The 11^{th} Romanian Master of Mathematics

Sexta, 22 de fevereiro de 2019

PROBLEMA 1
Amy e Bob jogam um jogo. No começo, Amy escreve um inteiro positivo no quadro. Depois os jogadores jogam em turnos, Bob joga primeiro. Nos turnos de Bob, Bob troca o número n no quadro por um número da forma $n-a^2$, onde a é um inteiro positivo escolhido por Bob. Nos turnos de Amy, Amy troca o número n no quadro por um número da forma n^k, onde k é um inteiro positivo escolhido por Amy. Bob ganha se o número no quadro se tornar zero. Amy consegue prevenir a vitória de Bob?

PROBLEMA 2
Seja $ABCD$ um trapézio isósceles, com $AB||CD$. Seja E o ponto médio de AC. Sejam ω e Ω os circumcírculos dos triângulos ABE e CDE, respectivamente. Seja P a intersecção da tangente a ω em A com a tangente a Ω em D. Prove que PE é tangente a Ω.

PROBLEMA 3
Dado qualquer real positivo ϵ, prove que existem apenas finitos inteiros positivos v para os quais a seguinte propriedade é falsa:

Qualquer grafo com v vertices e pelo menos $(1+\epsilon)v$ arestas tem dois ciclos distintos de tamanhos iguais.

(Lembre-se de que a noção de um ciclo simples não permite a repetição de vértices em um ciclo.)

Sábado, 23 de fevereiro de 2019

PROBLEMA 4
Prove que para todo inteiro positivo n existe um polígono (não necessariamente convexo) sem três vértices colineares, o que admite exatamente n triangulações diferentes.

PROBLEMA 5
Determine todas as funções $f : \mathbb{R} \to \mathbb{R}$ que satisfazem

$$f(x + yf(x)) + f(xy) = f(x) + f(2019y),$$

para todos os números reais x e y.

PROBLEMA 6
Ache todos os pares de inteiros (c, d), ambos maiores que 1, com a seguinte propriedade: Para qualquer polinômio mônico Q de grau d com coeficientes inteiros e para qualquer primo $p > c(2c+1)$, existe um conjunto S com no máximo $\left(\frac{2c-1}{2c+1}\right)p$ inteiros, tal que

$$\bigcup_{s\in S} \{s, Q(s), Q(Q(s)), Q(Q(Q(s))), \dots\}$$

é um sistema completo de resíduos módulo p (i.e., intersecta com todas as classes de resíduos módulo p).

The 10^{TH} Romanian Master of Mathematics

Fevereiro de 2018

PROBLEMA 1
Let $ABCD$ be a cyclic quadrangle and let P be a point on the side AB. The diagonal AC crosses the segment DP at Q. The parallel through P to CD crosses the extension of the side BC beyond B at K, and the parallel through Q to BD crosses the extension of the side BC beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.

PROBLEMA 2
Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^{9} = Q(x)^{21} + Q(x)^{20}.$$

PROBLEMA 3
Ann and Bob play a game on an infinite checkered plane making moves in turn. Ann makes the first move. A move consists in orienting any unit grid-segment that has not been oriented before. If at some stage some oriented segments form an oriented cycle, Bob wins. Does Bob have a strategy that guarantees him to win?

Fevereiro de 2018

PROBLEMA 4
Let a, b, c, d be positive integers such that $ad \neq bc$ and $\gcd(a, b, c, d) = 1$. Prove that, as n runs through the positive integers, the values $\gcd(an + b, cn + d)$ may achieve form the set of all positive divisors of some integer.

PROBLEMA 5
Let n be positive integer and fix $2n$ distinct points on a circumference. Split these points into n pairs and join the points in each pair by an arrow (i.e., an oriented line segment). The resulting configuration is *good* if no two arrows cross, and there are no arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ such that $ABCD$ is a convex quadrangle *oriented clockwise*. Determine the number of good configurations.

PROBLEMA 6
Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω cross ℓ at Y and Z. Prove that, as X traces Ω, the circle XYZ is tangent to two fixed circles.

