Croatia TST 2016, #2

PROBLEM 1

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y:

$$f(x^2) + xf(y) = f(x)f(x + f(y)) .$$

PROBLEM 2

Let S be a set of $N \geq 3$ points in the plane. Assume that no 3 points in S are collinear. The segments with both endpoints in S are colored in two colors.

Prove that there is a set of $N - 1$ segments of the same color which don't intersect except in their endpoints such that no subset of them forms a polygon with positive area.

PROBLEM 3

Let ABC be an acute triangle with circumcenter O. Points E and F are chosen on segments OB and OC such that $BE = OF$. If M is the midpoint of the arc EOA and N is the midpoint of the arc AOF, prove that $\sphericalangle ENO + \sphericalangle OMF = 2\sphericalangle BAC$.

PROBLEM 4

Let $p > 10^9$ be a prime number such that $4p + 1$ is also prime.

Prove that the decimal expansion of $\frac{1}{4p+1}$ contains all the digits $0, 1, \ldots, 9$.