NL/2015 - Folha 1/2

$a, b, \in \mathbb{Z}$ t.q. $a! + b! \mid a!b!$.

Prove que $3a \geq 2b+2$.

Se $a = 1 \Rightarrow 1 + b! \mid b!$ Absurdo $\Rightarrow a \neq 1$,
$b \neq 1$

Se $a \geq b \Rightarrow 3a \geq 2b + a \geq 2b + 2$. ok!

Logo, $a < b$. $\Rightarrow$

$\Rightarrow b = a + c$. (Quero $a \geq 2c+2$)

$a! + (a+c)! \mid a! \cdot (a+c)! \Leftrightarrow a!\left(1 + (a+1)(a+2)\cdots(a+c)\right) \mid a!(a+c)! \Leftrightarrow$

$\Leftrightarrow 1 + (a+1)(a+2)\cdots(a+c) \mid (a+c)! \Leftrightarrow$

$\Leftrightarrow 1 + (a+1)(a+2)\cdots(a+c) \mid a!$ (1)

Mas, $\forall x \leq c$, $x \mid (a+1)\cdots(a+c) \Rightarrow \left(x, (a+1)\cdots(a+c) + 1\right) = 1$.

Logo (1) $\Leftrightarrow 1 + (a+1)(a+2)\cdots(a+c) \mid (c+1)(c+2)\cdots a$. $\Rightarrow$

$\Rightarrow (c+1)\cdots a \geq 1 + (a+1)(a+2)\cdots(a+c) \Rightarrow (c+1)\cdots a > (a+1)\cdots(a+c)$

Porém, todo elemento do LE é menor que o do LD e, #elem(LE) = $a-c$
e #elem(LD) = c. $\Rightarrow a - c > c \Rightarrow a > 2c$.

Basta mostrar que $a \neq 2c+1$.

Suponha que $a = 2c+1 \Rightarrow b = a + c = 3c+1$.

$\Rightarrow 1 + (2c+2)\cdots(3c+1) \mid (c+1)\cdots(2c+1)$

Mas, $3c \mid (2c+2)\cdots(3c+1) \Rightarrow c \mid (2c+2)\cdots(3c+1) \Rightarrow \left(c, (2c+2)\cdots(3c+1) + 1\right) = 1 \Rightarrow$

$\Rightarrow 1 + (2c+2)\cdots(3c+1) \mid (c+1)\cdots(2c-1)(2c+1)$.

$$\Rightarrow (c+1)\cdots(2c-1)\cdot(2c+1) > (2c+2)\cdots(3c+1).$$

Mas, $\left.\begin{array}{l} c+1 < 2c+2 \\ \vdots \\ 2c-1 < 3c \\ 2c+1 < 3c+1 \end{array}\right\} \Rightarrow \begin{array}{c} (c+1)\cdots(2c-1)(2c+1) \\ \wedge \\ (2c+2)\cdots(3c+1) \end{array}$. Absurdo!

Logo: $a \geq 2c+2 \Rightarrow$

$$\Rightarrow 3a \geq 2b+2$$

$\square$