# IMC (1994 – 2021)

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Todo e qualquer feedback, especialmente sobre erros neste livreto (mesmo erros tipográficos pequenos), é apreciado. Você também pode contribuir enviando suas soluções (de preferência, formatadas em TEX).

Você pode enviar comentários e soluções para zeusdanmou+tex@gmail.com.

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## Day 1

**PROBLEMA 1**

Let $A$ be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is unique real $n \times n$ matrix $X$ that satisfies the equation $X + AX + XA^2 = A$.

(b) Express $X$ in terms of $A$

**PROBLEMA 2**

Let $n$ and $k$ be fixed positive integers , and $a$ be arbitrary nonnegative integer . Choose a random $k$-element subset $X$ of $\{1, 2, ..., k+a\}$ uniformly (i.e., all k-element subsets are chosen with the same probability) and, independently of $X$, choose random n-elements subset $Y$ of $\{1, 2, .., k+a+n\}$ uniformly. Prove that the probability $P\left(\min(Y) > \max(X)\right)$ does not depend on $a$.

**PROBLEMA 3**

We say that a positive real number $d$ is *good* if there exists an infinite squence $a_1, a_2, a_3, ... \in (0, d)$ such that for each positive integer $n$, the points $a_1, a_2, ..., a_n$ partition the interval $[0, d]$ into segments of length at most $\frac{1}{n}$ each. Find $\sup\{d : d \text{ is good}\}$.

**PROBLEMA 4**

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$ , there exists a function $g : \mathbb{R} \to (0, \infty)$ such that for every pair $(x, y)$ of real numbers, if $|x - y| < \min\{g(x), g(y)\}$, then $|f(x) - f(y)| < \varepsilon$ Prove that $f$ is pointwise limit of a squence of continuous $\mathbb{R} \to \mathbb{R}$ functions i.e., there is a squence $h_1, h_2, ...,$ of continuous $\mathbb{R} \to \mathbb{R}$ such that $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$

## Day 2

**PROBLEMA 5**
Let $A$ be a real $n \times n$ matrix and suppose that for every positive integer $m$ there exists a real symmetric matrix $B$ such that

$$2021B = A^m + B^2.$$

Prove that $|\det A| \leq 1$.

**PROBLEMA 6**
For a prime number $p$, let $GL_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible $2 \times 2$ matrices of residues modulo $p$, and let $S_p$ be the symmetric group (the group of all permutations) on $p$ elements. Show that there is no injective group homomorphism $\phi : GL_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

**PROBLEMA 7**
Let $D \subseteq \mathbb{C}$ be an open set containing the closed unit disk $\{z : |z| \leq 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$|f(0)| \leq \max_{|z|=1} |f(z)p(z)|$$

**PROBLEMA 8**
Let $n$ be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?

## Day 1

**PROBLEMA 1**
Let $n$ be a positive integer. Compute the number of words $w$ that satisfy the following three properties.

1. $w$ consists of $n$ letters from the alphabet $\{a, b, c, d\}$.
2. $w$ contains an even number of $a$'s
3. $w$ contains an even number of $b$'s.

For example, for $n = 2$ there are 6 such words: $aa, bb, cc, dd, cd, dc$.

**PROBLEMA 2**
$A, B$ are $n \times n$ matrices such that $\operatorname{rank}(AB - BA + I) = 1$. Prove that $\operatorname{tr}(ABAB) - \operatorname{tr}(A^2B^2) = \frac{1}{2}n(n-1)$.

**PROBLEMA 3**
Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\varepsilon \in (0, 1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\varepsilon^{1-d}$ vertices such that

$$(1 - \varepsilon)K \subseteq L \subseteq K.$$

Official definitions: For a real $\alpha$, a set $T \in \mathbb{R}^d$ is a *convex polytope with at most vertices*, if $T$ is a convex hull of a set $X \in \mathbb{R}^d$ of at most $\alpha$ points, i.e. $T = \{\sum\limits_{x \in X} t_x x | t_x \geq 0, \sum\limits_{x \in X} t_x = 1\}$. Define $\alpha K = \{\alpha x | x \in K\}$. A set $T \in \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.

**PROBLEMA 4**
A polynomial $p$ with real coefficients satisfies $p(x + 1) - p(x) = x^{100}$ for all $x \in \mathbb{R}$. Prove that $p(1 - t) \geq p(t)$ for $0 \leq t \leq 1/2$.

## Day 2

**PROBLEMA 5**
Find all twice continuously differentiable functions $f : \mathbb{R} \to (0, \infty)$ satisfying $f''(x)f(x) \geq 2f'(x)^2$.

**PROBLEMA 6**
Ache todos os primos $p$ tais que existe um único $a \in \{0, 1, 2, \dots, p-1\}$ para o qual $a^3 - 3a + 1 \equiv 0 \pmod{p}$.

**PROBLEMA 7**
Let $G$ be a group and $n \geq 2$ be an integer. Let $H_1, H_2$ be 2 subgroups of $G$ that satisfy

$$[G : H_1] = [G : H_2] = n \text{ and } [G : (H_1 \cap H_2)] = n(n-1).$$

Prove that $H_1, H_2$ are conjugate in $G$.

Official definitions: $[G : H]$ denotes the index of the subgroup of $H$, i.e. the number of distinct left cosets $xH$ of $H$ in $G$. The subgroups $H_1, H_2$ are conjugate if there exists $g \in G$ such that $g^{-1}H_1g = H_2$.

**PROBLEMA 8**
Compute $\lim\limits_{n\to\infty} \frac{1}{\log\log n} \sum\limits_{k=1}^{n} (-1)^k \binom{n}{k} \log k$.

## Day 1

**PROBLEMA 1**
Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

**PROBLEMA 2**
A four-digit number $YEAR$ is called *very good* if the system

$$\begin{aligned} Yx + Ey + Az + Rw &= Y \\ Rx + Yy + Ez + Aw &= E \\ Ax + Ry + Yz + Ew &= A \\ Ex + Ay + Rz + Yw &= R \end{aligned}$$

of linear equations in the variables $x, y, z$ and $w$ has at least two solutions. Find all very good $YEAR$s in the 21st century.

(The 21st century starts in 2001 and ends in 2100.)

**PROBLEMA 3**
Let $f : (-1,1) \to \mathbb{R}$ be a twice differentiable function such that

$$2f'(x) + xf''(x) \geqslant 1 \quad \text{for } x \in (-1,1).$$

Prove that

$$\int_{-1}^{1} xf(x)dx \geqslant \frac{1}{3}.$$

**PROBLEMA 4**
Let $(n+3)a_{n+2} = (6n+9)a_{n+1} - na_n$ and $a_0 = 1$ and $a_1 = 2$. Prove that all the terms of the sequence are integers.

**PROBLEMA 5**
Determine whether there exist an odd positive integer $n$ and $n \times n$ matrices $A$ and $B$ with integer entries, that satisfy the following conditions:

- $\det(B) = 1$;
- $AB = BA$;
- $A^4 + 4A^2B^2 + 16B^4 = 2019I$.

(Here $I$ denotes the $n \times n$ identity matrix.)

## Day 2

**PROBLEMA 6**

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $g$ is differentiable. Assume that $(f(0) - g'(0))(g'(1) - f(1)) > 0$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = g'(c)$.

**PROBLEMA 7**

Let $C = \{4, 6, 8, 9, 10, \ldots\}$ be the set of composite positive integers. For each $n \in C$ let $a_n$ be the smallest positive integer $k$ such that $k!$ is divisible by $n$. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n.$$

**PROBLEMA 8**

Let $x_1, \ldots, x_n$ be real numbers. For any set $I \subset \{1, 2, \ldots, n\}$ let $s(I) = \sum_{i \in I} x_i$. Assume that the function $I \to s(I)$ takes on at least $1.8^n$ values where $I$ runs over all $2^n$ subsets of $\{1, 2, \ldots, n\}$. Prove that the number of sets $I \subset \{1, 2, \ldots, n\}$ for which $s(I) = 2019$ does not exceed $1.7^n$.

**PROBLEMA 9**

Determine all positive integers $n$ for which there exist $n \times n$ real invertible matrices $A$ and $B$ that satisfy $AB - BA = B^2 A$.

**PROBLEMA 10**

2019 points are chosen at random, independently, and distributed uniformly in the unit disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Let $C$ be the convex hull of the chosen points. Which probability is larger: that $C$ is a polygon with three vertices, or a polygon with four vertices?

## Day 1

**PROBLEMA 1**
Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

- There is a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge;
- $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

**PROBLEMA 2**
Does there exist a field such that its multiplicative group is isomorphism to its additive group?

**PROBLEMA 3**
Determine all rational numbers $a$ for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

**PROBLEMA 4**
Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f(b) - f(a) = (b-a)f'(\sqrt{ab}) \qquad \text{for all} \qquad a, b > 0.$$

**PROBLEMA 5**
Let $p$ and $q$ be prime numbers with $p < q$. Suppose that in a convex polygon $P_1, P_2, \dots, P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$P_1P_2 + P_2P_3 + \cdots + P_kP_{k+1} \geqslant \frac{k^3 + k}{2}$$

holds for every integer $k$ with $1 \leqslant k \leqslant p$.

## Day 2

**PROBLEMA 6**
Let $k$ be a positive integer. Find the smallest positive integer $n$ for which there exists $k$ nonzero vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$ such that for every pair $i, j$ of indices with $|i-j| > 1$ the vectors $v_i$ and $v_j$ are orthogonal.

**PROBLEMA 7**
Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 = 0$ and

$$a_{n+1}^3 = a_n^2 - 8 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|.$$

**PROBLEMA 8**
Let $\Omega = \{(x, y, z) \in \mathbb{Z}^3 : y + 1 \geqslant x \geqslant y \geqslant z \geqslant 0\}$. A frog moves along the points of $\Omega$ by jumps of length 1. For every positive integer $n$, determine the number of paths the frog can take to reach $(n, n, n)$ starting from $(0, 0, 0)$ in exactly $3n$ jumps.

**PROBLEMA 9**
Determine all pairs $P(x), Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2 + 1$ and $Q(x)$ divides $P(x)^2 + 1$.

**PROBLEMA 10**
For $R > 1$ let $\mathcal{D}_R = \{(a, b) \in \mathbb{Z}^2 : 0 < a^2 + b^2 < R\}$. Compute

$$\lim_{R \to \infty} \sum_{(a,b) \in \mathcal{D}_R} \frac{(-1)^{a+b}}{a^2 + b^2}.$$

## Day 1

**PROBLEMA 1**
Determine all complex numbers $\lambda$ for which there exists a positive integer $n$ and a real $n \times n$ matrix $A$ such that $A^2 = A^T$ and $\lambda$ is an eigenvalue of $A$.

**PROBLEMA 2**
Let $f : \mathbb{R} \to (0, \infty)$ be a differentiabe function, and suppose that there exists a constant $L > 0$ such that

$$|f'(x) - f'(y)| \leq L|x - y|$$

for all $x, y$. Prove that

$$(f'(x))^2 < 2Lf(x)$$

holds for all $x$.

**PROBLEMA 3**
For any positive integer $m$, denote by $P(m)$ the product of positive divisors of $m$ (e.g $P(6) = 36$). For every positive integer $n$ define the sequence

$$a_1(n) = n, \qquad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \ldots, 2016)$$

Determine whether for every set $S \subset \{1, 2, \ldots, 2017\}$, there exists a positive integer $n$ such that the following condition is satisfied: *For every with , the number is a perfect square if and only if .*

**PROBLEMA 4**
There are $n$ people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group $S$ of people such that at least $n/2017$ persons in $S$ have exactly two friends in $S$.

**PROBLEMA 5**
Let $k$ and $n$ be positive integers with $n \geq k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_0$$

be a polynomial with complex coefficients such that

$$c_0c_{n-2} = c_1c_{n-3} = \cdots = c_{n-2}c_0 = 0$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

## Day 2

**PROBLEMA 6**
Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)\,\mathrm{d}x = L.$$

**PROBLEMA 7**
Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer $n$, let

$$q_n(x) = (x+1)^n p(x) + x^n p(x+1).$$

Prove that there are only finitely many numbers $n$ such that all roots of $q_n(x)$ are real.

