# IMO Shortlist

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| | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 09 | 08 | 07 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A1 | | | | | | | | | | | |
| A2 | | | | | | | | | | | |
| A3 | | | | | | | | | | | |
| A4 | | | | | | | | | | | |
| A5 | | | | | | | | | | | |
| A6 | | | | | | | | | | | |
| A7 | | | $\times$ | $\times$ | $\times$ | | | | | | |
| A8 | | | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | | $\times$ | $\times$ | $\times$ |
| A9 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| A10 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| C1 | | | | | | | | | | | |
| C2 | | | | | | | | | | | |
| C3 | | | | | | | | | | | |
| C4 | | | | | | | | | | | |
| C5 | | | | | | | | | | | |
| C6 | | | | | | | | | | | |
| C7 | | | | | | | | | | $\times$ | |
| C8 | | | $\times$ | | | $\times$ | $\times$ | $\times$ | | $\times$ | |
| C9 | $\times$ | $\times$ | $\times$ | | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| C10 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| G1 | | | | | | | | | | | |
| G2 | | | | | | | | | | | |
| G3 | | | | | | | | | | | |
| G4 | | | | | | | | | | | |
| G5 | | | | | | | | | | | |
| G6 | | | | | | | | | | | |
| G7 | | | | | $\times$ | | | | | | |
| G8 | | | | $\times$ | $\times$ | | | $\times$ | | $\times$ | |
| G9 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| G10 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| N1 | | | | | | | | | | | |
| N2 | | | | | | | | | | | |
| N3 | | | | | | | | | | | |
| N4 | | | | | | | | | | | |
| N5 | | | | | | | | | | | |
| N6 | | | | | | | | | | | |
| N7 | | | | | | | | $\times$ | | $\times$ | |
| N8 | | | | | $\times$ | | | $\times$ | $\times$ | $\times$ | $\times$ |
| N9 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| N10 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |

# Álgebra

**Problema 1.** Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

**Problema 2.** Determine todos os inteiros $n \geq 3$ para os quais existem números reais $a_1, a_2, \dots, a_{n+2}$, tais que $a_{n+1} = a_1$, $a_{n+2} = a_2$ e

$$a_i a_{i+1} + 1 = a_{i+2}$$

para $i = 1, 2, \dots, n$.

**Problema 3.** Given any set $S$ of positive integers, show that at least one of the following two assertions is true:

1. there exist distinct finite subsets $F$ and $G$ of $S$ such that $\sum_{x \in F} 1/x = \sum_{x \in G} 1/x$;
2. there exists a positive rational number $r < 1$ such that $\sum_{x \in F} 1/x \neq r$, for all infinite subsets $F$ of $S$.

**Problema 4.** Let $a_0, a_1, a_2, \dots$ be a sequence of real numbers such that $a_0 = 0$, $a_1 = 1$, and for every $n \geq 2$ there exists $1 \leq k \leq n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximal value of $a_{2018} - a_{2017}$.

**Problema 5.** Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

**Problema 6.** Let $m, n \geq 2$ be integers. Let $f(x_1, \dots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \dots, x_n) = \left\lfloor \frac{x_1 + \cdots + x_n}{m} \right\rfloor \quad \text{for every} x_1, \dots, x_n \in \{0, 1, \dots m-1\}.$$

Prove that the total degree of $f$ is at least $n$.

**Problema 7.** Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}}$$

where $a, b, c, d$ are nonnegative real numbers which satisfy $a + b + c + d = 100$

# Combinatória

**Problema 1.** Let $n \geq 3$ be an integer. Prove that there exists a set $S$ of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set $S$ can be partitioned into two subsets with equal sum of elements, with one of the subsets of cardinality $m$.

**Problema 2.** Guilherme and Zeus play a game on a $20 \times 20$ chessboard. In the beginning the board is empty. In every turn, Zeus places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then, Guilherme places a white queen on an empty square. The game gets finished when somebody cannot move.

Find the maximal positive $K$ such that, regardless of the strategy of Guilherme, Zeus can put at least $K$ knights on the board.

**Problema 3.** Seja $n$ um inteiro positivo. Guilherme executa uma sequência de movimentos numa fita que consiste em $n+1$ quadrados enfileirados, numerados de 0 a $n$, da esquerda pra direita. Inicialmente, $n$ perdras são colocadas no quadrado 0, e os outros quadrados ficam vazios. Em cada turno, Guilherme escolhe qualquer quadrado não vazio (com $k$ pedras), tira uma dessas pedras e move ela para a direita no máximo $k$ quadrados (a pedra deve continuar na fita). O objetivo de Guilherme é mover todas as $n$ pedras para o quadrado $n$.

Prove que Guilherme não alcança seu objetivo com menos que

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

movimentos.

**Problema 4.** Um triângulo *anti-Pascal* é uma disposição de números em forma de triângulo equilátero tal que, exceto para os números na última linha, cada número é o módulo da diferença entre os dois números imediatamente abaixo dele. Por exemplo, a seguinte disposição de números é um triângulo anti-Pascal com quatro linhas que contém todos os inteiros de 1 até 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Determine se existe um triângulo anti-Pascal com 2018 linhas que contenha todos os inteiros de 1 até $1+2+\cdots+2018$.

**Problema 5.** Let $k$ be a positive integer. The organising committee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournment, the committee has to pay 1 coin to the hotel. The organizers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

**Problema 6.** Let $a$ and $b$ be distinct positive integers. The following infinite process takes place on an initially empty board.

1. If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.

2. If no such pair exists, we write down two times the number 0.

Prove that, no matter how we make the choices in $(i)$, operation $(ii)$ will be performed only finitely many times.

**Problema 7.** Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coulored twice – once for each of the two circles that cross at that point. If two colourings agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

# Geometria

**Problema 1.** Seja $\Gamma$ o circuncírculo do triângulo acutângulo $ABC$. Os pontos $D$ e $E$ estão sobre os segmentos $AB$ e $AC$, respectivamente, de modo que $AD = AE$. As mediatrizes de $BD$ e $CE$ intersectam os arcos menores $AB$ e $AC$ de $\Gamma$ nos pontos $F$ e $G$, respectivamente. Prove que as retas $DE$ e $FG$ são paralelas (ou são a mesma reta).