The 9^{TH} Romanian Master of Mathematics

Sexta, 24 de fevereiro de 2017

PROBLEMA 1 (a) Prove that every positive integer n can be written uniquely in the form

$$n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j},$$

where $k \geq 0$ and $0 \leq m_1 < m_2 \cdots < m_{2k+1}$ are integers.

This number k is called *weight* of n.

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

PROBLEMA 2
Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k \leq n$ and $k+1$ distinct integers $x_1, x_2, \cdots, x_{k+1}$ such that

$$P(x_1) + P(x_2) + \cdots + P(x_k) = P(x_{k+1}).$$

Note. A polynomial is *monic* if the coefficient of the highest power is one.

PROBLEMA 3
Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1, \ldots, A_k$ of X is tight if the union $A_1 \cup \cdots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.

Note. A subset A of X is *proper* if $A \neq X$. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

Sábado, 25 de fevereiro de 2017

PROBLEMA 4
In the Cartesian plane, let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the graphs of the quadratic functions $f_1(x) = p_1x^2 + q_1x + r_1$ and $f_2(x) = p_2x^2 + q_2x + r_2$, where $p_1 > 0 > p_2$. The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ cross at distinct points A and B. The four tangents to $\mathcal{G}_1$ and $\mathcal{G}_2$ at A and B form a convex quadrilateral which has an inscribed circle. Prove that the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same axis of symmetry.

PROBLEMA 5
Fix an integer $n \geq 2$. An $n \times n$ sieve is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A stick is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

PROBLEMA 6
Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points on the segments AB, BC, CD, and DA, respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.

The 8$^{\text{TH}}$ Romanian Master of Mathematics

Fevereiro de 2016

PROBLEMA 1
Let ABC be a triangle and let D be a point on the segment BC, $D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E. The circle ACD meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

PROBLEMA 2
Given positive integers m and $n \geq m$, determine the largest number of dominoes (1×2 or 2×1 rectangles) that can be placed on a rectangular board with m rows and $2n$ columns consisting of cells (1×1 squares) so that:

(i) each domino covers exactly two adjacent cells of the board;

(ii) no two dominoes overlap;

(iii) no two form a 2×2 square; and

(iv) the bottom row of the board is completely covered by n dominoes.

PROBLEMA 3
A *cubic sequence* is a sequence of integers given by $a_n = n^3 + bn^2 + cn + d$, where b, c and d are integer constants and n ranges over all integers, including negative integers.

(a) Show that there exists a cubic sequence such that the only terms of the sequence which are squares of integers are a_{2015} and a_{2016}.

(b) Determine the possible values of $a_{2015} \cdot a_{2016}$ for a cubic sequence satisfying the condition in part (a).

Fevereiro de 2016

PROBLEMA 4
Let x and y be positive real numbers such that $x + y^{2016} \geq 1$. Prove that $x^{2016} + y > 1 - \frac{1}{100}$.

PROBLEMA 5
A convex hexagon $A_1B_1A_2B_2A_3B_3$ is inscribed in a circle Ω of radius R. The diagonals A_1B_2, A_2B_3, and A_3B_1 concur at X. For $i = 1, 2, 3$, let ω_i be the circle tangent to the segments XA_i and XB_i, and to the arc A_iB_i of Ω not containing other vertices of the hexagon; let r_i be the radius of ω_i.

(a) Prove that $R \geq r_1 + r_2 + r_3$.

(b) If $R = r_1 + r_2 + r_3$, prove that the six points where the circles ω_i touch the diagonals A_1B_2, A_2B_3, $A_3B_1$are concyclic.

PROBLEMA 6
A set of n points in Euclidean 3-dimensional space, no four of which are coplanar, is partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$. An $\mathcal{AB}$-tree is a configuration of $n - 1$ segments, each of which has an endpoint in $\mathcal{A}$ and an endpoint in $\mathcal{B}$, and such that no segments form a closed polyline. An $\mathcal{AB}$-tree is transformed into another as follows: choose three distinct segments A_1B_1, B_1A_2, and A_2B_2 in the $\mathcal{AB}$-tree such that A_1 is in $\mathcal{A}$ and $|A_1B_1| + |A_2B_2| > |A_1B_2| + |A_2B_1|$, and remove the segment A_1B_1 to replace it by the segment A_1B_2. Given any $\mathcal{AB}$-tree, prove that every sequence of successive transformations comes to an end (no further transformation is possible) after finitely many steps.