**PROBLEMA 8**
Define the sequence $A_1, A_2, \ldots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \ldots)$$

where $I_m$ is the $m \times m$ identity matrix.

Prove that $A_n$ has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, respectively.

**PROBLEMA 9**
Define the sequence $f_1, f_2, \ldots : [0, 1) \to \mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1 = 1; \qquad f'_{n+1} = f_n f_{n+1} \quad \text{on } (0, 1), \quad \text{and} \quad f_{n+1}(0) = 1.$$

Show that $\lim\limits_{n\to\infty} f_n(x)$ exists for every $x \in [0, 1)$ and determine the limit function.

**PROBLEMA 10**
Let $K$ be an equilateral triangle in the plane. Prove that for every $p > 0$ there exists an $\varepsilon > 0$ with the following property: If $n$ is a positive integer, and $T_1, \ldots, T_n$ are non-overlapping triangles inside $K$ such that each of them is homothetic to $K$ with a negative ratio, and

$$\sum_{\ell=1}^{n} \operatorname{area}(T_\ell) > \operatorname{area}(K) - \varepsilon,$$

then

$$\sum_{\ell=1}^{n} \operatorname{perimeter}(T_\ell) > p.$$

## Day 1

**PROBLEMA 1**
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose that $f$ has infinitely many zeros, but there is no $x \in (a, b)$ with $f(x) = f'(x) = 0$. (a) Prove that $f(a)f(b) = 0$. (b) Give an example of such a function on $[0, 1]$.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

**PROBLEMA 2**
Let $k$ and $n$ be positive integers. A sequence $(A_1, \dots, A_k)$ of $n \times n$ real matrices is *preferred* by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \le i \le k$, but $A_iA_j = 0$ for $1 \le i, j \le k$ with $i \neq j$. Show that $k \le n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each $n$.

(Proposed by Fedor Petrov, St. Petersburg State University)

**PROBLEMA 3**
Let $n$ be a positive integer. Also let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \dots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_ib_i - b_i^2}{a_i + b_i} \le \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i - \left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n}(a_i + b_i)}$$

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(Proposed by Daniel Strzelecki, Nicolaus Copernicus University in Toruń, Poland)

**PROBLEMA 4**
Let $n \ge k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties: (i) $\mathcal{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly $k$ elements; (ii) for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$. Prove that $\mathcal{F}$ contains at least three sets with at least $n$ elements.

(Proposed by Fedor Petrov, St. Petersburg State University)

**PROBLEMA 5**
Let $S_n$ denote the set of permutations of the sequence $(1, 2, \dots, n)$. For every permutation $\pi = (\pi_1, \dots, \pi_n) \in S_n$, let $\mathrm{inv}(\pi)$ be the number of pairs $1 \le i < j \le n$ with $\pi_i > \pi_j$; i. e. the number of inversions in $\pi$. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\mathrm{inv}(\pi)$ is divisible by $n + 1$. Prove that there exist infinitely many primes $p$ such that $f(p - 1) > \frac{(p-1)!}{p}$, and infinitely many primes $p$ such that $f(p - 1) < \frac{(p-1)!}{p}$.

(Proposed by Fedor Petrov, St. Petersburg State University)

## Day 2

**PROBLEMA 6**

Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\sum\limits_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \leq 2.$$

(Proposed by Gerhard J. Woeginger, The Netherlands)

**PROBLEMA 7**

Today, Ivan the Confessor prefers continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(x) + f(y) \geq |x-y|$ for all pairs $x, y \in [0,1]$. Find the minimum of $\int_0^1 f$ over all preferred functions.

(Proposed by Fedor Petrov, St. Petersburg State University)

**PROBLEMA 8**

Let $n$ be a positive integer, and denote by $\mathbb{Z}_n$ the ring of integers modulo $n$. Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying the following three properties:

(i) $f(x) \neq x$,

(ii) $f(f(x)) = x$,

(iii) $f(f(f(x+1)+1)+1) = x$ for all $x \in \mathbb{Z}_n$.

Prove that $n \equiv 2 \pmod 4$.

(Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Germany)

**PROBLEMA 9**

Let $k$ be a positive integer. For each nonnegative integer $n$, let $f(n)$ be the number of solutions $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + ... + |x_k| \leq n$. Prove that for every $n \geq 1$, we have $f(n-1)f(n+1) \leq f(n)^2$.

(Proposed by Esteban Arreaga, Renan Finder and José Madrid, IMPA, Rio de Janeiro)

**PROBLEMA 10**

Let $A$ be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \leq \frac{n}{\ln 2} \|A\|^{n-1}.$$

(Here $\|B\| = \sup\limits_{\|x\| \leq 1} \|Bx\|$ for every $n \times n$ matrix $B$ and $\|x\| = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

(Proposed by Ian Morris and Fedor Petrov, St. Petersburg State University)

## Day 1

**PROBLEMA 1**
For any integer $n \geq 2$ and two $n \times n$ matrices with real entries $A,\ B$ that satisfy the equation

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

prove that $\det(A) = \det(B)$.

Does the same conclusion follow for matrices with complex entries?

(Proposed by Zbigniew Skoczylas, Wroclaw University of Technology)

**PROBLEMA 2**
For a positive integer $n$, let $f(n)$ be the number obtained by writing $n$ in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \leq \frac{n^2}{4}.$$

When does equality hold?

(Proposed by Stephan Wagner, Stellenbosch University)

**PROBLEMA 3**
Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and $F(n) = \frac{5}{2}F(n-1) - F(n-2)$ for $n \geq 2$.

Determine whether or not $\displaystyle\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$ is a rational number.

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

**PROBLEMA 4**
Determine whether or not there exist 15 integers $m_1, \ldots, m_{15}$ such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16). \tag{1}$$

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

**PROBLEMA 5**
Let $n \geq 2$, let $A_1, A_2, \ldots, A_{n+1}$ be $n+1$ points in the $n$-dimensional Euclidean space, not lying on the same hyperplane, and let $B$ be a point strictly inside the convex hull of $A_1, A_2, \ldots, A_{n+1}$. Prove that $\angle A_i B A_j > 90^\circ$ holds for at least $n$ pairs $(i, j)$ with $1 \leq i < j \leq n+1$.

Proposed by Géza Kós, Eötvös University, Budapest

## Day 2

**PROBLEMA 6**
Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)} < 2.$$

Proposed by Ivan Krijan, University of Zagreb

**PROBLEMA 7**
Compute

$$\lim_{A\to+\infty} \frac{1}{A} \int_{1}^{A} A^{\frac{1}{x}}\, dx.$$

Proposed by Jan Šustek, University of Ostrava

**PROBLEMA 8**
Consider all $26^{26}$ words of length 26 in the Latin alphabet. Define the *weight* of a word as $1/(k+1)$, where $k$ is the number of letters not used in this word. Prove that the sum of the weights of all words is $3^{75}$.

Proposed by Fedor Petrov, St. Petersburg State University

**PROBLEMA 9**
An $n \times n$ complex matrix $A$ is called *t-normal* if $AA^t = A^t A$ where $A^t$ is the transpose of $A$. For each $n$, determine the maximum dimension of a linear space of complex $n \times n$ matrices consisting of t-normal matrices.

Proposed by Shachar Carmeli, Weizmann Institute of Science

**PROBLEMA 10**
Let $n$ be a positive integer, and let $p(x)$ be a polynomial of degree $n$ with integer coefficients. Prove that

$$\max_{0\le x\le 1} \left|p(x)\right| > \frac{1}{e^n}.$$

Proposed by Géza Kós, Eötvös University, Budapest

## Day 1

**PROBLEMA 1**
Determine all pairs $(a, b)$ of real numbers for which there exists a unique symmetric $2 \times 2$ matrix $M$ with real entries satisfying $\text{trace}(M) = a$ and $\det(M) = b$.

(Proposed by Stephan Wagner, Stellenbosch University)

**PROBLEMA 2**
Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \dots)$$

Find all pairs $(\alpha, \beta)$ of positive real numbers such that $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{n^\alpha} = \beta$.

(Proposed by Tomas Barta, Charles University, Prague)

**PROBLEMA 3**
Let $n$ be a positive integer. Show that there are positive real numbers $a_0, a_1, \dots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \dots \pm a_1 x \pm a_0$$

has $n$ distinct real roots.

(Proposed by Stephan Neupert, TUM, München)

**PROBLEMA 4**
Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with $1 < p_1 < \dots < p_k$. Prove that $e_1$ is an even number. A number $n$ is *perfect* if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of $n$.

(Proposed by Javier Rodrigo, Universidad Pontificia Comillas)

**PROBLEMA 5**
Let $A_1 A_2 \dots A_{3n}$ be a closed broken line consisting of $3n$ lines segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $i = 1, 2, \dots, 3n$, the triangle $A_i A_{i+1} A_{i+2}$ has counterclockwise orientation and $\angle A_i A_{i+1} A_{i+2} = 60$ž, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3}{2}n^2 - 2n + 1$

## Day 2

**PROBLEMA 6**
For a positive integer $x$, denote its $n^{\text{th}}$ decimal digit by $d_n(x)$, i.e. $d_n(x) \in \{0, 1, \dots, 9\}$ and $x = \sum_{n=1}^{\infty} d_n(x)10^{n-1}$. Suppose that for some sequence $(a_n)_{n=1}^{\infty}$, there are only finitely many zeros in the sequence $(d_n(a_n))_{n=1}^{\infty}$. Prove that there are infinitely many positive integers that do not occur in the sequence $(a_n)_{n=1}^{\infty}$.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

**PROBLEMA 7**
Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \le i < j \le n} a_{ii}a_{jj} \ge \sum_{1 \le i < j \le n} \lambda_i \lambda_j$$

and determine all matrices for which equality holds.

(Proposed by Matrin Niepel, Comenius University, Bratislava)

**PROBLEMA 8**
Let $f(x) = \frac{\sin x}{x}$, for $x > 0$, and let $n$ be a positive integer. Prove that $|f^{(n)}(x)| < \frac{1}{n+1}$, where $f^{(n)}$ denotes the $n^{\text{th}}$ derivative of $f$.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

**PROBLEMA 9**
We say that a subset of $\mathbb{R}^n$ is $k$-*almost contained* by a hyperplane if there are less than $k$ points in that set which do not belong to the hyperplane. We call a finite set of points $k$-*generic* if there is no hyperplane that $k$-almost contains the set. For each pair of positive integers $(k, n)$, find the minimal number of $d(k, n)$ such that every finite $k$-generic set in $\mathbb{R}^n$ contains a $k$-generic subset with at most $d(k, n)$ elements.

(Proposed by Shachar Carmeli, Weizmann Inst. and Lev Radzivilovsky, Tel Aviv Univ.)

**PROBLEMA 10**
For every positive integer $n$, denote by $D_n$ the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_j \neq j$ for every $1 \le j \le n$. For $1 \le k \le \frac{n}{2}$, denote by $\Delta(n, k)$ the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_i = k + i$ for every $1 \le i \le k$ and $x_j \neq j$ for every $1 \le j \le n$. Prove that

$$\Delta(n, k) = \sum_{i=0}^{k=1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)}$$

(Proposed by Combinatorics; Ferdowsi University of Mashhad, Iran; Mirzavaziri)

## Day 1

**PROBLEMA 1**
Let $A$ and $B$ be real symmetric matrixes with all eigenvalues strictly greater than 1. Let $\lambda$ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

**PROBLEMA 2**
Let $f : \mathcal{R} \to \mathcal{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove there exists $\xi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that

$$f''(\xi) = f(\xi)\left(1 + 2\tan^2\xi\right).$$

**PROBLEMA 3**
There are $2n$ students in a school ($n \in \mathbb{N}, n \geqslant 2$). Each week $n$ students go on a trip. After several trips the following condition was fulfiled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

**PROBLEMA 4**
Let $n \geqslant 3$ and let $x_1, x_2, ..., x_n$ be nonnegative real numbers. Define $A = \sum_{i=1}^{n} x_i, B = \sum_{i=1}^{n} x_i^2, C = \sum_{i=1}^{n} x_i^3$. Prove that:

$$(n+1)\,A^2B + (n-2)\,B^2 \geqslant A^4 + (2n-2)\,AC.$$

**PROBLEMA 5**
Does there exist a sequence $(a_n)$ of complex numbers such that for every positive integer $p$ we have that $\sum_{n=1}^{+\infty} a_n^p$ converges if and only if $p$ is not a prime?