**Problema 2.** Seja $ABC$ um triângulo com $AB = AC$, e seja $M$ o ponto médio de $BC$. Seja $P$ um ponto tal que $PB < PC$ e $PA$ paralelo a $BC$. Sejam $X$ e $Y$ pontos nas retas $PB$ e $PC$, respectivamente, tal que $B$ cai no segmento $PX$, $C$ cai no segmento $PY$, e $\angle PXM = \angle PYM$. Prove que o quadrilátero $APXY$ é cíclico.

**Problema 3.** A circle $\omega$ of radius 1 is given. A collection $T$ of triangles is called *good* if the following conditions hold:

1. each triangle from $T$ is inscribled in $\omega$;
2. no two triangles from $T$ have a common interior point.

Determine all the positive real numbers $t$ such that, for each positive integer $n$, there exists a good collection of $n$ triangles, each of perimeter greater than $t$.

**Problema 4.** A point $T$ is chosen inside a triangle $ABC$. Let $A_1$, $B_1$, and $C_1$ be the reflections of $T$ in $BC$, $CA$ and $AB$, respectively. Let $\Omega$ be the circumcircle of $A_1B_1C_1$. The lines $A_1T$, $B_1T$ and $C_1T$ meet $\Omega$ again at $A_2$, $B_2$ and $C_2$, respectively. Prove that the lines $AA_2$, $BB_2$ and $CC_2$ are concurrent on $\Omega$.

**Problema 5.** Let $ABC$ be a triancle with circumcircle $\omega$ and incentre $I$. A line $\ell$ intersects the lines $AI$, $BI$ and $CI$ at points $D$, $E$, and $F$, respectively, distinct from the points $A$, $B$, $C$ and $I$. The perpendicular bissectors of the segments $AD$, $BE$, and $CF$ determine a triangle $\Theta$. Show that the circumcircle of the triangle $\Theta$ is tangent to $\omega$.

**Problema 6.** Um quadrilátero convexo $ABCD$ satisfaz $AB \cdot CD = BC \cdot DA$. O ponto $X$ está no interior de $ABCD$ de modo que

$$\angle XAB = \angle XCD \quad \text{e} \quad \angle XBC = \angle XDA$$

Prove que $\angle BXA + \angle DXC = 180°$.

**Problema 7.** Let $O$ be the circumcentre, and $\Omega$ be the circumcircle of an acute-angled triangle $ABC$. Let $P$ be an arbitrary point on $\Omega$, distict from $A$, $B$, $C$, and their antipodes in $\Omega$. Denote the circumcentres od the triangles $AOP$, $BOP$, and $COP$ by $O_A$, $O_B$, and $O_C$, respectively. The lines $\ell_A$, $\ell_B$, and $\ell_C$ are perpendicular to $BC$, $CA$, and $AB$ pass through $O_A$, $O_B$, and $O_C$, respectively. Prove that the circumcircle of the triangle formed by $\ell_A$, $\ell_B$, and $\ell_C$ is tangent to the line $OP$.

# Teoria dos Números

**Problema 1.** Determine all pairs $(m, n)$ of positive integers for which there exists a positive integer $s$ such that $sm$ and $sn$ have an equal number of divisors.

**Problema 2.** Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

Each number in the table is congruent to 1 modulo $n$.The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to $n$ modulo $n^2$.

Let $R_i$ be the product of the numbers in the $i^{\text{th}}$ row, and $C_j$ be the product of the number in the $j^{\text{th}}$ column. Prove that the sums $R_1 + \dots R_n$ and $C_1 + \dots C_n$ are congruent modulo $n^4$.

**Problema 3.** Define the sequence $a_0, a_1, \dots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

**Problema 4.** Sejam $a_1, a_2, \dots$ uma sequência infinita de inteiros positivos. Suponha que existe um inteiro $N > 1$ tal que, para cada $n \geq N$, o número

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n+1}}{a_n} + \frac{a_n}{a_1}$$

é um inteiro. Prove que existe um inteiro positivo $M$ tal que $am = a_{m+1}$ para todo $m \geq M$.

**Problema 5.** Four positive integers $x$, $y$, $z$ and $t$ satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both $xy$ and $zt$ are perfect squares?

**Problema 6.** Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m+n)|f(m) + f(n)$ for all pairs $m, n$ of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of $f$.

**Problema 7.** Let $n \geq 2018$ be an integer, and let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}$$

forms an arithmetic progression. Prove that the terms of the sequence are equal.

# Álgebra

**Problema 1.** Let $a_1, a_2, \ldots, a_n, k$ and $M$ be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$ , prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

**Problema 2.** Let $q$ be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

- In the first line, Gugu writes down every number of the form $a - b$ , where $a$ and $b$ are two (not necessarily distinct) numbers on his napkin.
- In the second line, Gugu writes down every number of the form $qab$ , where $a$ and $b$ are two (not necessarily distinct) numbers from the first line.
- In the third line, Gugu writes down every number of the form $a^2 + b^2 - c^2 - d^2$ , where $a, b, c, d$ are four (not necessarily distinct) numbers from the first line.

Determine all values of $q$ such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

**Problema 3.** Seja $S$ um conjunto finito, e seja $\mathcal{A}$ o conjunto de todas as funções de $S$ em $S$. Seja $f$ um elemento de $\mathcal{A}$, e seja $T = f(S)$ a imagem de $S$ pela função $f$. Supponha que $f \circ g \circ f \neq g \circ f \circ g$ para todo $g$ em $\mathcal{A}$ com $g \neq f$. Mostre que $f(T) = T$.

**Problema 4.** A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n} (a_i + a_j) \qquad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant $M$ such that $|a_n| \leq M$ for all positive integers $n$.

