Olimpíada Polonesa de Matemática

Todo e qualquer feedback, especialmente sobre erros neste livreto (mesmo erros tipográficos pequenos), é apreciado. Você também pode contribuir enviando suas soluções (de preferência, formatadas em TEX).
Você pode enviar comentários e soluções para zeusdanmou+tex@gmail.com.
A versão mais atualizada desse arquivo (provavelmente) está disponível clicando aqui. Última atualização: 23 de dezembro de 2021.

Primeiro Dia

PROBLEMA 1
Let p_i for $i = 1, 2, ..., k$ be a sequence of smallest consecutive prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 3$ etc.). Let $N = p_1 \cdot p_2 \cdot ... \cdot p_k$. Prove that in a set $\{1, 2, ..., N\}$ there exist exactly $\frac{N}{2}$ numbers which are divisible by odd number of primes p_i.

PROBLEMA 2
Let n be an integer. For pair of integers $0 \leq i,\ j \leq n$ there exist real number $f(i, j)$ such that:

1. $f(i, i) = 0$ for all integers $0 \leq i \leq n$, and;
2. $0 \leq f(i, l) \leq 2 \max\{f(i, j), f(j, k), f(k, l)\}$ for all integers $i,\ j,\ k,\ l$ satisfying $0 \leq i \leq j \leq k \leq l \leq n$.

Prove that

$$f(0, n) \leq 2 \sum_{k=1}^{n} f(k-1, k).$$

PROBLEMA 3
Let ω be the circumcircle of a triangle ABC. Let P be any point on ω different than the verticies of the triangle. Line AP intersects BC at D, BP intersects AC at E and CP intersects AB at F. Let X be the projection of D onto line passing through midpoints of AP and BC, Y be the projection of E onto line passing through BP and AC and let Z be the projection of F onto line passing through midpoints of CP and AB. Let Q be the circumcenter of triangle XYZ. Prove that all possible points Q, corresponding to different positions of P lie on one circle.

Segundo Dia

PROBLEMA 4
Prove that for every pair of positive real numbers a, b and for every positive integer n,

$$(a+b)^n - a^n - b^n \geq \frac{2^n - 2}{2^{n-2}} \cdot ab(a+b)^{n-2}.$$

PROBLEMA 5
A convex hexagon $ABCDEF$ is given where $\measuredangle FAB + \measuredangle BCD + \measuredangle DEF = 360^\circ$ and $\measuredangle AEB = \measuredangle ADB$. Suppose the lines AB and DE are not parallel. Prove that the circumcenters of the triangles $\triangle AFE, \triangle BCD$ and the intersection of the lines AB and DE are collinear.

PROBLEMA 6
Given an integer $d \geq 2$ and a circle ω. Hansel drew a finite number of chords of circle ω. The following condition is fulfilled: each end of each chord drawn is at least an end of d different drawn chords. We assume that the chords with a common end intersect.

(a) Prove that there is a drawn chord which intersects at least $\frac{d^2}{8}$ other drawn chords.

(b) Prove that there is a drawn chord which intersects at least $\frac{d^2}{4}$ other drawn chords.

Primeiro Dia

PROBLEMA 1
Let ABC be an acute triangle. Points X and Y lie on the segments AB and AC, respectively, such that $AX = AY$ and the segment XY passes through the orthocenter of the triangle ABC. Lines tangent to the circumcircle of the triangle AXY at points X and Y intersect at point P. Prove that points A, B, C, P are concyclic.

PROBLEMA 2
Let p a prime number and r an integer such that $p|r^7 - 1$. Prove that if there exist integers a, b such that $p|r + 1 - a^2$ and $p|r^2 + 1 - b^2$, then there exist an integer c such that $p|r^3 + 1 - c^2$.

PROBLEMA 3
$n \geq 3$ guests met at a party. Some of them know each other but there is no quartet of different guests a, b, c, d such that in pairs $\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}$ guests know each other but in pairs $\{a, c\}, \{b, d\}$ guests don't know each other. We say a nonempty set of guests X is an *ingroup*, when guests from X know each other pairwise and there are no guests not from X knowing all guests from X. Prove that there are at most $\frac{n(n-1)}{2}$ different ingroups at that party.

Segundo Dia

PROBLEMA 4
Let n, k, ℓ be positive integers and $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ an injection such that $\sigma(x) - x \in \{k, -\ell\}$ for all $x \in \{1, 2, \ldots, n\}$. Prove that $k + \ell | n$.

PROBLEMA 5
The sequence $a_1, a_2, \ldots, a_n$ of positive real numbers satisfies the following conditions:

$$\sum_{i=1}^{n} \frac{1}{a_i} \leq 1 \quad \text{and} \quad a_i \leq a_{i-1} + 1$$

for all $i \in \{1, 2, \ldots, n\}$, where a_0 is an integer. Prove that

$$n \leq 4a_0 \cdot \sum_{i=1}^{n} \frac{1}{a_i}$$

PROBLEMA 6
Denote by Ω the circumcircle of the acute triangle ABC. Point D is the midpoint of the arc BC of Ω not containing A. Circle ω centered at D is tangent to the segment BC at point E. Tangents to the circle ω passing through point A intersect line BC at points K and L such that points B, K, L, C lie on the line BC in that order. Circle γ_1 is tangent to the segments AL and BL and to the circle Ω at point M. Circle γ_2 is tangent to the segments AK and CK and to the circle Ω at point N. Lines KN and LM intersect at point P. Prove that $\sphericalangle KAP = \sphericalangle EAL$.

Primeiro Dia

PROBLEMA 1
An acute triangle ABC in which $AB < AC$ is given. The bisector of $\angle BAC$ crosses BC at D. Point M is the midpoint of BC. Prove that the line though centers of circles escribed on triangles ABC and ADM is parallel to AD.

PROBLEMA 2
A subset S of size n of a plane consisting of points with both coordinates integer is given, where n is an odd number. The injective function $f\colon S \to S$ satisfies the following: for each pair of points $A, B \in S$, the distance between points $f(A)$ and $f(B)$ is not smaller than the distance between points A and B. Prove there exists a point X such that $f(X) = X$.

PROBLEMA 3
Find all real numbers c for which there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for each $x, y \in \mathbb{R}$ it's true that

$$f(f(x) + f(y)) + cxy = f(x + y).$$

Segundo Dia

PROBLEMA 4
Let n be a positive integer. Suppose there are exactly M squarefree integers k such that $\left\lfloor \frac{n}{k} \right\rfloor$ is odd in the set $\{1, 2, \ldots, n\}$. Prove M is odd.

An integer is *squarefree* if it is not divisible by any square other than 1.

PROBLEMA 5
An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C, respectively. The line tangent in point A to the circle escribed on ABC crosses BC at P. The line parallel to BC that goes through point A crosses EF at Q. Prove PQ is perpendicular to the median from A of triangle ABC.

PROBLEMA 6
A prime $p > 3$ is given. Let K be the number of such permutations $(a_1, a_2, \ldots, a_p)$ of $\{1, 2, \ldots, p\}$ such that

$$a_1a_2 + a_2a_3 + \ldots + a_{p-1}a_p + a_pa_1$$

is divisible by p. Prove $K + p$ is divisible by p^2.

Primeiro Dia

PROBLEMA 1

Points P and Q lie respectively on sides AB and AC of a triangle ABC and $BP = CQ$. Segments BQ and CP cross at R. Circumscribed circles of triangles BPR and CQR cross again at point S different from R. Prove that point S lies on the bisector of angle BAC.

PROBLEMA 2

A sequence $(a_1, a_2, \ldots, a_k)$ consisting of pairwise distinct squares of an $n \times n$ chessboard is called a *cycle* if $k \geq 4$ and squares a_i and a_{i+1} have a common side for all $i = 1, 2, \ldots, k$, where $a_{k+1} = a_1$. Subset X of this chessboard's squares is *mischievous* if each cycle on it contains at least one square in X.