## Day 2

**PROBLEMA 6**
Let $z$ be a complex number with $|z+1|>2$. Prove that $\left|z^3+1\right|>1$.

**PROBLEMA 7**
Let $p, q$ be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1}(-1)^{\left\lfloor \frac{k}{p}\right\rfloor+\left\lfloor \frac{k}{q}\right\rfloor}=\begin{cases}0 & \text{if } pq \text{ is even}\\ 1 & \text{if } pq \text{ odd}\end{cases}$$

**PROBLEMA 8**
Suppose that $v_1, v_2, ..., v_d$ are unit vectors in $\mathbb{R}^d$. Prove that there exists a unitary vector $u$ such that $|u \cdot v_i| \leq \dfrac{1}{\sqrt{d}}$ for $i=1,2,...,d$.

*Note.* Here $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$.

**PROBLEMA 9**
Does there exist an infinite set $M$ consisting of positive integers such that for any $a, b \in M$ with $a<b$ the sum $a+b$ is square-free? *Note.* A positive integer is called square-free if no perfect square greater than 1 divides it.

**PROBLEMA 10**
Consider a circular necklace with 2013 beads. Each bead can be paintes either green or white. A painting of the necklace is called *good* if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.

*Note.* Two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.

## Day 1

**PROBLEMA 1**
For every positive integer $n$, let $p(n)$ denote the number of ways to express $n$ as a sum of positive integers. For instance, $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1.$$

Also define $p(0) = 1$.

Prove that $p(n) - p(n-1)$ is the number of ways to express $n$ as a sum of integers each of which is strictly greater than 1.

**PROBLEMA 2**
Let $n$ be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

**PROBLEMA 3**
Given an integer $n > 1$, let $S_n$ be the group of permutations of the numbers 1, 2, 3, $\ldots$, $n$. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group $S_n$. It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group $S_n$. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

**PROBLEMA 4**
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies $f'(t) > f(f(t))$ for all $t \in \mathbb{R}$. Prove that $f(f(f(t))) \leq 0$ for all $t \geq 0$.

**PROBLEMA 5**
Let $a$ be a rational number and let $n$ be a positive integer. Prove that the polynomial $X^{2^n}(X+a)^{2^n} + 1$ is irreducible in the ring $\mathbb{Q}[X]$ of polynomials with rational coefficients.

## Day 2

**PROBLEMA 6**
Consider a polynomial

$$f(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1x + a_0.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients $a_0, a_1, \ldots, a_{2011}$ and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values. Homer's goal is to make $f(x)$ divisible by a fixed polynomial $m(x)$ and Albert's goal is to prevent this. (a) Which of the players has a winning strategy if $m(x) = x - 2012$? (b) Which of the players has a winning strategy if $m(x) = x^2 + 1$?

**PROBLEMA 7**
Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$, and

$$a_{n+1} = \frac{na_n^2}{1 + (n+1)a_n}, \quad \forall n \geq 1.$$

Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

**PROBLEMA 8**
Is the set of positive integers $n$ such that $n! + 1$ divides $(2012n)!$ finite or infinite?

**PROBLEMA 9**
Let $n \geq 2$ be an integer. Find all real numbers $a$ such that there exist real numbers $x_1, x_2, \ldots, x_n$ satisfying

$$x_1(1 - x_2) = x_2(1 - x_3) = \cdots = x_n(1 - x_1) = a.$$

**PROBLEMA 10**
Let $c \geq 1$ be a real number. Let $G$ be an Abelian group and let $A \subset G$ be a finite set satisfying $|A + A| \leq c|A|$, where $X + Y := \{x + y | x \in X, y \in Y\}$ and $|Z|$ denotes the cardinality of $Z$. Prove that

$$|\underbrace{A + A + \cdots + A}_{k}| \leq c^k |A|$$

for every positive integer $k$.

## Day 1

**PROBLEMA 1**

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point $x$ is called a *shadow* point if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that • all the points of the open interval $I = (a, b)$ are shadow points; • $a$ and $b$ are not shadow points. Prove that a) $f(x) \leq f(b)$ for all $a < x < b$; b) $f(a) = f(b)$.

**PROBLEMA 2**

Does there exist a real $3 \times 3$ matrix $A$ such that $\operatorname{tr}(A) = 0$ and $A^2 + A^t = I$? ($\operatorname{tr}(A)$ denotes the trace of $A$, $A^t$ the transpose of $A$, and $I$ is the identity matrix.)

**PROBLEMA 3**

Let $p$ be a prime number. Call a positive integer $n$ interesting if

$$x^n - 1 = (x^p - x + 1)f(x) + pg(x)$$

for some polynomials $f$ and $g$ with integer coefficients. a) Prove that the number $p^p - 1$ is interesting. b) For which $p$ is $p^p - 1$ the minimal interesting number?

**PROBLEMA 4**

Let $A_1, A_2, \ldots, A_n$ be finite, nonempty sets. Define the function

$$f(t) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}|}.$$

Prove that $f$ is nondecreasing on $[0, 1]$. ($|A|$ denotes the number of elements in $A$.)

**PROBLEMA 5**

Let $n$ be a positive integer and let $V$ be a $(2n-1)$-dimensional vector space over the two-element field. Prove that for arbitrary vectors $v_1, \ldots, v_{4n-1} \in V$, there exists a sequence $1 \leq i_1 < \cdots < i_{2n} \leq 4n - 1$ of indices such that $v_{i_1} + \cdots + v_{i_{2n}} = 0$.

## Day 2

**PROBLEMA 6**
Let $(a_n) \subset (\frac{1}{2}, 1)$. Define the sequence $x_0 = 0, x_{n+1} = \dfrac{a_{n+1} + x_n}{1 + a_{n+1}x_n}$. Is this sequence convergent? If yes find the limit.

**PROBLEMA 7**
An alien race has three genders: male, female and emale. A married triple consists of three persons, one from each gender who all like each other. Any person is allowed to belong to at most one married triple. The feelings are always mutual ( if $x$ likes $y$ then $y$ likes $x$). The race wants to colonize a planet and sends $n$ males, $n$ females and $n$ emales. Every expedition member likes at least $k$ persons of each of the two other genders. The problem is to create as many married triples so that the colony could grow.

a) Prove that if $n$ is even and $k \geq 1/2$ then there might be no married triple. b) Prove that if $k \geq 3n/4$ then there can be formed $n$ married triple ( i.e. everybody is in a triple).

**PROBLEMA 8**
Calculate $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) \ln\left(1 + \frac{1}{2n}\right) \ln\left(1 + \frac{1}{2n+1}\right)$.

**PROBLEMA 9**
Let $f$ be a polynomial with real coefficients of degree $n$. Suppose that $\dfrac{f(x) - f(y)}{x - y}$ is an integer for all $0 \leq x < y \leq n$. Prove that $a - b | f(a) - f(b)$ for all distinct integers $a, b$.

**PROBLEMA 10**
Let $F = A_0A_1...A_n$ be a convex polygon in the plane. Define for all $1 \leq k \leq n - 1$ the operation $f_k$ which replaces $F$ with a new polygon $f_k(F) = A_0A_1..A_{k-1}A'_kA_{k+1}...A_n$ where $A'_k$ is the symmetric of $A_k$ with respect to the perpendicular bisector of $A_{k-1}A_{k+1}$. Prove that $(f_1 \circ f_2 \circ f_3 \circ ... \circ f_{n-1})^n(F) = F$.

## Day 1

**PROBLEMA 1**
Let $0 < a < b$. Prove that $\int_a^b (x^2+1)e^{-x^2}dx \geq e^{-a^2} - e^{-b^2}$.

**PROBLEMA 2**
Compute the sum of the series $\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + ...$

**PROBLEMA 3**
Define the sequence $x_1, x_2, ...$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute $\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n}{x_{n+1}}$.

**PROBLEMA 4**
Let $a, b$ be two integers and suppose that $n$ is a positive integer for which the set $\mathbb{Z} \setminus \{ax^n + by^n \mid x, y \in \mathbb{Z}\}$ is finite. Prove that $n = 1$.

**PROBLEMA 5**
Suppose that $a, b, c$ are real numbers in the interval $[-1, 1]$ such that $1 + 2abc \geq a^2 + b^2 + c^2$. Prove that $1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$ for all positive integers $n$.

## Day 2

**PROBLEMA 6**

$(a)$ A sequence $x_1, x_2, \ldots$ of real numbers satisfies

$$x_{n+1} = x_n \cos x_n \text{ for all } n \geq 1.$$

Does it follows that this sequence converges for all initial values $x_1$? (5 points)

$(b)$ A sequence $y_1, y_2, \ldots$ of real numbers satisfies

$$y_{n+1} = y_n \sin y_n \text{ for all } n \geq 1.$$

Does it follows that this sequence converges for all initial values $y_1$? (5 points)

**PROBLEMA 7**

Let $a_0, a_1, \ldots, a_n$ be positive real numbers such that $a_{k+1} - a_k \geq 1$ for all $k = 0, 1, \ldots, n-1$. Prove that

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right)\cdots\left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right)\left(1 + \frac{1}{a_1}\right)\cdots\left(1 + \frac{1}{a_n}\right).$$

**PROBLEMA 8**

Denote by $S_n$ the group of permutations of the sequence $(1, 2, \ldots, n)$. Suppose that $G$ is a subgroup of $S_n$, such that for every $\pi \in G \setminus \{e\}$ there exists a unique $k \in \{1, 2, \ldots, n\}$ for which $\pi(k) = k$. (Here $e$ is the unit element of the group $S_n$.) Show that this $k$ is the same for all $\pi \in G \setminus \{e\}$.

**PROBLEMA 9**

Let $A$ be a symmetric $m \times m$ matrix over the two-element field all of whose diagonal entries are zero. Prove that for every positive integer $n$ each column of the matrix $A^n$ has a zero entry.

**PROBLEMA 10**

Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number $p$ and every real number $y$.

## Day 1

**PROBLEMA 1**
Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(r) \leq g(r) \quad \forall r \in \mathbb{Q}$$

Does this imply $f(x) \leq g(x) \quad \forall x \in \mathbb{R}$ if

(a) $f$ and $g$ are non-decreasing?

(b) $f$ and $g$ are continuous?

**PROBLEMA 2**
Let $A, B, C$ be real square matrices of the same order, and suppose $A$ is invertible. Prove that

$$(A - B)C = BA^{-1} \implies C(A - B) = A^{-1}B$$

**PROBLEMA 3**
In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to $n$ and let $a_i$ be the number of friends of the $i^{\text{th}}$ resident. Suppose that

$$\sum_{i=1}^{n} a_i^2 = n^2 - n$$

Let $k$ be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of $k$.

**PROBLEMA 4**
Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \geq c_1 \geq \cdots \geq c_n \geq 0$ is a sequence of real numbers which form a convex sequence. (That is $2c_k \leq c_{k-1} + c_{k+1}$ for every $k = 1, 2, \cdots, n-1$ ) and consider the polynomial

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \cdots + c_n a_n z^n$$

Prove that :

$$\max_{|z| \leq 1} q(z) \leq \max_{|z| \leq 1} p(z)$$

**PROBLEMA 5**
Let $n$ be a positive integer. An $n-$*simplex* in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \cdots, P_n$, called its vertices, which do not all belong to the same hyperplane. For every $n$-simplex $\mathcal{S}$ we denote by $v(\mathcal{S})$ the volume of $\mathcal{S}$, and we write $C(\mathcal{S})$ for the center of the unique sphere containing all the vertices of $\mathcal{S}$. Suppose that $P$ is a point inside an $n$-simplex $\mathcal{S}$. Let $\mathcal{S}_i$ be the $n$-simplex obtained from $\mathcal{S}$ by replacing its $i^{\text{th}}$ vertex by $P$. Prove that :

$$\sum_{j=0}^{n} v(\mathcal{S}_j) C(\mathcal{S}_j) = v(\mathcal{S}) C(\mathcal{S})$$

## Day 2

**PROBLEMA 6**
Let $\ell$ be a line and $P$ be a point in $\mathbb{R}^3$. Let $S$ be the set of points $X$ such that the distance from $X$ to $\ell$ is greater than or equal to two times the distance from $X$ to $P$. If the distance from $P$ to $\ell$ is $d > 0$, find Volume($S$).

