(VO 2004)

Multiplicando por $(a^x bc+1)(ab^x c+1)(abc^x+1)$ em ambos os lados, obtemos:

$$\underbrace{\sum_{cyc}(a^{x+2}+1)(ab^x c+1)(abc^x+1)}_{=LE} \geq \underbrace{3\cdot\prod_{cyc}(a^x bc+1)}_{=LD}$$

$$LE = \sum_{cyc}\left[a^{x+4}\cdot b^{x+1}\cdot c^{x+1}\right] +$$

$$+\sum_{cyc}\left[a^{x+3}b^x c + a^{x+3}bc^x\right] + \sum_{cyc}\left[a^2 b^{x+1}c^{x+1}\right]$$

$$+\sum_{cyc}\left[ab^x c + abc^x\right] + \sum_{cyc}\left[a^{x+2}\right]$$

$$+\sum_{cyc} 1 . \Rightarrow$$

$$\Rightarrow LE = \frac{1}{2}\sum_{sym}\left[a^{x+4}b^{x+1}c^{x+1}\right] +$$

$$+\sum_{sym}\left[a^{x+3}b^x c\right] + \frac{1}{2}\sum_{sym}\left[a^{x+1}b^{x+1}c^2\right]$$

$$+\sum_{sym}\left[a^x bc\right] + \frac{1}{2}\sum_{sym}\left[a^{x+2}\right]$$

$$+3$$

$$LD = 3 \cdot a^{x+2} \cdot b^{x+2} \cdot c^{x+2} +$$

$$+ 3 \cdot \sum_{cyc} \left[a^{x+1} b^{x+1} c^{2} \right]$$

$$+ 3 \cdot \sum_{cyc} \left[a^{x} b c \right]$$

$$+ 3 \cdot 1$$

$$LD = \frac{1}{2} \sum_{sym} \left[a^{x+2} \cdot b^{x+2} \cdot c^{x+2} \right] +$$

$$+ \frac{3}{2} \sum_{sym} \left[a^{x+1} b^{x+1} c^{2} \right]$$

$$+ \frac{3}{2} \sum_{sym} \left[a^{x} b c \right]$$

$$+ 3$$

Como $(x+4, x+1, x+1) \succ (x+2, x+2, x+2)$

$$\Rightarrow \sum_{sym} [a^{x+4} \cdot b^{x+1} \cdot c^{x+1}] \geq \sum_{sym} [a^{x+2} b^{x+2} c^{x+2}] . \qquad (I)$$

Como $(x+3, x, 1) \succ (x+1, x+1, 2)$

$$\Rightarrow \sum_{sym} [a^{x+3} b^{x} c] \geq \sum_{sym} [a^{x+1} b^{x+1} c^{2}] . \qquad (II)$$

Como $(x+2, 0, 0) \succ (x, 1, 1)$

$$\Rightarrow \sum_{sym} [a^{x+2}] \geq \sum_{sym} [a^{x} bc] . \qquad (III)$$

Desse modo, $\frac{1}{2}(I) + (II) + \frac{1}{2}(III) \Rightarrow$

$\Rightarrow \quad LE \geq LD.$

$\square$

Obs: $\sum_{cyc}[a^x b^y c^z] := a^x b^y c^z + a^y b^z c^x + a^z b^x c^y$

(somatório cíclico, ou seja, passa pelas rotações)

$\sum_{sym}[a^x b^y c^z] := a^x b^y c^z + a^x b^z c^y + a^y b^x c^z + a^y b^z c^x + a^z b^x c^y + a^z b^y c^x$

(somatório simétrico, ou seja, passa pelas permutações):

$(x_1, x_2, \dots, x_n) \succ (y_1, y_2, \dots, y_n)$

$\Leftrightarrow$

$x_1 \geq y_1$

$x_1 + x_2 \geq y_1 + y_2$

$\vdots$

$x_1 + \dots + x_{n-1} \geq y_1 + \dots + y_{n-1}$

$x_1 + \dots + x_n = y_1 + \dots + y_n$.

Desigualdade de Muirhead:

Se $(x_1, x_2, \dots, x_n) \succ (y_1, \dots, y_n)$,

então:

$$\sum_{sym}[a_1^{x_1} a_2^{x_2} \dots a_n^{x_n}] \geq \sum_{sym}[a_1^{y_1} a_2^{y_2} \dots a_n^{y_n}]$$