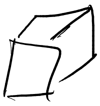
Problema 1 (2016 PUMaC Combinatorics A, 4 ↗). A knight is placed at the origin of the Cartesian plane. Each turn, the knight moves in an chess L-shape (2 units parallel to one axis and 1 unit parallel to the other) to one of eight possible location, chosen at random. After 2016 such turns, what is the expected value of the square of the distance of the knight from the origin?

Valor ~~esperado~~ → médio

Def: → $\mathbb{E}(X) = \sum_{x} x \cdot \mathbb{P}(X=x)$.

R$1 ↗  → CARA → R$2 ; → coroa → R$0 } $\mathbb{E}(J) = 1\cdot\frac{1}{2} + (-1)\cdot\frac{1}{2} = 0$.

 D_6

$$\mathbb{E}(D_6) = \sum_{i=1}^{6} i\, \underbrace{\mathbb{P}(D_6 = i)}_{1/6} = \frac{1}{6}(1+2+\dots+6) = \frac{7}{2}.$$

 D_6 D_6'

$$\rightarrow \mathbb{E}(D_6 + D_6') = \mathbb{E}(D_6) + \mathbb{E}(D_6')$$

• D_6, D_6' são independentes!

	1	2	3	4	5	6
1	1+1	1+2	...			
2						
3				3+4		
4						
5						
6						6+6

Problema 1 (2016 PUMaC Combinatorics A, 4 ↗). A knight is placed at the origin of the Cartesian plane. Each turn, the knight moves in an chess L-shape (2 units parallel to one axis and 1 unit parallel to the other) to one of eight possible location, chosen at random. After 2016 such turns, what is the expected value of the square of the distance of the knight from the origin?

Thm (Mágica): X, Y Variáveis aleatórias

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

mesmo que eles não sejam independentes.

Prova: Usar a definição

$$\mathbb{E}(X+Y) = \sum_z z \cdot \mathbb{P}(X+Y=z)$$

$$= \sum_z z \sum_x \mathbb{P}(X=x \text{ e } Y=z-x)$$

$$= \sum_z \sum_x z \cdot \mathbb{P}(X=x \text{ e } Y=z-x)$$

$$= \sum_x \sum_z z \, \mathbb{P}(X=x \text{ e } Y=z-x)$$

$$= \sum_x \sum_y (x+y) \, \mathbb{P}(X=x \text{ e } Y=y)$$

$$= \sum_y \sum_x y \, \mathbb{P}(X=x \text{ e } Y=y) + \sum_x \sum_y x \, \mathbb{P}(X=x \text{ e } Y=y)$$

Problema 1 (2016 PUMaC Combinatorics A, 4). A knight is placed at the origin of the Cartesian plane. Each turn, the knight moves in an chess L-shape (2 units parallel to one axis and 1 unit parallel to the other) to one of eight possible location, chosen at random. After ~~2016~~ k such turns, what is the expected value of the square of the distance of the knight from the origin?

$$= \sum_y y \underbrace{\boxed{\sum_x \mathbb{P}(X=x \text{ e } Y=y)}}_{\mathbb{P}(Y=y)} + \sum_x x \underbrace{\boxed{\sum_y \mathbb{P}(X=x \text{ e } Y=y)}}_{\mathbb{P}(X=x)}$$

$$= \sum_y y\,\mathbb{P}(Y=y) + \sum_x x\,\mathbb{P}(X=x)$$

$$= \mathbb{E}[Y] + \mathbb{E}[X]$$

$$\mathbb{E}[d^2] = \mathbb{E}[x^2 + y^2] = \mathbb{E}[(x)^2] + \mathbb{E}[y^2]$$

$+5/2$ $\qquad$ $+5/2$

$\Delta x = \pm 1$ ou ± 2 (em cada movimento)

x_i

$$\mathbb{E}[x_{i+1}^2] = \frac{(x_i-1)^2 + (x_i+1)^2 + (x_i-2)^2 + (x_i+2)^2}{4}$$

$$= x_i^2 + 5/2$$

Problema 1 (2016 PUMaC Combinatorics A, 4 ↗). A knight is placed at the origin of the Cartesian plane. Each turn, the knight moves in an chess L-shape (2 units parallel to one axis and 1 unit parallel to the other) to one of eight possible location, chosen at random. After 2016 such turns, what is the expected value of the square of the distance of the knight from the origin?

