Problemas do HMMT November 2020

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1 Introdução: Valor Esperado

Definição 1.1 (Valor Esperado)
Dada uma variável aleatória X, o valor esperado de X, denotado por $\mathbb{E}[X]$, é

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}[X = x].$$

Exemplo 1.1
Seja D_6 a variável aleatória definida pelo valor da face superior de um dado de 6 faces.

$$\mathbb{E}[D_6] = \frac{7}{2}.$$

Teorema 1.2 (Linearidade da Esperança)
Dadas variáveis aleatórias X_1 e X_2 (que podem ser dependentes!),

$$\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2].$$

Problema 1.1
No Pensi, existem n alunos, e cada um dos alunos possui uma etiqueta com seu nome. Guilherme recolhe e embaralha as etiquetas, entregando aleatoriamente uma etiqueta para cada aluno. Seja S o número de alunos que recebem a etiqueta com seu próprio nome. Prove que o valor esperado de S é 1.

Problema 1.2
Em um berçário, 2006 bebês sentam em um círculo. De repente, cada bebê cutuca aleatóriamente o bebê imediatamente a sua direita ou o bebê imediatamente a sua esquerda. Qual é o valor esperado do número de bebês que não foram cutucados?

Problema 1.3 (NIMO 4.3)
One day, a bishop and a knight were on squares in the same row of an infinite chessboard, when a huge meteor storm occurred, placing a meteor in each square on the chessboard independently and randomly with probability p. Neither the bishop nor the knight were hit, but their movement may have been obstructed by the meteors. For what value of p is the expected number of valid squares that the bishop can move to (in one move) equal to the expected number of squares that the knight can move to (in one move)?

Problema 1.4 (SJSU M179 Midterm)
Prove que qualquer subgrafo de $K_{n,n}$, o grafo bipartido completo com n vértices em cada partição, que possui pelo menos $n^2 - n + 1$ arestas possui um pareamento perfeito, i.e., n arestas disjuntas, i.e., possui um subgrafo em que todos os vértices tem grau 1.

2 Não-Combinatória

Problema 2.1 (HMMT November 2020, Guts, 7)

Compute the maximum number of sides of a polygon that is the cross-section of a regular hexagonal prism.

Problema 2.2 (HMMT November 2020, Guts, 18)

Suppose Harvard Yard is a 17×17 square. There are 14 dorms located on the perimeter of the Yard. If s is the minimum distance between two dorms, the maximum possible value of s can be expressed as $a - \sqrt{b}$, where a, b are positive integers. Compute $100a + b$.

Problema 2.3 (HMMT November 2020, Guts, 25)

Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers where $a_1 = \sum_{i=0}^{100} i!$ and $a_i + a_{i+1}$ is an odd perfect square for all $i \geq 1$. Compute the smallest possible value of a_{100}.

3 Combinatória

Problema 3.1 (HMMT November 2020, Guts, 9)

A fair coin is flipped eight times in a row. Let p be the probability that there is exactly one pair of consecutive flips that are both heads and exactly one pair of consecutive flips that are both tails. If $p = \frac{a}{b}$, where a, b are relatively prime positive integers, compute $100a + b$.

Problema 3.2 (HMMT November 2020, Guts, 12)

In a single-elimination tournment consisting of $2^9 = 512$ teams, there is a strict ordering on the skill levels of the teams, but Joy does not know that ordering. The teams are randomly put into a bracket and they play out the tournament, where the better team always beats the worse team. Joy is then given the results of all 511 matches and must write a list of teams such that she can guarantee that the third-best team is on that list. What is the minimum possible length of Joy's list?

Problema 3.3 (HMMT November 2020, Guts, 13)

Wendy is playing darts with a circular dartboard of radius 20. Whenever she throws a dart, it lands uniformly at random on the dartboard. At the start of the game, there are 2020 darts placed randomly on the board. Every turn, she takes the dart farthest from the center and throws it at the board again. What is the expected number of darts she has to throw before all the darts are within 10 units of the center?

Problema 3.4 (HMMT November 2020, Guts, 23)

Two points are chosen inside the square $\{(x, y) \mid 0 \leq x, y \leq 1\}$ uniformly at random, and a unit square is drawn centered at each point with edges parallel to the coordinate axes. The expected area of the union of the two squares can be expressed as $\frac{a}{b}$, where a, b are relatively prime. Compute $100a + b$.

Problema 3.5 (HMMT November 2020, Guts, 28)

Bernie has 2020 marbles and 2020 bags labeled $B_1, B_2, \ldots B_{2020}$ in which he randomly distributes the marbles (each marble is placed in a random bag independently). If E is the expected number of integers $1 \leq i \leq 2020$ such that B_i has at least i marbles, compute the closest integer to $1000E$.