Nível I

Problema 1 Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\ldots99}_{321 \text{ digits}}.$$

Find the sum of the digits of N.

Problema 2 Jenn randomly chooses a number J from $1, 2, 3, \ldots, 19, 20$. Bela then randomly chooses a number B from $1, 2, 3, \ldots, 19, 20$ distinct from J. The value of $B - J$ is at least 2 with a probability that can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 3 In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

Problema 4 A soccer team has 22 available players. A fixed set of 11 players starts the game, while the other 11 are available as substitutes. During the game, the coach may make as many as 3 substitutions, where any one of the 11 players in the game is replaced by one of the substitutes. No player removed from the game may reenter the game, although a substitute entering the game may be replaced later. No two substitutions can happen at the same time. The players involved and the order of substitutions matter. Let n be the number of ways the coach can make substitutions during the game (including the possibility of making no substitutions). Find the remainder when n is divided by 1000.

Problema 5 A moving particle starts at the point $(4, 4)$ and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a, b), it moves at random to one of the points $(a-1, b)$, $(a, b-1)$, or $(a-1, b-1)$, each with probability $\frac{1}{3}$, independently of its previous moves. The probability that it will hit the coordinate axes at $(0, 0)$ is $\frac{m}{3^n}$, where m and n are positive integers, and m is not divisible by 3. Find $m + n$.

Problema 6 In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

Problema 7 There are positive integers x and y that satisfy the system of equations

$$\begin{aligned} \log_{10} x + 2\log_{10}(\gcd(x, y)) &= 60 \\ \log_{10} y + 2\log_{10}(\operatorname{lcm}(x, y)) &= 570. \end{aligned}$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

Problema 8 Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 9 Let $\tau(n)$ denote the number of positive integer divisors of n. Find the sum of the six least positive integers n that are solutions to $\tau(n) + \tau(n+1) = 7$.

Problema 10 For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x - z_1)^3 (x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_j \right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 11 In $\triangle ABC$, the sides have integer lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An *excircle* of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

Problema 12 Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.

Problema 13 Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

Problema 14 Find the least odd prime factor of $2019^8 + 1$.

Problema 15 Let $\overline{AB}$ be a chord of circle ω, and let P be a point on chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Nível II

Problema 1 Points $C \neq D$ lie on the same side of line AB so that $\triangle ABC$ and $\triangle BAD$ are congruent with $AB = 9$, $BC = AD = 10$, and $CA = DB = 17$. The intersection of these two triangular regions has area $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 2 Lily pads $1, 2, 3, \ldots$ lie in a row on a pond. A frog makes a sequence of jumps starting on pad 1. From any pad k the frog jumps to either pad $k + 1$ or pad $k + 2$ chosen randomly and independently with probability $\frac{1}{2}$. The probability that the frog visits pad 7 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problema 3 Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following systems of equations:

$$abc = 70,$$
$$cde = 71,$$
$$efg = 72.$$

Problema 4 A standard six-sided fair die is rolled four times. The probability that the product of all four numbers rolled is a perfect square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 5 Four ambassadors and one advisor for each of them are to be seated at a round table with 12 chairs numbered in order from 1 to 12. Each ambassador must sit in an even-numbered chair. Each advisor must sit in a chair adjacent to his or her ambassador. There are N ways for the 8 people to be seated at the table under these conditions. Find the remainder when N is divided by 1000.

Problema 6 In a Martian civilization, all logarithms whose bases are not specified are assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56$$
$$\log_{\log(x)}(x) = 54$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.

Problema 7 Triangle ABC has side lengths $AB = 120$, $BC = 220$, and $AC = 180$. Lines ℓ_A, ℓ_B, and ℓ_C are drawn parallel to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, such that the intersection of ℓ_A, ℓ_B, and ℓ_C with the interior or $\triangle ABC$ are segments of length 55, 45, and 15, respectively. Find the perimeter of the triangle whose sides lie on ℓ_A, ℓ_B, and ℓ_C.

Problema 8 The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and $f(\frac{1+\sqrt{3}i}{2}) = 2015 + 2019\sqrt{3}i$. Find the remainder when $f(1)$ is divided by 1000.

Problema 9 Call a positive integer n k-*pretty* if n has exactly k positive divisors and n is divisible by k. For example, 18 is 6-*pretty*. Let S be the sum of positive integers less than 2019 that are 20-*pretty*. Find $\frac{S}{20}$.

Problema 10 There is a unique angle θ between $0°$ and $90°$ such that for nonnegative integers n, the value of $\tan(2^n\theta)$ is positive when n is a multiple of 3, and negative otherwise. The degree measure of θ is $\frac{p}{q}$, where p and q are relatively prime integers. Find $p + q$.

Problema 11 Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 12 For $n \geq 1$ call a finite sequence $(a_1, a_2 \ldots a_n)$ of positive integers [i]progressive[/i] if $a_i < a_{i+1}$ and a_i divides a_{i+1} for all $1 \leq i \leq n - 1$. Find the number of progressive sequences such that the sum of the terms in the sequence is equal to 360.

Problema 13 Regular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ is inscribed in a circle of area 1. Point P lies inside the circle so that the region bounded by $\overline{PA_1}$, $\overline{PA_2}$, and the minor arc $\widehat{A_1A_2}$ of the circle has area $\frac{1}{7}$, while the region bounded by $\overline{PA_3}$, $\overline{PA_4}$, and the minor arc $\widehat{A_3A_4}$ of the circle has area $\frac{1}{9}$. There is a positive integer n such that the area of the region bounded by $\overline{PA_6}$, $\overline{PA_7}$, and the minor arc $\widehat{A_6A_7}$ is equal to $\frac{1}{8} - \frac{\sqrt{2}}{n}$. Find n.