The 7th Romanian Master of Mathematics

Fevereiro de 2015

PROBLEMA 1
Does there exist an infinite sequence of positive integers $a_1, a_2, a_3, ...$ such that a_m and a_n are coprime if and only if $|m - n| = 1$?

PROBLEMA 2
For an integer $n \geq 5$, two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is *legal* if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

PROBLEMA 3
A finite list of rational numbers is written on a blackboard. In an *operation*, we choose any two numbers a, b, erase them, and write down one of the numbers

$$a + b,\ a - b,\ b - a,\ a \times b,\ a/b \text{ (if } b \neq 0),\ b/a \text{ (if } a \neq 0).$$

Prove that, for every integer $n > 100$, there are only finitely many integers $k \geq 0$, such that, starting from the list

$$k + 1,\ k + 2,\ \ldots,\ k + n,$$

it is possible to obtain, after $n - 1$ operations, the value $n!$.

Fevereiro de 2015

PROBLEMA 4
Let ABC be a triangle, and let D be the point where the incircle meets side BC. Let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the angle bisector of $\angle BAC$.

PROBLEMA 5
Let $p \geq 5$ be a prime number. For a positive integer k, let $R(k)$ be the remainder when k is divided by p, with $0 \leq R(k) \leq p - 1$. Determine all positive integers $a < p$ such that, for every $m = 1, 2, \cdots, p - 1$,

$$m + R(ma) > a.$$

PROBLEMA 6
Given a positive integer n, determine the largest real number μ satisfying the following condition: for every set C of $4n$ points in the interior of the unit square U, there exists a rectangle T contained in U such that

- the sides of T are parallel to the sides of U;
- the interior of T contains exactly one point of C;
- the area of T is at least μ.

The 6th Romanian Master of Mathematics

Fevereiro de 2013

PROBLEMA 1
For a positive integer a, define a sequence of integers $x_1, x_2, \dots$ by letting $x_1 = a$ and $x_{n+1} = 2x_n + 1$ for $n \geq 1$. Let $y_n = 2^{x_n} - 1$. Determine the largest possible k such that, for some positive integer a, the numbers $y_1, \dots, y_k$ are all prime.

PROBLEMA 2
Does there exist a pair (g, h) of functions $g, h : \mathbb{R} \to \mathbb{R}$ such that the only function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbb{R}$ is identity function $f(x) \equiv x$?

PROBLEMA 3
Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ, and let K be the common point of the segment MR and the circle ω. Prove that the circumcircle of the triangle KPQ and ω are tangent to one another.

Fevereiro de 2013

PROBLEMA 4
Let P and P' be two convex quadrilateral regions in the plane (regions contain their boundary). Let them intersect, with O a point in the intersection. Suppose that for every line ℓ through O the segment $\ell \cap P$ is strictly longer than the segment $\ell \cap P'$. Is it possible that the ratio of the area of P' to the area of P is greater than 1.9?

PROBLEMA 5
Given a positive integer $k \geq 2$, set $a_1 = 1$ and, for every integer $n \geq 2$, let a_n be the smallest $x > a_{n-1}$ such that:

$$x = 1 + \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor.$$

Prove that every prime occurs in the sequence $a_1, a_2, \dots$.

PROBLEMA 6
$2n$ distinct tokens are placed at the vertices of a regular $2n$-gon, with one token placed at each vertex. A *move* consists of choosing an edge of the $2n$-gon and interchanging the two tokens at the endpoints of that edge. Suppose that after a finite number of moves, every pair of tokens have been interchanged exactly once. Prove that some edge has never been chosen.

