USA TST 2020, #1

PROBLEMA 1
Choose positive integers $b_1, b_2, \dots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \geq r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

PROBLEMA 2
Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

PROBLEMA 3
Let $\alpha \geq 1$ be a real number. Hephaestus and Poseidon play a turn-based game on an infinite grid of unit squares. Before the game starts, Poseidon chooses a finite number of cells to be flooded. Hephaestus is building a levee, which is a subset of unit edges of the grid (called walls) forming a connected, non-self-intersecting path or loop. More formally, there must exist lattice points $A_0, A_1, \dots, A_k$, pairwise distinct except possibly $A_0 = A_k$, such that the set of walls is exactly $\{A_0A_1, A_1A_2, \dots, A_{k-1}A_k\}$. Once a wall is built it cannot be destroyed; in particular, if the levee is a closed loop (i.e. $A_0 = A_k$) then Hephaestus cannot add more walls. Since each wall has length 1, the length of the levee is k.

The game then begins with Hephaestus moving first. On each of Hephaestus's turns, he adds one or more walls to the levee, as long as the total length of the levee is at most αn after his nth turn. On each of Poseidon's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well. Hephaestus wins if the levee forms a closed loop such that all flooded cells are contained in the interior of the loop — hence stopping the flood and saving the world. For which α can Hephaestus guarantee victory in a finite number of turns no matter how Poseidon chooses the initial cells to flood?

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PROBLEMA 4

For a finite simple graph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G.

Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

PROBLEMA 5

Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions:

- $m > 1$ and $\gcd(m, n) = 1$;
- the numbers $P(0)$, $P^2(0)$, $\ldots$, $P^{m-1}(0)$ are not divisible by n; and
- $P^m(0)$ is divisible by n.

Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

PROBLEMA 6

Let $P_1P_2 \cdots P_{100}$ be a cyclic 100-gon and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $\overline{P_{i-2}P_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points $Q_1, Q_2, \ldots, Q_{100}$ are concyclic.