Problema 14 Find the sum of all positive integers n such that, given an unlimited supply of stamps of denominations 5, n, and $n+1$ cents, 91 cents is the greatest postage that cannot be formed.

Problema 15 In acute triangle ABC points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive relatively prime integers. Find $m + n$.

Nível I

Problema 1 Let S be the number of ordered pairs of integers (a, b) with $1 \leq a \leq 100$ and $b \geq 0$ such that the polynomial $x^2 + ax + b$
can be factored into the product of two (not necessarily distinct)
linear factors with integer coefficients. Find the remainder when S is divided by 1000.

Problema 2 The number n can be written in base 14 as $\underline{a}\ \underline{b}\ \underline{c}$, can be written in base 15 as $\underline{a}\ \underline{c}\ \underline{b}$, and can be written in base 6 as $\underline{a}\ \underline{c}\ \underline{a}\ \underline{c}$, where $a > 0$. Find the base-10 representation of n.

Problema 3 Kathy has 5 red cards and 5 green cards. She shuffles the 10 cards and lays out 5
of the cards in a row in a random order. She will be happy if and only
if all the red cards laid out are adjacent and all the green cards laid
out are adjacent. For example, card orders $RRGGG$, $GGGGR$, or $RRRRR$ will make Kathy happy, but $RRRGR$ will not. The probability that Kathy will be happy is $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 4 In $\triangle ABC$, $AB = AC = 10$ and $BC = 12$. Point D lies strictly between A and B on $\overline{AB}$ and point E lies strictly between A and C on $\overline{AC}$ so that $AD = DE = EC$. Then AD can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problema 5 For each ordered pair of real numbers (x, y) satisfying

$$\log_2(2x + y) = \log_4(x^2 + xy + 7y^2)$$

there is a real number K such that

$$\log_3(3x + y) = \log_9(3x^2 + 4xy + Ky^2).$$

Find the product of all possible values of K.

Problema 6 Let N be the number of complex numbers z with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000.

Problema 7 A right hexagonal prism has height 2. The bases are regular hexagons with side length 1. Any 3 of the 12 vertices determine a triangle. Find the number of these triangles that are isosceles (including equilateral triangles).

Problema 8 Let $ABCDEF$ be an equiangular hexagon such that $AB = 6, BC = 8, CD = 10$, and $DE = 12$. Denote d the diameter of the largest circle that fits inside the hexagon. Find d^2.

Problema 9 Find the number of four-element subsets of $\{1, 2, 3, 4, \ldots, 20\}$ with the property that two distinct elements of a subset have a sum of 16, and two distinct elements of a subset have a sum of 24. For example, $\{3, 5, 13, 19\}$ and $\{6, 10, 20, 18\}$ are two such subsets.

Problema 10 The wheel shown below consists of
two circles and five spokes, with a label at each point where a spoke
meets a circle. A bug walks along the wheel, starting at point A.
At every step of the process, the bug walks from one labeled point to
an adjacent labeled point. Along the inner circle the bug only walks in a
counterclockwise direction, and along the outer circle the bug only
walks in a clockwise direction. For example, the bug could travel along
the path $AJABCHCHIJA$, which has 10 steps. Let n be the number of paths with 15 steps that begin and end at point A. Find the remainder when n is divided by 1000.A

Problema 11 Find the least positive integer n such that when 3^n is written in base 143, its two right-most digits in base 143 are 01.

Problema 12 For every subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Problema 13 Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

Problema 14 Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E.

Problema 15 David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin\varphi_A = \frac{2}{3}$, $\sin\varphi_B = \frac{3}{5}$, and $\sin\varphi_C = \frac{6}{7}$. All three quadrilaterals have the same area K, which can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Nível II

Problema 1 Points A, B, and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C, Paul starting at B and running toward C, and Eve starting at C and running toward A. When Paul meets Eve, he turns around and runs toward A. Paul and Ina both arrive at B at the same time. Find the number of meters from A to B.

Problema 2 Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

Problema 3 Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

Problema 4 In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, that total area of the six triangular regions. Then $K = \frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

Problema 5 Suppose that x, y, and z are complex numbers such that $xy = -80-320i$, $yz = 60$, and $zx = -96+24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x+y+z = a+bi$. Find a^2+b^2.

Problema 6 A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a-2)x^2 + (-4a+3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 7 Triangle ABC has sides $AB = 9, BC = 5\sqrt{3}$, and $AC = 12$. Points $A = P_0, P_1, P_2, \ldots, P_{2450} = B$ are on segment $\overline{AB}$ with P_k between P_{k-1} and P_{k+1} for $k = 1, 2, \ldots, 2449$, and points $A = Q_0, Q_1, Q_2, \ldots, Q_{2450} = C$ for $k = 1, 2, \ldots, 2449$. Furthermore, each segment $\overline{P_kQ_k}, k = 1, 2, \ldots, 2449$, is parallel to $\overline{BC}$. The segments cut the triangle into 2450 regions, consisting of 2449 trapezoids and 1 triangle. Each of the 2450 regions have the same area. Find the number of segments $\overline{P_kQ_k}, k = 1, 2, \ldots, 2450$, that have rational length.

Problema 8 A frog is positioned at the origin in the coordinate plane. From the point (x, y), the frog can jump to any of the points $(x+1, y), (x+2, y), (x, y+1)$, or $(x, y+2)$. Find the number of distinct sequences of jumps in which the frog begins at $(0, 0)$ and ends at $(4, 4)$.

Problema 9 Octagon $ABCDEFGH$ with side lengths $AB = CD = EF = GH = 10$ and $BC = DE = FG = HA = 11$ is formed by removing four $6-8-10$ triangles from the corners of a 23×27 rectangle with side $\overline{AH}$ on a short side of the rectangle, as shown. Let J be the midpoint of $\overline{HA}$, and partition the octagon into 7 triangles by drawing segments $\overline{JB}$, $\overline{JC}$, $\overline{JD}$, $\overline{JE}$, $\overline{JF}$, and $\overline{JG}$. Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.A

Problema 10 Find the number of functions $f(x)$ from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$ that satisfy $f(f(x)) = f(f(f(x)))$ for all x in $\{1, 2, 3, 4, 5\}$.