**Problema 5.** An integer $n \geq 3$ is given. We call an $n$-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \ldots, y_n$ of these numbers, we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \cdots + y_{n-1} y_n \geq -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leq i < j \leq n} x_i x_j \geq K$$

holds for every Shiny $n$-tuple $(x_1, x_2, \ldots, x_n)$.

**Problema 6.** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers $x$ and $y$,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

**Problema 7.** Let $a_0, a_1, a_2, \ldots$ be a sequence of integers and $b_0, b_1, b_2, \ldots$ be a sequence of positive integers such that $a_0 = 0, a_1 = 1$, and

$$a_{n+1} = \begin{cases} a_n b_n + a_{n-1} & \text{if } b_{n-1} = 1 \\ a_n b_n - a_{n-1} & \text{if } b_{n-1} > 1 \end{cases} \qquad \text{for } n = 1, 2, \ldots.$$

for $n = 1, 2, \ldots$. Prove that at least one of the two numbers $a_{2017}$ and $a_{2018}$ must be greater than or equal to 2017.

**Problema 8.** A function $f : \mathbb{R} \to \mathbb{R}$ has the following property:

$$\text{For every } x, y \in \mathbb{R} \text{ such that } (f(x)+y)(f(y)+x) > 0, \text{ we have } f(x) + y = f(y) + x.$$

Prove that $f(x) + y \leq f(y) + x$ whenever $x > y$.

# Combinatória

**Problema 1.** A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

**Problema 2.** Let $n$ be a positive integer. Define a chameleon to be any sequence of $3n$ letters, with exactly $n$ occurrences of each of the letters $a, b$, and $c$. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon $X$, there exists a chameleon $Y$ such that $X$ cannot be changed to $Y$ using fewer than $3n^2/2$ swaps.

**Problema 3.** Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

Choose any number of the form $2^j$, where $j$ is a non-negative integer, and put it into an empty cell. Choose two (not necessarily adjacent) cells with the same number in them; denote that number by $2^j$. Replace the number in one of the cells with $2^{j+1}$ and erase the number in the other cell.

At the end of the game, one cell contains $2^n$, where $n$ is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of $n$.

**Problema 4.** An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following $N$ conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

$\vdots$

($N$) no one stands between the two shortest players.

Show that this is always possible.

**Problema 5.** A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, $A_0$, and the hunter's starting point, $B_0$ are the same. After $n-1$ rounds of the game, the rabbit is at point $A_{n-1}$ and the hunter is at point $B_{n-1}$. In the $n^{\text{th}}$ round of the game, three things occur in order: The rabbit moves invisibly to a point $A_n$ such that the distance between $A_{n-1}$ and $A_n$ is exactly 1. A tracking device reports a point $P_n$ to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between $P_n$ and $A_n$ is at most 1. The hunter moves visibly to a point $B_n$ such that the distance between $B_{n-1}$ and $B_n$ is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after $10^9$ rounds, she can ensure that the distance between her and the rabbit is at most 100?

**Problema 6.** Let $n > 1$ be a given integer. An $n \times n \times n$ cube is composed of $n^3$ unit cubes. Each unit cube is painted with one colour. For each $n \times n \times 1$ box consisting of $n^2$ unit cubes (in any of the three possible orientations), we consider the set of colours present in that box (each colour is listed only once). This way, we get $3n$ sets of colours, split into three groups according to the orientation.

It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of $n$, the maximal possible number of colours that are present.

**Problema 7.** For any finite sets $X$ and $Y$ os positive integers, denote by $f_X(k)$ the $k^{\text{th}}$ smallest positive integer not in $X$, and let

$$X * Y = X \cup \{f_X(y) : y \in Y\}.$$

Let $A$ be a set of $a > 0$ positive integers and let $B$ be a set of $b > 0$ positive integers. Prove that if $A * B = B * A$, then

$$\underbrace{A * (A * \cdots (A * (A * A)) \cdots )}_{\text{A appears } b \text{ times}} = \underbrace{B * (B * \cdots (B * (B * B)) \cdots )}_{\text{B appears } a \text{ times}}.$$

**Problema 8.** Let $n$ be a given positive integer. In the Cartesian plane, each lattice point with nonnegative coordinates initially contains a butterfly, and there are no other butterflies. The neighborhood of a lattice point $c$ consists of all lattice points within the axis-aligned $(2n+1) \times (2n+1)$ square entered at $c$ , apart from $c$ itself. We call a butterfly lonely , crowded , or comfortable , depending on whether the number of butterflies in its neighborhood $N$ is respectively less than, greater than, or equal to half of the number of lattice points in $N$ . Every minute, all lonely butterflies fly away simultaneously. This process goes on for as long as there are any lonely butterflies. Assuming that the process eventually stops, determine the number of comfortable butterflies at the final state.

## Geometria

**Problema 1.** Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from $E$ to $BC$ and the line segments $AC$ and $BD$ are concurrent.

**Problema 2.** Let $R$ and $S$ be different points on a circle $\Omega$ such that $RS$ is not a diameter. Let $\ell$ be the tangent line to $\Omega$ at $R$. Point $T$ is such that $S$ is the midpoint of the line segment $RT$. Point $J$ is chosen on the shorter arc $RS$ of $\Omega$ so that the circumcircle $\Gamma$ of triangle $JST$ intersects $\ell$ at two distinct points. Let $A$ be the common point of $\Gamma$ and $\ell$ that is closer to $R$. Line $AJ$ meets $\Omega$ again at $K$. Prove that the line $KT$ is tangent to $\Gamma$.

**Problema 3.** Let $O$ be the circumcenter of an acute triangle $ABC$. Line $OA$ intersects the altitudes of $ABC$ through $B$ and $C$ at $P$ and $Q$, respectively. The altitudes meet at $H$. Prove that the circumcenter of triangle $PQH$ lies on a median of triangle $ABC$.