Determine all real numbers C with the following property: for each integer $n \geq 2$, on an $n \times n$ chessboard there exists a mischievous subset consisting of at most Cn^2 squares.

PROBLEMA 3

Integers $a_1, a_2, \ldots, a_n$ satisfy

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

If m is the number of distinct prime factors of $a_1 a_2 \cdots a_n$, then prove that

$$(a_1 a_2 \cdots a_n)^{m-1} \geq (n!)^m.$$

Segundo Dia

PROBLEMA 4

Prove that the set of positive integers $\mathbb{Z}^+$ can be represented as a sum of five pairwise distinct subsets with the following property: each 5-tuple of numbers of form $(n, 2n, 3n, 4n, 5n)$, where $n \in \mathbb{Z}^+$, contains exactly one number from each of these five subsets.

PROBLEMA 5

Point M is the midpoint of BC of a triangle ABC, in which $AB = AC$. Point D is the orthogonal projection of M on AB. Circle ω is inscribed in triangle ACD and tangent to segments AD and AC at K and L respectively. Lines tangent to ω which pass through M cross line KL at X and Y, where points X, K, L and Y lie on KL in this specific order. Prove that points M, D, X and Y are concyclic.

PROBLEMA 6

Three sequences $(a_0, a_1, \ldots, a_n)$, $(b_0, b_1, \ldots, b_n)$, $(c_0, c_1, \ldots, c_{2n})$ of non-negative real numbers are given such that for all $0 \leq i, j \leq n$ we have $a_i b_j \leq (c_{i+j})^2$. Prove that

$$\sum_{i=0}^{n} a_i \cdot \sum_{j=0}^{n} b_j \leq \left(\sum_{k=0}^{2n} c_k \right)^2.$$

Primeiro Dia

PROBLEMA 1
Let p be a certain prime number. Find all non-negative integers n for which polynomial $P(x) = x^4 - 2(n+p)x^2 + (n-p)^2$ may be rewritten as product of two quadratic polynomials $P_1,\ P_2 \in \mathbb{Z}[X]$.

PROBLEMA 2
Let $ABCD$ be a quadrilateral circumscribed on the circle ω with center I. Assume $\angle BAD + \angle ADC < \pi$. Let $M,\ N$ be points of tangency of ω with $AB,\ CD$ respectively. Consider a point $K \in MN$ such that $AK = AM$. Prove that ID bisects the segment KN.

PROBLEMA 3
Let $a,\ b \in \mathbb{Z}_+$. Denote $f(a,b)$ the number sequences $s_1,\ s_2,\ ...,\ s_a,\ s_i \in \mathbb{Z}$ such that $|s_1| + |s_2| + ... + |s_a| \leq b$. Show that $f(a,b) = f(b,a)$.

Segundo Dia

PROBLEMA 4
Let k, n be odd positve integers greater than 1. Prove that if there a exists natural number a such that $k|2^a+1,\ n|2^a-1$, then there is no natural number b satisfying $k|2^b-1,\ n|2^b+1$.

PROBLEMA 5
There are given two positive real number $a < b$. Show that there exist positive integers $p,\ q,\ r,\ s$ satisfying following conditions: 1. $a < \frac{p}{q} < \frac{r}{s} < b$. 2. $p^2 + q^2 = r^2 + s^2$.

PROBLEMA 6
Let I be an incenter of $\triangle ABC$. Denote $D,\ S \neq A$ intersections of AI with $BC,\ O(ABC)$ respectively. Let $K,\ L$ be incenters of $\triangle DSB,\ \triangle DCS$. Let P be a reflection of I with the respect to KL. Prove that $BP \perp CP$.

Primeiro Dia

PROBLEMA 1
In triangle ABC the angle $\angle A$ is the smallest. Points D, E lie on sides AB, AC so that $\angle CBE = \angle DCB = \angle BAC$. Prove that the midpoints of AB, AC, BE, CD lie on one circle.

PROBLEMA 2
Let P be a polynomial with real coefficients. Prove that if for some integer k $P(k)$ isn't integral, then there exist infinitely many integers m, for which $P(m)$ isn't integral.

PROBLEMA 3
Find the biggest natural number m such that among any five 500-element subsets of $\{1, 2, \ldots, 1000\}$ there exist two sets, whose intersection contains at least m numbers.

Segundo Dia

PROBLEMA 4
Solve the system

$$\begin{cases} x + y + z = 1 \\ x^5 + y^5 + z^5 = 1 \end{cases}$$

in real numbers.

PROBLEMA 5
Prove that diagonals of a convex quadrilateral are perpendicular if and only if inside of the quadrilateral there is a point, whose orthogonal projections on sides of the quadrilateral are vertices of a rectangle.

PROBLEMA 6
Prove that for each positive integer a there exists such an integer $b > a$, for which $1 + 2^a + 3^a$ divides $1 + 2^b + 3^b$.

Primeiro Dia

PROBLEMA 1
Let $k, n \geq 1$ be relatively prime integers. All positive integers not greater than $k+n$ are written in some order on the blackboard. We can swap two numbers that differ by k or n as many times as we want. Prove that it is possible to obtain the order $1, 2, \ldots, k+n-1, k+n$.

PROBLEMA 2
Let $k \geq 2$, $n \geq 1$, $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_n$ be integers such that $1 < a_1 < a_2 < \cdots < a_k < b_1 < b_2 < \cdots < b_n$. Prove that if $a_1 + a_2 + \cdots + a_k > b_1 + b_2 + \cdots + b_n$, then $a_1 \cdot a_2 \cdot \ldots \cdot a_k > b_1 \cdot b_2 \cdot \ldots \cdot b_n$.

PROBLEMA 3
A tetrahedron $ABCD$ with acute-angled faces is inscribed in a sphere with center O. A line passing through O perpendicular to plane ABC crosses the sphere at point D' that lies on the opposide side of plane ABC than point D. Line DD' crosses plane ABC in point P that lies inside the triangle ABC. Prove, that if $\angle APB = 2\angle ACB$, then $\angle ADD' = \angle BDD'$.

Segundo Dia

PROBLEMA 4
Denote the set of positive rational numbers by $\mathbb{Q}_+$. Find all functions $f : \mathbb{Q}_+ \to \mathbb{Q}_+$ that satisfy

$$\underbrace{f(f(f(\ldots f(f}_{n}(q))\ldots))) = f(nq)$$

for all integers $n \geq 1$ and rational numbers $q > 0$.

PROBLEMA 5
Find all pairs (x, y) of positive integers that satisfy

$$2^x + 17 = y^4$$

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PROBLEMA 6
In an acute triangle ABC point D is the point of intersection of altitude h_a and side BC, and points M, N are orthogonal projections of point D on sides AB and AC. Lines MN and AD cross the circumcircle of triangle ABC at points P, Q and A, R. Prove that point D is the center of the incircle of PQR.

Primeiro Dia

PROBLEMA 1
Find all solutions of the following equation in integers $x, y : x^4 + y = x^3 + y^2$

PROBLEMA 2
There are given integers a and b such that a is different from 0 and the number $3 + a + b^2$ is divisible by $6a$. Prove that a is negative.

PROBLEMA 3
Given is a quadrilateral $ABCD$ in which we can inscribe circle. The segments AB, BC, CD and DA are the diameters of the circles $o1, o2, o3$ and $o4$, respectively. Prove that there exists a circle tangent to all of the circles $o1, o2, o3$ and $o4$.

