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Você pode enviar comentários e soluções para `zeusdanmou+tex@gmail.com`.
A versão mais atualizada desse arquivo (provavelmente) está disponível clicando aqui. Última atualização: 13 de julho de 2020.

Dia I

PROBLEMA 1
Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

PROBLEMA 2
An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A *beam* is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

- The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
- No two beams have intersecting interiors.
- The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

What is the smallest positive number of beams that can be placed to satisfy these conditions?

PROBLEMA 3
Let p be an odd prime. An integer x is called a *quadratic non-residue* if p does not divide $x - t^2$ for any integer t.

Denote by A the set of all integers a such that $1 \leq a < p$, and both a and $4 - a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

Dia II

PROBLEMA 4
Suppose that (a_1, b_1), (a_2, b_2), $\ldots$, (a_{100}, b_{100}) are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

PROBLEMA 5
A finite set S of points in the coordinate plane is called *overdetermined* if $|S| \geq 2$ and there exists a nonzero polynomial $P(t)$, with real coefficients and of degree at most $|S| - 2$, satisfying $P(x) = y$ for every point $(x, y) \in S$.

For each integer $n \geq 2$, find the largest integer k (in terms of n) such that there exists a set of n distinct points that is *not* overdetermined, but has k overdetermined subsets.

PROBLEMA 6
Let $n \geq 2$ be an integer. Let $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$ be $2n$ real numbers such that

$$0 = x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n$$

$$\text{and } \ 1 = x_1^2 + x_2^2 + ... + x_n^2 = y_1^2 + y_2^2 + ... + y_n^2.$$

Prove that

$$\sum_{i=1}^{n} (x_i y_i - x_i y_{n+i-1}) \geq \frac{2}{\sqrt{n-1}}.$$

Dia I

PROBLEMA 1
Seja $\mathbb{N}$ o conjunto dos inteiros positivos. Uma função $f : \mathbb{N} \to \mathbb{N}$ satisfaz a equação

$$\underbrace{f(f(\ldots f}_{f(n)\text{ vezes}}(n)\ldots)) = \frac{n^2}{f(f(n))}$$

para todos os inteiros positivos n. Sabendo disso, determine todos os possíveis valores de $f(1000)$.

PROBLEMA 2
Seja $ABCD$ um quadrilátero cíclico que satisfaz $AD^2 + BC^2 = AB^2$. As diagonais de $ABCD$ intersectam em E. Seja P um ponto no segmento $\overline{AB}$ que satisfaça $\angle APD = \angle BPC$. Mostre que a reta PE bisecta $\overline{CD}$.

PROBLEMA 3
Seja K o conjunto de inteiros positivos que não contém o dígito 7 em sua representação decimal. Ache todos os polinômios f com coeficientes inteiros não-negativos tais que $f(n) \in K$ para todo $n \in K$.

Dia II

PROBLEMA 4
Seja n um inteiro não negativo. Determine a quantidade de maneiras que é possível escolher $(n+1)^2$ conjuntos $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, para inteiros i, j com $0 \le i, j \le n$, tal que:

- para todo $0 \le i, j \le n$, o conjunto $S_{i,j}$ possui $i + j$ elementos; e
- $S_{i,j} \subseteq S_{k,l}$ sempre que $0 \le i \le k \le n$ e $0 \le j \le l \le n$.

PROBLEMA 5
Dois números racionais $\frac{m}{n}$ e $\frac{n}{m}$ são escritos num quadro, onde m e n são inteiros positivos primos entre si. A qualquer momento, Evan pode pegar escolher dois números x e y escritos no quadro e escrever sua média aritmética $\frac{x+y}{2}$ ou sua média harmônica $\frac{2xy}{x+y}$ no quadro. Ache todos os pares m e n tal que Evan pode escreber o número 1 no quadro com um número finitos passos.

