Let k and n be positive integers and let

$$S = \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid 0 \le a_k \le \cdots \le a_1 \le n, a_1 + \cdots + a_k = \cancel{n} k\}.$$

Determine, with proof, the value of

$$T = \sum_{(a_1,\ldots,a_k)\in S} \overbrace{\binom{n}{a_1}\binom{a_1}{a_2}\cdots\binom{a_{k-1}}{a_k}}^{\alpha(\vec{a})}$$

in terms of k and n, where the sum is over all k-tuples in S.

$K=1$. $S = \{(n)\}$. $T = \binom{n}{n} = 1$

$K=2$. $S = \{(n-j, j),\ j = 0, \ldots, \lfloor n/2 \rfloor\}$

$n=1$: $S = \{(1,0)\}$. $T = \binom{1}{1}\binom{1}{0} = 1$

$n=2$: $S = \{(2,0), (1,1)\}$. $T = \binom{2}{2}\binom{2}{0} + \binom{2}{1}\binom{1}{1} = 3$.

$n=3$: $S = \{(3,0), (2,1)\}$ $T = \binom{3}{3}\binom{3}{0} + \binom{3}{2}\binom{2}{1} = 7$

$k \backslash n$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	3	7	19	51		
3	1	3	10	31			
4	1	3	10				
5	1	3	10				

Se $n < K$: $T(n,K) = T(n,n)$

$(\underbrace{\quad\quad}_{n}, 0, 0, 0)$

$T(3,3) = 10$ $S(3,3) = \{(3,0,0), (2,1,0), (1,1,1)\}$
$\downarrow 1 \quad \downarrow 6 \quad \downarrow 3$

$T(4,3) = 31$ $S(4,3) = \{(4,0,0), (3,1,0), (2,2,0), (2,1,1)\}$
$\downarrow 1 \quad \downarrow 12 \quad \downarrow 6 \quad 12$

$\vec{a} = (a_1, a_2, \ldots, a_K)$.

$$\alpha(\vec{a}) = \binom{n}{a_1}\binom{a_1}{a_2}\cdots\binom{a_{K-1}}{a_K}$$

$$= \#(\underset{A_0}{[n]} \supset A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_K \text{ t.q. } \textstyle\sum a_i = \cancel{n} K)$$

Seja $f(l) = \max\{i \text{ tq } l \in A_i\}$.

CONTAGEM DUPLA $\Rightarrow$ $\displaystyle\sum_{l\in[n]} f(l) = \sum |A_i|$, $\qquad \sum f(i) = \cancel{n} K \iff \sum a_i = \cancel{n} K$

(l, A_i)

Claim:

$$\sum_{\vec{a}\in S} \alpha(\vec{a}) = \#(\underset{A_0}{[n]} \supset A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_K \text{ t.q. } \textstyle\sum f(l) = \cancel{n} K)$$

$$= \#(f : [n] \to \{0, \ldots, K\} \text{ t.q. } \textstyle\sum f(l) = \cancel{n} K)$$

$$g(x) = (1 + x + x^2 + \cdots + x^k)^n$$
$$= \left(\frac{x^{k+1} - 1}{x - 1} \right)^n =$$

O coeficiente de x^k é o que a gente quer

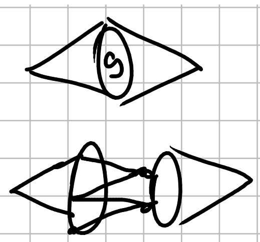
$$= \#\left(f : [n] \to \mathbb{Z}_{\geq 0} \quad \text{t.q. } \sum f(i) = k \right)$$

$\underbrace{\bullet\bullet\bullet\bullet}_{k} \; \underbrace{|||||}_{n-1}$

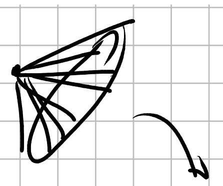
$$= \binom{k+n-1}{k}.$$

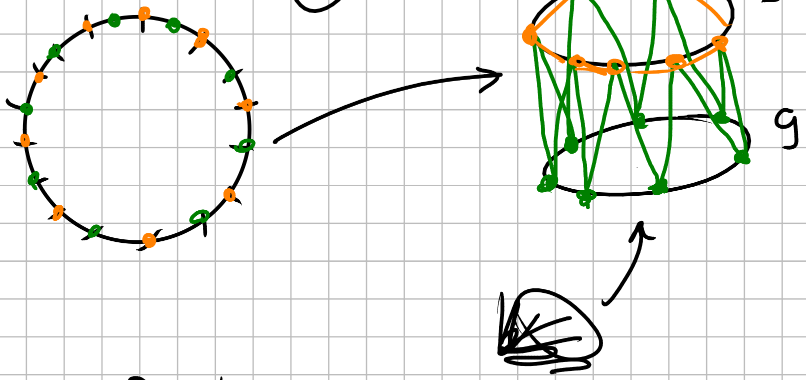
4. A convex polyhedron has n faces that are all congruent triangles with angles $36°$, $72°$, and $72°$. Determine, with proof, the maximum possible value of n.