Segundo Dia

PROBLEMA 4
Given is a tetrahedron $ABCD$ in which $AB = CD$ and the sum of measures of the angles BAD and BCD equals 180 degrees. Prove that the measure of the angle BAD is larger than the measure of the angle ADC.

PROBLEMA 5
Let k,m and n be three different positive integers. Prove that

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) \leq kmn - (k + m + n).$$

PROBLEMA 6
For each positive integer n determine the maximum number of points in space creating the set A which has the following properties: 1) the coordinates of every point from the set A are integers from the range $[0, n]$ 2) for each pair of different points $(x_1, x_2, x_3), (y_1, y_2, y_3)$ belonging to the set A it is satisfied at least one of the following inequalities $x_1 < y_1, x_2 < y_2, x_3 < y_3$ and at least one of the following inequalities $x_1 > y_1, x_2 > y_2, x_3 > y_3$.

Primeiro Dia

PROBLEMA 1
Decide, whether exists positive rational number w, which isn't integer, such that w^w is a rational number.

PROBLEMA 2
Determine all pairs (m, n) of positive integers, for which cube K with edges of length n, can be build in with cuboids of shape $m \times 1 \times 1$ to create cube with edges of length $n + 2$, which has the same center as cube K.

PROBLEMA 3
Triangle ABC with $AB = AC$ is inscribed in circle o. Circles o_1 and o_2 are internally tangent to circle o in points P and Q, respectively, and they are tangent to segments AB and AC, respectively, and they are disjoint with the interior of triangle ABC. Let m be a line tangent to circles o_1 and o_2, such that points P and Q lie on the opposite side than point A. Line m cuts segments AB and AC in points K and L, respectively. Prove, that intersection point of lines PK and QL lies on bisector of angle BAC.

Segundo Dia

PROBLEMA 4
n players $(n \geq 4)$ took part in the tournament. Each player played exactly one match with every other player, there were no draws. There was no four players (A, B, C, D), such that A won with B, B won with C, C won with D and D won with A. Determine, depending on n, maximum number of trios of players (A, B, C), such that A won with B, B won with C and C won with A. (Attention: Trios (A, B, C), (B, C, A) and (C, A, B) are the same trio.)

PROBLEMA 5
Point O is a center of circumcircle of acute triangle ABC, bisector of angle BAC cuts side BC in point D. Let M be a point such that, $MC \perp BC$ and $MA \perp AD$. Lines BM and OA intersect in point P. Show that circle of center in point P passing through a point A is tangent to line BC.

PROBLEMA 6
Show that for any positive real numbers a, b, c true is inequality: $\left(\frac{a-b}{c}\right)^2 + \left(\frac{b-c}{a}\right)^2 + \left(\frac{c-a}{b}\right)^2 \geq 2\sqrt{2}\left(\frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b}\right)$.

Primeiro Dia

PROBLEMA 1
Find all integers $n \geq 1$ such that there exists a permutation $(a_1, a_2, ..., a_n)$ of $(1, 2, ..., n)$ such that $a_1 + a_2 + ... + a_k$ is divisible by k for $k = 1, 2, ..., n$

PROBLEMA 2
The incircle of triangle ABC is tangent to BC, CA, AB at D, E, F respectively. Consider the triangle formed by the line joining the midpoints of AE, AF, the line joining the midpoints of BF, BD, and the line joining the midpoints of CD, CE. Prove that the circumcenter of this triangle coincides with the circumcenter of triangle ABC.

PROBLEMA 3
Let $n \geq 3$ be an odd integer. Determine how many real solutions there are to the set of n equations

$$\begin{cases} x_1(x_1+1) = x_2(x_2-1) \\ x_2(x_2+1) = x_3(x_3-1) \\ \qquad\vdots \\ x_n(x_n+1) = x_1(x_1-1) \end{cases}$$

Segundo Dia

PROBLEMA 4
Determine all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$f(x)f(y) = g(x)g(y) + g(x) + g(y).$$

PROBLEMA 5
In a tetrahedron $ABCD$, the four altitudes are concurrent at H. The line DH intersects the plane ABC at P and the circumsphere of $ABCD$ at $Q \neq D$. Prove that $PQ = 2HP$.

PROBLEMA 6
Prove that it is impossible for polynomials $f_1(x), f_2(x), f_3(x), f_4(x) \in \mathbb{Q}[x]$ to satisfy

$$f_1^2(x) + f_2^2(x) + f_3^2(x) + f_4^2(x) = x^2 + 7.$$

Primeiro Dia

PROBLEMA 1
The integer number $n > 1$ is given and a set $S \subset \{0, 1, 2, \ldots, n-1\}$ with $|S| > \frac{3}{4}n$. Prove that there exist integer numbers a, b, c such that the remainders after the division by n of the numbers:

$$a, b, c, a+b, b+c, c+a, a+b+c$$

belong to S.

PROBLEMA 2
Positive rational number a and b satisfy the equality

$$a^3 + 4a^2b = 4a^2 + b^4.$$

Prove that the number $\sqrt{a} - 1$ is a square of a rational number.

PROBLEMA 3
$ABCD$ is a parallelogram in which angle DAB is acute. Points A, P, B, D lie on one circle in exactly this order. Lines AP and CD intersect in Q. Point O is the circumcenter of the triangle CPQ. Prove that if $D \neq O$ then the lines AD and DO are perpendicular.

Segundo Dia

PROBLEMA 4
On the side BC of the triangle ABC there are two points D and E such that $BD < BE$. Denote by p_1 and p_2 the perimeters of triangles ABC and ADE respectively. Prove that

$$p_1 > p_2 + 2 \cdot \min\{BD, EC\}.$$

PROBLEMA 5
Prime number $p > 3$ is congruent to 2 modulo 3. Let $a_k = k^2 + k + 1$ for $k = 1, 2, \ldots, p-1$. Prove that product $a_1 a_2 \ldots a_{p-1}$ is congruent to 3 modulo p.

PROBLEMA 6
Real number $C > 1$ is given. Sequence of positive real numbers $a_1, a_2, a_3, \ldots$, in which $a_1 = 1$ and $a_2 = 2$, satisfy the conditions

$$a_{mn} = a_m a_n,$$

$$a_{m+n} \leq C(a_m + a_n),$$

for $m, n = 1, 2, 3, \ldots$. Prove that $a_n = n$ for $n = 1, 2, 3, \ldots$.

Primeiro Dia

PROBLEMA 1
Each vertex of a convex hexagon is the center of a circle whose radius is equal to the shorter side of the hexagon that contains the vertex. Show that if the intersection of all six circles (including their boundaries) is not empty, then the hexagon is regular.

PROBLEMA 2
Let S be a set of all points of a plane whose coordinates are integers. Find the smallest positive integer k for which there exists a 60-element subset of set S with the following condition satisfied for any two elements A, B of the subset there exists a point C contained in S such that the area of triangle ABC is equal to k .

PROBLEMA 3
Let P, Q, R be polynomials of degree at least 1 with integer coefficients such that for any real number x holds: $P(Q(x)) = Q(R(x)) = R(P(x))$. Show that the polynomials P, Q, R are equal.

Segundo Dia

PROBLEMA 4
Let $x_1, x_2, .., x_n$ be non-negative numbers whose sum is 1 . Show that there exist numbers $a_1, a_2, \ldots, a_n$ chosen from amongst $0, 1, 2, 3, 4$ such that $a_1, a_2, \ldots, a_n$ are different from $2, 2, \ldots, 2$ and $2 \leq a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq 2 + \frac{2}{3^n-1}$.

PROBLEMA 5
A sphere is inscribed in tetrahedron $ABCD$ and is tangent to faces BCD, CAD, ABD, ABC at points P, Q, R, S respectively. Segment PT is the sphere's diameter, and lines TA, TQ, TR, TS meet the plane BCD at points A', Q', R', S'. respectively. Show that A is the center of a circumcircle on the triangle $S'Q'R'$.

