Treinamento para Provas de Velocidade em Equipe, #2

Alunos Matematicamente Internacionais

Instruções:

- Tamanho esperado da equipe: entre 6 e 8 pessoas.
- Tempo disponível: 80 mintuos.

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Round 1

Problema 1.1 Let

$$a_k = 0.\overbrace{0\ldots0}^{k-1\,0's}1\overbrace{0\ldots0}^{k-1\,0's}1$$

The value of $\sum_{k=1}^{\infty} a_k$ can be expressed as a rational number $\frac{p}{q}$ in simplest form. Find $p+q$.

Problema 1.2 There are five dots arranged in a line from left to right. Each of the dots is colored from one of five colors so that no 3 consecutive dots are all the same color. How many ways are there to color the dots?

Problema 1.3 Frist Campus Center is located 1 mile north and 1 mile west of Fine Hall. The area within 5 miles of Fine Hall that is located north and east of Frist can be expressed in the form $\frac{a}{b}\pi - c$, where a, b, c are positive integers and a and b are relatively prime. Find $a+b+c$.

Problema 1.4 Find the number of positive integers $n < 2018$ such that $25^n + 9^n$ is divisible by 13.

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Round 2

Problema 2.1 If $a_1, a_2, \ldots$ is a sequence of real numbers such that for all n,

$$\sum_{k=1}^{n} a_k \left(\frac{k}{n}\right)^2 = 1,$$

find the smallest n such that $a_n < \frac{1}{2018}$.

Problema 2.2 In an election between A and B, during the counting of the votes, neither candidate was more than 2 votes ahead, and the vote ended in a tie, 6 votes to 6 votes. Two votes for the same candidate are indistinguishable. In how many orders could the votes have been counted? One possibility is AABBABBABABA.

Problema 2.3 Let $\overline{AD}$ be a diameter of a circle. Let point B be on the circle, point C on $\overline{AD}$ such that A, B, C form a right triangle at C. The value of the hypotenuse of the triangle is 4 times the square root of its area. If $\overline{BC}$ has length 30, what is the length of the radius of the circle?

Problema 2.4 For a positive integer n, let $f(n)$ be the number of (not necessarily distinct) primes in the prime factorization of k. For example, $f(1) = 0, f(2) = 1$, and $f(4) = f(6) = 2$. let $g(n)$ be the number of positive integers $k \leq n$ such that $f(k) \geq f(j)$ for all $j \leq n$. Find $g(1) + g(2) + \ldots + g(100)$.

Round 3

Problema 3.1 Let $x_0, x_1, \ldots$ be a sequence of real numbers such that $x_n = \frac{1+x_{n-1}}{x_{n-2}}$ for $n \geq 2$.

Find the number of ordered pairs of positive integers (x_0, x_1) such that the sequence gives $x_{2018} = \frac{1}{1000}$.

Problema 3.2 Alex starts at the origin O of a hexagonal lattice. Every second, he moves to one of the six vertices adjacent to the vertex he is currently at. If he ends up at X after 2018 moves, then let p be the probability that the shortest walk from O to X (where a valid move is from a vertex to an adjacent vertex) has length 2018. Then p can be expressed as $\frac{a^m-b}{c^n}$, where a, b, and c are positive integers less than 10; a and c are not perfect squares; and m and n are positive integers less than 10000. Find $a+b+c+m+n$.

Problema 3.3 Let $\triangle ABC$ satisfy $AB = 17, AC = \frac{70}{3}$ and $BC = 19$. Let I be the incenter of $\triangle ABC$ and E be the excenter of $\triangle ABC$ opposite A. (Note: this means that the circle tangent to ray AB beyond B, ray AC beyond C, and side BC is centered at E.) Suppose the circle with diameter IE intersects AB beyond B at D. If $BD = \frac{a}{b}$ where a, b are coprime positive integers, find $a+b$.

Problema 3.4 What is the largest integer $n < 2018$ such that for all integers $b > 1$, n has at least as many 1's in its base-4 representation as it has in its base-b representation?

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Round 4

Problema 4.1 Suppose real numbers a, b, c, d satisfy $a+b+c+d = 17$ and $ab+bc+cd+da = 46$. If the minimum possible value of $a^2+b^2+c^2+d^2$ can be expressed as a rational number $\frac{p}{q}$ in simplest form, find $p+q$.

Problema 4.2 If a and b are selected uniformly from $\{0, 1, \ldots, 511\}$ without replacement, the expected number of 1's in the binary representation of $a+b$ can be written in simplest from as $\frac{m}{n}$. Compute $m+n$.

Problema 4.3 Triangle ABC has $\angle A = 90°$, $\angle C = 30°$, and $AC = 12$. Let the circumcircle of this triangle

be W. Define D to be the point on arc BC not containing A so that $\angle CAD = 60°$. Define points E and F to be the foots of the perpendiculars from D to lines AB and AC, respectively. Let J be the intersection of line EF with W, where J is on the minor arc AC. The line DF intersects W at H other than D. The area of the triangle FHJ is in the form $\frac{a}{b}(\sqrt{c}-\sqrt{d})$ for positive integers a, b, c, d, where a, b are relatively prime, and the sum of a, b, c, d is minimal. Find $a+b+c+d$.

Problema 4.4 Let n be a positive integer. Let $f(n)$ be the probability that, if divisors a, b, c of n are selected uniformly at random with replacement, then $\gcd(a, \mathrm{lcm}(b, c)) = \mathrm{lcm}(a, \gcd(b, c))$. Let $s(n)$ be the sum of the distinct prime divisors of n. If $f(n) < \frac{1}{2018}$, compute the smallest possible value of $s(n)$.