**Problema 4.** In triangle $ABC$, let $\omega$ be the excircle opposite to $A$. Let $D, E$ and $F$ be the points where $\omega$ is tangent to $BC, CA$, and $AB$, respectively. The circle $AEF$ intersects line $BC$ at $P$ and $Q$. Let $M$ be the midpoint of $AD$. Prove that the circle $MPQ$ is tangent to $\omega$.

**Problema 5.** Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments $AA_1, BB_1$, and $CC_1$ have the same perpendicular bisector. Let the diagonals $AC_1$ and $A_1C$ meet at $D$, and denote by $\omega$ the circle $ABC$. Let $\omega$ intersect the circle $A_1BC_1$ again at $E \neq B$. Prove that the lines $BB_1$ and $DE$ intersect on $\omega$.

**Problema 6.** Let $n \geq 3$ be an integer. Two regular $n$-gons $\mathcal{A}$ and $\mathcal{B}$ are given in the plane. Prove that the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.

(That is, prove that there exists a line separating those vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary from the other vertices of $\mathcal{A}$.)

**Problema 7.** A convex quadrilateral $ABCD$ has an inscribed circle with center $I$. Let $I_a, I_b, I_c$ and $I_d$ be the incenters of the triangles $DAB, ABC, BCD$ and $CDA$, respectively. Suppose that the common external tangents of the circles $AI_bI_d$ and $CI_bI_d$ meet at $X$, and the common external tangents of the circles $BI_aI_c$ and $DI_aI_c$ meet at $Y$. Prove that $\angle XIY = 90°$.

**Problema 8.** There are 2017 mutually external circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A tangent segment is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn.

Find all possible numbers of tangent segments when Luciano stops drawing.

# Teoria dos Números

**Problema 1.** For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of $a_0$ such that there exists a number $A$ such that $a_n = A$ for infinitely many values of $n$ .

**Problema 2.** Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index $i$ in the set $\{1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element $a_i$ from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1} 10^{p-1} = \sum_{i=0}^{p-1} a_i .10^i$$

.

The goal of Eduardo is to make $M$ divisible by $p$ , and the goal of Fernando is to prevent this.
Prove that Eduardo has a winning strategy.

**Problema 3.** Determiner all integers $n \geq 2$ having the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by $n$ , there exists an index $1 \leq i \leq n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \ldots + a_{i+n-1}$$

is divisible by $n$ . Here, we let $a_i = a_{i-n}$ when $i > n$ .

**Problema 4.** Call a rational number short if it has finitely many digits in its decimal expansion. For a positive integer $m$ , we say that a positive integer $t$ is $m-$ tastic if there exists a number $c \in \{1, 2, 3, \ldots, 2017\}$ such that $\dfrac{10^t - 1}{c \cdot m}$ is short, and such that $\dfrac{10^k - 1}{c \cdot m}$ is not short for any $1 \leq k < t$ . Let $S(m)$ be the the set of $m-$ t astic numbers. Consider $S(m)$ for $m = 1, 2, \ldots$. What is the maximum number of elements in $S(m)$ ?

**Problema 5.** Find all pairs $(p, q)$ of prime numbers which $p > q$ and

$$\frac{(p+q)^{p+q}(p-q)^{p-q} - 1}{(p+q)^{p-q}(p-q)^{p+q} - 1}$$

is an integer.

**Problema 6.** Find the smallest positive integer $n$ or show no such $n$ exists, with the following property: there are infinitely many distinct $n$ - tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \quad \text{and} \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

**Problema 7.** An ordered pair $(x, y)$ of integers is a primitive point if the greatest common divisor of $x$ and $y$ is 1 . Given a finite set $S$ of primitive points, prove that there exist a positive integer $n$ and integers $a_0, a_1, \ldots, a_n$ such that, for each $(x, y)$ in $S$ , we have:

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n = 1.$$

**Problema 8.** Let $p$ be an odd prime number and $\mathbb{Z}_{>0}$ be the set of positive integers. Suppose that a function $f : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0, 1\}$ satisfies the following properties:

- $f(1, 1) = 0$.
- $f(a, b) + f(b, a) = 1$ for any pair of relatively prime positive integers $a, b$ not both equal to 1;
- $f(a + b, b) = f(a, b)$ for any pair of relatively prime positive integers $(a, b)$ .

Prove that

$$\sum_{n=1}^{p-1} f(n^2, p) \geqslant \sqrt{2p} - 2.$$

# Álgebra

**Problema 1.** Let $a$, $b$, $c$ be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

**Problema 2.** Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers $a_1, a_2, a_3, a_4, a_5$ (not necessarily distinct), one can always choose distinct subscripts $i, j, k, l$ such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leq C.$$

**Problema 3.** Find all positive integers $n$ such that the following statement holds: Suppose real numbers $a_1$, $a_2$, $\ldots$, $a_n$, $b_1$, $b_2$, $\ldots$, $b_n$ satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \ldots, n$. Then there exists $\varepsilon_1$, $\varepsilon_2$, $\ldots$, $\varepsilon_n$, each of which is either $-1$ or 1, such that

$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \leq 1.$$

**Problema 4.** Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

**Problema 5.** Consider fractions $\frac{a}{b}$ where $a$ and $b$ are positive integers.

(a) Prove that for every positive integer $n$, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n}+1$.

(b) Show that there are infinitely many positive integers $n$ such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n}$.

**Problema 6.** The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of $k$ for which it is possible to erase exactly $k$ of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

**Problema 7.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and for all $x, y \in \mathbb{R}$,

$$f(x+y)^2 = 2f(x)f(y) + \max\left\{f(x^2+y^2), f(x^2) + f(y^2)\right\}.$$

**Problema 8.** Find the largest real constant $a$ such that for all $n \geq 1$ and for all real numbers $x_0, x_1, ..., x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$ we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geq a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$

## Combinatória

**Problema 1.** The leader of an IMO team chooses positive integers $n$ and $k$ with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an $n$-digit binary string, and the deputy leader writes down all $n$-digit binary strings which differ from the leader's in exactly $k$ positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down $001, 111$ and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of $n$ and $k$) needed to guarantee the correct answer?