PROBLEMA 6
Ache todos os polinômios P com coeficientes reais tal que

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

é válido para todos reais não nulos x, y, z satisfazendo $2xyz = x + y + z$.

Dia I

PROBLEMA 1
Sejam a, b, c reais positivos tais que $a+b+c=4\sqrt[3]{abc}$. Prove que

$$2\left(ab+bc+ca\right)+4\min\left(a^2,b^2,c^2\right)\geq a^2+b^2+c^2.$$

PROBLEMA 2
Ache todas as funções $f:(0,\infty)\rightarrow(0,\infty)$ tais que

$$f\left(x+\frac{1}{y}\right)+f\left(y+\frac{1}{z}\right)+f\left(z+\frac{1}{x}\right)=1$$

para todos $x, y, z>0$ com $xyz=1$.

PROBLEMA 3
Dado um inteiro $n\geq 2$, seja $\{a_1,a_2,\ldots,a_m\}$ o conjunto dos inteiros menores que n que são primos com n. Prove que, se todo primo que divide m também divide n, então $a_1^k+a_2^k+\cdots+a_m^k$ é divisível por m para todo inteiro positivo k.

Dia II

PROBLEMA 4
Seja p um primo, e sejam $a_1,\ldots,a_p$ inteiros. Mostre que existe um inteiro k tal que os números

$$a_1+k, a_2+2k,\ldots,a_p+pk$$

produz pelo menos $\frac{1}{2}p$ restos distintos quando divididos por p.

PROBLEMA 5
In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle PAC=90°$.

PROBLEMA 6
Let a_n be the number of permutations $(x_1,x_2,\ldots,x_n)$ of the numbers $(1,2,\ldots,n)$ such that the n ratios $\frac{x_k}{k}$ for $1\leq k\leq n$ are all distinct. Prove that a_n is odd for all $n\geq 1$.

Dia I

PROBLEMA 1
Prove that there are infinitely many distinct pairs (a, b) of relatively prime integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

PROBLEMA 2
Let $m_1, m_2, \ldots, m_n$ be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an A-*inversion* of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

- $a_i \geq w_i > w_j$
- $w_j > a_i \geq w_i$, or
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \ldots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly k B-inversions.

PROBLEMA 3
Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Dia II

PROBLEMA 4
Let P_1, P_2, $\ldots$, P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, $\ldots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

PROBLEMA 5
Let $\mathbb{Z}$ denote the set of all integers. Find all real numbers $c > 0$ such that there exists a labeling of the lattice points $(x, y) \in \mathbb{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and
- for each label i, the distance between any two points labeled i is at least c^i.

PROBLEMA 6
Find the minimum possible value of

$$\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$.

Dia I

PROBLEMA 1
Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

PROBLEMA 2
Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

PROBLEMA 3
Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

Dia II

PROBLEMA 4
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

PROBLEMA 5
An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overleftrightarrow{MN}$ and $\overleftrightarrow{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

PROBLEMA 6
Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard.

The wizard has $2n$ cards; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order.

You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is winnable if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds.

For which values of n and k is the game winnable?

Dia I

PROBLEMA 1
Solve in integers the equation
$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

PROBLEMA 2
Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

PROBLEMA 3
Let $S = \{1, 2, \ldots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,
$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

Dia II

PROBLEMA 4
Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i,k), (i,l), (j,k), (j,l)$ for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively,j or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

PROBLEMA 5
Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

PROBLEMA 6
Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the set A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$. (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are equivalent, but $\{1, 1, 2, 3\}$ and $\{1, 2, 3\}$ differ.)

Dia I

PROBLEMA 1
Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

PROBLEMA 2
Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

PROBLEMA 3
Prove that there exists an infinite set of points

$$\ldots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \ldots$$

in the plane with the following property: For any three distinct integers a, b, and c, points P_a, P_b, and P_c are collinear if and only if $a + b + c = 2014$.

Dia II

PROBLEMA 4
Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

PROBLEMA 5
Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of the angle $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.