$X=\{0\}$ $\quad E[X]=0, \quad E[X^2]=0.$

$X=\{-1,+1\}$ $\quad E[X]=0, \quad E[X^2]=1.$

$X=\{1,3\}$ $\quad E[X]=2, \quad E[X^2]=5$

Variância

$$E[d_{i+1}^2]=E[d_i^2]+5.$$

$$E[d_{2016}^2]=2016\cdot 5=\boxed{10080}$$

$K=0, \quad K=1, \quad K=2$

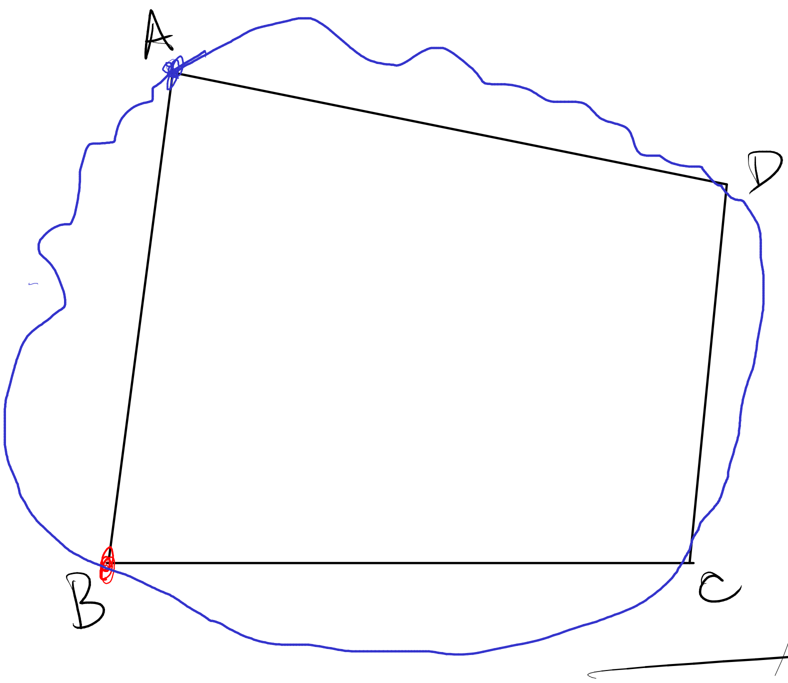
$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 5 \quad 10 \quad \Longrightarrow \quad 10080$

8^{2016}

$$E[d^2]=\frac{1}{8^{2016}}\cdot\sum_{\text{possib.}}(d^2).$$

Problema 2 (Romênia). $ABCD$ é um quadrilátero convexo inscrito em um círculo Γ. Mostre que existe $P \in \Gamma$ tal que $PA + PC = PB + PD$.



$PA + PC$ mínimo?

$P \equiv A$ ou $P \equiv C$ $\quad 4R\,\mathrm{sen}\,\alpha \cos\alpha = 2R\,\mathrm{sen}\,2\alpha$

$PA + PC \longrightarrow AC$

$PA + PC$ ~~máximo?~~

P é ponto médio de $\widehat{AC}$.

$PA + PC \rightarrow 4R \cdot \mathrm{sen}\,\alpha$

$PB + PD \rightarrow BD = 4R\,\mathrm{sen}\,\beta \cdot \cos\beta$

$PB + PD \rightarrow 4R\,\mathrm{sen}\,\beta$

$$\left.\begin{array}{ll} P: & PA + PC \geq PB + PD \\ Q: & QA + QC \leq QB + QD \end{array}\right\} \text{ ACABOU!}$$

S.P.G.: $>$

$$\left.\begin{array}{ll} P \to A: & AC > AB + AD \\ Q \to C: & AC > CB + CD \end{array}\right\} 2AC > (AB + BC) + (AD + DC)$$

Absurdo!

Problema 3 (2014 ELMO Shortlist, N2 ↗). Define the Fibanocci sequence recursively by $F_1 = 1$, $F_2 = 1$ and $F_{i+2} = F_i + F_{i+1}$ for all i. Prove that for all integers $b, c > 1$, there exists an integer n such that the sum of the digits of F_n when written in base b is greater than c.

Feito pelo Rodriguinho -

Problema 4 (OBM 2007, 2). Para quantos números inteiros c, $-2007 \le c \le 2007$, existe um inteiro x tal que $x^2 + c$ é múltiplo de 2^{2007}?

Quantos elementos de $\{-2007, \dots, -1, 0, 1, \dots, 2007\}$ são resíduos quadráticos mod 2^{2007}?

$2: \{0, 1\}$

$2^2: \{0, 1, \cancel{2}, \cancel{3}\}$

$2^3: \{0, 1, 4, \cancel{5}\}$ $\quad 2^2 \cdot 2, \quad 2^2 \cdot 3$

$2^4: \{0, 1, 4, \cancel{8}, 9, \cancel{12}\}$ $\quad 2^2 \cdot 5$ $\quad$ não r.q. (mod 8)

$2^5: \{0, 1, 4, 9, 16, 17, \cancel{20}, 25\}$

$2^6: \{0, 1, 4, 9, 16, 17, 25, \cancel{32}, 33, 36, 41, \cancel{48}, 49, 57\}$

$2^4 \cdot 2 \qquad 2^4 \cdot 3$

n ímpar, só corta $2^{n-3} \cdot 5$

n par, só corta $2^{n-2} \cdot 2$ e $2^{n-2} \cdot 3$

Thm (JM): Se m é r.q. (2^{n-1}) e $v_2(m) \le n-4$, então $m, m + 2^{n-1}$ são r.q. (2^n)

Problema 4 (OBM 2007, 2). Para quantos números inteiros c, $-2007 \leq c \leq 2007$, existe um inteiro x tal que $x^2 + c$ é múltiplo de 2^{2007}?