Problema 11 Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k.

Problema 12 Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of $\triangle APB$ and $\triangle CPD$ equals the sum of the areas of $\triangle BPC$ and $\triangle APD$. Find the area of quadrilateral $ABCD$.

Problema 13 Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 14 The incircle of ω of $\triangle ABC$ is tangent to $\overline{BC}$ at X. Let $Y \neq X$ be the other intersection of $\overline{AX}$ with ω. Points P and Q lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{PQ}$ is tangent to ω at Y. Assume that $AP = 3, PB = 4, AC = 8$, and $AQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 15 Find the number of functions f from $\{0, 1, 2, 3, 4, 5, 6\}$ to the integers such that $f(0) = 0$, $f(6) = 12$, and

$$|x - y| \leq |f(x) - f(y)| \leq 3|x - y|$$

for all x and y in $\{0, 1, 2, 3, 4, 5, 6\}$.

Nível I

Problema 1 Fifteen distinct points are designated on $\triangle ABC$: the 3 vertices A, B, and C; 3 other points on side $\overline{AB}$; 4 other points on side $\overline{BC}$; and 5 other points on side $\overline{CA}$. Find the number of triangles with positive area whose vertices are among these 15 points.

Problema 2 When each of 702, 787, and 855 is divided by the positive integer m, the remainder is always the positive integer r. When each of 412, 722, and 815 is divided by the positive integer n, the remainder is always the positive integer $s \neq r$. Fine $m+n+r+s$.

Problema 3 For a positive integer n, let d_n be the units digit of $1+2+\cdots+n$. Find the remainder when

$$\sum_{n=1}^{2017} d_n$$

is divided by 1000.

Problema 4 A pyramid has a triangular base with side lengths 20, 20, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. The volume of the pyramid is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Problema 5 A rational number written in base eight is $\underline{a}\underline{b}.\underline{c}\underline{d}$, where all digits are nonzero. The same number in base twelve is $\underline{b}\underline{b}.\underline{b}\underline{a}$. Find the base-ten number $\underline{a}\underline{b}\underline{c}$.

Problema 6 A circle is circumscribed around an isosceles triangle whose two congruent angles have degree measure x.

Two points are chosen independently and uniformly at random on the circle, and a chord is drawn between them. The probability that the chord intersects the triangle is $\frac{14}{25}$. Find the difference between the largest and smallest possible values of x.

Problema 7 For nonnegative integers a and b with $a+b \leq 6$, let $T(a,b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a,b)$, where a and b are nonnegative integers with $a+b \leq 6$. Find the remainder when S is divided by 1000.

Problema 8 Two real numbers a and b are chosen independently and uniformly at random from the interval $(0,75)$. Let O and P be two points on the plane with $OP = 200$. Let Q and R be on the same side of line OP such that the degree measures of $\angle POQ$ and $\angle POR$ are a and b respectively, and $\angle OQP$ and $\angle ORP$ are both right angles. The probability that $QR \leq 100$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 9 Let $a_{10} = 10$, and for each integer $n > 10$ let $a_n = 100a_{n-1} + n$. Find the least $n > 10$ such that a_n is a multiple of 99.

Problema 10 Let $z_1 = 18+83i$, $z_2 = 18+39i$, and $z_3 = 78+99i$, where $i = \sqrt{-1}$. Let z be the unique complex number with the properties that $\frac{z_3-z_1}{z_2-z_1} \cdot \frac{z-z_2}{z-z_3}$ is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

Problema 11 Consider arrangements of the 9 numbers $1,2,3,\ldots,9$ in a 3×3 array. For each such arrangement, let a_1, a_2, and a_3 be the medians of the numbers in rows 1, 2, and 3 respectively, and let m be the median of $\{a_1, a_2, a_3\}$. Let Q be the number of arrangements for which $m = 5$. Find the remainder when Q is divided by 1000.

Problema 12 Call a set S product-free if there do not exist $a,b,c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16,20\}$ are product-free, whereas the sets $\{4,16\}$ and $\{2,8,16\}$ are not product-free. Find the number of product-free subsets of the set $\{1,2,3,4,5,6,7,8,9,10\}$.

Problema 13 For every $m \geq 2$, let $Q(m)$ be the least positive integer with the following property: For every $n \geq Q(m)$, there is always a perfect cube k^3 in the range $n < k^3 \leq m\cdot n$. Find the remainder when

$$\sum_{m=2}^{2017} Q(m)$$

is divided by 1000.

Problema 14 Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

Problema 15 The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.

Nível II

Problema 1 Find the number of subsets of $\{1,2,3,4,5,6,7,8\}$ that are subsets of neither $\{1,2,3,4,5\}$ nor $\{4,5,6,7,8\}$.

Problema 2 Teams T_1, T_2, T_3, and T_4 are in the playoffs. In the semifinal matches, T_1 plays T_4 and T_2 plays T_3. The winners of those two matches will play each other in the final match to determine the champion. When T_i plays T_j, the probability that T_i wins is $\frac{i}{i+j}$, and the outcomes of all the matches are independent. The probability that T_4 will be the champion is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problema 3 A triangle has vertices $A(0,0)$, $B(12,0)$, and $C(8,10)$. The probability that a randomly chosen point inside the triangle is closer to vertex B than to either vertex A or vertex C can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problema 4 Find the number of positive integers less than or equal to 2017 whose base-three representation contains no digit equal to 0.

Problema 5 A set contains four numbers. The six pairwise sums of distinct elements of the set, in no particular order, are 189, 320, 287, 234, x, and y. Find the greatest possible value of $x+y$.