PROBLEMA 6
Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \ldots n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

Dia I

PROBLEMA 1

In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

PROBLEMA 2

For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

PROBLEMA 3

Let n be a positive integer. There are $\frac{n(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the black side up. An operation is to choose a line parallel to the sides of the triangle, and flipping all the marks on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

Dia II

PROBLEMA 4

Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x + xyz}, \sqrt{y + xyz}, \sqrt{z + xyz}) = \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

PROBLEMA 5

Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

PROBLEMA 6

Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

Dia I

PROBLEMA 1
Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \ldots, a_n$ with $\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n)$, there exist three that are the side lengths of an acute triangle.

PROBLEMA 2
A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

PROBLEMA 3
Determine which integers $n > 1$ have the property that there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of nonzero integers such that the equality

$$a_k + 2a_{2k} + \ldots + na_{nk} = 0$$

holds for every positive integer k.

Dia II

PROBLEMA 4
Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

PROBLEMA 5
Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

PROBLEMA 6
For integer $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \ldots + x_n = 0, \qquad \text{and} \qquad x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \ldots, n\}$, define

$$S_A = \sum_{i \in A} x_i.$$

(If A is the empty set, then $S_A = 0$.)

Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \ldots, x_n, \lambda$ does equality hold?

Dia I

PROBLEMA 1
Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

PROBLEMA 2
An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding $2m$ to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

PROBLEMA 3
In hexagon $ABCDEF$, which is nonconvex but not self-intersecting, no pair of opposite sides are parallel. The internal angles satisfy $\angle A = 3\angle D$, $\angle C = 3\angle F$, and $\angle E = 3\angle B$. Furthermore $AB = DE$, $BC = EF$, and $CD = FA$. Prove that diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

Dia II

PROBLEMA 4
Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

PROBLEMA 5
Let P be a given point inside quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP, \quad \angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

PROBLEMA 6
Let A be a set with $|A| = 225$, meaning that A has 225 elements. Suppose further that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \ldots \cup A_{11}| \geq 165$, and give an example for which equality holds.

Dia I

PROBLEMA 1
Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by$P$, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

PROBLEMA 2
There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

PROBLEMA 3
The 2010 positive numbers $a_1, a_2, \ldots, a_{2010}$ satisfy the inequality $a_i a_j \leq i + j$ for all distinct indices i, j. Determine, with proof, the largest possible value of the product $a_1 a_2 \ldots a_{2010}$.

Dia II

PROBLEMA 4
Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

PROBLEMA 5
Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

PROBLEMA 6
A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k at most one of the pairs (k, k) and $(-k, -k)$ is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.

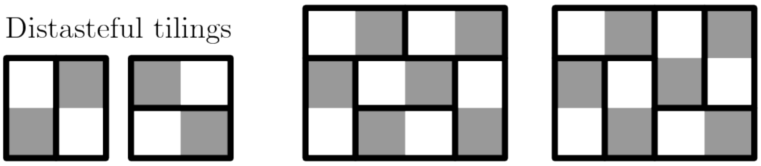
Dia I

PROBLEMA 1
Given circles ω_1 and ω_2 intersecting at points X and let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line

PROBLEMA 2
Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \ldots, n-1, n\}$ which does not contain three elements c (not necessarily distinct) satisfying

PROBLEMA 3
We define a *chessboard polygon* to be a polygon whose sides are situated along lines of the form $x = a$ or where a and b are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping 1×2 rectangles. Finally, a *tasteful tiling* is one which avoids the two configurations of dominoes shown on the left below. Two tilings of a 3×4 rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.

Distasteful tilings



a) Prove that if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully.
b) Prove that such a tasteful tiling is unique.