**PROBLEMA 7**
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1, f'(0) = 0$ and for all $x \in [0, \infty)$, it satisfies

$$f''(x) - 5f'(x) + 6f(x) \geq 0$$

Prove that, for all $x \in [0, \infty)$,

$$f(x) \geq 3e^{2x} - 2e^{3x}$$

**PROBLEMA 8**
Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be two $n \times n$ matrices such that

$$A^2B + BA^2 = 2ABA$$

Prove there exists $k \in \mathbb{N}$ such that

$$(AB - BA)^k = \mathbf{0}_n$$

Here $\mathbf{0}_n$ is the null matrix of order $n$.

**PROBLEMA 9**
Let $p$ be a prime number and $\mathbf{W} \subseteq \mathbb{F}_p[x]$ be the smallest set satisfying the following :

- $x + 1 \in \mathbf{W}$ and $x^{p-2} + x^{p-3} + \cdots + x^2 + 2x + 1 \in \mathbf{W}$
- For $\gamma_1, \gamma_2$ in $\mathbf{W}$, we also have $\gamma(x) \in \mathbf{W}$, where $\gamma(x)$ is the remainder $(\gamma_1 \circ \gamma_2)(x) \pmod{x^p - x}$.

How many polynomials are in $\mathbf{W}$?

**PROBLEMA 10**
Let $\mathbb{M}$ be the vector space of $m \times p$ real matrices. For a vector subspace $S \subseteq \mathbb{M}$, denote by $\delta(S)$ the dimension of the vector space generated by all columns of all matrices in $S$. Say that a vector subspace $T \subseteq \mathbb{M}$ is a *covering matrix space* if

$$\bigcup_{A \in T, A \neq \mathbf{0}} \ker A = \mathbb{R}^p$$

Such a $T$ is minimal if it doesn't contain a proper vector subspace $S \subset T$ such that $S$ is also a covering matrix space.

- Let $T$ be a minimal covering matrix space and let $n = \dim(T)$ Prove that

$$\delta(T) \leq \binom{n}{2}$$

- Prove that for every integer $n$ we can find $m$ and $p$, and a minimal covering matrix space $T$ as above such that $\dim T = n$ and $\delta(T) = \binom{n}{2}$

## Day 1

**PROBLEMA 1**
Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) - f(y) \in \mathbb{Q} \quad \text{for all} \quad x - y \in \mathbb{Q}$$

**PROBLEMA 2**
Denote by $\mathbb{V}$ the real vector space of all real polynomials in one variable, and let $\gamma : \mathbb{V} \to \mathbb{R}$ be a linear map. Suppose that for all $f, g \in \mathbb{V}$ with $\gamma(fg) = 0$ we have $\gamma(f) = 0$ or $\gamma(g) = 0$. Prove that there exist $c, x_0 \in \mathbb{R}$ such that

$$\gamma(f) = cf(x_0) \quad \forall f \in \mathbb{V}$$

**PROBLEMA 3**
Let $p$ be a polynomial with integer coefficients and let $a_1 < a_2 < \cdots < a_k$ be integers. Given that $p(a_i) \neq 0 \forall\ i = 1, 2, \cdots, k$.

- Prove $\exists\ a \in \mathbb{Z}$ such that
$$p(a_i) \mid p(a) \ \ \forall i = 1, 2, \ldots, k$$

- Does there exist $a \in \mathbb{Z}$ such that
$$\prod_{i=1}^{k} p(a_i) \mid p(a)?$$

**PROBLEMA 4**
We say a triple of real numbers $(a_1, a_2, a_3)$ is *better* than another triple $(b_1, b_2, b_3)$ when exactly two out of the three following inequalities hold: $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$. We call a triple of real numbers *special* when they are nonnegative and their sum is 1.

For which natural numbers $n$ does there exist a collection $S$ of special triples, with $|S| = n$, such that any special triple is bettered by at least one element of $S$?

**PROBLEMA 5**
Does there exist a finite group $G$ with a normal subgroup $H$ such that $|\text{Aut } H| > |\text{Aut } G|$? Disprove or provide an example. Here the notation $|\text{Aut } X|$ for some group $X$ denotes the number of isomorphisms from $X$ to itself.

**PROBLEMA 6**
For a permutation $\sigma \in S_n$ with $(1, 2, \ldots, n) \mapsto (i_1, i_2, \ldots, i_n)$, define

$$D(\sigma) = \sum_{k=1}^{n} |i_k - k|$$

Let
$$Q(n, d) = |\{\sigma \in S_n : D(\sigma) = d\}|$$

Show that when $d \geq 2n$, $Q(n, d)$ is an even number.

## Day 2

**PROBLEMA 7**
Let $n, k$ be positive integers and suppose that the polynomial $x^{2k} - x^k + 1$ divides $x^{2n} + x^n + 1$. Prove that $x^{2k} + x^k + 1$ divides $x^{2n} + x^n + 1$.

**PROBLEMA 8**
Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

**PROBLEMA 9**
Let $n$ be a positive integer. Prove that $2^{n-1}$ divides

$$\sum_{0 \le k < n/2} \binom{n}{2k+1} 5^k.$$

**PROBLEMA 10**
Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients, and let $f(x), g(x) \in \mathbb{Z}[x]$ be nonconstant polynomials such that $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$. Prove that if the polynomial $f(x) - 2008$ has at least 81 distinct integer roots, then the degree of $g(x)$ is greater than 5.

**PROBLEMA 11**
Let $n$ be a positive integer, and consider the matrix $A = (a_{ij})_{1 \le i,j \le n}$ where $a_{ij} = 1$ if $i + j$ is prime and $a_{ij} = 0$ otherwise. Prove that $|\det A| = k^2$ for some integer $k$.

**PROBLEMA 12**
Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, let $d > 0$, and suppose that $S$ is a set of points (not necessarily countable) in $\mathcal{H}$ such that the distance between any two distinct points in $S$ is equal to $d$. Show that there is a point $y \in \mathcal{H}$ such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : \ x \in S \right\}$$

is an orthonormal system of vectors in $\mathcal{H}$.

## Day 1

**PROBLEMA 1**
Let $f$ be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer $k$. Prove that all coefficients of $f$ are divisible by 5.

**PROBLEMA 2**
Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose $n^2$ entries are precisely the numbers $1, 2, \ldots, n^2$?

**PROBLEMA 3**
Call a polynomial $P(x_1, \ldots, x_k)$ *good* if there exist $2 \times 2$ real matrices $A_1, \ldots, A_k$ such that

$$P(x_1, \ldots, x_k) = \det\left(\sum_{i=1}^{k} x_i A_i\right).$$

Find all values of $k$ for which all homogeneous polynomials with $k$ variables of degree 2 are good. (A polynomial is homogeneous if each term has the same total degree.)

**PROBLEMA 4**
Let $G$ be a finite group. For arbitrary sets $U, V, W \subset G$, denote by $N_{UVW}$ the number of triples $(x, y, z) \in U \times V \times W$ for which $xyz$ is the unity . Suppose that $G$ is partitioned into three sets $A, B$ and $C$ (i.e. sets $A, B, C$ are pairwise disjoint and $G = A \cup B \cup C$). Prove that $N_{ABC} = N_{CBA}$.

**PROBLEMA 5**
Let $n$ be a positive integer and $a_1, \ldots, a_n$ be arbitrary integers. Suppose that a function $f : \mathbb{Z} \to \mathbb{R}$ satisfies $\sum_{i=1}^{n} f(k + a_i l) = 0$ whenever $k$ and $l$ are integers and $l \neq 0$. Prove that $f = 0$.

**PROBLEMA 6**
How many nonzero coefficients can a polynomial $P(x)$ have if its coefficients are integers and $|P(z)| \leq 2$ for any complex number $z$ of unit length?

## Day 2

**PROBLEMA 7**

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of $f$ can be moved to the graph of $cf$ using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers $a$ and $b$?

**PROBLEMA 8**

Let $x$, $y$ and $z$ be integers such that $S = x^4 + y^4 + z^4$ is divisible by 29. Show that $S$ is divisible by $29^4$.

**PROBLEMA 9**

Let $C$ be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$. (A set is closed if its complement is a union of open intervals. A function $g$ is nondecreasing if $g(x) \leq g(y)$ for all $x \leq y$.)

**PROBLEMA 10**

Let $n > 1$ be an odd positive integer and $A = (a_{ij})_{i,j=1..n}$ be the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j \equiv \pm 2 \pmod{n} \\ 0 & \text{otherwise} \end{cases} .$$

Find $\det A$.

**PROBLEMA 11**

For each positive integer $k$, find the smallest number $n_k$ for which there exist real $n_k \times n_k$ matrices $A_1, A_2, \ldots, A_k$ such that all of the following conditions hold:

(1) $A_1^2 = A_2^2 = \ldots = A_k^2 = 0$,

(2) $A_i A_j = A_j A_i$ for all $1 \leq i, j \leq k$, and

(3) $A_1 A_2 \ldots A_k \neq 0$.

**PROBLEMA 12**

Let $f \neq 0$ be a polynomial with real coefficients. Define the sequence $f_0, f_1, f_2, \ldots$ of polynomials by $f_0 = f$ and $f_{n+1} = f_n + f_n'$ for every $n \geq 0$. Prove that there exists a number $N$ such that for every $n \geq N$, all roots of $f_n$ are real.

## Day 1

**PROBLEMA 1**
Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements. (a) If f is continuous and range(f)=$\mathbb{R}$ then f is monotonic (b) If f is monotonic and range(f)=$\mathbb{R}$ then f is continuous (c) If f is monotonic and f is continuous then range(f)=$\mathbb{R}$

**PROBLEMA 2**
Find the number of positive integers x satisfying the following two conditions: 1. $x < 10^{2006}$ 2. $x^2 - x$ is divisible by $10^{2006}$

**PROBLEMA 3**
Let $A$ be an $nxn$ matrix with integer entries and $b_1, b_2, ..., b_k$ be integers satisfying $detA = b_1 \cdot b_2 \cdot ... \cdot b_k$. Prove that there exist $nxn$-matrices $B_1, B_2, ..., B_k$ with integers entries such that $A = B_1 \cdot B_2 \cdot ... \cdot B_k$ and $detB_i = b_i$ for all $i = 1, ..., k$.

**PROBLEMA 4**
Let f be a rational function (i.e. the quotient of two real polynomials) and suppose that $f(n)$ is an integer for infinitely many integers n. Prove that f is a polynomial.

**PROBLEMA 5**
Let $a, b, c, d$ three strictly positive real numbers such that

$$a^2 + b^2 + c^2 = d^2 + e^2,$$

$$a^4 + b^4 + c^4 = d^4 + e^4.$$

Compare

$$a^3 + b^3 + c^3$$

with

$$d^3 + e^3,$$

**PROBLEMA 6**
Find all sequences $a_0, a_1, \ldots, a_n$ of real numbers such that $a_n \neq 0$, for which the following statement is true:

If $f : \mathbb{R} \to \mathbb{R}$ is an $n$ times differentiable function and $x_0 < x_1 < \ldots < x_n$ are real numbers such that $f(x_0) = f(x_1) = \ldots = f(x_n) = 0$ then there is $h \in (x_0, x_n)$ for which

$$a_0 f(h) + a_1 f'(h) + \ldots + a_n f^{(n)}(h) = 0.$$

## Day 2

**PROBLEMA 7**
Let $V$ be a convex polygon. (a) Show that if $V$ has $3k$ vertices, then $V$ can be triangulated such that each vertex is in an odd number of triangles. (b) Show that if the number of vertices is not divisible with 3, then $V$ can be triangulated such that exactly 2 vertices have an even number of triangles.

