Treinamento para Provas de Velocidade em Equipe, #1
a.k.a., "Mini-Guts"

Instruções:

- Tamanho esperado da equipe: 8 pessoas.
- Tempo disponível: 65 minutos.

Round 1 (5 points each)

Problem 1.1 Find the number of pairs of real numbers (x, y) such that $x^4 + y^4 = 4xy - 2$.

Problem 1.2 Define a function given the following 2 rules: for prime p, $f(p) = p + 1$; and for positive integers a and b, $f(ab) = f(a) \cdot f(b)$. For how many positive integers $n \leq 100$ is $f(n)$ divisible by 3?

Problem 1.3 Let a sequence be defined as follows: $a_0 = 1$, and for $n > 0$, a_n is $\frac{1}{3}a_{n-1}$ and is $\frac{1}{9}a_{n-1}$ with probability $\frac{1}{2}$. If the expected value of $\sum_{n=0}^{\infty} a_n$ can be expressed in simplest form as $\frac{p}{q}$, what is $p + q$?

Round 2 (7 points each)

Problem 2.1 Compute the period (i.e. length of the repeating part) of the decimal expansion of $\frac{1}{729}$.

Problem 2.2 Let ABC be a triangle with side lengths $13, 14, 15$. The points on the interior of ABC with distance at least 1 from each side are shaded. The area of the shaded region can be written in simplest form as $\frac{m}{n}$. Find $m + n$.

Problem 2.3 Sophie has 20 indistinguishable pairs of socks in a laundry bag. She pulls them out one at a time. After pulling out 30 socks, the expected number of unmatched socks among the socks that she has pulled out can be expressed in simplest form as $\frac{m}{n}$. Find $m + n$.

Round 3 (10 points each)

Problem 3.1 The number 400000001 can be written as $p \cdot q$, where p and q are prime numbers. Find the sum of the prime factors of $p + q - 1$.

Problem 3.2 Some number of regular polygons meet at a point on the plane such that the polygons' interiors do not overlap, but the polygons fully surround the point (i.e. a sufficiently small circle centered at the point would be contained in the union of the polygons). What is the largest possible number of sides in any of the polygons?

Problem 3.3 Let $0 \leq a, b, c, d \leq 10$. For how many ordered quadruples (a, b, c, d) is $ad - bc$ a multiple of 11?

Round 4 (12 points each)

Problem 4.1 Let w and h be positive integers and define $N(w,h)$ to be the number of ways of arranging wh people of distinct heights for a photoshoot in such a way that they form w columns of h people, with the people of each column sorted by height (i.e. shortest at the front, tallest at the back). Find the largest value of $N(w,h)$ that divides 1008.

Problem 4.2 Let x be a real number such that $\tan^{-1}(x) + \tan^{-1}(3x) = \frac{\pi}{6}$ and $0 < x < \frac{\pi}{6}$. Then x^2 may be written as $\frac{a+b\sqrt{c}}{d}$ for a, b, c, d integers with $d > 0$, $\gcd(a^2, b^2, c, d^2) = 1$ and c squarefree. Find $a + b + c + d$.

Problem 4.3 Let k be the largest integer such that 2^k divides

$$\left(\prod_{n=1}^{25}\left(\sum_{i=0}^{n}\binom{n}{i}\right)^2\right)\left(\prod_{n=1}^{25}\left(\sum_{i=0}^{n}\binom{n}{i}^2\right)\right).$$

Find k.

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Round 5 (15 points each)

Problem 5.1 Find the number of nonzero terms of the polynomial $P(x)$ if

$$x^{2018} + x^{2017} + x^{2016} + x^{999} + 1 = (x^4 + x^3 + x^2 + x + 1)P(x).$$

Problem 5.2 Compute the smallest positive integer n that is a multiple of 29 with the property that for every positive integer that is relatively prime to n, $k^n \equiv 1 \pmod{n}$.

Problem 5.3 Kite $ABCD$ has right angles at B and D, and $AB < BC$. Points $E \in AB$ and $F \in AD$ satisfy $AE = 4$, $EF = 7$, and $FA = 5$. If $AB = 8$ and points X lies outside $ABCD$ while satisfying $XE - XF = 1$ and $XE + XF + 2XA = 23$, then XA may be written as $\frac{a-b\sqrt{c}}{d}$ for a, b, c, d positive integers with $\gcd(a^2, b^2, c, d^2) = 1$ and c squarefree. Find $a + b + c + d$.

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Round 6 (20 points each)

Problem 6.1 Let a, b, and c be such that the coefficient of the $x^a y^b z^c$ term in the expansion of $(x + 2y + 3z)^{100}$ is maximal (no other term has a strictly larger coefficient). Find the sum of all possible values of $1,000,000a + 1,000b + c$.

Problem 6.2 The triangle ABC satisfies $AB = 10$ and has angles $\angle A = 75°$, $\angle B = 60°$, and $\angle C = 45°$. Let I_A be the center of the excircle opposite A, and let D, E be the circumcenters of triangle BCI_A and ACI_A respectively. If O is the circumcenter of triangle ABC, then the area of triangle EOD can be written as $\frac{a\sqrt{b}}{c}$ for square-free b and coprime a, c. Find the value of $a + b + c$.

Problem 6.3 If a and b are positive integers such that $3\sqrt{2 + \sqrt{2 + \sqrt{3}}} = a \cos \frac{\pi}{b}$, find $a + b$.