Dia II

PROBLEMA 4
For $n \geq 2$ let $a_1, a_2, \ldots a_n$ be positive real numbers such that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2.$$

Prove that

PROBLEMA 5
Trapezoid with is inscribed in circle ω and point G lies inside triangle Rays AG and BG meet ω again at points P and respectively. Let the line through G parallel to $\overline{AB}$ intersects $\overline{BD}$ and $\overline{BC}$ at points R and respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects

PROBLEMA 6
Let $s_1, s_2, s_3, \ldots$ be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \ldots$. Suppose that $t_1, t_2, t_3, \ldots$ is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and

Dia I

PROBLEMA 1
Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0 k_1 \ldots k_n - 1$ is the product of two consecutive integers.

PROBLEMA 2
Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

PROBLEMA 3
Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that

$$|x| + \left| y + \frac{1}{2} \right| < n.$$

A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \ldots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates). Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

Dia II

PROBLEMA 4
Let $\mathcal{P}$ be a convex polygon with n sides, $n \geq 3$. Any set of $n - 3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n - 2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

PROBLEMA 5
Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

PROBLEMA 6
At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

Dia I

PROBLEMA 1
Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \leq a_k \leq k-1$ for which $a_1 + a_2 + ... + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, ...$ Prove that for any n the sequence $a_1, a_2, ...$ eventually becomes constant.

PROBLEMA 2
A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

PROBLEMA 3
Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

Dia II

PROBLEMA 4
An animal with n cells is a connected figure consisting of n equal-sized cells[1].

A dinosaur is an animal with at least 2007 cells. It is said to be primitive it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

(1) Animals are also called polyominoes. They can be defined inductively. Two cells are adjacent if they share a complete edge. A single cell is an animal, and given an animal with n cells, one with $n+1$ cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells.

PROBLEMA 5
Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

PROBLEMA 6
Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω. Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \leq R^3 ,$$

with equality if and only if triangle ABC is equilateral.

Dia I

PROBLEMA 1
Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

if and only if s is not a divisor of $p-1$.

Note: For x a real number, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.

PROBLEMA 2
For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

PROBLEMA 3
For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence

$$\{p\left(f\left(n^2\right)\right) - 2n\}_{n \geq 0}$$

is bounded above. (In particular, this requires $f\left(n^2\right) \neq 0$ for $n \geq 0$.)

Dia II

PROBLEMA 4
Find all positive integers n such that there are $k \geq 2$ positive rational numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \ldots + a_k = a_1 \cdot a_2 \cdots a_k = n$.

PROBLEMA 5
A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

PROBLEMA 6
Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

Dia I

PROBLEMA 1
Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

PROBLEMA 2
Prove that the system

$$x^6 + x^3 + x^3y + y = 147^{157}$$
$$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$

has no solutions in integers x, y, and z.

PROBLEMA 3
Let ABC be an acute-angled triangle, and let P and Q be two points on its side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle.

Dia II

PROBLEMA 4
Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table?

(The table is stable if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

PROBLEMA 5
Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side.

Prove that there exist at least two balancing lines.

PROBLEMA 6
For m a positive integer, let $s(m)$ be the sum of the digits of m. For $n \geq 2$, let $f(n)$ be the minimal k for which there exists a set S of n positive integers such that $s\left(\sum_{x\in X} x\right) = k$ for any nonempty subset $X \subset S$. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$

Dia I

PROBLEMA 1
Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60 degrees. Prove that
$$\frac{1}{3}|AB^3 - AD^3| \leq |BC^3 - CD^3| \leq 3|AB^3 - AD^3|.$$
When does equality hold?

PROBLEMA 2
Suppose $a_1, \ldots, a_n$ are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:
(a) For $i = 1, \ldots, n$, $a_i \in S$.
(b) For $i, j = 1, \ldots, n$ (not necessarily distinct), $a_i - a_j \in S$.
(c) For any integers $x, y \in S$, if $x + y \in S$, then $x - y \in S$.
Prove that S must be equal to the set of all integers.

PROBLEMA 3
For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar, but noncongruent, polygons?