**PROBLEMA 8**
Find all functions $f : \mathbb{R} \to R$ such that for any $a < b$, $f([a, b])$ is an interval of length $b - a$

**PROBLEMA 9**
Compare $\tan(\sin x)$ with $\sin(\tan x)$, for $x \in \left]0, \frac{\pi}{2}\right[$.

**PROBLEMA 10**
Let $v_0$ be the zero ector and let $v_1, ..., v_{n+1} \in \mathbb{R}^n$ such that the Euclidian norm $|v_i - v_j|$ is rational for all $0 \leq i, j \leq n+1$. Prove that $v_1, ..., v_{n+1}$ are linearly dependent over the rationals.

**PROBLEMA 11**
Show that there are an infinity of integer numbers $m, n$, with $gcd(m, n) = 1$ such that the equation $(x + m)^3 = nx$ has 3 different integer sollutions.

**PROBLEMA 12**
The scores of this problem were: one time 17/20 (by the runner-up) one time 4/20 (by Andrei Negut) one time 1/20 (by the winner) the rest had zero... just to give an idea of the difficulty.

Let $A_i, B_i, S_i$ $(i = 1, 2, 3)$ be invertible real $2 \times 2$ matrices such that

- not all $A_i$ have a common real eigenvector,
- $A_i = S_i^{-1} B_i S_i$ for $i = 1, 2, 3$,
- $A_1 A_2 A_3 = B_1 B_2 B_3 = I$.

Prove that there is an invertible $2 \times 2$ matrix $S$ such that $A_i = S^{-1} B_i S$ for all $i = 1, 2, 3$.

## Day 1

**PROBLEMA 1**
Let $A$ be a $n \times n$ matrix such that $A_{ij} = i + j$. Find the rank of $A$.

RemarkNot asked in the contest: is diagonalisable since real symetric matrix it is not difficult to find its eigenvalues.

**PROBLEMA 2**
2) all elements in 0,1,2; B[n] = number of rows with no 2 sequent 0's; A[n] with no 3 sequent elements the same; prove —A[n+1]—=3.—B[n]—

**PROBLEMA 3**
3) $f$ cont diff, $R \rightarrow ]0, +\infty[$, prove $| \int_0^1 f^3 - f(0)^2 \int_0^1 f| \leq \max_{[0,1]} |f'| (\int_0^1 f)^2$

**PROBLEMA 4**
4) find all polynom with coeffs a permutation of $[1, ..., n]$ and all roots rational

**PROBLEMA 5**
5) f twice cont diff, $|f''(x) + 2xf'(x) + (x^2 + 1)f(x)| \leq 1$. prove $\lim_{x \to +\infty} f(x) = 0$

**PROBLEMA 6**
6) $G$ group, $G_m$ and $G_n$ commutative subgroups being the $m$ and $n$ th powers of the elements in $G$. Prove $G_{gcd(m,n)}$ is commutative.

## Day 2

**PROBLEMA 7**
1. Let $f(x) = x^2 + bx + c$, M = x — —f(x)—¡1. Prove $|M| \leq 2\sqrt{2}$ (—...— = length of interval(s))

**PROBLEMA 8**
Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(f(x))^n$ is a polynomial for every integer $n \geq 2$. Is $f$ also a polynomial?

**PROBLEMA 9**
What is the maximal dimension of a linear subspace $V$ of the vector space of real $n \times n$ matrices such that for all $A$ in $B$ in $V$, we have trace$(AB) = 0$ ?

**PROBLEMA 10**
Let $f : \mathbb{R} \to \mathbb{R}$ be a three times differentiable function. Prove that there exists $w \in [-1, 1]$ such that

$$\frac{f'''(w)}{6} = \frac{f(1)}{2} - \frac{f(-1)}{2} - f'(0).$$

**PROBLEMA 11**
Find all $r > 0$ such that when $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, $\|\text{grad } f(0,0)\| = 1$, $\|\text{grad } f(u) - \text{grad } f(v)\| \leq \|u - v\|$, then the max of $f$ on the disk $\|u\| \leq r$, is attained at exactly one point.

**PROBLEMA 12**
6. If $p, q$ are rationals, $r = p + \sqrt{7}q$, then prove there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z) - (\pm I_2)$ for which $\frac{ar+b}{cr+d} = r$ and $det(A) = 1$

## Day 1

**PROBLEMA 1**
Let $S$ be an infinite set of real numbers such that $|x_1 + x_2 + \cdots + x_n| \leq 1$ for all finite subsets $\{x_1, x_2, \ldots, x_n\} \subset S$. Show that $S$ is countable.

**PROBLEMA 2**
Let $f_1(x) = x^2 - 1$, and for each positive integer $n \geq 2$ define $f_n(x) = f_{n-1}(f_1(x))$. How many distinct real roots does the polynomial $f_{2004}$ have?

**PROBLEMA 3**
Let $A_n$ be the set of all the sums $\displaystyle\sum_{k=1}^{n} \arcsin x_k$, where $n \geq 2$, $x_k \in [0,1]$, and $\displaystyle\sum_{k=1}^{n} x_k = 1$.

a) Prove that $A_n$ is an interval.

b) Let $a_n$ be the length of the interval $A_n$. Compute $\displaystyle\lim_{n\to\infty} a_n$.

**PROBLEMA 4**
Suppose $n \geq 4$ and let $S$ be a finite set of points in the space $(\mathbb{R}^3)$, no four of which lie in a plane. Assume that the points in $S$ can be colored with red and blue such that any sphere which intersects $S$ in at least 4 points has the property that exactly half of the points in the intersection of $S$ and the sphere are blue. Prove that all the points of $S$ lie on a sphere.

**PROBLEMA 5**
Let $S$ be a set of $\dbinom{2n}{n} + 1$ real numbers, where $n$ is an positive integer. Prove that there exists a monotone sequence $\{a_i\}_{1 \leq i \leq n+2} \subset S$ such that

$$|x_{i+1} - x_1| \geq 2|x_i - x_1|,$$

for all $i = 2, 3, \ldots, n+1$.

**PROBLEMA 6**
For every complex number $z$ different from 0 and 1 we define the following function

$$f(z) := \sum \frac{1}{\log^4 z}$$

where the sum is over all branches of the complex logarithm.

a) Prove that there are two polynomials $P$ and $Q$ such that $f(z) = \dfrac{P(z)}{Q(z)}$ for all $z \in \mathbb{C} - \{0, 1\}$.

b) Prove that for all $z \in \mathbb{C} - \{0, 1\}$ we have

$$f(z) = \frac{z^3 + 4z^2 + z}{6(z-1)^4}.$$

## Day 2

**PROBLEMA 7**
Let $A$ be a real $4 \times 2$ matrix and $B$ be a real $2 \times 4$ matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Find $BA$.

**PROBLEMA 8**
Let $f, g : [a, b] \to [0, \infty)$ be two continuous and non-decreasing functions such that each $x \in [a, b]$ we have

$$\int_a^x \sqrt{f(t)}\ dt \le \int_a^x \sqrt{g(t)}\ dt \ \text{ and } \ \int_a^b \sqrt{f(t)}\ dt = \int_a^b \sqrt{g(t)}\ dt.$$

Prove that

$$\int_a^b \sqrt{1 + f(t)}\ dt \ge \int_a^b \sqrt{1 + g(t)}\ dt.$$

**PROBLEMA 9**
Let $D$ be the closed unit disk in the plane, and let $z_1, z_2, \ldots, z_n$ be fixed points in $D$. Prove that there exists a point $z$ in $D$ such that the sum of the distances from $z$ to each of the $n$ points is greater or equal than $n$.

**PROBLEMA 10**
For $n \ge 1$ let $M$ be an $n \times n$ complex array with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, with multiplicities $m_1, m_2, \ldots, m_k$ respectively. Consider the linear operator $L_M$ defined by $L_M X = MX + XM^T$, for any complex $n \times n$ array $X$. Find its eigenvalues and their multiplicities. ($M^T$ denotes the transpose matrix of $M$).

**PROBLEMA 11**
Prove that

$$\int_0^1 \int_0^1 \frac{dx\ dy}{\frac{1}{x} + |\log y| - 1} \le 1.$$

**PROBLEMA 12**
For $n \ge 0$ define the matrices $A_n$ and $B_n$ as follows: $A_0 = B_0 = (1)$, and for every $n > 0$ let

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \text{ and } B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{pmatrix}.$$

Denote by $S(M)$ the sum of all the elements of a matrix $M$. Prove that $S(A_n^{k-1}) = S(A_k^{n-1})$, for all $n, k \ge 2$.

## Day 1

**PROBLEMA 1**

(a) Let $a_1, a_2, ...$ be a sequenceof reals with $a_1 = 1$ and $a_{n+1} > \frac{3}{2}a_n$ for all $n$. Prove that $\lim_{n\to\infty} \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$ exists. (finite or infinite)

(b) Prove that for all $\alpha > 1$ there is a sequence $a_1, a_2, ...$ with the same properties such that $\lim_{n\to\infty} \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha$

**PROBLEMA 2**

Let $a_1, a_2, ..., a_{51}$ be non-zero elements of a field of characteristic $p$. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence $b_1, ..., b_{51}$. If this new sequence is a permutation of the original one, find all possible values of $p$.

**PROBLEMA 3**

Let $A \in \mathbb{R}^{n\times n}$ such that $3A^3 = A^2 + A + I$. Show that the sequence $A^k$ converges to an idempotent matrix. (idempotent: $B^2 = B$)

**PROBLEMA 4**

Determine the set of all pairs (a,b) of positive integers for which the set of positive integers can be decomposed into 2 sets A and B so that $a \cdot A = b \cdot B$.

**PROBLEMA 5**

Let $g : [0, 1] \to \mathbb{R}$ be a continuous function and let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of functions defined by $f_0(x) = g(x)$ and

$$f_{n+1}(x) = \frac{1}{x}\int_0^x f_n(t)dt.$$

Determine $\lim_{n\to\infty} f_n(x)$ for every $x \in (0, 1]$.

**PROBLEMA 6**

Let $p = \sum_{k=0}^{n} a_k X^k \in R[X]$ a polynomial such that all his roots lie in the half plane $\{z \in C | Re(z) < 0\}$. Prove that $a_k a_{k+3} < a_{k+1}a_{k+2}$, for every k=0,1,2...,n-3.

## Day 2

**PROBLEMA 7**
Let $A, B \in \mathbb{R}^{n\times n}$ such that $AB + B + A = 0$. Prove that $AB = BA$.

**PROBLEMA 8**
Evaluate $\lim_{x\to 0^+} \int_x^{2x} \frac{\sin^m(t)}{t^n} dt$. $(m, n \in \mathbb{N})$

**PROBLEMA 9**
Let $A$ be a closed subset of $\mathbb{R}^n$ and let $B$ be the set of all those points $b \in \mathbb{R}^n$ for which there exists exactly one point $a_0 \in A$ such that $|a_0 - b| = \inf_{a\in A} |a - b|$. Prove that $B$ is dense in $\mathbb{R}^n$; that is, the closure of $B$ is $\mathbb{R}^n$

**PROBLEMA 10**
Find all the positive integers $n$ for which there exists a Family $\mathcal{F}$ of three-element subsets of $S = \{1, 2, ..., n\}$ satisfying

(i) for any two different elements $a, b \in S$ there exists exactly one $A \in \mathcal{F}$ containing both $a$ and $b$;

(ii) if $a, b, c, x, y, z$ are elements of $S$ such that $\{a, b, x\}, \{a, c, y\}, \{b, c, z\} \in \mathcal{F}$, then $\{x, y, z\} \in \mathcal{F}$ .

**PROBLEMA 11**
a) Show that for each function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$, there exists a function $g : \mathbb{Q} \to \mathbb{R}$ with $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{Q}$. b) Find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for which there is no function $g : \mathbb{Q} \to \mathbb{R}$ such that $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

**PROBLEMA 12**
Let $(a_n)$ be the sequence defined by $a_0 = 1, a_{n+1} = \sum_{k=0}^{n} \dfrac{a_k}{n-k+2}$. Find the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{2^k},$$

if it exists.

