0×0

$\downarrow$

1×0

$\downarrow$

2×0

$1 \times 0 \quad 0 \times 1$

$\searrow \quad \swarrow$

1×1

$(a-1) \times b \quad a \times (b-1)$

$\searrow \quad \swarrow$

$a \times b$

Problema 3

Serafina and Florência play tennis. The player to first win at least four points and at least two more than the other player wins. We know that Serafina gets a point each time with probability $p \le \frac{1}{2}$, independent of the game so far. Prove that the probability that Serafina wins is at most $2p^2$.

$3 \times 3 \rightarrow 4 \times 3 \leftarrow 4 \times 2$ ← ACABOU AQUI

$\underline{4 \times 3}$

$$\underset{\text{na 7a rodada}}{\mathbb{P}(4 \times 3)} = p \cdot \underset{\text{na 6a rodada}}{\mathbb{P}(3 \times 3)}$$

$$\mathbb{P}(\text{Na } 7^a \text{ rodada, o placar estar } 4 \times 3) = p \cdot \underbrace{\mathbb{P}(\text{NA } 6^a \text{ rodada} \dots 3 \times 3)}$$

6ª rodada: 3×3 → 4×2 → Jogo ACABA AQUI

7ª rodada: 4×3

F \ S	0	1	2	3	4
0	1	p	p^2	p^3	p^4
1	$(1-p)$	$2p(1-p)$	$3p^2(1-p)$	$4p^3(1-p)$	$4p^4(1-p)$
2	$(1-p)^2$	$3p(1-p)^2$	$6p^2(1-p)^2$	$10p^3(1-p)^2$	$10p^4(1-p)^2$
3	$(1-p)^3$	$4p(1-p)^3$	$10p^2(1-p)^3$	$20p^3(1-p)^3$	
4	$(1-p)^4$	$4p(1-p)^4$	$10p^2(1-p)^4$		

$\rightarrow \dfrac{p^2}{1-2p+2p^2}$

$$\mathbb{P}(S \text{ ganhar}) =$$

$4 \times 0 \rightarrow p^4 \; +$

$4 \times 1 \rightarrow 4p^4(1-p) \; +$

$4 \times 2 \quad 10p^4(1-p)^2 \; +$

$3 \times 3 \rightarrow (20p^3(1-p)^3) \cdot \left(\dfrac{p^2}{1-2p+2p^2}\right)$

$$= p^4\left(\underbrace{1 + 4 - 4p + 10 - 20p + 10p^2}_{15 - 24p + 10p^2} + \frac{20p(1-p)^3}{1-2p+2p^2}\right)$$

$$= \frac{p^4}{1-2p+2p^2}\left(\begin{array}{l} 15 - 30p + 30p^2 \\ - 24p + 48p^2 - 48p^3 \\ + 10p^2 - 20p^3 + \cancel{20p^4} \\ + 20p - 60p^2 + 60p^3 - \cancel{20p^4} \end{array}\right)$$

$$= \frac{p^4(15 - 34p + 28p^2 - 8p^3)}{(1-2p+2p^2)}$$

$$\le \frac{p^2(15 - 34p + 28p^2 - 8p^3)}{2}$$

$$\le \frac{p^2\left((1-2p)^3 + 6(1-2p)^2 + 4(1-2p) + 4\right)}{2}$$

$$\le 2p^2$$

yay!

3(b):

$\boxed{\#S - \#F}$

delta

-2

-1 $\;\; 1-p$

0 $\;\; p$

$+1$

$+2$

$$\mathbb{P}(S \text{ ganhar}: \Delta = -1) = p \cdot \mathbb{P}(\dots 0) + (1-p)\,\cancel{\mathbb{P}(\dots -2)}^{\,0}$$

$$\mathbb{P}(S \text{ ganhar}: \Delta = 0) = p\,\mathbb{P}(\dots +1) + (1-p)\,\mathbb{P}(\dots -1)$$