Dia II

PROBLEMA 4
Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

PROBLEMA 5
Let $a, b, c > 0$. Prove that $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$.

PROBLEMA 6
A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that
$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$
Prove that $ABCD$ is an isosceles trapezoid.

Dia I

PROBLEMA 1
Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

PROBLEMA 2
A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

PROBLEMA 3
Let $n \neq 0$. For every sequence of integers

$$A = a_0, a_1, a_2, \ldots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \ldots, n$, define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

Dia II

PROBLEMA 4
Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

PROBLEMA 5
Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

PROBLEMA 6
At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003^{2003}. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Dia I

PROBLEMA 1
Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:
(a) the union of any two white subsets is white;
(b) the union of any two black subsets is black;
(c) there are exactly N white subsets.

PROBLEMA 2
Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2\cot \frac{B}{2}\right)^2 + \left(3\cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

PROBLEMA 3
Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Dia II

PROBLEMA 4
Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

PROBLEMA 5
Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \leq i < k)$.

PROBLEMA 6
I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of the sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.

Dia I

PROBLEMA 1
Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.

PROBLEMA 2
Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

PROBLEMA 3
Let $a, b, c \geq 0$ and satisfy

$$a^2 + b^2 + c^2 + abc = 4.$$

Show that

$$0 \leq ab + bc + ca - abc \leq 2.$$

Dia II

PROBLEMA 4
Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.

PROBLEMA 5
Let S be a set of integers (not necessarily positive) such that

(a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$;

(b) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S.

Prove that S is the set of all integers.

PROBLEMA 6
Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Dia I

PROBLEMA 1
Call a real-valued function f very convex if

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers x and y. Prove that no very convex function exists.

PROBLEMA 2
Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA, respectively. Prove that all triangles in S are isosceles and similar to one another.

PROBLEMA 3
A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand. Find, as a function of $R, W,$ and B, the minimal total penalty a player can amass and all the ways in which this minimum can be achieved.

Dia II

PROBLEMA 4
Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

PROBLEMA 5
Let $A_1A_2A_3$ be a triangle and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exist circles $\omega_2, \omega_3, \ldots, \omega_7$ such that for $k = 2, 3, \ldots, 7$, ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1}, where $A_{n+3} = A_n$ for all $n \geq 1$. Prove that $\omega_7 = \omega_1$.

PROBLEMA 6
Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_ia_j, b_ib_j\} \leq \sum_{i,j=1}^{n} \min\{a_ib_j, a_jb_i\}.$$

Dia I

PROBLEMA 1
Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

PROBLEMA 2
Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

PROBLEMA 3
Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$$\left\{\frac{ra}{p}\right\} + \left\{\frac{rb}{p}\right\} + \left\{\frac{rc}{p}\right\} + \left\{\frac{rd}{p}\right\} = 2$$

for any integer r not divisible by p. Prove that at least two of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are divisible by p.

(Note: $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.)

Dia II

PROBLEMA 4
Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \qquad \text{and} \qquad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

PROBLEMA 5
The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

PROBLEMA 6
Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

Dia I

PROBLEMA 1
Suppose that the set $\{1, 2, \cdots, 1998\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \le i \le 999)$ so that for all i, $|a_i - b_i|$ equals 1 or 6. Prove that the sum

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{999} - b_{999}|$$

ends in the digit 9.

PROBLEMA 2
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ $(B \in \mathcal{C}_2)$. Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

PROBLEMA 3
Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \ge n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \ge n^{n+1}.$$

Dia II

PROBLEMA 4
A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

PROBLEMA 5
Prove that for each $n \ge 2$, there is a set S of n integers such that $(a - b)^2$ divides ab for every distinct $a, b \in S$.

PROBLEMA 6
Let $n \ge 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2 \ldots A_n$ for which exactly k of the quadrilaterals $A_iA_{i+1}A_{i+2}A_{i+3}$ have an inscribed circle. (Here $A_{n+j} = A_j$.)