle A$.

PROBLEMA 5
Let r_1, r_2, …, r_m be a given set of m positive rational numbers such that $\sum_{k=1}^{m} r_k = 1$. Define the function f by $f(n) = n - \sum_{k=1}^{m} [r_k n]$ for each positive integer n. Determine the minimum and maximum values of $f(n)$. Here $[x]$ denotes the greatest integer less than or equal to x.

PROBLEMA 1
Let $f(x) = \frac{9^x}{9^x+3}$. Evaluate $\sum_{i=1}^{1995} f\left(\frac{i}{1996}\right)$.

PROBLEMA 2
Let $\{a, b, c\} \in \mathbb{R}^+$. Prove that $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$.

PROBLEMA 3
Define a boomerang as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180°. Let C be a convex polygon with s sides. The interior region of C is the union of q quadrilaterals, none of whose interiors overlap each other. b of these quadrilaterals are boomerangs. Show that $q \geq b + \frac{s-2}{2}$.

PROBLEMA 4
Let n be a constant positive integer. Show that for only non-negative integers k, the Diophantine equation $\sum_{i=1}^{n} x_i^3 = y^{3k+2}$ has infinitely many solutions in the positive integers x_i, y.

PROBLEMA 5
u is a real parameter such that $0 < u < 1$.

For $0 \leq x \leq u$, $f(x) = 0$.

For $u \leq x \leq n$, $f(x) = 1 - \left(\sqrt{ux} + \sqrt{(1-u)(1-x)}\right)^2$.

The sequence $\{u_n\}$ is define recursively as follows: $u_1 = f(1)$ and $u_n = f(u_{n-1})$ $\forall n \in \mathbb{N}, n \neq 1$.

Show that there exists a positive integer k for which $u_k = 0$.

PROBLEMA 1
Evaluate $\sum_{n=1}^{1994}\left((-1)^n\cdot\left(\frac{n^2+n+1}{n!}\right)\right)$.

PROBLEMA 2
Prove that $(\sqrt{2}-1)^n$ $\forall n\in\mathbb{Z}^+$ can be represented as $\sqrt{m}-\sqrt{m-1}$ for some $m\in\mathbb{Z}^+$.

PROBLEMA 3
25 men sit around a circular table. Every hour there is a vote, and each must respond yes or no. Each man behaves as follows: on the n^{th}, vote if his response is the same as the response of at least one of the two people he sits between, then he will respond the same way on the $(n+1)^{\text{th}}$ vote as on the n^{th} vote; but if his response is different from that of both his neighbours on the n^{th} vote, then his response on the $(n+1)^{\text{th}}$ vote will be different from his response on the n^{th} vote. Prove that, however everybody responded on the first vote, there will be a time after which nobody's response will ever change.

PROBLEMA 4
Let AB be a diameter of a circle Ω and P be any point not on the line through AB. Suppose that the line through PA cuts Ω again at U, and the line through PB cuts Ω at V. Note that in case of tangency, U may coincide with A or V might coincide with B. Also, if P is on Ω then $P=U=V$. Suppose that $|PU|=s|PA|$ and $|PV|=t|PB|$ for some $0\leq s,t\in\mathbb{R}$. Determine $\cos\angle APB$ in terms of s,t.

PROBLEMA 5
Let ABC be an acute triangle. Let AD be the altitude on BC, and let H be any interior point on AD. Lines BH,CH, when extended, intersect AC,AB at E,F respectively. Prove that $\angle EDH=\angle FDH$.

PROBLEMA 1
Determine a triangle for which the three sides and an altitude are four consecutive integers and for which this altitude partitions the triangle into two right triangles with integer sides. Show that there is only one such triangle.

PROBLEMA 2
Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence

$$x,\ x+1,\ x+2,\ x+3,\ldots.$$

PROBLEMA 3
In triangle ABC, the medians to the sides $\overline{AB}$ and $\overline{AC}$ are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.

PROBLEMA 4
Anumber of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a single and that between a boy and a girl was called a mixed single. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

PROBLEMA 5
Let $y_1, y_2, y_3, \ldots$ be a sequence such that $y_1 = 1$ and, for $k > 0$, is defined by the relationship:

$$y_{2k} = \begin{cases} 2y_k & \text{if } k \text{ is even} \\ 2y_k + 1 & \text{if } k \text{ is odd} \end{cases}$$

$$y_{2k+1} = \begin{cases} 2y_k & \text{if } k \text{ is odd} \\ 2y_k + 1 & \text{if } k \text{ is even} \end{cases}$$

Show that the sequence takes on every positive integer value exactly once.

PROBLEMA 1
Prove that the product of the first n natural numbers is divisible by the sum of the first n natural numbers if and only if $n+1$ is not an odd prime.

PROBLEMA 2
For $x, y, z \geq 0$, establish the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$

and determine when equality holds.

PROBLEMA 3
In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$.

PROBLEMA 4
Solve the equation

$$x^2 + \frac{x^2}{(x+1)^2} = 3$$

PROBLEMA 5
A deck of $2n+1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n+1$ cards are placed in a row, with the joker in the middle. For each k with $1 \leq k \leq n$, the two cards numbered k have exactly $k-1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?

PROBLEMA 1
Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.

PROBLEMA 2
Let n be a fixed positive integer. Find the sum of all positive integers with the property that in base 2 each has exactly $2n$ digits, consisting of n 1's and n 0's. (The first digit cannot be 0.)

PROBLEMA 3
Let C be a circle and P a given point in the plane. Each line through P which intersects C determines a chord of C. Show that the midpoints of these chords lie on a circle.

PROBLEMA 4
Can ten distinct numbers $a_1, a_2, b_1, b_2, b_3, c_1, c_2, d_1, d_2, d_3$ be chosen from $\{0, 1, 2, \ldots, 14\}$, so that the 14 differences $|a_1 - b_1|$, $|a_1 - b_2|$, $|a_1 - b_3|$, $|a_2 - b_1|$, $|a_2 - b_2|$, $|a_2 - b_3|$, $|c_1 - d_1|$, $|c_1 - d_2|$, $|c_1 - d_3|$, $|c_2 - d_1|$, $|c_2 - d_2|$, $|c_2 - d_3|$, $|a_1 - c_1|$, and $|a_2 - c_2|$ are all distinct?

PROBLEMA 5
The sides of an equilateral triangle ABC are divided into n equal parts ($n \geq 2$). For each point on a side, we draw the lines parallel to other sides of the triangle ABC, e.g. for $n = 3$ we have the following diagram:A

For each $n \geq 2$, find the number of existing parallelograms.