**Problema 2.** Find all positive integers $n$ for which all positive divisors of $n$ can be put into the cells of a rectangular table under the following constraints:

- each cell contains a distinct divisor;
- the sums of all rows are equal;
- and the sums of all columns are equal.

**Problema 3.** Let $n$ be a positive integer relatively prime to 6. We paint the vertices of a regular $n$-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

**Problema 4.** Find all integers $n$ for which each cell of $n \times n$ table can be filled with one of the letters $I, M$ and $O$ in such a way that:

- in each row and each column, one third of the entries are $I$, one third are $M$ and one third are $O$ and;
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are $I$, one third are $M$ and one third are $O$.

Note: The rows and columns of an $n \times n$ table are each labelled 1 to $n$ in a natural order. Thus each cell corresponds to a pair of positive integer $(i, j)$ with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of first type consists all cells $(i, j)$ for which $i + j$ is a constant, and the diagonal of this second type consists all cells $(i, j)$ for which $i - j$ is constant.

**Problema 5.** Let $n \geq 3$ be a positive integer. Find the maximum number of diagonals in a regular $n$-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

**Problema 6.** There are $n \geq 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands $X$ and $Y$. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected to a ferry route to exactly one of $X$ and $Y$, a new route between this island and the other of $X$ and $Y$ is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

**Problema 7.** There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n - 1$ times. Every time he claps,each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if $n$ is odd.

(b) Prove that Geoff can never fulfill his wish if $n$ is even.

**Problema 8.** Let $n$ be a positive integer. Determine the smallest positive integer $k$ with the following property: it is possible to mark $k$ cells on a $2n \times 2n$ board so that there exists a unique partition of the board into $1 \times 2$ and $2 \times 1$ dominoes, none of which contain two marked cells.

## Geometria

**Problema 1.** Triangle $BCF$ has a right angle at $B$. Let $A$ be the point on line $CF$ such that $FA = FB$ and $F$ lies between $A$ and $C$. Point $D$ is chosen so that $DA = DC$ and $AC$ is the bisector of $\angle DAB$. Point $E$ is chosen so that $EA = ED$ and $AD$ is the bisector of $\angle EAC$. Let $M$ be the midpoint of $CF$. Let $X$ be the point such that $AMXE$ is a parallelogram. Prove that $BD, FX$ and $ME$ are concurrent.

**Problema 2.** Let $ABC$ be a triangle with circumcircle $\Gamma$ and incenter $I$ and let $M$ be the midpoint of $\overline{BC}$. The points $D$, $E$, $F$ are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects $\Gamma$ at a point $X$ other than $A$. Prove that lines $XD$ and $AM$ meet on $\Gamma$.

**Problema 3.** Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset $S$ of the plane is said to be nice if:

(i) there is a point $T$ in $S$ such that for every point $Q$ in $S$, the segment $TQ$ lies entirely in $S$;

(ii) for any triangle $P_1P_2P_3$, there exists a unique point $A$ in $S$ and a permutation $\sigma$ of the indices $\{1, 2, 3\}$ for which triangles $ABC$ and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets $S$ and $S'$ of the set $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

**Problema 4.** Let $ABC$ be a triangle with $AB = AC \neq BC$ and let $I$ be its incentre. The line $BI$ meets $AC$ at $D$, and the line through $D$ perpendicular to $AC$ meets $AI$ at $E$. Prove that the reflection of $I$ in $AC$ lies on the circumcircle of triangle $BDE$.

**Problema 5.** Let $D$ be the foot of perpendicular from $A$ to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle $ABC$. A circle $\omega$ with centre $S$ passes through $A$ and $D$, and it intersects sides $AB$ and $AC$ at $X$ and $Y$ respectively. Let $P$ be the foot of altitude from $A$ to $BC$, and let $M$ be the midpoint of $BC$. Prove that the circumcentre of triangle $XSY$ is equidistant from $P$ and $M$.

**Problema 6.** Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90°$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet $AC$ at $E$ and $F$ respectively, and meet each other at point $P$. Let $M$ be the midpoint of $AC$ and let $\omega$ be the circumcircle of triangle $BPD$. Segments $BM$ and $DM$ intersect $\omega$ again at $X$ and $Y$ respectively. Denote by $Q$ the intersection point of lines $XE$ and $YF$. Prove that $PQ \perp AC$.

**Problema 7.** Let $I$ be the incentre of a non-equilateral triangle $ABC$, $I_A$ be the $A$-excentre, $I'_A$ be the reflection of $I_A$ in $BC$, and $l_A$ be the reflection of line $AI'_A$ in $AI$. Define points $I_B$, $I'_B$ and line $l_B$ analogously. Let $P$ be the intersection point of $l_A$ and $l_B$.

(a) Prove that $P$ lies on line $OI$ where $O$ is the circumcentre of triangle $ABC$.

(b) Let one of the tangents from $P$ to the incircle of triangle $ABC$ meet the circumcircle at points $X$ and $Y$. Show that $\angle XIY = 120°$.

**Problema 8.** Let $A_1, B_1$ and $C_1$ be points on sides $BC$, $CA$ and $AB$ of an acute triangle $ABC$ respectively, such that $AA_1$, $BB_1$ and $CC_1$ are the internal angle bisectors of triangle $ABC$. Let $I$ be the incentre of triangle $ABC$, and $H$ be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geq AI + BI + CI.$$

# Teoria dos Números

**Problema 1.** For any positive integer $k$, denote the sum of digits of $k$ in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

**Problema 2.** Let $\tau(n)$ be the number of positive divisors of $n$. Let $\tau_1(n)$ be the number of positive divisors of $n$ which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

**Problema 3.** A set of postive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of $b$ such that there exists a non-negative integer $a$ for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

**Problema 4.** Let $n, m, k$ and $l$ be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that $m = 1$ and $l = 2k$ ; or $l|k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

**Problema 5.** Let $a$ be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let $A$ be the set of positive integers $k$ for which the equation admits a solution in $\mathbb{Z}^2$ with $x > \sqrt{a}$, and let $B$ be the set of positive integers for which the equation admits a solution in $\mathbb{Z}^2$ with $0 \leq x < \sqrt{a}$. Show that $A = B$.

