► **PROBLEMA 1 (Banco IMO 2011, G2)**

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let $O_1$ and $r_1$ be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define $O_2, O_3, O_4$ and $r_2, r_3, r_4$ in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

*Solução.* Here, we are using oriented segments.

Let $M$ be the intersection of the diagonals. Let $B_i$ be the other intersection of $A_{i+2}M$ with the circumference $(A_{i+1}A_{i+2}A_{i+3})$. Thus, by Power of Point,

$$MB_i \cdot MA_{i+2} = MA_{i+1} \cdot MA_{i+3}.$$

We have

$$\begin{aligned}
\frac{1}{O_iA_i^2 - r_i^2} &= \frac{1}{A_iB_i \cdot A_iA_{i+2}} \\
&= \frac{1}{(A_iM + MB_i) \cdot A_iA_{i+2}} \\
&= \frac{MA_{i+2}}{(-MA_i \cdot MA_{i+2} + MB_i \cdot MA_{i+2})A_iA_{i+2}} \\
&= \frac{MA_{i+2}}{A_iA_{i+2}(MA_{i+1}MA_{i+3} - MA_iMA_{i+2})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{4}\left(\frac{1}{O_iA_i^2 - r_i^2}\right) &= \left(\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_3A_3^2 - r_3^2}\right) + \left(\frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_4A_4^2 - r_4^2}\right) \\
&= \frac{A_1M + MA_3}{A_1A_3 \cdot (MA_2 \cdot MA_4 - MA_1 \cdot MA_3)} + \frac{A_2M + MA_4}{A_2A_4 \cdot (MA_1 \cdot MA_3 - MA_2 \cdot MA_4)} \\
&= \frac{1}{MA_2 \cdot MA_4 - MA_1 \cdot MA_3} + \frac{1}{MA_1 \cdot MA_3 - MA_2 \cdot MA_4} \\
&= 0.
\end{aligned}$$