Problemas Sortidos de Combinatória

Guilherme Zeus Dantas e Moura
zeusdanmou@gmail.com

Instruções: Decidam em equipe qual problema chama mais atenção e tentem fazer. Os problemas *não* estão em ordem.

Problema 1 (Torneio das Cidades 1999, Junior A, 6). A rook is allowed to move one cell either horizontally or vertically. After 64 moves the rook visited all cells of the 8×8 chessboard and returned back to the initial cell. Prove that the number of moves in the vertical direction and the number of moves in the horizontal direction cannot be equal.

Problema 2 (Irã 1997-98). An $n \times n$ table is filled with numbers $-1, 0, 1$ in such a manner that every row and column contain exactly one 1 and one -1. Prove that the rows and columns can be reordered so that in the resulting table each number has been replaced with its negative.

Problema 3 (OBM 2014, 5). Em cada casa de um tabuleiro $2m \times 2n$ está escrito um inteiro. A operação permitida é tomar três casas formando um L-triminó (ou seja, uma casa C e outras duas casas com um lado em comum com C, um na horizontal e outro na vertical) e somar 1 ao inteiro em cada uma das três casas. Determine a condição necessária e suficiente, em função de m, n e dos números iniciais, para que seja possível deixar todos os números iguais.

Problema 4 (Banco IMO 2012, C2). Seja $n \geq 1$ um inteiro. Qual é a quantidade máxima de pares disjuntos de elementos do conjunto $\{1, 2, \ldots, n\}$ tal que, para quaisquer dois pares distintos, suas somas são distintas e não ultrapassam n.

Problema 5 (OBM 1995, 6 ↗). X has n elements. F is a family of subsets of X each with three elements, such that any two of the subsets have at most one element in common. Show that there is a subset of X with at least $\sqrt{2n}$ members which does not contain any members of F.

Problema 6. Let n be an positive integer. Find the smallest integer k with the following property:

Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n(n+1)$ and $0 \leq a_i \leq 1$ for all $i \in \{1, 2, \cdots, d\}$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in i^{th} group is at most i, for all $i \in \{1, 2, \cdots, k\}$.

Problema 7 (San Diego Power Contest Fall 2020-21, 7). Alice is wandering in the country of Wanderland. Wanderland consists of a finite number of cities, some of which are connected by two-way trains, such that Wanderland is connected: given any two cities, there is always a way to get from one to the other through a series of train rides.

Alice starts at Riverbank City and wants to end up at Conscious City. Every day, she picks a train going out of the city she is in uniformly at random among all of the trains, and then boards that train to the city it leads to. Show that the expected number of days it takes for her to reach Conscious City is finite.

Problema 8 (North Macedonia Junior 2020, 5). Let T be a triangle whose vertices have integer coordinates, such that each side of T contains exactly m points with integer coordinates. If the area of T is less than 2020, determine the largest possible value of m.

Problema 9 (CIIM 2020, 3 ↗). Let m, r, s, t be positive integers such that $m \geq s + 1$ and $r \geq t$. Consider m sets $A_1, A_2, \ldots, A_m$ with r elements each one. Suppose that, for each $1 \leq i \leq m$, there exist at least t elements of A_i, such that each one(element) belongs to (at least) s sets A_j where $j \neq i$. Determine the greatest quantity of elements in the following set $A_1 \cup A_2 \cup A_3 \cdots \cup A_m$.

Problema 10 (USEMO 2020, 2). Calvin and Hobbes play a game. First, Hobbes picks a family F of subsets of $\{1, 2, ..., 2020\}$, known to both players. Then, Calvin and Hobbes take turns choosing a number from $\{1, 2, ..., 2020\}$ which is not already chosen, with Calvin going first, until all numbers are taken (i.e., each player has 1010 numbers). Calvin wins if he has chosen all the elements of some member of F, otherwise Hobbes wins. What is the largest possible size of a family F that Hobbes could pick while still having a winning strategy?

Problema 11 (Ibero 1994, 3). In each square of an $n \times n$ grid there is a lamp. If the lamp is touched it changes its state every lamp in the same row and every lamp in the same column (the one that are on are turned off and viceversa). At the begin, all the lamps are off. Show that lways is possible, with an appropriated sequence of touches, that all the the lamps on the board end on and find, in function of n the minimal number of touches that are necessary to turn on every lamp.

Problema 12 (ELMO 2019, 3 ↗). Let $n \geq 3$ be a fixed integer. A game is played by n players sitting in a circle. Initially, each player draws three cards from a shuffled deck of $3n$ cards numbered $1, 2, \ldots, 3n$. Then, on each turn, every player simultaneously passes the smallest-numbered card in their hand one place clockwise and the largest-numbered card in their hand one place counterclockwise, while keeping the middle card.

Let T_r denote the configuration after r turns (so T_0 is the initial configuration). Show that T_r is eventually periodic with period n, and find the smallest integer m for which, regardless of the initial configuration, $T_m = T_{m+n}$.