PROBLEMA 6
Let n be a natural number equal or greater than 3 . A sequence of non-negative numbers $(c_0, c_1, \ldots, c_n)$ satisfies the condition: $c_pc_s + c_rc_t = c_{p+r}c_{r+s}$ for all non-negative p, q, r, s such that $p + q + r + s = n$. Determine all possible values of c_2 when $c_1 = 1$.

Primeiro Dia

PROBLEMA 1
In each cell of a matrix $n \times n$ a number from a set $\{1, 2, \dots, n^2\}$ is written — in the first row numbers $1, 2, \dots, n$, in the second $n+1, n+2, \dots, 2n$ and so on. Exactly n of them have been chosen, no two from the same row or the same column. Let us denote by a_i a number chosen from row number i. Show that:

$$\frac{1^2}{a_1} + \frac{2^2}{a_2} + \dots + \frac{n^2}{a_n} \geq \frac{n+2}{2} - \frac{1}{n^2+1}$$

PROBLEMA 2
A function $f : R^3 \to R$ for all reals a, b, c, d, e satisfies a condition:

$$f(a,b,c) + f(b,c,d) + f(c,d,e) + f(d,e,a) + f(e,a,b) = a + b + c + d + e$$

Show that for all reals $x_1, x_2, \dots, x_n$ $(n \geq 5)$ equality holds:

$$f(x_1,x_2,x_3) + f(x_2,x_3,x_4) + \dots + f(x_{n-1},x_n,x_1) + f(x_n,x_1,x_2) = x_1 + x_2 + \dots + x_n$$

PROBLEMA 3
In a convex pentagon $ABCDE$ in which $BC = DE$ following equalities hold:

$$\angle ABE = \angle CAB = \angle AED - 90^\circ, \qquad \angle ACB = \angle ADE$$

Show that $BCDE$ is a parallelogram.

Segundo Dia

PROBLEMA 4
Each point of a plane with both coordinates being integers has been colored black or white. Show that there exists an infinite subset of colored points, whose points are in the same color, having a center of symmetry.

[EDIT: added condition about being infinite - now it makes sense]

PROBLEMA 5
Let R be a parallelopiped. Let us assume that areas of all intersections of R with planes containing centers of three edges of R pairwisely not parallel and having no common points, are equal. Show that R is a cuboid.

PROBLEMA 6
Let S be a set of all positive integers which can be represented as $a^2 + 5b^2$ for some integers a, b such that $a \perp b$. Let p be a prime number such that $p = 4n + 3$ for some integer n. Show that if for some positive integer k the number kp is in S, then $2p$ is in S as well.

Here, the notation $a \perp b$ means that the integers a and b are coprime.

Primeiro Dia

PROBLEMA 1

1. In acute triangle ABC point O is circumcenter, segment CD is a height, point E lies on side AB and point M is a midpoint of CE. Line through M perpendicular to OM cuts lines AC and BC respectively in K, L. Prove that $\frac{LM}{MK} = \frac{AD}{DB}$

PROBLEMA 2

2. Positive integer will be called white, if it is equal to 1 or is a product of even number of primes (not necessarily distinct). Rest of the positive integers will be called black. Determine whether there exists a positive integer which sum of white divisors is equal to sum of black divisors

PROBLEMA 3

3. Plane is divided with horizontal and vertical lines into unit squares. Into each square we write a positive integer so that each positive integer appears exactly once. Determine whether it is possible to write numbers in such a way, that each written number is a divisor of a sum of its four neighbours.

Segundo Dia

PROBLEMA 4

4. Given is an integer $n \geq 1$. Find out the number of possible values of products $k \cdot m$, where k, m are integers satisfying $n^2 \leq k \leq m \leq (n+1)^2$.

PROBLEMA 5

5. In tetrahedron $ABCD$ following equalities hold: $\angle BAC + \angle BDC = \angle ABD + \angle ACD$ $\angle BAD + \angle BCD = \angle ABC + \angle ADC$ Prove that center of sphere circumscribed about ABCD lies on a line through midpoints of AB and CD.

PROBLEMA 6

6. Sequence $a_0, a_1, a_2, ...$ is determined by $a_0 = -1$ and $a_n + \frac{a_{n-1}}{2} + \frac{a_{n-2}}{3} + ... + \frac{a_1}{n} + \frac{a_0}{n+1} = 0$ for $n \geq 1$ Prove that $a_n > 0$ for $n \geq 1$

Primeiro Dia

PROBLEMA 1
Solve in reals:

$$
\begin{aligned}
a^2 &= b^3 + c^3 \\
b^2 &= c^3 + d^3 \\
c^2 &= d^3 + e^3 \\
d^2 &= e^3 + a^3 \\
e^2 &= a^3 + b^3
\end{aligned}
$$

PROBLEMA 2
Find all positive integers k for which number $3^k + 5^k$ is a power of some integer with exponent greater than 1.

PROBLEMA 3
Let $ABCDEF$ be a convex hexagon satisfying $AC = DF$, $CE = FB$ and $EA = BD$. Prove that the lines connecting the midpoints of opposite sides of the hexagon $ABCDEF$ intersect in one point.

Segundo Dia

PROBLEMA 4
Given a triplet we perform on it the following operation. We choose two numbers among them and change them into their sum and product, left number stays unchanged. Can we, starting from triplet $(3, 4, 5)$ and performing above operation, obtain again a triplet of numbers which are lengths of right triangle?

PROBLEMA 5
Tetrahedron $ABCD$ in which $AB = CD$ is given. Sphere inscribed in it is tangent to faces ABC and ABD respectively in K and L. Prove that if points K and L are centroids of faces ABC and ABD then tetrahedron $ABCD$ is regular.

PROBLEMA 6
Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or -1.

Primeiro Dia

PROBLEMA 1
Find all triplets (x, y, n) of positive integers which satisfy:

$$(x-y)^n = xy$$

PROBLEMA 2
The points A, B, C, D lie in this order on a circle o. The point S lies inside o and has properties $\angle SAD = \angle SCB$ and $\angle SDA = \angle SBC$. Line which in which angle bisector of $\angle ASB$ in included cut the circle in points P and Q. Prove $PS = QS$.

PROBLEMA 3
In a matrix $2n \times 2n$, $n \in N$, are $4n^2$ real numbers with a sum equal zero. The absolute value of each of these numbers is not greater than 1. Prove that the absolute value of a sum of all the numbers from one column or a row doesn't exceed n.

Segundo Dia

PROBLEMA 4
Given real $c > -2$. Prove that for positive reals $x_1, ..., x_n$satisfying:$\sum\limits_{i=1}^{n} \sqrt{x_i^2 + cx_ix_{i+1} + x_{i+1}^2} = \sqrt{c+2}\left(\sum\limits_{i=1}^{n} x_i\right)$
holds $c = 2$ or $x_1 = ... = x_n$

PROBLEMA 5
Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

has all of its terms relatively prime to m.

PROBLEMA 6
Let be a convex polygon with $n > 5$ vertices and area 1. Prove that there exists a convex hexagon inside the given polygon with area at least $\frac{3}{4}$

Primeiro Dia

PROBLEMA 1
A point D is taken on the side AB of a triangle ABC. Two circles passing through D and touching AC and BC at A and B respectively intersect again at point E. Let F be the point symmetric to C with respect to the perpendicular bisector of AB. Prove that the points D, E, F lie on a line.

PROBLEMA 2
Let P be a polynomial with integer coefficients such that there are two distinct integers at which P takes coprime values. Show that there exists an infinite set of integers, such that the values P takes at them are pairwise coprime.