Problema 6 Find the sum of all positive integers n such that $\sqrt{n^2+85n+2017}$ is an integer.

Problema 7 Find the number of integer values of k in the closed interval $[-500,500]$ for which the equation $\log(kx)=2\log(x+2)$ has exactly one real solution.

Problema 8 Find the number of positive integers n less than 2017 such that

$$1+n+\frac{n^2}{2!}+\frac{n^3}{3!}+\frac{n^4}{4!}+\frac{n^5}{5!}+\frac{n^6}{6!}$$

is an integer.

Problema 9 A special deck of cards contains 49 cards, each labeled with a number from 1 to 7 and colored with one of seven colors. Each number-color combination appears on exactly one card. Sharon will select a set of eight cards from the deck at random. Given that she gets at least one card of each color and at least one card with each number, the probability that Sharon can discard one of her cards and still have at least one card of each color and at least one card with each number is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problema 10 Rectangle $ABCD$ has side lengths $AB=84$ and $AD=42$. Point M is the midpoint of $\overline{AD}$, point N is the trisection point of $\overline{AB}$ closer to A, and point O is the intersection of $\overline{CM}$ and $\overline{DN}$. Point P lies on the quadrilateral $BCON$, and $\overline{BP}$ bisects the area of $BCON$. Find the area of $\triangle CDP$.

Problema 11 Five towns are connected by a system of roads. There is exactly one road connecting each pair of towns. Find the number of ways there are to make all the roads one-way in such a way that it is still possible to get from any town to any other town using the roads (possibly passing through other towns on the way).

Problema 12 Circle C_0 has radius 1, and the point A_0 is a point on the circle. Circle C_1 has radius $r<1$ and is internally tangent to C_0 at point A_0. Point A_1 lies on circle C_1 so that A_1 is located 90° counterclockwise from A_0 on C_1. Circle C_2 has radius r^2 and is internally tangent to C_1 at point A_1. In this way a sequence of circles $C_1, C_2, C_3, \dots$ and a sequence of points on the circles $A_1, A_2, A_3, \dots$ are constructed, where circle C_n has radius r^n and is internally tangent to circle C_{n-1} at point A_{n-1}, and point A_n lies on C_n 90° counterclockwise from point A_{n-1}, as shown in the figure below. There is one point B inside all of these circles. When $r=\frac{11}{60}$, the distance from the center of C_0 to B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.A_0

Problema 13 For each integer $n\geq 3$, let $f(n)$ be the number of 3-element subsets of the vertices of a regular n-gon that are the vertices of an isosceles triangle (including equilateral triangles). Find the sum of all values of n such that $f(n+1)=f(n)+78$.

Problema 14 A $10\times10\times10$ grid of points consists of all points in space of the form (i,j,k), where i, j, and k are integers between 1 and 10, inclusive. Find the number of different lines that contain exactly 8 of these points.

Problema 15 Tetrahedron $ABCD$ has $AD=BC=28$, $AC=BD=44$, and $AB=CD=52$. For any point X in space, define $f(X)=AX+BX+CX+DX$. The least possible value of $f(X)$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Nível I

Problema 1 For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \dots.$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Problema 2 Two dice appear to be standard dice with their faces numbered from 1 to 6, but each die is weighted so that the probability of rolling the number k is directly proportional to k. The probability of rolling a 7 with this pair of dice is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 3 A regular icosahedron is a 20-faced solid where each face is an equilateral triangle and five triangles meet at every vertex. The regular icosahedron shown below has one vertex at the top, one vertex at the bottom, an upper pentagon of five vertices all adjacent to the top vertex and all in the same horizontal plane, and a lower pentagon of five vertices all adjacent to the bottom vertex and all in another horizontal plane. Find the number of paths from the top vertex to the bottom vertex such that each part of a path goes downward or horizontally along an edge of the icosahedron, and no vertex is repeated.

Problema 4 A right prism with height h has bases that are regular hexagons with sides of length 12. A vertex A of the prism and its three adjacent vertices are the vertices of a triangular pyramid. The dihedral angle (the angle between the two planes) formed by the face of the pyramid that lies in a base of the prism and the face of the pyramid that does not contain A measures $60°$. Find h^2.

Problema 5 Anh read a book. On the first day she read n pages in t minutes, where n and t are positive integers. On the second day Anh read $n + 1$ pages in $t + 1$ minutes. Each day thereafter Anh read one more page than she read on the previous day, and it took her one more minute than on the previous day until she completely read the 374 page book. It took her a total of 319 minutes to read the book. Find $n + t$.

Problema 6 In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problema 7 For integers a and b consider the complex number

$$\frac{\sqrt{ab + 2016}}{ab + 100} - \left(\frac{\sqrt{|a + b|}}{ab + 100}\right) i.$$

Find the number of ordered pairs of integers (a, b) such that this complex number is a real number.

Problema 8 For a permutation $p = (a_1, a_2, \dots, a_9)$ of the digits $1, 2, \dots, 9$, let $s(p)$ denote the sum of the three 3-digit numbers $a_1a_2a_3$, $a_4a_5a_6$, and $a_7a_8a_9$. Let m be the minimum value of $s(p)$ subject to the condition that the units digit of $s(p)$ is 0. Let n denote the number of permutations p with $s(p) = m$. Find $|m - n|$.

Problema 9 Triangle ABC has $AB = 40$, $AC = 31$, and $\sin A = \frac{1}{5}$. This triangle is inscribed in rectangle $AQRS$ with B on $\overline{QR}$ and C on $\overline{RS}$. Find the maximum possible area of $AQRS$.

