Pawns and Rooks Problem

• = pawn
x = rook

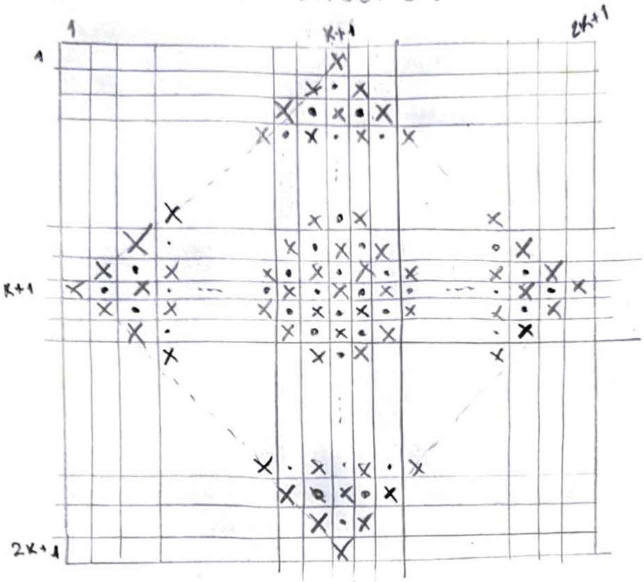
$2019 = 2K+1 = n$.

We claim that the answer is $K^2 = 1009^2$.

Lemma 1: $p = K^2$ is attainable.

Proof:

Here is the construction:



There are

$0+1+2+3+\dots+(K+1)+K+(K-1)+(K-2)+\dots+0$

$= (1+\dots+(K-1))$

$+$

$(1+\dots+K)$

$= \frac{(K-1)K}{2} + \frac{K(K+1)}{2} = K^2$.

Def: p_i is the number of pawns in the i-th line

r_i is the number of rooks in the i-th line

$\tilde{p}_i$ " " " pawns " " " column

$\tilde{r}_i$ " " " rooks " " " "

Lemma 2: $p_i + 1 = r_i$.

Proof: If $p_i < r_i - 1 \Rightarrow$ in the $r_i - 1$ spaces between the rooks, there is at least one of them which does not have a pawn. Abs!

thus, $p_i + 1 \geq r_i$

$\Rightarrow \sum (p_i + 1) \geq \sum r_i$

$\Rightarrow p + n \geq (p+n)$. However, the equality holds, thus the equality must hold for every i in $p_i + 1 \geq r_i$.

$\Rightarrow p_i + 1 = r_i$ (and $\tilde{p}_i + 1 = \tilde{r}_i$).

Corolacy1: The pieces in a line or column must alternate between + and •, starting and ending.

Corolary 2: A pawn must be attacked by a rook from all 4 sides.

Let a_i be the number of squares in the i-th line that are attacked by a rook from lines $1, 2, \dots,$ or $i-1$.

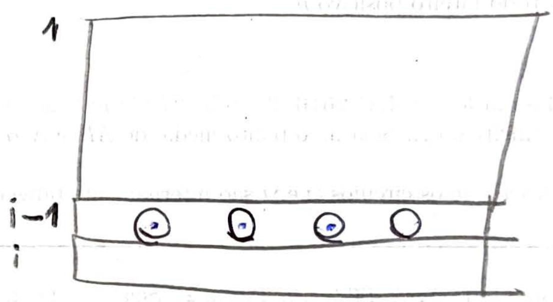
Lemmme 3: $p_i \leqslant a_i$, because every pawn is attaked by a rook above it.

Corolary: $r_i \leqslant a_i + 1$.

Let's prove that $a_i < i$.

$i=1$: $a_1 = 0 < 1$ OK!

Suppose $a_{i-1} < i-1$.



If the j-th square from the (i-1)-th line is attacked by a rook from $\leqslant i-2$.

Then

the j-th square from the i-th line is attacked

$\Longleftrightarrow$

the j-th square from the (i-1)th line does not have a pawn.

However, each pawn (out of p_{i-1}) from the (i-1)th line occupy one of the a_{i-1} attacked squares.

Thus

$$a_i = \#\left(\text{squares from the } i\text{-th line attacked by a rook from lines } 1, \dots, i-2, i-1\right)$$

$$= \left(\begin{array}{c} \#\left(\text{squares from the } i\text{-th line attacked by a rook from the } (i-1)\text{-th line}\right) \\ + \\ \#\left(\square \text{ from the } i\text{-th line attacked by a rook from lines } 1, 2, \dots, i-2\right) \end{array}\right)$$

$$= \left(\begin{array}{c} r_{i-1} \\ + \\ (a_{i-1} - p_{i-1}) \end{array}\right) = a_{i-1} + 1 < i.$$

$\square$

Thus, $a_i < i \Rightarrow p_i < i. \Rightarrow \boxed{p_i \leq i-1.}$

Using the same argument, $\boxed{p_i \leq n-i}$

$$\Rightarrow \sum_{i=1}^{n} p_i = \sum_{i=1}^{k+1} p_i + \sum_{i=k+2}^{2k+1} p_i \leq \sum_{i=1}^{k+1} (i-1) + \sum_{i=k+2}^{2k+1} (n-i) = k^2$$

$\Rightarrow$ # pawns $\leq k^2$.

i $n-i$

$\square$