## Day 1

**PROBLEMA 1**
A standard parabola is the graph of a quadratic polynomial $y = x^2 + ax + b$ with leading coëfficient 1. Three standard parabolas with vertices $V1, V2, V3$ intersect pairwise at points $A1, A2, A3$. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the $x$-axis.

Prove that standard parabolas with vertices $s(A1), s(A2), s(A3)$ intersect pairwise at the points $s(V1), s(V2), s(V3)$.

**PROBLEMA 2**
Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

**PROBLEMA 3**
Let $n$ be a positive integer and let $a_k = \dfrac{1}{\binom{n}{k}}, b_k = 2^{k-n},\ (k = 1..n)$.

Show that $\sum_{k=1}^{n} \dfrac{a_k - b_k}{k} = 0$.

**PROBLEMA 4**
Let $f : [a, b] \to [a, b]$ be a continuous function and let $p \in [a, b]$. Define $p_0 = p$ and $p_{n+1} = f(p_n)$ for $n = 0, 1, 2, ...$. Suppose that the set $T_p = \{p_n : n = 0, 1, 2, ...\}$ is closed, i.e., if $x \notin T_p$ then $\exists \delta > 0$ such that for all $x' \in T_p$ we have $|x' - x| \geq \delta$.

Show that $T_p$ has finitely many elements.

**PROBLEMA 5**
Prove or disprove the following statements: (a) There exists a monotone function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions $x$. (b) There exists a continuously differentiable function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions $x$.

**PROBLEMA 6**
For an $n \times n$ matrix with real entries let $||M|| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{||Mx||_2}{||x||_2}$, where $|| \cdot ||_2$ denotes the Euclidean norm on $\mathbb{R}^n$. Assume that an $n \times n$ matrxi $A$ with real entries satisfies $||A^k - A^{k-1}|| \leq \frac{1}{2002k}$ for all positive integers $k$. Prove that $||A^k|| \leq 2002$ for all positive integers $k$.

## Day 2

**PROBLEMA 7**
Compute the determinant of the $n \times n$ matrix $A = (a_{ij})_{ij}$,

$$a_{ij} = \begin{cases} (-1)^{|i-j|} & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

**PROBLEMA 8**
200 students participated in a math contest. They had 6 problems to solve. Each problem was correctly solved by at least 120 participants. Prove that there must be 2 participants such that every problem was solved by at least one of these two students.

**PROBLEMA 9**
For each $n \geq 1$ let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that $a_n \cdot b_n$ is an integer.

**PROBLEMA 10**
Let $OABC$ be a tetrahedon with $\angle BOC = \alpha, \angle COA = \beta$ and $\angle AOB = \gamma$. The angle between the faces $OAB$ and $OAC$ is $\sigma$ and the angle between the faces $OAB$ and $OBC$ is $\rho$. Show that $\gamma > \beta \cos \sigma + \alpha \cos \rho$.

**PROBLEMA 11**
Let $A$ be a complex $n \times n$ Matrix for $n > 1$. Let $A^H$ be the conjugate transpose of $A$. Prove that $A \cdot A^H = I_n$ if and only if $A = S \cdot (S^H)^{-1}$ for some complex Matrix $S$.

**PROBLEMA 12**
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function whose gradient $\nabla f$ exists at every point of $\mathbb{R}^n$ and satisfies the condition

$$\exists L > 0 \; \forall x_1, x_2 \in \mathbb{R}^n : \quad ||\nabla f(x_1) - \nabla f(x_2)|| \leq L||x_1 - x_2||.$$

Prove that

$$\forall x_1, x_2 \in \mathbb{R}^n : \quad ||\nabla f(x_1) - \nabla f(x_2)||^2 \leq L\langle \nabla f(x_1) - \nabla f(x_2), x_1 - x_2 \rangle.$$

## Day 1

**PROBLEMA 1**
Let $n$ be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, ..., n^2$ written in order, starting at the top left and moving along each row in turn left-to-right. (e.g. for $n = 3$ we get $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$) We choose $n$ entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

**PROBLEMA 2**
Let $r, s, t$ positive integers which are relatively prime and $a, b \in G$, $G$ a commutative multiplicative group with unit element $e$, and $a^r = b^s = (ab)^t = e$. (a) Prove that $a = b = e$. (b) Does the same hold for a non-commutative group $G$?

**PROBLEMA 3**
Find $\lim_{t \to 1^-} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n}$.

**PROBLEMA 4**
$p(x)$ is a polynomial of degree $n$ with every coefficient 0 or $\pm 1$, and $p(x)$ is divisible by $(x-1)^k$ for some integer $k > 0$. $q$ is a prime such that $\frac{q}{\ln q} < \frac{k}{\ln n+1}$. Show that the complex $q$-th roots of unity must be roots of $p(x)$.

**PROBLEMA 5**
Let $A$ be an $n \times n$ complex matrix such that $A \neq \lambda I_n$ for all $\lambda \in \mathbb{C}$. Prove that $A$ is similar to a matrix having at most one non-zero entry on the maindiagonal.

**PROBLEMA 6**
Suppose that the differentiable functions $a, b, f, g : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x) \geq 0, f'(x) \geq 0, g(x) \geq 0, g'(x) \geq 0 \text{ for all } x \in \mathbb{R},$$

$$\lim_{x \to \infty} a(x) = A \geq 0, \lim_{x \to \infty} b(x) = B \geq 0, \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty,$$

and

$$\frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} = b(x).$$

Prove that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{B}{A+1}$.

## Day 2

**PROBLEMA 7**
Let $r, s \geq 1$ be integers and $a_0, a_1, ..., a_{r-1}, b_0, b_1, ..., b_{s-1}$ be real non-negative numbers such that $(a_0 + a_1x + a_2x^2 + ... + a_{r-1}x^{r-1} + x^r)(b_0 + b_1x + b_2x^2 + ... + b_{s-1}x^{s-1} + x^s) = 1 + x + x^2 + ... + x^{r+s-1} + x^{r+s}$. Prove that each $a_i$ and each $b_j$ equals either 0 or 1.

**PROBLEMA 8**
Let $a_0 = \sqrt{2}, b_0 = 2, a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}, b_{n+1} = \frac{2b_n}{2+\sqrt{4+b_n^2}}$. a) Prove that the sequences $(a_n)$ and $(b_n)$ are decreasing and converge to 0. b) Prove that the sequence $(2^n a_n)$ is increasing, the sequence $(2^n b_n)$ is decreasing and both converge to the same limit. c) Prove that there exists a positive constant $C$ such that for all $n$ the following inequality holds: $0 < b_n - a_n < \frac{C}{8^n}$.

**PROBLEMA 9**
Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two of these points is strictly greater than $\sqrt{2}$.

**PROBLEMA 10**
Let $A = (a_{k,l})_{k,l=1,...,n}$ be a complex $n \times n$ matrix such that for each $m \in \{1, 2, ..., n\}$ and $1 \leq j_1 < ... < j_m$ the determinant of the matrix $(a_{j_k,j_l})_{k,l=1,...,n}$ is zero. Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the matrix $(a_{\sigma(k),\sigma(l)})_{k,l=1,...,n}$ has all of its nonzero elements above the diagonal.

**PROBLEMA 11**
Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, and such that

$$f(x + y) \geq f(x) + yf(f(x)) \text{ for all } x, y \in \mathbb{R}.$$

**PROBLEMA 12**
For each positive integer $n$, let $f_n(\vartheta) = \sin(\vartheta) \cdot \sin(2\vartheta) \cdot \sin(4\vartheta) \cdots \sin(2^n\vartheta)$. For each real $\vartheta$ and all $n$, prove that

$$|f_n(\vartheta)| \leq \frac{2}{\sqrt{3}} |f_n(\frac{\pi}{3})|$$

## Day 1

**PROBLEMA 1**
Does every monotone increasing function $f : [0, 1] \to [0, 1]$ have a fixed point? What about every monotone decreasing function?

**PROBLEMA 2**
Let $p(x) = x^5 + x$ and $q(x) = x^5 + x^2$, Find al pairs $(w, z) \in \mathbb{C} \times \mathbb{C}$, $w \neq z$ for which $p(w) = p(z), q(w) = q(z)$.

**PROBLEMA 3**
Let $A, B \in \mathbb{C}^{n \times n}$ with $\rho(AB - BA) = 1$. Show that $(AB - BA)^2 = 0$.

**PROBLEMA 4**
Let $(x_i)$ be a decreasing sequence of positive reals, then show that:

(a) for every positive integer $n$ we have $\sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}$.

(b) there is a constant C for which we have $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sqrt{\sum_{i=k}^{\infty} x_i^2} \leq C \sum_{i=1}^{\infty} x_i$.

**PROBLEMA 5**
Let $R$ be a ring of characteristic zero. Let $e, f, g \in R$ be idempotent elements (an element $x$ is called idempotent if $x^2 = x$) satisfying $e + f + g = 0$. Show that $e = f = g = 0$.

**PROBLEMA 6**
Let $f : \mathbb{R} \to ]0, +\infty[$ be an increasing differentiable function with $\lim_{x\to+\infty} f(x) = +\infty$ and $f'$ is bounded, and let $F(x) = \int_0^x f(t)dt$. Define the sequence $(a_n)$ recursively by $a_0 = 1, a_{n+1} = a_n + \frac{1}{f(a_n)}$ Define the sequence $(b_n)$ by $b_n = F^{-1}(n)$. Prove that $\lim_{x\to+\infty}(a_n - b_n) = 0$.

## Day 2

**PROBLEMA 7**
Show that a square may be partitioned into $n$ smaller squares for sufficiently large $n$. Show that for some constant $N(d)$, a $d$-dimensional cube can be partitioned into $n$ smaller cubes if $n \geq N(d)$.

**PROBLEMA 8**
Let $f$ be continuous and nowhere monotone on $[0,1]$. Show that the set of points on which $f$ obtains a local minimum is dense.

**PROBLEMA 9**
Let $p(z)$ be a polynomial of degree $n > 0$ with complex coefficients. Prove that there are at least $n+1$ complex numbers $z$ for which $p(z) \in \{0,1\}$.

**PROBLEMA 10**
Let $OABC$ be a tetrahedon with $\angle BOC = \alpha, \angle COA = \beta$ and $\angle AOB = \gamma$. The angle between the faces $OAB$ and $OAC$ is $\sigma$ and the angle between the faces $OAB$ and $OBC$ is $\rho$. Show that $\gamma > \beta \cos\sigma + \alpha \cos\rho$.

**PROBLEMA 11**
Find all functions $\mathbb{R}^+ \to \mathbb{R}^+$ for which we have for all $x, y \in \mathbb{R}^+$ that $f(x)f(yf(x)) = f(x+y)$.

**PROBLEMA 12**
Let $A$ be a real $n \times n$ Matrix and define $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ Prove or disprove that for any real polynomial $P(x)$ and any real matrices $A, B$, $P(e^{AB})$ is nilpotent if and only if $P(e^{BA})$ is nilpotent.

## Day 1

**PROBLEMA 1**
a) Show that $\forall n \in \mathbb{N}_0, \exists A \in \mathbb{R}^{n\times n} : A^3 = A + I$. b) Show that $\det(A) > 0, \forall A$ fulfilling the above condition.

**PROBLEMA 2**
Does there exist a bijective map $f : \mathbb{N} \to \mathbb{N}$ so that $\sum_{n=1}^{\infty} \frac{f(n)}{n^2}$ is finite?

**PROBLEMA 3**
Suppose that $f : \mathbb{R} \to \mathbb{R}$ fulfils $\left|\sum_{k=1}^{n} 3^k \left(f(x+ky) - f(x-ky)\right)\right| \leq 1$ for all $n \in \mathbb{N}, x, y \in \mathbb{R}$. Prove that $f$ is a constant function.

**PROBLEMA 4**
Find all strictly monotonic functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ for which $f\left(\frac{x^2}{f(x)}\right) = x$ for all $x$.

**PROBLEMA 5**
Suppose that $2n$ points of an $n \times n$ grid are marked. Show that for some $k > 1$ one can select $2k$ distinct marked points, say $a_1, ..., a_{2k}$, such that $a_{2i-1}$ and $a_{2i}$ are in the same row, $a_{2i}$ and $a_{2i+1}$ are in the same column, $\forall i$, indices taken mod 2n.