18 faces: Ex:



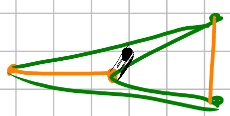
36 faces





Lema: $\text{grau}_p(v) \le 2$.

TODAS AS FACES SÃO TRIÂNGULOS $n = 2v - 4$.



$$\cancel{36}\cdot 5n \le \cancel{36}\cdot 9v$$

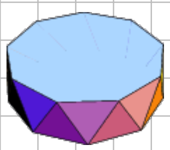
$$5n \le 9v$$

$$10v - 20 \le 9v$$

$$v \le 20.$$

$$n \le 36$$

yay.



5. For each positive real number α, define

$$\lfloor \alpha \mathbb{N} \rfloor := \{\lfloor \alpha m \rfloor \mid m \in \mathbb{N}\}.$$

α-Beatty sequence

Let n be a positive integer. A set $S \subseteq \{1, 2, \ldots, n\}$ has the property that: for each real $\beta > \cancel{0}\ 1$,

$$\text{if } S \subseteq \lfloor \beta \mathbb{N} \rfloor, \text{then } \{1, 2, \ldots, n\} \subseteq \lfloor \beta \mathbb{N} \rfloor. \qquad (*)$$

Determine, with proof, the smallest positive size of S.

OUTRO PROBLEMA c/ B.S. IMO SL 2019 N6

Se $\beta \leq 1$, então $\lfloor \beta \mathbb{N} \rfloor = \mathbb{N}$, logo $(*)$.

$n=2$ $\quad \lfloor 1.6\mathbb{N} \rfloor = \{1, 3, \ldots\}$ $\quad$ Logo $|S| \geq 2$.

$\lfloor 2\mathbb{N} \rfloor = \{2, 4, \ldots\}$

$n=3$ $\quad S = \{2,3\}$ funciona

$\alpha\mathbb{N}$

$n=4$ $\quad S = \{2,3,4\}$ funciona.

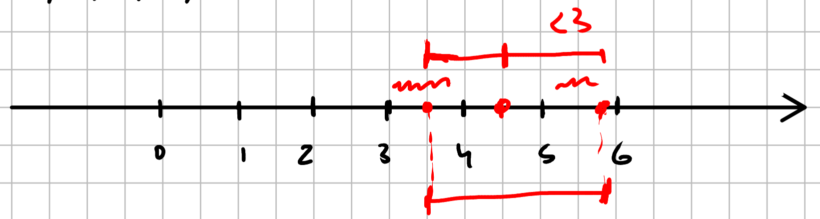
$\lfloor 1.5\mathbb{N} \rfloor = \{1, 3, 4, \ldots\} \Rightarrow 2 \in S$

$\lfloor 1.49\mathbb{N} \rfloor = \{1, 2, 4, \ldots\} \Rightarrow 3 \in S$

$\lfloor 1.25\mathbb{N} \rfloor = \{1, 2, 3, 5, \ldots\} \Rightarrow 4 \in S$ $\quad \alpha = \beta$

$\alpha > 1$

$n=5$ $\quad \{3,4,5\}$



$n = 2K+1$

$S = \{K+1, \ldots, 2K+1\}$

$S \subseteq \lfloor \beta \mathbb{N} \rfloor \Rightarrow K \in \lfloor \beta \mathbb{N} \rfloor$

$K < K\beta \leq d = \text{int}(\beta) < K+1$

Como $\beta > 1 \Rightarrow d = K\beta$.

$\lfloor K\beta \rfloor = K$

Seja $i \in \{K+1, \ldots, 2K+1\}$

$\beta = 1 + \frac{1}{i}$ $\quad \{1, \ldots, i-1, i+1, \ldots, 2i, 2i+2\}$

$n = 2K+1$

$S_K = \{K+1, \dots, 2K+1\}$

Se $S_K \subseteq \lfloor \beta N \rfloor \overset{?}{\Rightarrow}$ TODO MUNDO $2K+1$ $\subseteq \lfloor \beta N \rfloor$.

$\Downarrow$

$K \in \lfloor \beta N \rfloor$

$\Downarrow$

$S_{K-1} \subseteq \lfloor \beta N \rfloor \underset{H.I.}{\Rightarrow}$ TODO MUNDO ATÉ $2K-1$ $\in \lfloor \beta N \rfloor$

Suponha que $\sqrt{a} + \sqrt{b}$ é inteiro.

$a = x \cdot n^2$ $\qquad b = y \cdot m^2$

$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$ $\Rightarrow \sqrt{ab}$ é racional

$\Rightarrow \sqrt{ab}$ é inteiro

$\Rightarrow ab$ é $\square$

$\Rightarrow x = y$.

Logo $\sqrt{a} + \sqrt{b} = (n+m)\sqrt{x}$. $\in \mathbb{Z} \iff x = 1. \Rightarrow \sqrt{a}, \sqrt{b} \in \mathbb{Z}$.

6. Let scalene triangle ABC have circumcenter O and incenter I. Its incircle ω is tangent to sides BC, CA, and AB at D, E, and F, respectively. Let P be the foot of the altitude from D to EF, and let line DP intersect ω again at $Q \neq D$. The line OI intersects the altitude from A toBC at T. Given that $OI \parallel BC$, show that $PQ = PT$.