The 5th Romanian Master of Mathematics

Fevereiro de 2012

PROBLEMA 1
Given a finite number of boys and girls, a *sociable set of boys* is a set of boys such that every girl knows at least one boy in that set; and a *sociable set of girls* is a set of girls such that every boy knows at least one girl in that set. Prove that the number of sociable sets of boys and the number of sociable sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

PROBLEMA 2
Given a non-isosceles triangle ABC, let D, E, and F denote the midpoints of the sides BC, CA, and AB respectively. The circle BCF and the line BE meet again at P, and the circle ABE and the line AD meet again at Q. Finally, the lines DP and FQ meet at R. Prove that the centroid G of the triangle ABC lies on the circle PQR.

PROBLEMA 3
Each positive integer is coloured red or blue. A function f from the set of positive integers to itself has the following two properties:

(a) if $x \leq y$, then $f(x) \leq f(y)$; and

(b) if x, y and z are (not necessarily distinct) positive integers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.

Prove that there exists a positive number a such that $f(x) \leq ax$ for all positive integers x.

Fevereiro de 2012

PROBLEMA 4
Prove that there are infinitely many positive integers n such that $2^{2^n+1} + 1$ is divisible by n but $2^n + 1$ is not.

PROBLEMA 5
Given a positive integer $n \geq 3$, colour each cell of an $n \times n$ square array with one of $\lfloor (n+2)^2/3 \rfloor$ colours, each colour being used at least once. Prove that there is some 1×3 or 3×1 rectangular subarray whose three cells are coloured with three different colours.

PROBLEMA 6
Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

The 4^{th} Romanian Master of Mathematics

Fevereiro de 2011

PROBLEMA 1
Prove that there exist two functions $f, g : \mathbb{R} \to \mathbb{R}$, such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.

PROBLEMA 2
Determine all positive integers n for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

(1) for each integer k, the number $f(k)$ is an integer if and only if k is not divisible by n;

(2) the degree of f is less than n.

PROBLEMA 3
A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

Fevereiro de 2011

PROBLEMA 4
Given a positive integer $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, we write $\Omega(n)$ for the total number $\sum_{i=1}^{s} \alpha_i$ of prime factors of n, counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ (so, for example, $\lambda(12) = \lambda(2^2 \cdot 3^1) = (-1)^{2+1} = -1$).

Prove the following two claims:

(i) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = +1$;

(ii) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = -1$.

PROBLEMA 5
For every $n \geq 3$, determine all the configurations of n distinct points $X_1, X_2, \ldots, X_n$ in the plane, with the property that for any pair of distinct points X_i, X_j there exists a permutation σ of the integers $\{1, \ldots, n\}$, such that $d(X_i, X_k) = d(X_j, X_{\sigma(k)})$ for all $1 \leq k \leq n$.

(We write $d(X, Y)$ to denote the distance between points X and Y.)

PROBLEMA 6
The cells of a square 2011×2011 array are labelled with the integers $1, 2, \ldots, 2011^2$, in such a way that every label is used exactly once. We then identify the left-hand and right-hand edges, and then the top and bottom, in the normal way to form a torus (the surface of a doughnut).

Determine the largest positive integer M such that, no matter which labelling we choose, there exist two neighbouring cells with the difference of their labels at least M.

(Cells with coordinates (x, y) and (x', y') are considered to be neighbours if $x = x'$ and $y - y' \equiv \pm 1 \pmod{2011}$, or if $y = y'$ and $x - x' \equiv \pm 1 \pmod{2011}$.)

The 3rd Romanian Master of Mathematics

Fevereiro de 2010

PROBLEMA 1
For a finite non empty set of primes P, let $m(P)$ denote the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P.

(i) Show that $|P| \leq m(P)$, with equality if and only if $\min(P) > |P|$.

(ii) Show that $m(P) < (|P|+1)(2^{|P|}-1)$.