**PROBLEMA 6**
(a) Let $p > 1$ a real number. Find a real constant $c_p$ for which the following statement holds: If $f : [-1,1] \to \mathbb{R}$ is a continuously differentiable function with $f(1) > f(-1)$ and $|f'(y)| \leq 1 \forall y \in [-1,1]$, then $\exists x \in [-1,1] : f'(x) > 0$ so that $\forall y \in [-1,1] : |f(y) - f(x)| \leq c_p \sqrt[p]{f'(x)}|y - x|$.

(b) What if $p = 1$?

## Day 2

**PROBLEMA 7**
Let $R$ be a ring where $\forall a \in R : a^2 = 0$. Prove that $abc + abc = 0$ for all $a, b, c \in R$.

**PROBLEMA 8**
We roll a regular 6-sided dice $n$ times. What is the probabilty that the total number of eyes rolled is a multiple of 5?

**PROBLEMA 9**
Let $x_i \geq -1$ and $\sum_{i=1}^{n} x_i^3 = 0$. Prove $\sum_{i=1}^{n} x_i \leq \frac{n}{3}$.

**PROBLEMA 10**
Prove that there's no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x)^2 \geq f(x+y)\,(f(x)+y)$ for all $x, y > 0$.

**PROBLEMA 11**
Let $S$ be the set of words made from the letters $a, b$ and $c$. The equivalence relation $\sim$ on $S$ satisfies

$$uu \sim u$$

$$u \sim v \Rightarrow uw \sim vw \text{ and } wu \sim wv$$

for all words $u, v$ and $w$. Prove that every word in $S$ is equivalent to a word of length $\leq 8$.

**PROBLEMA 12**
Let $A$ be a subset of $\mathbb{Z}/n\mathbb{Z}$ with at most $\frac{\ln(n)}{100}$ elements. Define $f(r) = \sum_{s \in A} e^{\frac{2\pi i r s}{n}}$. Show that for some $r \neq 0$ we have $|f(r)| \geq \frac{|A|}{2}$.

## Day 1

**PROBLEMA 1**
Let $V$ be a 10-dimensional real vector space and $U_1, U_2$ two linear subspaces such that $U_1 \subseteq U_2, \dim U_1 = 3, \dim U_2 = 6$. Let $\varepsilon$ be the set of all linear maps $T : V \to V$ which have $T(U_1) \subseteq U_1, T(U_2) \subseteq U_2$. Calculate the dimension of $\varepsilon$. (again, all as real vector spaces)

**PROBLEMA 2**
Consider the following statement: for any permutation $\pi_1 \neq \mathbb{I}$ of $\{1, 2, ..., n\}$ there is a permutation $\pi_2$ such that any permutation on these numbers can be obtained by a finite compostion of $\pi_1$ and $\pi_2$.

(a) Prove the statement for $n = 3$ and $n = 5$. (b) Disprove the statement for $n = 4$.

**PROBLEMA 3**
Let $f(x) = 2x(1 - x), x \in \mathbb{R}$ and denote $f_n = f \circ f \circ ... \circ f$, $n$ times. (a) Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$. (b) Now compute $\int_0^1 f_n(x)dx$.

**PROBLEMA 4**
The function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(0) = 2, f'(0) = -2, f(1) = 1$. Prove that there is a $\xi \in ]0, 1[$ for which we have $f(\xi) \cdot f'(\xi) + f''(\xi) = 0$.

**PROBLEMA 5**
Let $P$ be a polynomial of degree $n$ with only real zeros and real coefficients. Prove that for every real $x$ we have $(n - 1)(P'(x))^2 \geq nP(x)P''(x)$. When does equality occur?

**PROBLEMA 6**
Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $xf(y) + yf(x) \leq 1$ for every $x, y \in [0, 1]$. (a) Show that $\int_0^1 f(x)dx \leq \frac{\pi}{4}$. (b) Find such a funtion for which equality occurs.

## Day 2

**PROBLEMA 7**
$V$ is a real vector space and $f, f_i : V \to \mathbb{R}$ are linear for $i = 1, 2, ..., k$. Also $f$ is zero at all points for which all of $f_i$ are zero. Show that $f$ is a linear combination of the $f_i$.

**PROBLEMA 8**
$S$ ist the set of all cubic polynomials $f$ with $|f(\pm 1)| \le 1$ and $|f(\pm \frac{1}{2})| \le 1$. Find $\sup_{f\in S} \max_{-1\le x\le 1} |f''(x)|$ and all members of $f$ which give equality.

**PROBLEMA 9**
Given $0 < c < 1$, we define $f(x) = \begin{cases} \frac{x}{c} & x \in [0, c] \\ \frac{1-x}{1-c} & x \in [c, 1] \end{cases}$ Let $f^n(x) = f(f(...f(x)))$ . Show that for each positive integer $n$, $f^n$ has a non-zero finite nunber of fixed points which aren't fixed points of $f^k$ for $k < n$.

**PROBLEMA 10**
Let $S_n = \{1, 2, ..., n\}$. How many functions $f : S_n \to S_n$ satisfy $f(k) \le f(k + 1)$ and $f(k) = f(f(k + 1))$ for $k < n$?

**PROBLEMA 11**
$S$ is a family of balls in $\mathbb{R}^n$ $(n > 1)$ such that the intersection of any two contains at most one point. Show that the set of points belonging to at least two members of $S$ is countable.

**PROBLEMA 12**
$f : (0, 1) \to [0, \infty)$ is zero except at a countable set of points $a_1, a_2, a_3, ...$ . Let $b_n = f(a_n)$. Show that if $\sum b_n$ converges, then $f$ is differentiable at at least one point. Show that for any sequence $b_n$ of non-negative reals with $\sum b_n = \infty$ , we can find a sequence $a_n$ such that the function $f$ defined as above is nowhere differentiable.

## Day 1

**PROBLEMA 1**
Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive reals with $\lim\limits_{n\to+\infty} \epsilon_n = 0$. Find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \epsilon_n\right)$$

**PROBLEMA 2**
Let $a_n$ be a sequence of reals. Suppose $\sum a_n$ converges. Do these sums converge aswell?

(a) $a_1 + a_2 + (a_4 + a_3) + (a_8 + ... + a_5) + (a_{16} + ... + a_9) + ...$

(b) $a_1 + a_2 + (a_3) + (a_4) + (a_5 + a_7) + (a_6 + a_8) + (a_9 + a_{11} + a_{13} + a_{15}) + (a_{10} + a_{12} + a_{14} + a_{16}) + (a_{17} + a_{19} + ...$

**PROBLEMA 3**
Let $A, B \in \mathbb{R}^{n\times n}$ with $A^2 + B^2 = AB$. Prove that if $BA - AB$ is invertible then $3|n$.

**PROBLEMA 4**
Let $\alpha$ be a real number, $1 < \alpha < 2$.

(a) Show that $\alpha$ can uniquely be represented as the infinte product

$$\alpha = \left(1 + \frac{1}{n_1}\right)\left(1 + \frac{1}{n_2}\right)\cdots$$

with $n_i$ positive integers satisfying $n_i^2 \leq n_{i+1}$.

(b) Show that $\alpha \in \mathbb{Q}$ iff from some $k$ onwards we have $n_{k+1} = n_k^2$.

**PROBLEMA 5**
For postive integer $n$ consider the hyperplane

$$R_0^n = x = (x_1 x_2 ... x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0$$

and the lattice

$$Z_0^n = \{y \in R_0^n : \ (\forall i : y_i \in \mathbb{N})\}$$

Define the quasi-norm in $\mathbb{R}^n$ by $\|x\|_p = \sqrt[p]{\sum\limits_{i=1}^{n} |x_i|^p}$ if $0 < p < \infty$ and $\|x\|_\infty = \max\limits_i |x_i|$.

(a) If $x \in R_0^n$ so that $\max x_i - \min x_i \leq 1$ then prove that $\forall p \in [1, \infty], \forall y \in Z_0^n$ we have $\|x\|_p \leq \|x + y\|_p$ (b) Prove that for every $p \in ]0, 1[$, there exist $n \in \mathbb{N}, x \in R_0^n, y \in Z_0^n$ with $\max x_i - \min x_i \leq 1$ and $\|x\|_p > \|x + y\|_p$

**PROBLEMA 6**
Suppose $F$ is a family of finite subsets of $\mathbb{N}$ and for any 2 sets $A, B \in F$ we have $A \cap B \neq \emptyset$.

(a) Is it true that there is a finite subset $Y$ of $\mathbb{N}$ such that for any $A, B \in F$ we have $A \cap B \cap Y \neq \emptyset$? (b) Is the above true if we assume that all members of $F$ have the same size?

## Day 2

**PROBLEMA 7**
Let $f \in C^3(\mathbb{R})$ nonnegative function with $f(0) = f'(0) = 0, f''(0) > 0$. Define $g(x)$ as follows:

$$\left\{ \begin{array}{lll} g(x) = (\frac{\sqrt{f(x)}}{f'(x)})' & \text{for} & x \neq 0 \\ g(x) = 0 & \text{for} & x = 0 \end{array} \right.$$

(a) Show that $g$ is bounded in some neighbourhood of 0. (b) Is the above true for $f \in C^2(\mathbb{R})$?

**PROBLEMA 8**
Let $M \in GL_{2n}(K)$, represented in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

Show that $\det M. \det H = \det A$.

**PROBLEMA 9**
Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(\log n)}{n^\alpha}$ converges iff $\alpha > 0$.

**PROBLEMA 10**
(a) Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ be a linear mapping. Prove that $\exists! C \in \mathbb{R}^{n\times n}$ such that $f(A) = Tr(AC), \forall A \in \mathbb{R}^{n\times n}$.

(b) Suppose in addtion that $\forall A, B \in \mathbb{R}^{n\times n} : f(AB) = f(BA)$. Prove that $\exists \lambda \in \mathbb{R} : f(A) = \lambda Tr(A)$

**PROBLEMA 11**
Let $X$ be an arbitrary set and $f$ a bijection from $X$ to $X$. Show that there exist bijections $g,\ g' : X \to X$ s.t. $f = g \circ g',\ g \circ g = g' \circ g' = 1_X$.

**PROBLEMA 12**
Let $f : [0, 1] \to \mathbb{R}$ continuous. We say that $f$ crosses the axis at $x$ if $f(x) = 0$ but $\exists y, z \in [x-\epsilon, x+\epsilon] : f(y) < 0 < f(z)$ for any $\epsilon$.

(a) Give an example of a function that crosses the axis infinitely often.

(b) Can a continuous function cross the axis uncountably often?

## Day 1

**PROBLEMA 1**
Let $A = (a_{ij}) \in M_{(n+1)\times(n+1)}(\mathbb{R})$ with $a_{ij} = a + |i-j|d$, where $a$ and $d$ are fixed real numbers. Calculate $\det(A)$.

**PROBLEMA 2**
Evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x} dx,$$

where $n$ is a natural number.

**PROBLEMA 3**
The linear operator $A$ on a finite-dimensional vector space $V$ is called an involution if $A^2 = I$, where $I$ is the identity operator. Let $\dim V = n$. i) Prove that for every involution $A$ on $V$, there exists a basis of $V$ consisting of eigenvectors of $A$. ii) Find the maximal number of distinct pairwise commuting involutions on $V$.

**PROBLEMA 4**
Let $a_1 = 1$, $a_n = \frac{1}{n}\sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$. Show that i) $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 2^{-\frac{1}{2}}$; ii) $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \frac{2}{3}$

**PROBLEMA 5**
i) Let $a, b$ be real numbers such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every $x \in [0,1]$. Prove that

$$\lim_{n\to\infty} n \int_0^1 (1+ax+bx^2)^n dx = \begin{cases} -\frac{1}{a} & \text{if } a < 0, \\ \infty & \text{if } a \geq 0. \end{cases}$$

ii) Let $f : [0,1] \to [0,\infty)$ be a function with a continuous second derivative and let $f''(x) \leq 0$ for every $x \in [0,1]$. Suppose that $L = \lim_{n\to\infty} n \int_0^1 (f(x))^n dx$ exists and $0 < L < \infty$. Prove that $f'$ has a constant sign and $\min_{x\in[0,1]} |f'(x)| = L^{-1}$.