PROBLEMA 3
On a tournament with $n \geq 3$ participants, every two participants played exactly one match and there were no draws. A three-element set of participants is called a *draw-triple* if they can be enumerated so that the first defeated the second, the second defeated the third, and the third defeated the first. Determine the largest possible number of draw-triples on such a tournament.

Segundo Dia

PROBLEMA 4
Let real numbers a, b, c. Prove that $\sqrt{2(a^2+b^2)} + \sqrt{2(b^2+c^2)} + \sqrt{2(c^2+a^2)} \geq \sqrt{3(a+b)^2 + 3(b+c)^2 + 3(c+a)^2}$.

PROBLEMA 5
Find the greatest possible number of lines in space that all pass through a single point and the angle between any two of them is the same.

PROBLEMA 6
An integer $m > 1$ is given. The infinite sequence $(x_n)_{n\geq 0}$ is defined by $x_i = 2^i$ for $i < m$ and $x_i = x_{i-1} + x_{i-2} + \cdots + x_{i-m}$ for $i \geq m$. Find the greatest natural number k such that there exist k successive terms of this sequence which are divisible by m.

Primeiro Dia

PROBLEMA 1
In an acute-angled triangle ABC, CD is the altitude. A line through the midpoint M of side AB meets the rays CA and CB at K and L respectively such that $CK = CL$. Point S is the circumcenter of the triangle CKL. Prove that $SD = SM$.

PROBLEMA 2
Let $0 < a < 1$ be a real number. Prove that for all finite, strictly increasing sequences $k_1, k_2, \ldots, k_n$ of non-negative integers we have the inequality

$$\left(\sum_{i=1}^{n} a^{k_i}\right)^2 < \frac{1+a}{1-a}\sum_{i=1}^{n} a^{2k_i}.$$

PROBLEMA 3
Find all polynomials W with integer coefficients satisfying the following condition: For every natural number $n, 2^n - 1$ is divisible by $W(n)$.

Segundo Dia

PROBLEMA 4
A prime number p and integers x, y, z with $0 < x < y < z < p$ are given. Show that if the numbers x^3, y^3, z^3 give the same remainder when divided by p, then $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

PROBLEMA 5
The sphere inscribed in a tetrahedron $ABCD$ touches face ABC at point H. Another sphere touches face ABC at O and the planes containing the other three faces at points exterior to the faces. Prove that if O is the circumcenter of triangle ABC, then H is the orthocenter of that triangle.

PROBLEMA 6
Let n be an even positive integer. Show that there exists a permutation $(x_1, x_2, \ldots, x_n)$ of the set $\{1, 2, \ldots, n\}$, such that for each $i \in \{1, 2, \ldots, n\}, x_{i+1}$ is one of the numbers $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$, where $x_{n+1} = x_1$.

Primeiro Dia

PROBLEMA 1
Find all the natural numbers a, b, c such that:
1) a^2+1 and b^2+1 are primes 2) $(a^2+1)(b^2+1) = (c^2+1)$

PROBLEMA 2
On sides AC and BC of acute-angled triangle ABC rectangles with equal areas $ACPQ$ and $BKLC$ were built exterior. Prove that midpoint of PL, point C and center of circumcircle are collinear.

PROBLEMA 3
Three non-negative integers are written on a blackboard. A move is to replace two of the integers k, m by $k+m$ and $|k-m|$. Determine whether we can always end with triplet which has at least two zeros

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Segundo Dia

PROBLEMA 4
$x_1, ..., x_n$ are non-negative reals and $n \geq 3$. Prove that at least one of the following inequalities is true:

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2},$$

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_{i-2}} \geq \frac{n}{2}.$$

PROBLEMA 5
There is given a triangle ABC in a space. A sphere does not intersect the plane of ABC. There are 4 points K, L, M, P on the sphere such that AK, BL, CM are tangent to the sphere and $\frac{AK}{AP} = \frac{BL}{BP} = \frac{CM}{CP}$. Show that the sphere touches the circumsphere of $ABCP$.

PROBLEMA 6
k is a positive integer. The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = k+1$, $a_{n+1} = a_n^2 - ka_n + k$. Show that a_m and a_n are coprime (for $m \neq n$).

Primeiro Dia

PROBLEMA 1
Prove the following inequality:

$x_1 + 2x_2 + 3x_3 + ... + nx_n \leq \frac{n(n-1)}{2} + x_1 + x_2^2 + x_3^3 + ... + x_n^n$
where $\forall_{x_i} x_i > 0$

PROBLEMA 2
Given a regular tetrahedron $ABCD$ with edge length 1 and a point P inside it. What is the maximum value of $|PA| + |PB| + |PC| + |PD|$.

PROBLEMA 3
A sequence $x_0 = A$ and $x_1 = B$ and $x_{n+2} = x_{n+1} + x_n$ is called a Fibonacci type sequence. Call a number C a repeated value if $x_t = x_s = c$ for t different from s. Prove one can choose A and B to have as many repeated value as one likes but never infinite.

Segundo Dia

PROBLEMA 4
Assume that a, b are integers and n is a natural number. $2^n a + b$ is a perfect square for every n.Prove that $a = 0$.

PROBLEMA 5
Let $ABCD$ be a parallelogram and let K and L be points on the segments BC and CD, respectively, such that $BK \cdot AD = DL \cdot AB$. Let the lines DK and BL intersect at P. Show that $\measuredangle DAP = \measuredangle BAC$.

PROBLEMA 6
Given positive integers $n_1 < n_2 < ... < n_{2000} < 10^{100}$. Prove that we can choose from the set $\{n_1, ..., n_{2000}\}$ nonempty, disjont sets A and B which have the same number of elements, the same sum and the same sum of squares.

Primeiro Dia

PROBLEMA 1

Find number of solutions in non-negative reals to the following equations:

$$\begin{aligned} x_1 + x_n^2 &= 4x_n \\ x_2 + x_1^2 &= 4x_1 \\ &\;... \\ x_n + x_{n-1}^2 &= 4x_{n-1} \end{aligned}$$

PROBLEMA 2

Let a triangle ABC satisfy $AC = BC$; in other words, let ABC be an isosceles triangle with base AB. Let P be a point inside the triangle ABC such that $\angle PAB = \angle PBC$. Denote by M the midpoint of the segment AB. Show that $\angle APM + \angle BPC = 180°$.

PROBLEMA 3

The sequence $p_1, p_2, p_3, ...$ is defined as follows. p_1 and p_2 are primes. p_n is the greatest prime divisor of $p_{n-1} + p_{n-2} + 2000$. Show that the sequence is bounded.

Segundo Dia

PROBLEMA 4

$PA_1A_2...A_n$ is a pyramid. The base $A_1A_2...A_n$ is a regular n-gon. The apex P is placed so that the lines PA_i all make an angle $60^{\cdot}$ with the plane of the base. For which n is it possible to find B_i on PA_i for $i = 2, 3, ..., n$ such that $A_1B_2 + B_2B_3 + B_3B_4 + ... + B_{n-1}B_n + B_nA_1 < 2A_1P$?

PROBLEMA 5

In the unit squre For the given natural number $n \geq 2$ find the smallest number k that from each set of k unit squares of the nxn chessboard one can achoose a subset such that the number of the unit squares contained in this subset an lying in a row or column of the chessboard is even

PROBLEMA 6

Show that the only polynomial of odd degree satisfying $p(x^2 - 1) = p(x)^2 - 1$ for all x is $p(x) = x$

Primeiro Dia

PROBLEMA 1

Let D be a point on the side BC of a triangle ABC such that $AD > BC$. Let E be a point on the side AC such that $\frac{AE}{EC} = \frac{BD}{AD-BC}$. Show that $AD > BE$.