Thm: Se m é r.q. $\Rightarrow m = 2^{2\alpha} t$, com y ímpar.

$m = x^2$. $x = 2^{\alpha} y$ $m = 2^{2\alpha} \cdot (y^2)$.

Prova (thm Jm): Seja $m \equiv x^2 \ (2^{n-1})$. $x = 2^{\alpha} \cdot y$. (Sup. $\alpha \leq \frac{n-4}{2}$)

$$\left(2^{n-\alpha-2} + x\right)^2 \underset{(2^n)}{\equiv} 2^{2n-2\alpha-4} + 2^{n-\alpha-1} \cdot x + x^2$$

$$\underset{(2^n)}{\equiv} 2^{n-1} + x^2,$$

e $x^2 \equiv x^2$ (OK!)

Para n ímpar $m = 2^{\alpha} \cdot y$

Falta

$\alpha = n-3$ $\alpha = n-2$, $\alpha = n-1$

$m \equiv 2^{n-2} (2^{n-1})$ OK p/ind $m \equiv 2^{n-1} \equiv \left(2^{\frac{n-1}{2}}\right)^2$ OK,

$m \equiv 2^{n-3}$, OK! $m \equiv 3 \cdot 2^{n-3}$, $m \equiv 5 \cdot 2^{n-3}$ $m \equiv 7 \cdot 2^{n-3}$

$m \equiv 3 \cdot 2^{n-3} (2^{n-1})$ OK, p/ind $m \equiv 3 \cdot 2^{n-2} (2^{n-1})$ OK p/ind

(*) $m \equiv 5 \cdot 2^{n-3} \equiv x^2 \pmod{2^n}$

Logo, $x \equiv 2^{\frac{n-3}{2}} \cdot y$.

$\Rightarrow 5 \cdot 2^{n-3} \equiv 2^{n-3} \cdot y^2 \pmod{2^n}$

$\Rightarrow 5 \equiv y^2 \pmod 8$. FALSO!

Mas $5 \cdot 2^{n-3} \equiv 2^{n-3} \pmod{2^{n-2}}$ ✓

Logo, cortou!

Para n par, $m = 2^\alpha \cdot y$.

$\alpha = n-3$; $\alpha = n-2$; $\alpha = n-1$.

$m \equiv 2^{n-3} \pmod{2^{n-2}}$ OK! p/ n ímpar

$m \equiv 2^{n-2} \pmod{2^n}$ OK!

Pq é ímpar

$m = \left(2^{\frac{n-2}{2}}\right)^2$.

$m \equiv 3 \cdot 2^{n-2} \pmod{2^n}$

$x^2 \equiv 3 \cdot 2^{n-2} \pmod{2^n}$

$2^{n-2} \cdot y^2 \equiv 3 \cdot 2^{n-2} \pmod{2^n}$

$y^2 \equiv 3 \pmod 4$ FALSO $\Rightarrow$ Cortou!

$m \equiv 2^{n-1} \pmod{2^n}$ não é

$m \equiv 0 \pmod{2^{n-1}}$ é

Cortou!

Terminando!

$2007 = (11111010111)_2$ $\qquad 2^{2007}$

$-2007 = (\dots 1\,1100000101001)_2$ $\qquad 2^{2007}$

• Se 1º dg por 1.1

$(\times\times\times\ \times\times\ \times\times\times\ 001)_2$

$2^8-4-1=251$

$2^8-4-1=251$

• Se 3º alg pro 1

2007 | 1 1 1 1 1 1 0 1 0 1 1 1

$(\times\times\times\times\times\ 00100)$

−2007 | 0 0 0 0 0 0 1 0 1 0 0 1

$0000\dots \rightarrow 2^6-1 = 63$ casos

$1111\dots \rightarrow 2^6-2 = 62$ casos

• Se 5º

$(\times\times\times\times\ 0010000)$

$\dots 0000 \rightarrow 16$ casos

$\dots 1111 \rightarrow 15$ casos

• Se 7º

$(\times\times 0010000000)$ → 0000: 4 casos; 1111: 4 casos

Se 9º

$(0010000\,00\,00)$ ← 2 casos

Se 11º

$(1000000000\,00) \longrightarrow$ 2 casos

$(000000000000) \longrightarrow$ 1 caso.

Soma final: $1+4+8+31+62+63+502 = \boxed{671 \text{ casos}}$

Lema:

m é r.q. (mod 2^n)

$\Leftrightarrow$

$\boxed{m = 2^{\alpha} \cdot y, \text{ com } \alpha \text{ par e } y = 8k+1.}$

-2007, $+2007$

$\underline{\alpha = 0}$: $\{8k+1\}$

$\underline{\alpha = 2}$: $\{32k+4\}$

$\underline{\alpha = 4}$: $\{128k+16\}$

$\alpha =$

$\vdots$

$\alpha = 12$ $\quad\times$