Problema 10 A strictly increasing sequence of positive integers $a_1, a_2, a_3, \dots$ has the property that for every positive integer k, the subsequence $a_{2k-1}, a_{2k}, a_{2k+1}$ is geometric and the subsequence $a_{2k}, a_{2k+1}, a_{2k+2}$ is arithmetic. Suppose that $a_{13} = 2016$. Find a_1.

Problema 11 Let $P(x)$ be a nonzero polynomial such that $(x - 1)P(x + 1) = (x + 2)P(x)$ for every real x, and $(P(2))^2 = P(3)$. Then $P(\frac{7}{2}) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 12 Find the least positive integer m such that $m^2 - m + 11$ is a product of at least four not necessarily distinct primes.

Problema 13 Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line $y = 24$. A fence is located at the horizontal line $y = 0$. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a

point where $y = 0$,
with equal likelihoods he chooses one of three directions where he
either jumps parallel to the fence or jumps away from the fence, but he
never chooses the direction that would have him cross over the fence to
where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

Problema 14 Centered at each lattice point in the coordinate plane are a circle of radius $\frac{1}{10}$ and a square with sides of length $\frac{1}{5}$ whose sides are parallel to the coordinate axes. The line segment from $(0, 0)$ to $(1001, 429)$ intersects m of the squares and n of the circles. Find $m + n$.

Problema 15 Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

Nível II

Problema 1 Initially Alex, Betty, and Charlie had a total of 444 peanuts. Charlie had the most peanuts, and Alex had the least. The three numbers of peanuts that each person had form a geometric progression. Alex eats 5 of his peanuts, Betty eats 9 of her peanuts, and Charlie eats 25 of his peanuts. Now the three numbers of peanuts that each person has form an arithmetic progression. Find the number of peanuts Alex had initially.

Problema 2 There is a 40% chance of rain on Saturday and a 30% of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. The probability that it rains at least one day this weekend is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Problema 3 Let x, y and z be real numbers satisfying the system

$$\log_2(xyz - 3 + \log_5 x) = 5$$

$$\log_3(xyz - 3 + \log_5 y) = 4$$

$$\log_4(xyz - 3 + \log_5 z) = 4.$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

Problema 4 An $a \times b \times c$ rectangular box is built from $a \cdot b \cdot c$ unit cubes. Each unit cube is colored red, green, or yellow. Each of the a layers of size $1 \times b \times c$ parallel to the $(b \times c)$-faces of the box contains exactly 9 red cubes, exactly 12 green cubes, and some yellow cubes. Each of the b layers of size $a \times 1 \times c$ parallel to the $(a \times c)$-faces of the box contains exactly 20 green cubes, exactly 25 yellow cubes, and some red cubes. Find the smallest possible volume of the box.

Problema 5 Triangle ABC_0 has a right angle at C_0. Its side lengths are pairwise relatively prime positive integers, and its perimeter is p. Let C_1 be the foot of the altitude to $\overline{AB}$, and for $n \geq 2$, let C_n be the foot of the altitude to $\overline{C_{n-2}B}$ in $\triangle C_{n-2}C_{n-1}B$. The sum $\sum\limits_{n=1}^{\infty} C_{n-1}C_n = 6p$. Find p.

Problema 6 For polynomial $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$, define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then $\sum\limits_{i=0}^{50} |a_i| = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 7 Squares $ABCD$ and $EFGH$ have a common center and $\overline{AB} \parallel \overline{EF}$. The area of $ABCD$ is 2016, and the area of $EFGH$ is a smaller positive integer. Square $IJKL$ is constructed so that each of its vertices lies on a side of $ABCD$ and each vertex of $EFGH$ lies on a side of $IJKL$. Find the difference between the largest and smallest possible integer values of the area of $IJKL$.

Problema 8 Find the number of sets $\{a, b, c\}$ of three distinct positive integers with the property that the product of a, b, and c is equal to the product of $11, 21, 31, 41, 51$, and 61.

Problema 9 The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k.

Problema 10 Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4, PQ = 3, QB = 6, BT = 5$, and $AS = 7$, then $ST = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 11 For positive integers N and k, define N to be k-nice if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

Problema 12 The figure below shows a ring
made of six small sections which you are to paint on a wall. You have
four paint colors available and will paint each of the six sections a
solid color. Find the number of ways you can choose to paint each of the
six sections if no two adjacent section can be painted with the same
color.[asy]
size(3cm);
draw(unitcircle);
draw(scale(0.6)*unitcircle);
for(int i = 0; i ¡ 6; ++i)
draw(dir(60*i)–0.6*dir(60*i));
[/asy]

Problema 13 Beatrix is going to place six rooks on a 6×6 chessboard where both the rows and columns are labelled 1 to 6; the rooks are placed so that no two rooks are in the same row or the same column. The value of a square is the sum of its row number and column number. The score of an arrangement of rooks is the least value of any occupied square. The average score over all valid configurations is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problema 14 Equilateral $\triangle ABC$ has side length 600. Points P and Q lie outside of the plane of $\triangle ABC$ and are on the opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a $120°$ dihedral angle (The angle between the two planes). There is a point O whose distance from each of A, B, C, P and Q is d. Find d.

Problema 15 For $1 \le i \le 215$ let $a_i = \frac{1}{2^i}$ and $a_{216} = \frac{1}{2^{215}}$. Let $x_1, x_2, \ldots, x_{216}$ be positive real numbers such that

$$\sum_{i=1}^{216} x_i = 1 \quad \text{and} \quad \sum_{1 \le i < j \le 216} x_i x_j = \frac{107}{215} + \sum_{i=1}^{216} \frac{a_i x_i^2}{2(1 - a_i)}.$$

The maximum possible value of $x_2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Nível I

Problema 1 The expressions $A = 1\times2+3\times4+5\times6+\cdots+37\times38+39$ and $B = 1+2\times3+4\times5+\cdots+36\times37+38\times39$ are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between integers A and B.

