IMO SL 2011 - G2

Here, we are using ordered segments.

Let M be the intersection of the diagonals.

Let B_i be the other intersection of $A_{i+2}M$ with $(A_{i+1}A_{i+2}A_{i+3})$.

Thus: Power of point: $MB_i \cdot MA_{i+2} = MA_{i+1} \cdot MA_{i+3}$

We have:

$$\frac{1}{O_iA_i^2 - r^2} = \frac{1}{A_iB_i \cdot A_iA_{i+2}} = \frac{1}{(A_iM + MB_i)\, A_iA_{i+2}} =$$

$$= \frac{MA_{i+2}}{(-MA_i \cdot MA_{i+2} + MB_i \cdot MA_{i+2})\, A_iA_{i+2}} = \frac{MA_{i+2}}{A_iA_{i+2}\,(MA_{i+1} \cdot MA_{i+3} - MA_i \cdot MA_{i+2})}$$

So:

$$\left(\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_3A_3^2 - r_3^2}\right) + \left(\frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_4A_4^2 - r_4^2}\right) =$$

$$= \left(\frac{A_1M + MA_3}{A_1A_3\,(MA_2 \cdot MA_4 - MA_1 MA_3)}\right) + \left(\frac{A_2M + MA_4}{A_2A_4\,(MA_1 MA_3 - MA_2 MA_4)}\right)$$

$$= \left(\frac{1}{MA_2 MA_4 - MA_1 MA_3}\right) + \left(\frac{1}{MA_1 MA_3 - MA_2 MA_4}\right) = 0$$

$\square$