Banco de Problemas para a Tutoria

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1. Determine todos os polinômios $P(x)$ com coeficientes reais que satisfazem

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

para todo real x com $|x| \le 1$.

Esboço. A gente acha que a resposta são todos os polinômios da forma

$$P(T_8(\tfrac{x}{\sqrt{2}})),$$

onde T_8 é o 8º polinômio de Chebyshev e P é um polinômio qualquer.

$$\cos(8x) = T_8(\cos x)$$

2. In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

3. Two circles ω_1, ω_2 intersect each other at points A, B. Let PQ be a common tangent line of these two circles with $P \in \omega_1$ and $Q \in \omega_2$. An arbitrary point X lies on ω_1. Line AX intersects ω_2 for the second time at Y . Point $Y' \neq Y$ lies on ω_2 such that $QY = QY'$. Line $Y'B$ intersects ω_1 for the second time at X'. Prove that $PX = PX'$.

4. In triangle ABC, the incircle, with center I, touches the sides BC at point D. Line DI meets AC at X. The tangent line from X to the incircle (different from AC) intersects AB at Y. If YI and BC intersect at point Z, prove that $AB = BZ$.

5. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

6. Find the 2000^{th} digit in the square root of $N = 11\ldots1$, where N contains 1998 digits, all of them 1's.