**Problema 6.** Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers $m$ and $n$, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

**Problema 7.** Let $P = A_1A_2 \cdots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let $S$ be the area of $P$. An odd positive integer $n$ is given such that the squares of the side lengths of $P$ are integers divisible by $n$. Prove that $2S$ is an integer divisible by $n$.

**Problema 8.** Find all polynomials $P(x)$ of odd degree $d$ and with integer coefficients satisfying the following property: for each positive integer $n$, there exists $n$ positive integers $x_1, x_2, \ldots, x_n$ such that $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$ and $\frac{P(x_i)}{P(x_j)}$ is the $d$-th power of a rational number for every pair of indices $i$ and $j$ with $1 \leq i, j \leq n$.

# Álgebra

**Problema 1.** Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer $k$. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

**Problema 2.** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

**Problema 3.** Let $n$ be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leq r < s \leq 2n} (s - r - n)x_r x_s,$$

where $-1 \leq x_i \leq 1$ for all $i = 1, \cdots, 2n$.

**Problema 4.** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers $x$ and $y$.

**Problema 5.** Let $2\mathbb{Z} + 1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \mapsto 2\mathbb{Z} + 1$ satisfying

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x + y) + f(x - y)$$

for every $x, y \in \mathbb{Z}$.

**Problema 6.** Let $n$ be a fixed integer with $n \geq 2$. We say that two polynomials $P$ and $Q$ with real coefficients are block-similar if for each $i \in \{1, 2, \ldots, n\}$ the sequences

$$\begin{aligned} &P(2015i), P(2015i - 1), \ldots, P(2015i - 2014) \quad \text{and} \\ &Q(2015i), Q(2015i - 1), \ldots, Q(2015i - 2014) \end{aligned}$$

are permutations of each other.

(a) Prove that there exist distinct block-similar polynomials of degree $n + 1$.

(b) Prove that there do not exist distinct block-similar polynomials of degree $n$.

# Combinatória

**Problema 1.** In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let $A$ and $B$ be two towns, with $B$ to the right of $A$. We say that town $A$ can sweep town $B$ away if the right bulldozer of $A$ can move over to $B$ pushing off all bulldozers it meets. Similarly town $B$ can sweep town $A$ away if the left bulldozer of $B$ can move over to $A$ pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

**Problema 2.** We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points $A$ and $B$ in $\mathcal{S}$, there is a point $C$ in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points $A$, $B$ and $C$ in $\mathcal{S}$, there is no points $P$ in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of $n$ points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of $n$ points.

**Problema 3.** For a finite set $A$ of positive integers, a partition of $A$ into two disjoint nonempty subsets $A_1$ and $A_2$ is *good* if the least common multiple of the elements in $A_1$ is equal to the greatest common divisor of the elements in $A_2$. Determine the minimum value of $n$ such that there exists a set of $n$ positive integers with exactly 2015 good partitions.

**Problema 4.** Let $n$ be a positive integer. Two players $A$ and $B$ play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

- A player cannot choose a number that has been chosen by either player on any previous turn.
- A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
- The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player $A$ takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

**Problema 5.** The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

- $1 \leq a_j \leq 2015$ for all $j \geq 1$,
- $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

Prove that there exist two positive integers $b$ and $N$ for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leq 1007^2$$

for all integers $m$ and $n$ such that $n > m \geq N$.

**Problema 6.** Let $S$ be a nonempty set of positive integers. We say that a positive integer $n$ is clean if it has a unique representation as a sum of an odd number of distinct elements from $S$. Prove that there exist infinitely many positive integers that are not clean.

**Problema 7.** In a company of people some pairs are enemies. A group of people is called unsociable if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

## Geometria

**Problema 1.** Let $ABC$ be an acute triangle with orthocenter $H$. Let $G$ be the point such that the quadrilateral $ABGH$ is a parallelogram. Let $I$ be the point on the line $GH$ such that $AC$ bisects $HI$. Suppose that the line $AC$ intersects the circumcircle of the triangle $GCI$ at $C$ and $J$. Prove that $IJ = AH$.

**Problema 2.** Triangle $ABC$ has circumcircle $\Omega$ and circumcenter $O$. A circle $\Gamma$ with center $A$ intersects the segment $BC$ at points $D$ and $E$, such that $B$, $D$, $E$, and $C$ are all different and lie on line $BC$ in this order. Let $F$ and $G$ be the points of intersection of $\Gamma$ and $\Omega$, such that $A$, $F$, $B$, $C$, and $G$ lie on $\Omega$ in this order. Let $K$ be the second point of intersection of the circumcircle of triangle $BDF$ and the segment $AB$. Let $L$ be the second point of intersection of the circumcircle of triangle $CGE$ and the segment $CA$.

Suppose that the lines $FK$ and $GL$ are different and intersect at the point $X$. Prove that $X$ lies on the line $AO$.

**Problema 3.** Let $ABC$ be a triangle with $\angle C = 90°$, and let $H$ be the foot of the altitude from $C$. A point $D$ is chosen inside the triangle $CBH$ so that $CH$ bisects $AD$. Let $P$ be the intersection point of the lines $BD$ and $CH$. Let $\omega$ be the semicircle with diameter $BD$ that meets the segment $CB$ at an interior point. A line through $P$ is tangent to $\omega$ at $Q$. Prove that the lines $CQ$ and $AD$ meet on $\omega$.

**Problema 4.** Let $ABC$ be an acute triangle and let $M$ be the midpoint of $AC$. A circle $\omega$ passing through $B$ and $M$ meets the sides $AB$ and $BC$ at points $P$ and $Q$ respectively. Let $T$ be the point such that $BPTQ$ is a parallelogram. Suppose that $T$ lies on the circumcircle of $ABC$. Determine all possible values of $\frac{BT}{BM}$.

