Problemas Sortidos

Guilherme Zeus Dantas e Moura
zeusdanmou@gmail.com

1. Let a and b be positive integers with $a > b$. Suppose that

$$\sqrt{\sqrt{a}+\sqrt{b}}+\sqrt{\sqrt{a}-\sqrt{b}}$$

is an integer.

- Must $\sqrt{a}$ be an integer?
- Must $\sqrt{b}$ be an integer?

2. Let ABC be a right triangle with $\angle A = 90°$. A circle ω centered on BC is tangent to AB at D and AC at E. Let F and G be the intersections of ω and BC so that F lies between B and G. If lines DG and EF intersect at X, show that $AX = AD$.

3. Let k and n be positive integers and let

$$S = \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid 0 \le a_k \le \cdots \le a_1 \le n, a_1 + \cdots + a_k = k\}.$$

Determine, with proof, the value of

$$\sum_{(a_1,\ldots,a_k)\in S} \binom{n}{a_1}\binom{a_1}{a_2}\cdots\binom{a_{k-1}}{a_k}$$

in terms of k and n, where the sum is over all k-tuples in S.

4. A convex polyhedron has n faces that are all congruent triangles with angles $36°, 72°$, and $72°$. Determine, with proof, the maximum possible value of n.

5. For each positive real number α, define

$$\lfloor \alpha\mathbb{N} \rfloor := \{\lfloor \alpha m \rfloor \mid m \in \mathbb{N}\}.$$

Let n be a positive integer. A set $S \subseteq \{1, 2, \ldots, n\}$ has the property that: for each real $\beta > 0$,

$$\text{if } S \subseteq \lfloor \beta\mathbb{N} \rfloor, \text{then } \{1, 2, \ldots, n\} \subseteq \lfloor \beta\mathbb{N} \rfloor.$$

Determine, with proof, the smallest positive size of S.

6. Let scalene triangle ABC have circumcenter O and incenter I. Its incircle ω is tangent to sides BC, CA, and AB at D, E, and F, respectively. Let P be the foot of the altitude from D to EF, and let line DP intersect ω again at $Q \neq D$. The line OI intersects the altitude from A toBC at T. Given that $OI \parallel BC$, show that $PQ = PT$.