**PROBLEMA 6**
Upper content of a subset $E$ of the plane $\mathbb{R}^2$ is defined as

$$\mathcal{C}(E) = \inf\{\sum_{i=1}^{n} \mathrm{diam}(E_i)\}$$

where inf is taken over all finite families of sets $E_1, \ldots, E_n$ $n \in \mathbb{N}$, in $\mathbb{R}^2$ such that $E \subset \bigcup_{i=1}^n E_i$. Lower content of $E$ is defined as

$$\mathcal{K}(E) = \sup\{\mathrm{length}(L) |\, L \text{ is a closed line segment onto which } E \text{ can be contracted}\}$$

. Prove that i) $\mathcal{C}(L) = \mathrm{length}(L)$ if $L$ is a closed line segment; ii) $\mathcal{C}(E) \geq \mathcal{K}(E)$; iii) the equality in ii) is not always true even if $E$ is compact.

## Day 2

**PROBLEMA 7**
Prove that if $f : [0,1] \to [0,1]$ is a continuous function, then the sequence of iterates $x_{n+1} = f(x_n)$ converges if and only if

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0$$

**PROBLEMA 8**
Let $\theta$ be a positive real number. Show that if $k \in \mathbb{N}$ and both $\cosh k\theta$ and $\cosh(k+1)\theta$ are rational, then so is $\cosh\theta$.

**PROBLEMA 9**
Let $G$ be the subgroup of $GL_2(\mathbb{R})$ generated by $A$ and $B$, where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

. Let $H$ consist of the matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in $G$ for which $a_{11} = a_{22} = 1$. a) Show that $H$ is an abelian subgroup of $G$. b) Show that $H$ is not finitely generated.

**PROBLEMA 10**
Let $B$ be a bounded closed convex symmetric (with respect to the origin) set in $\mathbb{R}^2$ with boundary $\Gamma$. Let $B$ have the property that the ellipse of maximal area contained in $B$ is the disc $D$ of radius 1 centered at the origin with boundary $C$. Prove that $A \cap \Gamma \neq \emptyset$ for any arc $A$ of $C$ of length $l(A) \geq \frac{\pi}{2}$.

**PROBLEMA 11**
i) Prove that

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}$$

. ii) Prove that there is a positive constant $c$ such that for every $x \in [1,\infty)$ we have

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2} \right| \leq \frac{c}{x}$$

**PROBLEMA 12**
i) Prove that for every sequence $(a_n)_{n\in\mathbb{N}}$, such that $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n.$$

ii) Prove that for every $\epsilon > 0$ there exists a sequence $(b_n)_{n\in\mathbb{N}}$ such that $b_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n < \infty$ and

$$\sum_{n=1}^{\infty} (b_1 b_2 \cdots b_n)^{\frac{1}{n}} > (e - \epsilon) \sum_{n=1}^{\infty} b_n.$$

## Day 1

**PROBLEMA 1**
Let $X$ be a invertible matrix with columns $X_1, X_2 ..., X_n$. Let $Y$ be a matrix with columns $X_2, X_3, ..., X_n, 0$. Show that the matrices $A = YX^{-1}$ and $B = X^{-1}Y$ have rank $n-1$ and have only 0´s for eigenvalues.

**PROBLEMA 2**
Let $f$ be a continuous function on $[0,1]$ such that for every $x \in [0,1]$, we have $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$. Show that $\int_0^1 f(t)^2 dt \geq \frac{1}{3}$.

**PROBLEMA 3**
Let $f$ be twice continuously differentiable on $(0,\infty)$ such that $\lim_{x\to 0^+} f'(x) = -\infty$ and $\lim_{x\to 0^+} f''(x) = \infty$. Show that

$$\lim_{x\to 0^+} \frac{f(x)}{f'(x)} = 0.$$

**PROBLEMA 4**
Let $F : (1,\infty) \to \mathbb{R}$ be the function defined by

$$F(x) = \int_x^{x^2} \frac{dt}{\ln(t)}.$$

Show that $F$ is injective and find the set of values of $F$.

**PROBLEMA 5**
Let $A$ and $B$ be real $n \times n$ matrices. Assume there exist $n+1$ different real numbers $t_1, t_2, \ldots, t_{n+1}$ such that the matrices

$$C_i = A + t_i B, \ i = 1, 2, \ldots, n+1$$

are nilpotent. Show that both $A$ and $B$ are nilpotent.

**PROBLEMA 6**
Let $p > 1$. Show that there exists a constant $K_p > 0$ such that for every $x, y \in \mathbb{R}$ with $|x|^p + |y|^p = 2$, we have

$$(x-y)^2 \leq K_p(4 - (x+y)^2).$$

## Day 2

**PROBLEMA 7**

Let $A$ be a $3 \times 3$ real matrix such that the vectors $Au$ and $u$ are orthogonal for every column vector $u \in \mathbb{R}^3$. Prove that: a) $A^T = -A$. b) there exists a vector $v \in \mathbb{R}^3$ such that $Au = v \times u$ for every $u \in \mathbb{R}^3$, where $v \times u$ denotes the vector product in $\mathbb{R}^3$.

**PROBLEMA 8**

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $b_0 = 1$, $b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}$. Calculate

$$\sum_{n=1}^{\infty} b_n 2^n.$$

**PROBLEMA 9**

Let all roots of an $n$-th degree polynomial $P(z)$ with complex coefficients lie on the unit circle in the complex plane. Prove that all roots of the polynomial

$$2zP'(z) - nP(z)$$

lie on the same circle.

**PROBLEMA 10**

a) Prove that for every $\epsilon > 0$ there is a positive integer $n$ and real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\max_{x\in[-1,1]} |x - \sum_{k=1}^{n} \lambda_k x^{2k+1}| < \epsilon.$$

b) Prove that for every odd continuous function $f$ on $[-1, 1]$ and for every $\epsilon > 0$ there is a positive integer $n$ and real numbers $\mu_1, \ldots, \mu_n$ such that

$$\max_{x\in[-1,1]} |f(x) - \sum_{k=1}^{n} \mu_k x^{2k+1}| < \epsilon.$$

**PROBLEMA 11**

a) Prove that every function of the form

$$f(x) = \frac{a_0}{2} + \cos(x) + \sum_{n=2}^{N} a_n \cos(nx)$$

with $|a_0| < 1$ has positive as well as negative values in the period $[0, 2\pi)$. b) Prove that the function

$$F(x) = \sum_{n=1}^{100} \cos(n^{\frac{3}{2}} x)$$

has at least 40 zeroes in the interval $(0, 1000)$.

**PROBLEMA 12**

Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of continuous functions on the interval $[0, 1]$ such that

$$\int_0^1 f_m(x) f_n(x) dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and $\sup\{|f_n(x)| : x \in [0, 1] \,\text{and}\, n = 1, 2, \ldots\} < \infty$. Show that there exists no subsequence $(f_{n_k})$ of $(f_n)$ such that $\lim_{k\to\infty} f_{n_k}(x)$ exist for all $x \in [0, 1]$.

## Day 1

**PROBLEMA 1**

a) Let $A$ be a $n \times n$, $n \geq 2$, symmetric, invertible matrix with real positive elements. Show that $z_n \leq n^2 - 2n$, where $z_n$ is the number of zero elements in $A^{-1}$.

b) How many zero elements are there in the inverse of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 1 & 2 & \dots & \iddots \end{pmatrix}$$

**PROBLEMA 2**

Let $f \in C^1(a,b)$, $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a,b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

**PROBLEMA 3**

Given a set $S$ of $2n-1$, $n \in \mathbb{N}$, different irrational numbers. Prove that there are $n$ different elements $x_1, x_2, \dots, x_n \in S$ such that for all non-negative rational numbers $a_1, a_2, \dots, a_n$ with $a_1 + a_2 + \ldots + a_n > 0$ we have that $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ is an irrational number.

**PROBLEMA 4**

Let $\alpha \in \mathbb{R} \backslash \{0\}$ and suppose that $F$ and $G$ are linear maps (operators) from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying $F \circ G - G \circ F = \alpha F$.

a) Show that for all $k \in \mathbb{N}$ one has $F^k \circ G - G \circ F^k = \alpha k F^k$.

b) Show that there exists $k \geq 1$ such that $F^k = 0$.

**PROBLEMA 5**

a) Let $f \in C[0,b]$, $g \in C(\mathbb{R})$ and let $g$ be periodic with period $b$. Prove that $\int_0^b f(x) g(nx)\, \mathrm{d}x$ has a limit as $n \to \infty$ and

$$\lim_{n \to \infty} \int_0^b f(x) g(nx)\, \mathrm{d}x = \frac{1}{b} \int_0^b f(x)\, \mathrm{d}x \cdot \int_0^b g(x)\, \mathrm{d}x$$

b) Find

$$\lim_{n \to \infty} \int_0^\pi \frac{\sin x}{1 + 3\cos^2 nx}\, \mathrm{d}x$$

**PROBLEMA 6**

Let $f \in C^2[0,N]$ and $|f'(x)| < 1$, $f''(x) > 0$ for every $x \in [0,N]$. Let $0 \leq m_0 < m_1 < \cdots < m_k \leq N$ be integers such that $n_i = f(m_i)$ are also integers for $i = 0, 1, \dots, k$. Denote $b_i = n_i - n_{i-1}$ and $a_i = m_i - m_{i-1}$ for $i = 1, 2, \dots, k$.

a) Prove that

$$-1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_k}{a_k} < 1$$

b) Prove that for every choice of $A > 1$ there are no more than $N/A$ indices $j$ such that $a_j > A$.

c) Prove that $k \leq 3N^{2/3}$ (i.e. there are no more than $3N^{2/3}$ integer points on the curve $y = f(x)$, $x \in [0,N]$).

## Day 2

**PROBLEMA 7**
Let $f \in C^1[a,b]$, $f(a) = 0$ and suppose that $\lambda \in \mathbb{R}$, $\lambda > 0$, is such that

$$|f'(x)| \leq \lambda |f(x)|$$

for all $x \in [a,b]$. Is it true that $f(x) = 0$ for all $x \in [a,b]$?

**PROBLEMA 8**
Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$.

a) Prove that $f$ attains its minimum and its maximum.

b) Determine all points $(x,y)$ such that $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$ and determine for which of them $f$ has global or local minimum or maximum.

**PROBLEMA 9**
Let $f$ be a real-valued function with $n+1$ derivatives at each point of $\mathbb{R}$. Show that for each pair of real numbers $a$, $b$, $a < b$, such that

$$\ln\left(\frac{f(b) + f'(b) + \cdots + f^{(n)}(b)}{f(a) + f'(a) + \cdots + f^{(n)}(a)}\right) = b - a$$

there is a number $c$ in the open interval $(a,b)$ for which

$$f^{(n+1)}(c) = f(c)$$

**PROBLEMA 10**
Let $A$ be a $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \ldots (x - c_k)^{d_k}$$

where $c_1, c_2, \ldots, c_k$ are distinct (which means that $c_1$ appears $d_1$ times on the diagonal, $c_2$ appears $d_2$ times on the diagonal, etc. and $d_1 + d_2 + \ldots + d_k = n$).

Let $V$ be the space of all $n \times n$ matrices $B$ such that $AB = BA$. Prove that the dimension of $V$ is

$$d_1^2 + d_2^2 + \cdots + d_k^2$$

**PROBLEMA 11**
*problem 5.* Let $x_1, x_2, \ldots, x_k$ be vectors of $m$-dimensional Euclidean space, such that $x_1 + x_2 + \ldots + x_k = 0$. Show that there exists a permutation $\pi$ of the integers $\{1, 2, \ldots, k\}$ such that:

$$\left\|\sum_{i=1}^{n} x_{\pi(i)}\right\| \leq \left(\sum_{i=1}^{k} \|x_i\|^2\right)^{1/2}$$

for each $n = 1, 2, \ldots, k$. Note that $\|\cdot\|$ denotes the Euclidean norm. (18 points).

**PROBLEMA 12**
Find

$$\lim_{N\to\infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)}$$