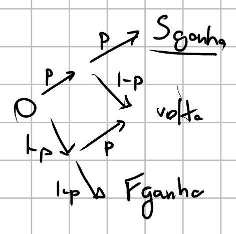
$$\mathbb{P}(S \text{ ganhar}: \Delta = +1) = p\,\underset{1}{\cancel{\mathbb{P}(\dots +2)}} + (1-p)\,\mathbb{P}(\dots 0)$$

$$\mathbb{P}(\dots 0) = p \cdot \left(p + (1-p)\mathbb{P}(\dots 0)\right) + (1-p)\left(p\,\mathbb{P}(\dots 0)\right)$$

$$(1 - 2p(1-p)) \cdot \mathbb{P}(\dots 0) = p^2$$

$$\mathbb{P}(\dots 0) = \frac{p^2}{1-2p+2p^2}$$

$$\mathbb{P}(S \text{ ganhar}: \Delta 0) = p^2 \cdot 1 + (1-p)^2 \cdot 0 + 2p(1-p)\,\mathbb{P}(\dots 0)$$



Problema 5

Let $N > 1$ and let $a_1, a_2, \ldots, a_N$ be nonnegative reals with sum at most 500. ~~Prove that there exist integers~~ $k \geq 1$ and $1 = n_0 < n_1 < \cdots < n_k = N$ such that

$$\sum_{i=1}^{k} n_i a_{n_{i-1}} < 2005.$$

$\Leftrightarrow (n_0, \ldots, n_k)$ funciona

$\underline{N=2}$ $\quad a_1 + a_2 \leq 500$

$k=1$, $1 = n_0 < n_1 = 2$: $\quad \sum_{i=1}^{1} n_1 a_{n_0} = 2 \cdot a_1 \leq 1000 < 2005$

$\underline{N=3}$ $\quad a_1 + a_2 + a_3 \leq 500$

$k=1$ ok!

$\underline{N=4}$ $\quad k=1$ ok!

$\underline{N=5}$ $\quad a_1 + a_2 + a_3 + a_4 \leq 500$

- $(1,5)$ não funciona $\Leftrightarrow 5a_1 \geq 2005$
- $(1,2,5)$ não funciona $\Leftrightarrow 2a_1 + 5a_2 \geq 2005$ $\quad \frac{7}{25}$
- $(1,3,5)$ não funciona $\Leftrightarrow 3a_1 + 5a_3 \geq 2005$ $\quad \frac{7}{25}$
- $(1,4,5)$ não funciona $\Leftrightarrow 4a_1 + 5a_4 \geq 2005$
- $(1,2,3,5)$ não funciona $\Leftrightarrow 2a_1 + 3a_2 + 5a_3 \geq 2005$ $\quad 1/5$
- $(1,2,4,5)$ não funciona $\Leftrightarrow 2a_1 + 4a_2 + 5a_4 \geq 2005$ $\quad 2/5$
- $(1,3,4,5)$ não funciona $\Leftrightarrow 3a_1 + 4a_3 + 5a_4 \geq 2005$
- $(1,2,3,4,5)$ não funciona $\Leftrightarrow 2a_1 + 3a_2 + 4a_3 + 5a_4 \geq 2005$

$(\times 51)$, $(\times 5)$

$$625(\Sigma) = (\overbrace{51 + 5(7 + 5(1+5))}^{236})\, 2005$$

$(\times 7)$, $(\times 5)$

$$\underbrace{74a_1}_{51} + 125(a_2 + a_3 + a_4) = \underbrace{(7 + 5(1+5))}\, 2005$$

$(\times 1)$, $(\times 5)$

$$12a_1 + \underbrace{18a_3}_{7} + 25(a_3 + a_4) = (5+1)\, 2005$$

$$N\left(\Sigma a_i\right) = \left(\Sigma x_i\right) \cdot 2005 \qquad 1 + \frac{2}{5} = \frac{7}{5}$$

$$1 + \frac{1}{5} + \frac{7}{25}$$

$N x_1 = N$

$N x_2 = x_1$

$N x_3 = x_1 + 2x_2$

$N x_4 = x_1 + x_2 + 3x_3$

$\vdots$

$N x_{N-1} = x_1 + x_2 + \cdots + (N-2) x_{N-2}$ $\quad +$

$$\Sigma a_i = \frac{(\Sigma x_i)}{N} \cdot 2005 > 500$$

$N^2 \quad x_1 = 1$

$N \quad N x_2 = 1$

$3 \quad N^2 x_3 = N + 2$

$N^3 x_4 = N^2 + 2N + 2 \quad = (N+1)^2 + 1$

$$N \cdot (\Sigma x_i) = N + (N-2)(\Sigma x_i - x_{N-1})$$

$$2(\Sigma x_i) = N - (N-2)x_{N-1}$$

$$\frac{(\Sigma x_i)}{N} = \frac{N - (N-2)x_{N-1}}{2N} =$$