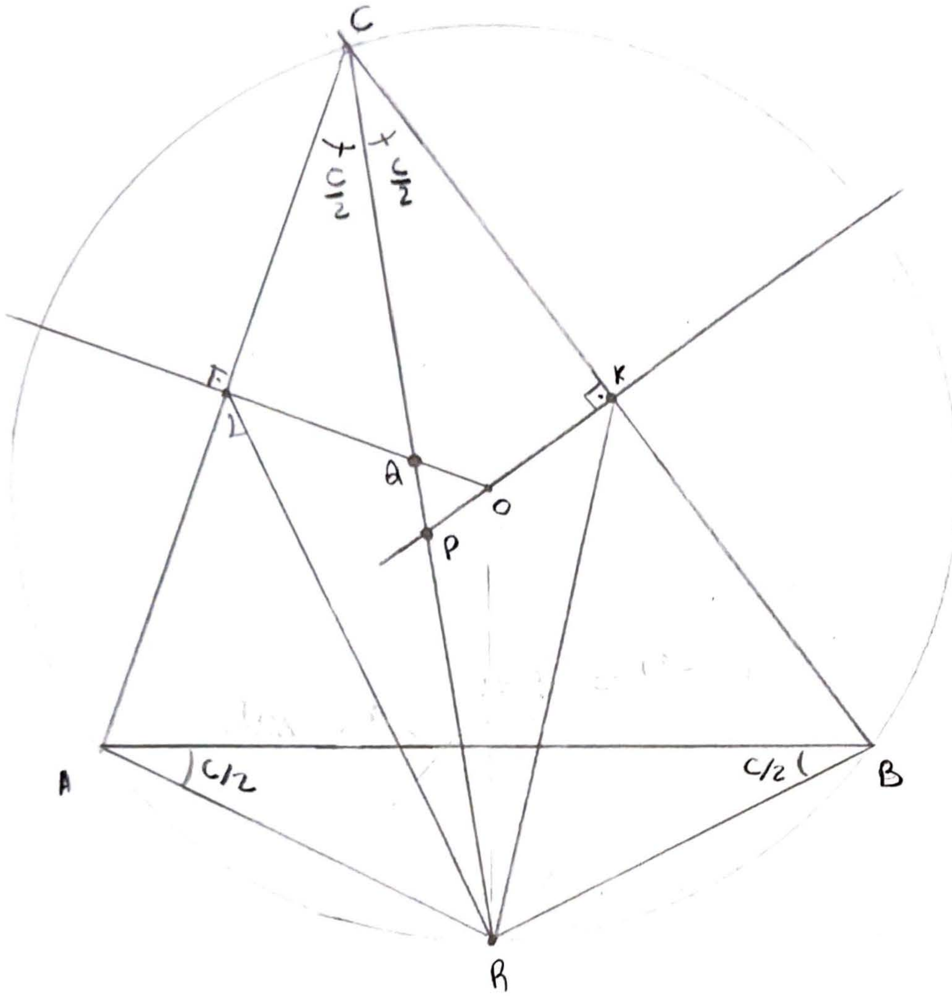
G1/2007 (IMO 2007/4)



- $\angle LQR = 90^\circ + \angle LCQ = 90^\circ + \angle RCP = \angle KPR$.

Thus, $[LQR] = [RPK] \iff \dfrac{LQ \cdot QR \cdot \operatorname{sen}(\angle LQR)}{2} = \dfrac{KP \cdot PR \cdot \operatorname{sen}(\angle KPR)}{2} \iff$

$$\iff \frac{LQ}{KP} = \frac{PR}{QR}.$$

But, $\Delta CLQ \sim \Delta CKP \Rightarrow \dfrac{LQ}{KP} = \dfrac{CQ}{CP}$.

And, $CR = 2R\operatorname{sen}(A + C/2) = 2R\operatorname{sen}(B + C/2)$
$= R(\operatorname{sen}(A + C/2) + \operatorname{sen}(B + C/2))$
$= R \cdot \operatorname{sen} 90^\circ \cdot \cos\left(\frac{A-B}{2}\right)$
$= R \cdot \cos\left(\frac{A-B}{2}\right)$

$CP = R \cdot \dfrac{\operatorname{sen}A}{\cos C}$; $CQ = R\dfrac{\operatorname{sen}B}{\cos C}$

$CP + CQ = \dfrac{R}{\cos C}(\operatorname{sen}A + \operatorname{sen}B) = \dfrac{R}{\cos C} \cdot \operatorname{sen}\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)$

$CP + CQ = R\cos\left(\frac{A-B}{2}\right) = CR \Rightarrow CQ = PR$ e $CP = QR$. $\Rightarrow \dfrac{LQ}{KP} = \dfrac{CQ}{CP} = \dfrac{PR}{QR}$

$\square$