Treinamento de Velocidade em Equipe

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Problema 1 (AIME I 2008, 1)

Of the students attending a school party, 60% of the students are girls, and 40% of the students like to dance. After these students are joined by 20 more boy students, all of whom like to dance, the party is now 58% girls. How many students now at the party like to dance?

Problema 2 (AIME I 2008, 2)

Square $AIME$ has sides of length 10 units. Isosceles triangle GEM has base EM, and the area common to triangle GEM and square $AIME$ is 80 square units. Find the length of the altitude to EM in $\triangle GEM$.

Problema 3 (AIME I 2008, 3)

Ed and Sue bike at equal and constant rates. Similarly, they jog at equal and constant rates, and they swim at equal and constant rates. Ed covers 74 kilometers after biking for 2 hours, jogging for 3 hours, and swimming for 4 hours, while Sue covers 91 kilometers after jogging for 2 hours, swimming for 3 hours, and biking for 4 hours. Their biking, jogging, and swimming rates are all whole numbers of kilometers per hour. Find the sum of the squares of Ed's biking, jogging, and swimming rates.

Problema 4 (AIME I 2008, 4)

There exist unique positive integers x and y that satisfy the equation $x^2 + 84x + 2008 = y^2$. Find $x + y$.

Problema 5 (AIME I 2008, 5)

A right circular cone has base radius r and height h. The cone lies on its side on a flat table. As the cone rolls on the surface of the table without slipping, the point where the cone's base meets the table traces a circular arc centered at the point where the vertex touches the table. The cone first returns to its original position on the table after making 17 complete rotations. The value of h/r can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Problema 6 (AIME I 2008, 6)

A triangular array of numbers has a first row consisting of the odd integers $1, 3, 5, \ldots, 99$ in increasing order. Each row below the first has one fewer entry than the row above it, and the bottom row has a single entry. Each entry in any row after the top row equals the sum of the two entries diagonally above it in the row immediately above it. How many entries in the array are multiples of 67?

$$\begin{array}{ccccccccccc} 1 & & 3 & & 5 & & \cdots & & 97 & & 99 \\ & 4 & & 8 & & 12 & & & & 196 & \\ & & & & & & \vdots & & & & \end{array}$$

Problema 7 (AIME I 2008, 7)

Let S_i be the set of all integers n such that $100i \le n < 100(i + 1)$. For example, S_4 is the set $400, 401, 402, \ldots, 499$. How many of the sets $S_0, S_1, S_2, \ldots, S_{999}$ do not contain a perfect square?

Problema 8 (AIME I 2008, 8)

Find the positive integer n such that

$$\arctan\frac{1}{3}+\arctan\frac{1}{4}+\arctan\frac{1}{5}+\arctan\frac{1}{n}=\frac{\pi}{4}.$$

Problema 9 (AIME I 2008, 9)

Ten identical crates each of dimensions 3 ft $\times$ 4 ft $\times$ 6 ft. The first crate is placed flat on the floor. Each of the remaining nine crates is placed, in turn, flat on top of the previous crate, and the orientation of each crate is chosen at random. Let $\frac{m}{n}$ be the probability that the stack of crates is exactly 41 ft tall, where m and n are relatively prime positive integers. Find m.

Problema 10 (AIME I 2008, 10)

Let $ABCD$ be an isosceles trapezoid with $\overline{AD}||\overline{BC}$ whose angle at the longer base $\overline{AD}$ is $\dfrac{\pi}{3}$. The diagonals have length $10\sqrt{21}$, and point E is at distances $10\sqrt{7}$ and $30\sqrt{7}$ from vertices A and D, respectively. Let F be the foot of the altitude from C to $\overline{AD}$. The distance EF can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

Problema 11 (AIME I 2008, 11)

Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

Problema 12 (AIME I 2008, 12)

On a long straight stretch of one-way single-lane highway, cars all travel at the same speed and all obey the safety rule: the distance from the back of the car ahead to the front of the car behind is exactly one car length for each 15 kilometers per hour of speed or fraction thereof (Thus the front of a car traveling 52 kilometers per hour will be four car lengths behind the back of the car in front of it.) A photoelectric eye by the side of the road counts the number of cars that pass in one hour. Assuming that each car is 4 meters long and that the cars can travel at any speed, let M be the maximum whole number of cars that can pass the photoelectric eye in one hour. Find the quotient when M is divided by 10.

Problema 13 (AIME I 2008, 13)

Let

$$p(x,y)=a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2+a_6x^3+a_7x^2y+a_8xy^2+a_9y^3.$$

Suppose that

$$\begin{aligned} p(0,0)&=p(1,0)=p(-1,0)=p(0,1)=p(0,-1)\\ &=p(1,1)=p(1,-1)=p(2,2)=0. \end{aligned}$$

There is a point $\left(\frac{a}{c},\frac{b}{c}\right)$ for which $p\left(\frac{a}{c},\frac{b}{c}\right)=0$ for all such polynomials, where a, b, and c are positive integers, a and c are relatively prime, and $c>1$. Find $a+b+c$.

Problema 14 (AIME I 2008, 14)

Let $\overline{AB}$ be a diameter of circle ω. Extend $\overline{AB}$ through A to C. Point T lies on ω so that line CT is tangent to ω. Point P is the foot of the perpendicular from A to line CT. Suppose $AB=18$, and let m denote the maximum possible length of segment BP. Find m^2.

Problema 15 (AIME I 2008, 15)

A square piece of paper has sides of length 100. From each corner a wedge is cut in the following manner: at each corner, the two cuts for the wedge each start at distance $\sqrt{17}$ from the corner, and they meet on the diagonal at an angle of $60°$ (see the figure below). The paper is then folded up along the lines joining the vertices of adjacent cuts. When the two edges of a cut meet, they are taped together. The result is a paper tray whose sides are not at right angles to the base. The height of the tray, that is, the perpendicular distance between the plane of the base and the plane formed by the upper edges, can be written in the form $\sqrt[n]{m}$, where m and n are positive integers, $m < 1000$, and m is not divisible by the nth power of any prime. Find $m + n$.