(The number $|P|$ is the size of set P)

PROBLEMA 2
For each positive integer n, find the largest real number C_n with the following property. Given any n real-valued functions $f_1(x), f_2(x), \cdots, f_n(x)$ defined on the closed interval $0 \leq x \leq 1$, one can find numbers $x_1, x_2, \cdots x_n$, such that $0 \leq x_i \leq 1$ satisfying

$$|f_1(x_1) + f_2(x_2) + \cdots f_n(x_n) - x_1 x_2 \cdots x_n| \geq C_n$$

PROBLEMA 3
Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_1 to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i, A_iA_{i+1}$ and $A_{i+1}A_{i+2}$ (indices are considered modulo 4 so $A_0 = A_4, A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

Fevereiro de 2010

PROBLEMA 4
Determine whether there exists a polynomial $f(x_1, x_2)$ with two variables, with integer coefficients, and two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the plane, satisfying the following conditions:

(i) A is an integer point (i.e a_1 and a_2 are integers);

(ii) $|a_1 - b_1| + |a_2 - b_2| = 2010$;

(iii) $f(n_1, n_2) > f(a_1, a_2)$ for all integer points (n_1, n_2) in the plane other than A;

(iv) $f(x_1, x_2) > f(b_1, b_2)$ for all integer points (x_1, x_2) in the plane other than B.

PROBLEMA 5
Let n be a given positive integer. Say that a set K of points with integer coordinates in the plane is connected if for every pair of points $R, S \in K$, there exists a positive integer ℓ and a sequence $R = T_0, T_1, T_2, \ldots, T_\ell = S$ of points in K, where each T_i is distance 1 away from T_{i+1}. For such a set K, we define the set of vectors

$$\Delta(K) = \{\overrightarrow{RS} \mid R, S \in K\}.$$

What is the maximum value of $|\Delta(K)|$ over all connected sets K of $2n+1$ points with integer coordinates in the plane?

PROBLEMA 6
Given a polynomial $f(x)$ with rational coefficients, of degree $d \geq 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \ldots$ as $f^0(\mathbb{Q}) = \mathbb{Q}$, $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \geq 0$. (Given a set S, we write $f(S)$ for the set $\{f(x) \mid x \in S\}$).

Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^{\infty} f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$, $n \geq 0$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

The 2^{nd} Romanian Master of Mathematics

Fevereiro de 2009

PROBLEMA 1
For $a_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$, and $n = \sum_{i=1}^{k} a_i$, let $d = \gcd(a_1, \ldots, a_k)$ denote the greatest common divisor of $a_1, \ldots, a_k$. Prove that

$$\frac{d}{n} \cdot \frac{n!}{\prod\limits_{i=1}^{k} (a_i!)}$$

is an integer.

PROBLEMA 2
A set S of points in space satisfies the property that all pairwise distances between points in S are distinct. Given that all points in S have integer coordinates (x, y, z) where $1 \le x, y, z \le n$, show that the number of points in S is less than $\min\left((n+2)\sqrt{\frac{n}{3}}, n\sqrt{6}\right)$.

PROBLEMA 3
Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that

$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$

denote by O_i the circumcenter of $\triangle A_jA_kA_l$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Assuming $\forall i A_i \neq O_i$, prove that the four lines A_iO_i are concurrent or parallel.

PROBLEMA 4
For a finite set X of positive integers, let $\Sigma(X) = \sum_{x \in X} \arctan \frac{1}{x}$. Given a finite set S of positive integers for which $\Sigma(S) < \frac{\pi}{2}$, show that there exists at least one finite set T of positive integers for which $S \subset T$ and $\Sigma(S) = \frac{\pi}{2}$.

The 1st Romanian Master of Mathematics

Fevereiro de 2008

PROBLEMA 1
Let ABC be an equilateral triangle and P in its interior. The distances from P to the triangle's sides are denoted by a^2, b^2, c^2 respectively, where $a, b, c > 0$. Find the locus of the points P for which a, b, c can be the sides of a non-degenerate triangle.

PROBLEMA 2
Prove that every bijective function $f : \mathbb{Z} \to \mathbb{Z}$ can be written in the way $f = u + v$ where $u, v : \mathbb{Z} \to \mathbb{Z}$ are bijective functions.

PROBLEMA 3
Let $a > 1$ be a positive integer. Prove that every non-zero positive integer N has a multiple in the sequence $(a_n)_{n\geq 1}$, $a_n = \left\lfloor \frac{a^n}{n} \right\rfloor$.

PROBLEMA 4
Consider a square of sidelength n and $(n+1)^2$ interior points. Prove that we can choose 3 of these points so that they determine a triangle (eventually degenerated) of area at most $\frac{1}{2}$.