Problema 2 The nine delegates to the
Economic Cooperation Conference include 2 officials from Mexico, 3
officials from Canada, and 4 officials from the United States. During
the opening session, three of the delegates fall asleep. Assuming that
the three sleepers were determined randomly, the probability that
exactly two of the sleepers are from the same country is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 3 There is a prime number p such that $16p+1$ is the cube of a positive integer. Find p.

Problema 4 Point B lies on line segment $\overline{AC}$ with $AB = 16$ and $BC = 4$. Points D and E lie on the same side of line AC forming equilateral triangles $\triangle ABD$ and $\triangle BCE$. Let M be the midpoint of $\overline{AE}$, and N be the midpoint of $\overline{CD}$. The area of $\triangle BMN$ is x. Find x^2.

Problema 5 In a drawer Sandy has 5 pairs of
socks, each pair a different color. On Monday Sandy selects two
individual socks at random from the 10 socks in the drawer. On Tuesday
Sandy selects 2 of the remaining 8 socks at random and on Wednesday two
of the reaining 6 socks at random. The probability that Wednesday is the
first day Sandy selects matching socks is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 6 Point A, B, C, D, and E are equally spaced on a minor arc of a circle. Points E, F, G, H, I and A are equally spaced on a minor arc of a second circle with center C as shown in the figure below. The angle $\angle ABD$ exceeds $\angle AHG$ by 12°. Find the degree measure of $\angle BAG.A$

Problema 7 In the diagram below, $ABCD$ is a square. Point E is the midpoint of $\overline{AD}$. Points F and G lie on $\overline{CE}$, and H and J lie on $\overline{AB}$ and $\overline{BC}$, respectively, so that $FGHJ$ is a square. Points K and L lie on $\overline{GH}$, and M and N lie on $\overline{AD}$ and $\overline{AB}$, respectively, so that $KLMN$ is a square. The area of $KLMN$ is 99. Find the area of $FGHJ$.

Problema 8 For positive integer n, let $s(n)$ denote the sum of the digits of n. Find the smallest positive integer n satisfying $s(n) = s(n+864) = 20$.

Problema 9 Let S be the set of all ordered triples of integers (a_1, a_2, a_3) with $1 \le a_1, a_2, a_3 \le 10$. Each ordered triple in S generates a sequence according to the rule $a_n = a_{n-1} \cdot |a_{n-2} - a_{n-3}|$ for all $n \ge 4$. Find the number of such sequences for which $a_n = 0$ for some n.

Problema 10 Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

Problema 11 Triangle ABC has positive integer side lengths with $AB = AC$. Let I be the intersection of the bisectors of $\angle B$ and $\angle C$. Suppose $BI = 8$. Find the smallest possible perimeter of $\triangle ABC$.

Problema 12 Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problema 13 With all angles measured in degrees, the product $\prod_{k=1}^{45} \csc^2(2k-1)^\circ = m^n$, where m and n are integers greater than 1. Find $m+n$.

Problema 14 For each integer $n \ge 2$, let $A(n)$ be the area of the region in the coordinate plane defined by the inequalities $1 \le x \le n$ and $0 \le y \le x \left\lfloor \sqrt{x} \right\rfloor$, where $\left\lfloor \sqrt{x} \right\rfloor$ is the greatest integer not exceeding $\sqrt{x}$. Find the number of values of n with $2 \le n \le 1000$ for which $A(n)$ is an integer.

Problema 15 A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge on one of the circular faces of the cylinder so that AB on that face measures 120°. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. The area of one of those unpainted faces is $a \cdot \pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a+b+c.A$

Nível II

Problema 1 Let N be the least positive integer that is both 22 percent less than one integer and 16 percent greater than another integer. Find the remainder when N is divided by 1000.

Problema 2 In a new school 40 percent of the students are freshmen, 30 percent are sophomores, 20 percent are juniors, and 10 percent are seniors. All freshmen are required to take Latin, and 80 percent of the sophomores, 50 percent of the juniors, and 20 percent of the seniors elect to take Latin. The probability that a randomly chosen Latin student is a sophomore is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 3 Let m be the least positive integer divisible by 17 whose digits sum to 17. Find m.

Problema 4 In an isosceles trapezoid, the parallel bases have lengths $\log 3$ and $\log 192$, and the altitude to these bases has length $\log 16$. The perimeter of the trapezoid can be written in the form $\log 2^p 3^q$, where p and q are positive integers. Find $p+q$.

Problema 5 Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the least positive integer n such that the probability that the two selected squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

Problema 6 Steve says to Jon, ”I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$ for some positive integers a and c. Can you tell me the values of a and c?”

After some calculations, Jon says, ”There is more than one such polynomial.”

Steve says, ”You're right. Here is the value of a.”He writes down a positive integer and asks, ”Can you tell me the value of c?”

Jon says, ”There are still two possible values of c.”

Find the sum of the two possible values of c.

Problema 7 Triangle ABC has side lengths $AB = 12$, $BC = 25$, and $CA = 17$. Rectangle $PQRS$ has vertex P on $\overline{AB}$, vertex Q on $\overline{AC}$, and vertices R and S on $\overline{BC}$. In terms of the side length $PQ = w$, the area of $PQRS$ can be expressed as the quadratic polynomial

$$\text{Area}(PQRS) = \alpha w - \beta \cdot w^2$$

Then the coefficient $\beta = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 8 Let a and b be positive integers satisfying $\frac{ab+1}{a+b} < \frac{3}{2}$. The maximum possible value of $\frac{a^3b^3+1}{a^3+b^3}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problema 9 A cylindrical barrel with radius 4 feet and height 10 feet is full of water. A solid cube with side length 8 feet is set into the barrel so that the diagonal of the cube is vertical. The volume of water thus displaced is v cubic feet. Find v^2.2 4

Problema 10 Call a permutation $a_1, a_2, \ldots, a_n$ quasi-increasing if $a_k \leq a_{k+1} + 2$ for each $1 \leq k \leq n-1$. For example, 54321 and 14253 are quasi-increasing permutations of the integers $1, 2, 3, 4, 5$, but 45123 is not. Find the number of quasi-increasing permutations of the integers $1, 2, \ldots, 7$.