PROBLEMA 2

Given 101 distinct non-negative integers less than 5050 show that one can choose four a, b, c, d such that $a+b-c-d$ is a multiple of 5050

PROBLEMA 3

Show that one can find 50 distinct positive integers such that the sum of each number and its digits is the same.

Segundo Dia

PROBLEMA 4

For which n do the equations have a solution in integers:

$$
\begin{aligned}
x_1^2 + x_2^2 + 50 &= 16x_1 + 12x_2 \\
x_2^2 + x_3^2 + 50 &= 16x_2 + 12x_3 \\
\cdots \quad \cdots \quad \cdots & \quad \cdots \quad \cdots \quad \cdots \\
x_{n-1}^2 + x_n^2 + 50 &= 16x_{n-1} + 12x_n \\
x_n^2 + x_1^2 + 50 &= 16x_n + 12x_1
\end{aligned}
$$

PROBLEMA 5

Prove that for any $2n$ real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, we have

$$\sum_{i<j} |a_i - a_j| + \sum_{i<j} |b_i - b_j| \leq \sum_{i,j \in [1,n]} |a_i - b_j|.$$

PROBLEMA 6

Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360°$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

Primeiro Dia

PROBLEMA 1
Find all solutions in positive integers to:

$$a + b + c = xyz$$
$$x + y + z = abc$$

PROBLEMA 2
F_n is the Fibonacci sequence $F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Find all pairs $m > k \geq 0$ such that the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = \frac{F_k}{F_m}$ and $x_{n+1} = \frac{2x_n - 1}{1 - x_n}$ for $x_n \neq 1$, or 1 if $x_n = 1$, contains the number 1

PROBLEMA 3
$PABCDE$ is a pyramid with $ABCDE$ a convex pentagon. A plane meets the edges PA, PB, PC, PD, PE in points A', B', C', D', E' distinct from A, B, C, D, E and P. For each of the quadrilaterals $ABB'A', BCC'B, CDD'C', DEE'D'$, $EAA'E'$ take the intersection of the diagonals. Show that the five intersections are coplanar.

Segundo Dia

PROBLEMA 4
Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, $a_n = a_{n-1} + a_{[n/2]}$. Does the sequence contain infinitely many multiples of 7?

PROBLEMA 5
The points D, E on the side AB of the triangle ABC are such that $\frac{AD}{DB}\frac{AE}{EB} = \left(\frac{AC}{CB}\right)^2$. Show that $\angle ACD = \angle BCE$.

PROBLEMA 6
S is a board containing all unit squares in the xy plane whose vertices have integer coordinates and which lie entirely inside the circle $x^2 + y^2 = 1998^2$. In each square of S is written $+1$. An allowed move is to change the sign of every square in S in a given row, column or diagonal. Can we end up with exactly one -1 and $+1$ on the rest squares by a sequence of allowed moves?

Primeiro Dia

PROBLEMA 1
The positive integers $x_1, x_2, ..., x_7$ satisfy $x_6 = 144$, $x_{n+3} = x_{n+2}(x_{n+1} + x_n)$ for $n = 1, 2, 3, 4$. Find x_7.

PROBLEMA 2
Find all real solutions to:

$$\begin{aligned} 3(x^2+y^2+z^2) &= 1 \\ x^2y^2+y^2z^2+z^2x^2 &= xyz(x+y+z)^3. \end{aligned}$$

PROBLEMA 3
In a tetrahedron $ABCD$, the medians of the faces ABD, ACD, BCD from D make equal angles with the corresponding edges AB, AC, BC. Prove that each of these faces has area less than or equal to the sum of the areas of the other two faces.

Comment

Segundo Dia

PROBLEMA 4
The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = 0$, $a_n = a_{[n/2]} + (-1)^{n(n+1)/2}$. Show that for any positive integer k we can find n in the range $2^k \leq n < 2^{k+1}$ such that $a_n = 0$.

PROBLEMA 5
$ABCDE$ is a convex pentagon such that $DC = DE$ and $\angle C = \angle E = 90^{\cdot}$. F is a point on the side AB such that $\frac{AF}{BF} = \frac{AE}{BC}$. Show that $\angle FCE = \angle ADE$ and $\angle FEC = \angle BDC$.

PROBLEMA 6
Given any n points on a unit circle show that at most $\frac{n^2}{3}$ of the segments joining two points have length $> \sqrt{2}$.

Primeiro Dia

PROBLEMA 1
Find all pairs (n, r) with n a positive integer and r a real such that $2x^2 + 2x + 1$ divides $(x + 1)^n - r$.

PROBLEMA 2
Let P be a point inside a triangle ABC such that $\angle PBC = \angle PCA < \angle PAB$. The line PB meets the circumcircle of triangle ABC at a point E (apart from B). The line CE meets the circumcircle of triangle APE at a point F (apart from E). Show that the ratio $\frac{|APEF|}{|ABP|}$ does not depend on the point P, where the notation $|P_1P_2...P_n|$ stands for the area of an arbitrary polygon $P_1P_2...P_n$.

PROBLEMA 3
a_i, x_i are positive reals such that $a_1 + a_2 + ... + a_n = x_1 + x_2 + ... + x_n = 1$. Show that

$$2\sum_{i<j} x_i x_j \leq \frac{n-2}{n-1} + \sum \frac{a_i x_i^2}{1 - a_i}$$

When do we have equality?

Segundo Dia

PROBLEMA 4
$ABCD$ is a tetrahedron with $\angle BAC = \angle ACD$ and $\angle ABD = \angle BDC$. Show that $AB = CD$.

PROBLEMA 5
Let $p(k)$ be the smallest prime not dividing k. Put $q(k) = 1$ if $p(k) = 2$, or the product of all primes $< p(k)$ if $p(k) > 2$. Define the sequence $x_0, x_1, x_2, ...$ by $x_0 = 1$, $x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$. Find all n such that $x_n = 111111$

PROBLEMA 6
From the set of all permutations f of $\{1, 2, ..., n\}$ that satisfy the condition: $f(i) \geq i - 1$ $i = 1, ..., n$ one is chosen uniformly at random. Let p_n be the probability that the chosen permutation f satisfies $f(i) \leq i + 1$ $i = 1, ..., n$ Find all natural numbers n such that $p_n > \frac{1}{3}$.

Primeiro Dia

PROBLEMA 1
Segments AC and BD meet at P, and $|PA| = |PD|$, $|PB| = |PC|$. O is the circumcenter of the triangle PAB. Show that OP and CD are perpendicular.

PROBLEMA 2
Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of positive rationals, such that $f(x + 1) = f(x) + 1$ and $f(x^3) = f(x)^3$ for all x.

PROBLEMA 3
Show that for real numbers $x_1, x_2, ..., x_n$ we have:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_i x_j}{i+j} \geq 0$$

When do we have equality?

Segundo Dia

PROBLEMA 4
The functions $f_0, f_1, f_2, ...$ are defined on the reals by $f_0(x) = 8$ for all x, $f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)}$. For all n solve the equation $f_n(x) = 2x$.

PROBLEMA 5
The base of a regular pyramid is a regular $2n$-gon $A_1A_2...A_{2n}$. A sphere passing through the top vertex S of the pyramid cuts the edge SA_i at B_i (for $i = 1, 2, ..., 2n$). Show that $\sum_{i=1}^{n} SB_{2i-1} = \sum_{i=1}^{n} SB_{2i}$.

PROBLEMA 6
Show that $(k^3)!$ is divisible by $(k!)^{k^2+k+1}$.

Primeiro Dia

PROBLEMA 1
Prove or disprove that there exist two tetrahedra T_1 and T_2 such that: (i) the volume of T_1 is greater than that of T_2; (ii) the area of any face of T_1 does not exceed the area of any face of T_2.