**Problema 5.** Let $ABC$ be a triangle with $CA \neq CB$. Let $D$, $F$, and $G$ be the midpoints of the sides $AB$, $AC$, and $BC$ respectively. A circle $\Gamma$ passing through $C$ and tangent to $AB$ at $D$ meets the segments $AF$ and $BG$ at $H$ and $I$, respectively. The points $H'$ and $I'$ are symmetric to $H$ and $I$ about $F$ and $G$, respectively. The line $H'I'$ meets $CD$ and $FG$ at $Q$ and $M$, respectively. The line $CM$ meets $\Gamma$ again at $P$. Prove that $CQ = QP$.

**Problema 6.** Let $ABC$ be an acute triangle with $AB > AC$. Let $\Gamma$ be its cirumcircle, $H$ its orthocenter, and $F$ the foot of the altitude from $A$. Let $M$ be the midpoint of $BC$. Let $Q$ be the point on $\Gamma$ such that $\angle HQA = 90°$ and let $K$ be the point on $\Gamma$ such that $\angle HKQ = 90°$. Assume that the points $A$, $B$, $C$, $K$ and $Q$ are all different and lie on $\Gamma$ in this order.

Prove that the circumcircles of triangles $KQH$ and $FKM$ are tangent to each other.

**Problema 7.** Let $ABCD$ be a convex quadrilateral, and let $P$, $Q$, $R$, and $S$ be points on the sides $AB$, $BC$, $CD$, and $DA$, respectively. Let the line segment $PR$ and $QS$ meet at $O$. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines $AC$,

$PQ$, and $RS$ are either concurrent or parallel to each other.

**Problema 8.** A triangulation of a convex polygon $\Pi$ is a partitioning of $\Pi$ into triangles by diagonals having no common points other than the vertices of the polygon. We say that a triangulation is a Thaiangulation if all triangles in it have the same area.

Prove that any two different Thaiangulations of a convex polygon $\Pi$ differ by exactly two triangles. (In other words, prove that it is possible to replace one pair of triangles in the first Thaiangulation with a different pair of triangles so as to obtain the second Thaiangulation.)

# Teoria dos Números

**Problema 1.** Determine all positive integers $M$ such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

**Problema 2.** Let $a$ and $b$ be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

**Problema 3.** Let $m$ and $n$ be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

**Problema 4.** Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence $a_n$ is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

**Problema 5.** Find all postive integers $(a, b, c)$ such that

$$ab - c, \quad bc - a, \quad ca - b$$

are all powers of 2.

**Problema 6.** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that $f$ has the following two properties:

(i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m) - m}{n} \in \mathbb{Z}_{>0}$;

(ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

**Problema 7.** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer $k$, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called $k$-good if $\gcd(f(m) + n, f(n) + m) \leq k$ for all $m \neq n$. Find all $k$ such that there exists a $k$-good function.

**Problema 8.** For every positive integer $n$ with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, define

$$\mho(n) = \sum_{i:\, p_i > 10^{100}} \alpha_i.$$

That is, $\mho(n)$ is the number of prime factors of $n$ greater than $10^{100}$, counted with multiplicity.

Find all strictly increasing functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$\mho(f(a) - f(b)) \leq \mho(a - b) \quad \text{for all integers } a \text{ and } b \text{ with } a > b.$$

# Álgebra

**Problema 1.** Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

**Problema 2.** Define the function $f : (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2} \\ x^2 & \text{if } x \geq \frac{1}{2} \end{cases}$$

Let $a$ and $b$ be two real numbers such that $0 < a < b < 1$. We define the sequences $a_n$ and $b_n$ by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$,

$b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer $n$ such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

**Problema 3.** For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \leq i \leq n} |x_1 + \cdots + x_i|.$$

Given $n$ real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price $D$. Greedy George, on the other hand, chooses $x_1$ such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses $x_2$ such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the $i$-th step he chooses $x_i$ among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price $G$.

Find the least possible constant $c$ such that for every positive integer $n$, for every collection of $n$ real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leq cD$.

**Problema 4.** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m) + n\big) + f(m) = f(n) + f(3m) + 2014$$

for all integers $m$ and $n$.

**Problema 5.** Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers $x$ and $y$ one has

$$|y^2 - P(x)| \leq 2|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leq 2|y|.$$

Determine all possible values of $P(0)$.

**Problema 6.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2$$

for all $n \in \mathbb{Z}$.

# Combinatória

**Problema 1.** Let $n$ points be given inside a rectangle $R$ such that no two of them lie on a line parallel to one of the sides of $R$. The rectangle $R$ is to be dissected into smaller rectangles with sides parallel to the sides of $R$ in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect $R$ into at least $n+1$ smaller rectangles.

**Problema 2.** We have $2^m$ sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are $a$ and $b$, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least $4^m$.

**Problema 3.** Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of $n^2$ unit squares. A configuration of $n$ rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer $k$ such that, for each peaceful configuration of $n$ rooks, there is a $k \times k$ square which does not contain a rook on any of its $k^2$ unit squares.

**Problema 4.** Construct a tetromino by attaching two $2 \times 1$ dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them $S$- and $Z$-tetrominoes, respectively.

Assume that a lattice polygon $P$ can be tiled with $S$-tetrominoes. Prove that no matter how we tile $P$ using only $S$- and $Z$-tetrominoes, we always use an even number of $Z$-tetrominoes.

**Problema 5.** A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions . Prove that for all sufficiently large $n$, in any set of $n$ lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant $c$.

**Problema 6.** We are given an infinite deck of cards, each with a real number on it. For every real number $x$, there is exactly one card in the deck that has $x$ written on it. Now two players draw disjoint sets $A$ and $B$ of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:

- The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.
- If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all $i$, then $A$ beats $B$.
- If three players draw three disjoint sets $A, B, C$ from the deck, $A$ beats $B$ and $B$ beats $C$ then $A$ also beats $C$.