Round 5

Problema 5.1 For $k \in \{0, 1, \ldots, 9\}$, let $\epsilon_k \in \{-1, 1\}$. If the minimum possible value of $\sum_{i=1}^{9} \sum_{j=0}^{i-1} \epsilon_i \epsilon_j 2^{i+j}$ is m, find $|m|$.

Problema 5.2 How many ways are there to color the 8 regions of a three-set Venn Diagram with 3 colors such that each color is used at least once? Two colorings are considered the same if one can be reached from the other by rotation and/or reflection.

Problema 5.3 Let $\triangle BC$ be a triangle with side lengths $AB = 9, BC = 10, CA = 11$. Let O be the circumcenter of $\triangle ABC$. Denote $D = AO \cap BC, E = BO \cap CA, F = CO \cap AB$. If $\frac{1}{AD} + \frac{1}{BE} + \frac{1}{FC}$ can be written in simplest form as $\frac{a\sqrt{b}}{c}$, find $a + b + c$.

Problema 5.4 Find the remainder when

$$\prod_{i=1}^{1903} (2^i + 5)$$

is divided by 1000.

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Round 6

Problema 6.1 Let a, b, c be non-zero real numbers that satisfy $\frac{1}{abc} + \frac{1}{a} + \frac{1}{c} = \frac{1}{b}$. The expression $\frac{4}{a^2+1} + \frac{4}{b^2+1} + \frac{7}{c^2+1}$ has a maximum value M. Find the sum of the numerator and denominator of the reduced form of M.

Problema 6.2 Michael is trying to drive a bus from his home, $(0, 0)$, to school, located at $(6, 6)$. There are horizontal and vertical roads at every line $x = 0, 1, \ldots, 6$ and $y = 0, 1, \ldots, 6$. The city has placed 6 roadblocks on lattice point intersections (x, y) with $0 \leq x, y \leq 6$. Michael notices that the only path he can take that only goes up and to the right is directly up from $(0, 0)$ to $(0, 6)$, and then right to $(6, 6)$. How many sets of 6 locations could the city have blocked?

Round 7

Problema 7.1 Let the sequence $\{a_n\}_{n=-2}^{\infty}$ satisfy $a_{-1} = a_{-2} = 0, a_0 = 1$, and for all non-negative integers n,

$$n^2 = \sum_{k=0}^{n} a_{n-k}a_{k-1} + \sum_{k=0}^{n} a_{n-k}a_{k-2}$$

Given a_{2018} is rational, find the maximum integer m such that 2^m divides the denominator of the reduced form of a_{2018}.

Problema 7.2 Frankie the Frog starts his morning at the origin in $\mathbb{R}^2$. He decides to go on a leisurely stroll, consisting of $3^1 + 3^{10} + 3^{11} + 3^{100} + 3^{111} + 3^{1000}$ moves, starting with the first move. On the nth move, he hops a distance of

$$\max\{k \in \mathbb{Z} : 3^k|n\} + 1,$$

then turns $90°$ counterclockwise. What is the square of the distance from his final position to the origin?

Problema 7.3 Let $ABCD$ be a parallelogram such that $AB = 35$ and $BC = 28$. Suppose that $BD \perp BC$. Let ℓ_1 be the reflection of AC across the angle bisector of $\angle BAD$, and let ℓ_2 be the line through B perpendicular to CD. ℓ_1 and ℓ_2 intersect at a point P. If PD can be expressed in simplest form as $\frac{m}{n}$, find $m + n$.

Problema 7.4 Find the smallest positive integer G such that there exist distinct positive integers a, b, c with the following properties: • $\gcd(a, b, c) = G$. • $\text{lcm}(a, b) = \text{lcm}(a, c) = \text{lcm}(b, c)$. • $\frac{1}{a} + \frac{1}{b}, \frac{1}{a} + \frac{1}{c}$, and $\frac{1}{b} + \frac{1}{c}$ are reciprocals of integers. • $\gcd(a, b) + \gcd(a, c) + \gcd(b, c) = 16G$.

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Round 8

Problema 8.1

$$\frac{p}{q} = \sum_{n=1}^{\infty} \frac{1}{2^{n+6}} \frac{(10 - 4\cos^2(\frac{\pi n}{24}))(1 - (-1)^n) - 3\cos(\frac{\pi n}{24})(1 + (-1)^n)}{25 - 16\cos^2(\frac{\pi n}{24})}$$

where p and q are relatively prime positive integers. Find $p + q$.

Problema 8.2 Let S_5 be the set of permutations of $\{1, 2, 3, 4, 5\}$, and let C be the convex hull of the set

$$\{(\sigma(1), \sigma(2), \ldots, \sigma(5)) \,|\, \sigma \in S_5\}.$$

Then C is a polyhedron. What is the total number of 2-dimensional faces of C?

Problema 8.3 Let ω be a circle. Let E be on ω and S outside ω such that line segment SE is tangent to ω. Let R be on ω. Let line SR intersect ω at B other than R, such that R is between S and B. Let I be the intersection of the bisector of $\angle ESR$ with the line tangent to ω at R; let A be the intersection of the bisector of $\angle ESR$ with ER. If the radius of the circumcircle of $\triangle EIA$ is 10, the radius of the circumcircle of $\triangle SAB$ is 14, and $SA = 18$, then IA can be expressed in simplest form as $\frac{m}{n}$. Find $m+n$.

Problema 8.4 Let p be a prime. Let $f(x)$ be the number of ordered pairs (a, b) of positive integers less than p, such that $a^b \equiv x \pmod{p}$. Suppose that there do not exist positive integers x and y, both less than p, such that $f(x) = 2f(y)$, and that the maximum value of f is greater than 2018. Find the smallest possible value of p.