PROBLEMA 2
Let X be the set of all lattice points in the plane (points (x, y) with $x, y \in \mathbb{Z}$). A path of length n is a chain $(P_0, P_1, ..., P_n)$ of points in X such that $P_{i-1}P_i = 1$ for $i = 1, ..., n$. Let $F(n)$ be the number of distinct paths beginning in $P_0 = (0, 0)$ and ending in any point P_n on line $y = 0$. Prove that $F(n) = \binom{2n}{n}$

PROBLEMA 3
Define

$$N = \sum_{k=1}^{60} e_k k^{k^k}$$

where $e_k \in \{-1, 1\}$ for each k. Prove that N cannot be the fifth power of an integer.

Segundo Dia

PROBLEMA 4
On the Cartesian plane consider the set V of all vectors with integer coordinates. Determine all functions $f : V \to \mathbb{R}$ satisfying the conditions: (i) $f(v) = 1$ for each of the four vectors $v \in V$ of unit length. (ii) $f(v + w) = f(v) + f(w)$ for every two perpendicular vectors $v, w \in V$ (Zero vector is considered to be perpendicular to every vector).

PROBLEMA 5
Two noncongruent circles k_1 and k_2 are exterior to each other. Their common tangents intersect the line through their centers at points A and B. Let P be any point of k_1. Prove that there is a diameter of k_2 with one endpoint on line PA and the other on PB.

PROBLEMA 6
If x, y, z are real numbers satisfying $x^2 + y^2 + z^2 = 2$, prove the inequality

$$x + y + z \leq 2 + xyz$$

When does equality occur?

Primeiro Dia

PROBLEMA 1
Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that satisfy

$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$$

PROBLEMA 2
Let $x_1, x_2, ..., x_n$ be positive numbers. Prove that

$$\sum_{i=1}^{n} \frac{x_i^2}{x_i^2+x_{i+1}x_{i+2}} \leq n-1$$

Where $x_{n+1}=x_1$ and $x_{n+2}=x_2$.

PROBLEMA 3
In a tournament, every two of the n players played exactly one match with each other (no draws). Prove that it is possible either (i) to partition the league in two groups A and B such that everybody in A defeated everybody in B; or (ii) to arrange all the players in a chain $x_1, x_2, ..., x_n, x_1$ in such a way that each player defeated his successor.

Segundo Dia

PROBLEMA 4
A triangle whose all sides have length not smaller than 1 is inscribed in a square of side length 1. Prove that the center of the square lies inside the triangle or on its boundary.

PROBLEMA 5
Suppose that (a_n) is a sequence of positive integers such that $\lim\limits_{n\to\infty} \dfrac{n}{a_n}=0$ Prove that there exists k such that there are at least 1990 perfect squares between $a_1+a_2+...+a_k$ and $a_1+a_2+...+a_{k+1}$.

PROBLEMA 6
Prove that for all integers $n>2$,

$$3 | \sum_{i=0}^{[n/3]} (-1)^i C_n^{3i}$$

Primeiro Dia

PROBLEMA 1

An even number of politicians are sitting at a round table. After a break, they come back and sit down again in arbitrary places. Show that there must be two people with the same number of people sitting between them as before the break..

Additional problem: Solve the problem when the number of people is in a form $6k+3$.

PROBLEMA 2

k_1, k_2, k_3 are three circles. k_2 and k_3 touch externally at P, k_3 and k_1 touch externally at Q, and k_1 and k_2 touch externally at R. The line PQ meets k_1 again at S, the line PR meets k_1 again at T. The line RS meets k_2 again at U, and the line QT meets k_3 again at V. Show that P, U, V are collinear.

PROBLEMA 3

The edges of a cube are labeled from 1 to 12. Show that there must exist at least eight triples (i, j, k) with $1 \leq i < j < k \leq 12$ so that the edges i, j, k are consecutive edges of a path. Also show that there exists labeling in which we cannot find nine such triples.

Segundo Dia

PROBLEMA 4

n, k are positive integers. A_0 is the set $\{1, 2, ..., n\}$. A_i is a randomly chosen subset of A_{i-1} (with each subset having equal probability). Show that the expected number of elements of A_k is $\frac{n}{2^k}$

PROBLEMA 5

Three circles of radius a are drawn on the surface of a sphere of radius r. Each pair of circles touches externally and the three circles all lie in one hemisphere. Find the radius of a circle on the surface of the sphere which touches all three circles.

PROBLEMA 6

Show that for positive reals a, b, c, d we have

$$\left(\frac{ab+ac+ad+bc+bd+cd}{6}\right)^3 \geq \left(\frac{abc+abd+acd+bcd}{4}\right)^2$$

Primeiro Dia

PROBLEMA 1
The real numbers $x_1, x_2, ..., x_n$ belong to the interval $(0, 1)$ and satisfy $x_1 + x_2 + ... + x_n = m + r$, where m is an integer and $r \in [0, 1)$. Show that $x_1^2 + x_2^2 + ... + x_n^2 \leq m + r^2$.

PROBLEMA 2
For a permutation $P = (p_1, p_2, ..., p_n)$ of $(1, 2, ..., n)$ define $X(P)$ as the number of j such that $p_i < p_j$ for every $i < j$. What is the expected value of $X(P)$ if each permutation is equally likely?

PROBLEMA 3
W is a polygon which has a center of symmetry S such that if P belongs to W, then so does P', where S is the midpoint of PP'. Show that there is a parallelogram V containing W such that the midpoint of each side of V lies on the border of W.

Segundo Dia

PROBLEMA 4
d is a positive integer and $f : [0, d] \to \mathbb{R}$ is a continuous function with $f(0) = f(d)$. Show that there exists $x \in [0, d-1]$ such that $f(x) = f(x + 1)$.

PROBLEMA 5
The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = a_2 = a_3 = 1$, $a_{n+3} = a_{n+2}a_{n+1} + a_n$. Show that for any positive integer r we can find s such that a_s is a multiple of r.

PROBLEMA 6
Find the largest possible volume for a tetrahedron which lies inside a hemisphere of radius 1.

Primeiro Dia

PROBLEMA 1
There are $n \geq 2$ points in a square side 1. Show that one can label the points $P_1, P_2, ..., P_n$ such that $\sum_{i=1}^{n} |P_{i-1} - P_i|^2 \leq 4$, where we use cyclic subscripts, so that P_0 means P_n.

PROBLEMA 2
A regular n-gon is inscribed in a circle radius 1. Let X be the set of all arcs PQ, where P, Q are distinct vertices of the n-gon. 5 elements $L_1, L_2, ..., L_5$ of X are chosen at random (so two or more of the L_i can be the same). Show that the expected length of $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ is independent of n.

PROBLEMA 3
$w(x)$ is a polynomial with integer coefficients. Let p_n be the sum of the digits of the number $w(n)$. Show that some value must occur infinitely often in the sequence $p_1, p_2, p_3, ...$.

Segundo Dia

PROBLEMA 4
Let S be the set of all tetrahedra which satisfy: (1) the base has area 1, (2) the total face area is 4, and (3) the angles between the base and the other three faces are all equal. Find the element of S which has the largest volume.

PROBLEMA 5
Find the smallest n such that $n^2 - n + 11$ is the product of four primes (not necessarily distinct).

PROBLEMA 6
A plane is tiled with regular hexagons of side 1. A is a fixed hexagon vertex. Find the number of paths P such that: (1) one endpoint of P is A, (2) the other endpoint of P is a hexagon vertex, (3) P lies along hexagon edges, (4) P has length 60, and (5) there is no shorter path along hexagon edges from A to the other endpoint of P.

Primeiro Dia

PROBLEMA 1
A square of side 1 is covered with m^2 rectangles. Show that there is a rectangle with perimeter at least $\frac{4}{m}$.

PROBLEMA 2
Find the maximum possible volume of a tetrahedron which has three faces with area 1.