Problema 11 The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 12 There are $2^{10} = 1024$

possible 10-letter strings in which each letter is either an A or a B.

Find the number of such strings that do not have more than 3 adjacent

letters that are identical.

Problema 13 Define the sequence $a_1, a_2, a_3, \ldots$ by $a_n = \sum_{k=1}^{n} \sin(k)$, where k represents radian measure. Find the index of the 100th term for which $a_n < 0$.

Problema 14 Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Problema 15 Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ so that line BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ, and the areas of $\triangle DBA$ and $\triangle ACE$ are equal. This common area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.D

Nível I

Problema 1 The 8 eyelets for the lace of a sneaker all lie on a rectangle, four equally spaced on each of the longer sides. The rectangle has a width of 50 mm and a length of 80 mm. There is one eyelet at each vertex of the rectangle. The lace itself must pass between the vertex eyelets along a width side of the rectangle and then crisscross between successive eyelets until it reaches the two eyelets at the other width side of the rectrangle as shown. After passing through these final eyelets, each of the ends of the lace must extend at least 200 mm farther to allow a knot to be tied. Find the minimum length of the lace in millimeters.[asy]

size(200);

defaultpen(linewidth(0.7));

path laceL=(-20,-30)..tension 0.75 ..(-90,-135)..(-102,-147)..(-152,-150)..tension 2 ..(-155,-140)..(-135,-40)..(-50,-4)..tension 0.8 ..origin;

path laceR=reflect((75,0),(75,-240))*laceL;

draw(origin–(0,-240)–(150,-240)–(150,0)–cycle,gray);

for(int i=0;i¡=3;i=i+1)

path circ1=circle((0,-80*i),5),circ2=circle((150,-80*i),5);

unfill(circ1); draw(circ1);

unfill(circ2); draw(circ2);

draw(laceL–(150,-80)–(0,-160)–(150,-240)–(0,-240)–(150,-160)–(0,-80)–(150,0),aceR,linewidth(1));[/asy]

Problema 2 An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

Problema 3 Find the number of rational numbers r, $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and denominator have a sum of 1000.

Problema 4 Jon and Steve ride their bicycles on a path that parallels two side-by-side train tracks running in the east/west direction. Jon rides east at 20 miles per hour, and Steve rides west at 20 miles per hour. Two trains of equal length traveling in opposite directions at constant but different speeds, each pass the two riders. Each train takes exactly 1 minute to go past Jon. The westbound train takes 10 times as long as the eastbound train to go past Steve. The length of each train is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 5 Let the set $S = \{P_1, P_2, \cdots, P_{12}\}$ consist of the twelve vertices of a regular 12-gon. A subset Q of S is called communal if there is a circle such that all points of Q are inside the circle, and all points of S not in Q are outside of the circle. How many communal subsets are there? (Note that the empty set is a communal subset.)

Problema 6 The graphs of $y = 3(x-h)^2 + j$ and $y = 2(x-h)^2 + k$ have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find h.

Problema 7 Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = \arg\left(\frac{w-z}{z}\right)$. The maximum possible value of $\tan^2\theta$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$. (Note that $\arg(w)$, for $w \neq 0$, denotes the measure of the angle that the ray from 0 to w makes with the positive real axis in the complex plane.

Problema 8 The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

Problema 9 Let $x_1 < x_2 < x_3$ be three real roots of equation $\sqrt{2014}x^3 - 4029x^2 + 2 = 0$. Find $x_2(x_1 + x_3)$.

Problema 10 A disk with radius 1 is externally tangent to a disk with radius 5. Let A be the point where the disks are tangent, C be the center of the smaller disk, and E be the center of the larger disk. While the larger disk remains fixed, the smaller disk is allowed to roll along the outside of the larger disk until the smaller disk has turned through an angle of 360°. That is, if the center of the smaller disk has moved to the point D, and the point on the smaller disk that began at A has now moved to point B, then $\overline{AC}$ is parallel to $\overline{BD}$. Then $\sin^2(\angle BEA) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 11 A token starts at the point $(0,0)$ of an xy-coordinate grid and them makes a sequence of six moves. Each move is 1
unit in a direction parallel to one of the coordinate axes. Each move
is selected randomly from the four possible directions and independently
of the other moves. The probability the token ends at a point on the
graph of $|y| = |x|$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problema 12 Let $A = \{1, 2, 3, 4\}$, and f and g be randomly chosen (not necessarily distinct) functions from A to A. The probability that the range of f and the range of g are disjoint is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Problema 13 On square $ABCD$, points E, F, G, and H lie on sides $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$, respectively, so that $\overline{EG} \perp \overline{FH}$ and $EG = FH = 34$. Segments $\overline{EG}$ and $\overline{FH}$ intersect at a point P, and the areas of the quadrilaterals $AEPH, BFPE, CGPF$, and $DHPG$ are in the ratio $269 : 275 : 405 : 411$. Find the area of square $ABCD.A$

Problema 14 Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4.$$

There are positive integers a, b, c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

Problema 15 In $\triangle ABC$, $AB = 3$, $BC = 4$, and $CA = 5$. Circle ω intersects $\overline{AB}$ at E and B, $\overline{BC}$ at B and D, and $\overline{AC}$ at F and G. Given that $EF = DF$ and $\frac{DG}{EG} = \frac{3}{4}$, length $DE = \frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. Find $a + b + c$.

