(IMO 2008 / P1) (Solution with complex numbers)

ω is the unit circle. $|a|=|b|=|c|=1$. $h=a+b+c$ $m=\frac{b+c}{2}$.

$$OA_1^2 = OM^2 + MA_1^2 = OM^2 + MH^2 = |m|^2 + |h-m|^2 = |m|^2 + |a+m|^2$$

$$= m\cdot\overline{m} + (a+m)\cdot(\overline{a+m}) = a\cdot\overline{a} + a\cdot\overline{m} + \overline{a}\cdot m + 2\cdot m\cdot\overline{m}$$

$$= 1 + \frac{1}{2}\cdot a\cdot(\overline{b}+\overline{c}) + \frac{1}{2}\overline{a}(b+c) + \frac{1}{2}(b+c)(\overline{b}+\overline{c}) =$$

$$= 1 + \frac{1}{2}a\overline{b} + \frac{1}{2}a\overline{c} + \frac{1}{2}\overline{a}b + \frac{1}{2}\overline{a}c + \frac{b\cdot\overline{b}}{2} + \frac{c\cdot\overline{c}}{2} + \frac{1}{2}\cdot b\overline{c} + \frac{1}{2}\overline{b}c =$$

$$= 2 + \frac{1}{2}(a\overline{b} + \overline{a}b + a\overline{c} + \overline{a}c + \overline{b}c + b\overline{c}),$$ which is symetric!

$OA_1^2 = \cdots OC_2^2 \Rightarrow A_1, \ldots, C_2,$ cyclic.

Problema 7 - (Geometria / Regis)

IMO 2008 / Problem 1). (Solution using vectors)

Let $O = \vec{0}$, $\vec{OA} = \vec{a}$; $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$, $\vec{OH} = \vec{a} + \vec{b} + \vec{c}$; $\vec{OM} = \dfrac{\vec{b} + \vec{c}}{2} = \vec{m}$

• $OA_1^2 = OM^2 + HA_1^2 = OM^2 + MH^2 = \vec{OM} \cdot \vec{OM} + \vec{MH} \cdot \vec{MH} =$

$$= \vec{m} \cdot \vec{m} + (\vec{a} + \vec{b} + \vec{c} - \vec{m}) \cdot (\vec{c} + \vec{b} + \vec{c} - \vec{m})$$

$$= \vec{m} \cdot \vec{m} + (\vec{a} + \vec{m})(a + \vec{m})$$

$$= \vec{a} \cdot \vec{a} + 2 \cdot \vec{a} \cdot \vec{m} + 2 \cdot \vec{m} \cdot \vec{m}$$

$$= \vec{a} \cdot \vec{a} + \vec{a}(\vec{b} + \vec{c}) + \frac{(\vec{b} + \vec{c})(\vec{b} + \vec{c})}{2}$$

$$= R^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \frac{\vec{b} \cdot \vec{b}}{2} + \frac{\vec{c} \cdot \vec{c}}{2} + \vec{b}\vec{c}$$

$$= 2R^2 + \vec{a}\vec{b} + \vec{b}\vec{c} + \vec{c}\vec{a}. \quad (\text{symmetric in } \vec{a}, \vec{b}, \vec{c})$$

Thus, $OA_1^2 = OA_2^2 = OB_1^2 = OB_2^2 = OC_1^2 = OC_2^2 \Rightarrow$

O is the center of a circle that passes through $A_1, A_2, B_1, B_2, C_1, C_2$, as desired. $\square$