PROBLEMA 3
p is a prime and m is a non-negative integer $< p - 1$. Show that $\sum_{j=1}^{p} j^m$ is divisible by p.

Segundo Dia

PROBLEMA 4
Find all n such that there is a real polynomial $f(x)$ of degree n such that $f(x) \geq f'(x)$ for all real x.

PROBLEMA 5
There is a chess tournament with $2n$ players ($n > 1$). There is at most one match between each pair of players. If it is not possible to find three players who all play each other, show that there are at most n^2 matches. Conversely, show that if there are at most n^2 matches, then it is possible to arrange them so that we cannot find three players who all play each other.

PROBLEMA 6
ABC is a triangle. The feet of the perpendiculars from B and C to the angle bisector at A are K, L respectively. N is the midpoint of BC, and AM is an altitude. Show that K, L, N, M are concyclic.

Primeiro Dia

PROBLEMA 1
Find the largest k such that for every positive integer n we can find at least k numbers in the set $\{n+1, n+2, ..., n+16\}$ which are coprime with $n(n+17)$.

PROBLEMA 2
Given a square side 1 and $2n$ positive reals $a_1, b_1, ..., a_n, b_n$ each ≤ 1 and satisfying $\sum a_i b_i \geq 100$. Show that the square can be covered with rectangles R_i with sides length (a_i, b_i) parallel to the square sides.

PROBLEMA 3
The function $f : R \to R$ satisfies $f(3x) = 3f(x) - 4f(x)^3$ for all real x and is continuous at $x = 0$. Show that $|f(x)| \leq 1$ for all x.

Segundo Dia

PROBLEMA 4
P is a point inside the triangle ABC is a triangle. The distance of P from the lines BC, CA, AB is d_a, d_b, d_c respectively. If r is the inradius, show that

$$\frac{2}{\frac{1}{d_a} + \frac{1}{d_b} + \frac{1}{d_c}} < r < \frac{d_a + d_b + d_c}{2}$$

PROBLEMA 5
$p(x, y)$ is a polynomial such that $p(cost, sint) = 0$ for all real t. Show that there is a polynomial $q(x, y)$ such that $p(x, y) = (x^2 + y^2 - 1)q(x, y)$.

PROBLEMA 6
There is a convex polyhedron with k faces. Show that if more than $k/2$ of the faces are such that no two have a common edge, then the polyhedron cannot have an inscribed sphere.

Primeiro Dia

PROBLEMA 1
Find the number of all real functions f which map the sum of n elements into the sum of their images, such that f^{n-1} is a constant function and f^{n-2} is not. Here $f^0(x) = x$ and $f^k = f \circ f^{k-1}$ for $k \geq 1$.

PROBLEMA 2
Let n be a positive integer. For all $i, j \in \{1, 2, ..., n\}$ define $a_{j,i} = 1$ if $j = i$ and $a_{j,i} = 0$ otherwise. Also, for $i = n + 1, ..., 2n$ and $j = 1, ..., n$ define $a_{j,i} = -\frac{1}{n}$. Prove that for any permutation p of the set $\{1, 2, ..., 2n\}$ the following inequality holds: $\sum_{j=1}^{n} |\sum_{k=1}^{n} a_{j,p}(k)| \geq \frac{n}{2}$

PROBLEMA 3
Let W be a regular octahedron and O be its center. In a plane P containing O circles $k_1(O, r_1)$ and $k_2(O, r_2)$ are chosen so that $k_1 \subset P \cap W \subset k_2$. Prove that $\frac{r_1}{r_2} \leq \frac{\sqrt{3}}{2}$

Segundo Dia

PROBLEMA 4
A coin is tossed n times, and the outcome is written in the form $(a_1, a_2, ..., a_n)$, where $a_i = 1$ or 2 depending on whether the result of the i-th toss is the head or the tail, respectively. Set $b_j = a_1 + a_2 + ... + a_j$ for $j = 1, 2, ..., n$, and let $p(n)$ be the probability that the sequence $b_1, b_2, ..., b_n$ contains the number n. Express $p(n)$ in terms of $p(n-1)$ and $p(n-2)$.

PROBLEMA 5
A regular hexagon of side 1 is covered by six unit disks. Prove that none of the vertices of the hexagon is covered by two (or more) discs.

PROBLEMA 6
Cities $P_1, ..., P_{1025}$ are connected to each other by airlines $A_1, ..., A_{10}$ so that for any two distinct cities P_k and P_m there is an airline offering a direct flight between them. Prove that one of the airlines can offer a round trip with an odd number of flights.

Primeiro Dia

PROBLEMA 1
On the plane are given a convex n-gon $P_1P_2....P_n$ and a point Q inside it, not lying on any of its diagonals. Prove that if n is even, then the number of triangles $P_iP_jP_k$ containing the point Q is even.

PROBLEMA 2
Let be given an irrational number a in the interval $(0,1)$ and a positive integer N. Prove that there exist positive integers p,q,r,s such that $\frac{p}{q} < a < \frac{r}{s}, \frac{r}{s} - \frac{p}{q} < \frac{1}{N}$, and $rq - ps = 1$.

PROBLEMA 3
Consider the following one-player game on an infinite chessboard. If two horizontally or vertically adjacent squares are occupied by a pawn each, and a square on the same line that is adjacent to one of them is empty, then it is allowed to remove the two pawns and place a pawn on the third (empty) square. Prove that if in the initial position all the pawns were forming a rectangle with the number of squares divisible by 3, then it is not possible to end the game with only one pawn left on the board.

Segundo Dia

PROBLEMA 4
Prove that if natural numbers a,b,c,d satisfy the equality $ab = cd$, then $\frac{gcd(a,c)gcd(a,d)}{gcd(a,b,c,d)} = a$

PROBLEMA 5
On the plane are given unit vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}$. Show that one can choose numbers $c_1, c_2, c_3 \in \{-1, 1\}$ such that the length of the vector $c_1\overrightarrow{a_1} + c_2\overrightarrow{a_2} + c_3\overrightarrow{a_3}$ is at least 2.

PROBLEMA 6
Prove that if all dihedral angles of a tetrahedron are acute, then all its faces are acute-angled triangles.

Primeiro Dia

PROBLEMA 1
Find a way of arranging n girls and n boys around a round table for which $d_n - c_n$ is maximum, where dn is the number of girls sitting between two boys and c_n is the number of boys sitting between two girls.

PROBLEMA 2
In a cyclic quadrilateral $ABCD$ the line passing through the midpoint of AB and the intersection point of the diagonals is perpendicular to CD. Prove that either the sides AB and CD are parallel or the diagonals are perpendicular

PROBLEMA 3
Find all pairs of positive numbers (x, y) which satisfy the system of equations $x^2 + y^2 = a^2 + b^2$ $x^3 + y^3 = a^3 + b^3$ where a and b are given positive numbers.

Segundo Dia

PROBLEMA 4
On a plane is given a finite set of points. Prove that the points can be covered by open squares $Q_1, Q_2, ..., Q_n$ such that $1 \leq \frac{N_j}{S_j} \leq 4$ for $j = 1, ..., n$, where N_j is the number of points from the set inside square Q_j and S_j is the area of Q_j.

PROBLEMA 5
Integers $x_0, x_1, ..., x_{n-1}, x_n = x_0, x_{n+1} = x_1$ satisfy the inequality $(-1)^{x_k} x_{k-1} x_{k+1} > 0$ for $k = 1, 2, ..., n$. Prove that the difference $\sum_{k=0}^{n-1} x_k - \sum_{k=0}^{n-1} |x_k|$ is divisible by 4.

PROBLEMA 6
Prove that the sum of dihedral angles in an arbitrary tetrahedron is greater than 2π