Nível II

Problema 1 Abe can paint the room in 15
hours, Bea can paint 50 percent faster than Abe, and Coe can paint twice
as fast as Abe. Abe begins to paint the room and works alone for the
first hour and a half. Then Bea joins Abe, and they work together until
half the room is painted. Then Coe joins Abe and Bea, and they work
together until the entire room is painted. Find the number of minutes
after Abe begins for the three of them to finish painting the room.

Problema 2 Arnold is studying the prevalence
of three health risk factors, denoted by A, B, and C. within a
population of men. For each of the three factors, the probability that a
randomly selected man in the population as only this risk factor (and
none of the others) is 0.1. For any two of the three factors, the
probability that a randomly selected man has exactly two of these two
risk factors (but not the third) is 0.14. The probability that a
randomly selected man has all three risk factors, given that he has A
and B is $\frac{1}{3}$. The probability that a man has none of the three risk factors given that he does not have risk factor A is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problema 3 A rectangle has sides of length a and 36. A hinge is installed at each vertex of the rectangle and at the midpoint of each side of length 36. The sides of length a
can be pressed toward each other keeping those two sides parallel so
the rectangle becomes a convex hexagon as shown. When the figure is a
hexagon with the sides of length a parallel and separated by a distance of 24, the hexagon has the same area as the original rectangle. Find a^2.[asy]
pair A,B,C,D,E,F,R,S,T,X,Y,Z;
dotfactor = 2;
unitsize(.1cm);
A = (0,0);
B = (0,18);
C = (0,36);
// don't look here
D = (12*2.236, 36);
E = (12*2.236, 18);
F = (12*2.236, 0);
draw(A–B–C–D–E–F–cycle);
dot(quot; quot;,A,NW);

dot(quot; quot;,B,NW);
dot(quot; quot;,C,NW);
dot(quot; quot;,D,NW);
dot(quot; quot;,E,NW);
dot(quot; quot;,F,NW);
//don't look here
R = (12*2.236 +22,0);
S = (12*2.236 + 22 - 13.4164,12);
T = (12*2.236 + 22,24);
X = (12*4.472+ 22,24);
Y = (12*4.472+ 22 + 13.4164,12);
Z = (12*4.472+ 22,0);
draw(R–S–T–X–Y–Z–cycle);
dot(quot; quot;,R,NW);
dot(quot; quot;,S,NW);
dot(quot; quot;,T,NW);
dot(quot; quot;,X,NW);
dot(quot; quot;,Y,NW);
dot(quot; quot;,Z,NW);
// sqrt180 = 13.4164
// sqrt5 = 2.236
[/asy]

Problema 4 The repeating decimals $0.abab\overline{ab}$ and $0.abcabc\overline{abc}$ satisfy

$$0.abab\overline{ab} + 0.abcabc\overline{abc} = \frac{33}{37},$$

where a, b, and c are (not necessarily distinct) digits. Find the three-digit number abc.

Problema 5 Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

Problema 6 Charles has two six-sided dice. One of the dice is fair, and the other die is biased so that it comes up six with probability $\frac{2}{3}$, and each of the other five sides has probability $\frac{1}{15}$.
Charles chooses one of the two dice at random and rolls it three times.
Given that the first two rolls are both sixes, the probability that the
third roll will also be a six is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problema 7 Let $f(x) = (x^2 + 3x + 2)^{\cos(\pi x)}$. Find the sum of all positive integers n for which

$$\left| \sum_{k=1}^{n} \log_{10} f(k) \right| = 1.$$

Problema 8 Circle C with radius 2 has diameter $\overline{AB}$. Circle D is internally tangent to circle C at A. Circle E is internally tangent to circle C, externally tangent to circle D, and tangent to $\overline{AB}$. The radius of circle D is three times the radius of circle E and can be written in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$.

Problema 9 Ten chairs are arranged in a
circle. Find the number of subsets of this set of chairs that contain at
least three adjacent chairs.

Problema 10 Let z be a complex number with $|z| = 2014$. Let P be the polygon in the complex plane whose vertices are z and every w such that $\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}$. Then the area enclosed by P can be written in the form $n\sqrt{3}$, where n is an integer. Find the remainder when n is divided by 1000.

Problema 11 In $\triangle RED, RD = 1, \angle DRE = 75^\circ$ and $\angle RED = 45^\circ$. Let M be the midpoint of segment $\overline{RD}$. Point C lies on side $\overline{ED}$ such that $\overline{RC} \perp \overline{EM}$. Extend segment $\overline{DE}$ through E to point A such that $CA = AR$. Then $AE = \frac{a-\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer. Find $a + b + c$.

Problema 12 Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A) + \cos(3B) + \cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

Problema 13 Ten adults enter a room, remove their shoes, and toss their shoes into a pile. Later, a child randomly pairs each left shoe with a right shoe without regard to which shoes belong together. The probability that for every positive integer $k < 5$, no collection of k pairs made by the child contains the shoes from exactly k of the adults is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 14 In $\triangle ABC$, $AB = 10$, $\angle A = 30^\circ$, and $\angle C = 45^\circ$. Let H, D, and M be points on line $\overline{BC}$ such that $\overline{AH} \perp \overline{BC}$, $\angle BAD = \angle CAD$, and $BM = CM$. Point N is the midpoint of segment $\overline{HM}$, and point P is on ray AD such that $\overline{PN} \perp \overline{BC}$. Then $AP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problema 15 For any integer $k \geq 1$, let $p(k)$ be the smallest prime which does not divide k. Define the integer function $X(k)$ to be the product of all primes less than $p(k)$ if $p(k) > 2$, and $X(k) = 1$ if $p(k) = 2$. Let $\{x_n\}$ be the sequence defined by $x_0 = 1$, and $x_{n+1}X(x_n) = x_n p(x_n)$ for $n \geq 0$. Find the smallest positive integer, t such that $x_t = 2090$.