How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets $A$ and $B$ such that $A$ beats $B$ according to one rule, but $B$ beats $A$ according to the other.

**Problema 7.** Let $M$ be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with $n$ segments so that each point in $M$ is the endpoint of exactly two segments. Then, at each step, one may choose two segments $AB$ and $CD$ sharing a common interior point and replace them by the segments $AC$ and $BD$ if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

**Problema 8.** A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine all possible first moves of the first player after which he has a winning strategy.

**Problema 9.** There are $n$ circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that $n$ must be odd.

# Geometria

**Problema 1.** Let $P$ and $Q$ be on segment $BC$ of an acute triangle $ABC$ such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let $M$ and $N$ be the points on $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$ and $Q$ is the midpoint of $AN$. Prove that the intersection of $BM$ and $CN$ is on the circumference of triangle $ABC$.

**Problema 2.** Let $ABC$ be a triangle. The points $K, L$, and $M$ lie on the segments $BC, CA$, and $AB$, respectively, such that the lines $AK, BL$, and $CM$ intersect in a common point. Prove that it is possible to choose two of the triangles $ALM, BMK$, and $CKL$ whose inradii sum up to at least the inradius of the triangle $ABC$.

**Problema 3.** Let $\Omega$ and $O$ be the circumcircle and the circumcentre of an acute-angled triangle $ABC$ with $AB > BC$. The angle bisector of $\angle ABC$ intersects $\Omega$ at $M \neq B$. Let $\Gamma$ be the circle with diameter $BM$. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect $\Gamma$ at points $P$ and $Q$, respectively. The point $R$ is chosen on the line $PQ$ so that $BR = MR$. Prove that $BR \parallel AC$.

(Here we always assume that an angle bisector is a ray.)

**Problema 4.** Consider a fixed circle $\Gamma$ with three fixed points $A, B$, and $C$ on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on $\Gamma$, let $M$ be the point on the segment $CP$ such that $CM = \lambda \cdot CP$ . Let $Q$ be the second point of intersection of the circumcircles of the triangles $AMP$ and $BMC$. Prove that as $P$ varies, the point $Q$ lies on a fixed circle.

**Problema 5.** Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point $H$ is the foot of the perpendicular from $A$ to $BD$. Points $S$ and $T$ lie on sides $AB$ and $AD$, respectively, such that $H$ lies inside triangle $SCT$ and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line $BD$ is tangent to the circumcircle of triangle $TSH$.

**Problema 6.** Let $ABC$ be a fixed acute-angled triangle. Consider some points $E$ and $F$ lying on the sides $AC$ and $AB$, respectively, and let $M$ be the midpoint of $EF$ . Let the perpendicular bisector of $EF$ intersect the line $BC$ at $K$, and let the perpendicular bisector of $MK$ intersect the lines $AC$ and $AB$ at $S$ and $T$ , respectively. We call the pair $(E, F)$ *interesting*, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs $(E_1, F_1)$ and $(E_2, F_2)$ are interesting. Prove that $\dfrac{E_1E_2}{AB} = \dfrac{F_1F_2}{AC}$.

**Problema 7.** Let $ABC$ be a triangle with circumcircle $\Omega$ and incentre $I$. Let the line passing through $I$ and perpendicular to $CI$ intersect the segment $BC$ and the arc $BC$ (not containing $A$) of $\Omega$ at points $U$ and $V$, respectively. Let the line passing through $U$ and parallel to $AI$ intersect $AV$ at $X$, and let the line passing through $V$ and parallel to $AI$ intersect $AB$ at $Y$. Let $W$ and $Z$ be the midpoints of $AX$ and $BC$, respectively. Prove that if the points $I$, $X$, and $Y$ are collinear, then the points $I$, $W$, and $Z$ are also collinear.

## Teoria dos Números

**Problema 1.** Let $n \geq 2$ be an integer, and let $A_n$ be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of $A_n$.

**Problema 2.** Determine all pairs $(x, y)$ of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

**Problema 3.** Para cada inteiro positivo $n$, o Banco de Cabo Verde produz moedas com valor $\frac{1}{n}$. Dada uma coleção finita de tais moedas (com valores não necessariamente distintos) com valor total de até $99 + \frac{1}{2}$, prove que é possível dividir essa coleção em 100 ou menos grupos, tal que cada grupo contém valor total menor ou igual a 1.

**Problema 4.** Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number $x$, $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$.)

**Problema 5.** Find all triples $(p, x, y)$ consisting of a prime number $p$ and two positive integers $x$ and $y$ such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of $p$.

**Problema 6.** Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with $a_1$ being prime and $a_1 \geq n + 2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$ . These points split $I$ into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by $a_1$.

**Problema 7.** Let $c \geq 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \geq 1$. Prove that for each integer $n \geq 2$ there exists a prime number $p$ dividing $a_n$ but none of the numbers $a_1, \ldots, a_{n-1}$.

**Problema 8.** For every real number $x$, let $||x||$ denote the distance between $x$ and the nearest integer.

Prove that for every pair $(a, b)$ of positive integers there exist an odd prime $p$ and a positive integer $k$ satisfying

$$\left|\left|\frac{a}{p^k}\right|\right| + \left|\left|\frac{b}{p^k}\right|\right| + \left|\left|\frac{a+b}{p^k}\right|\right| = 1.$$

# Álgebra

**Problema 1.** Let $n$ be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and $u_{k+1} = u_k + a_k u_{k-1}$, $v_{k+1} = v_k + a_{n-k} v_{k-1}$ for $k = 1, \ldots, n-1$.

Prove that $u_n = v_n$.

**Problema 2.** Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

**Problema 3.** Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;

(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

**Problema 4.** Let $n$ be a positive integer, and consider a sequence $a_1, a_2, \cdots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \cdots$ by defining $a_{n+i} = a_i$ for all $i \geq 1$. If

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

and

$$a_{a_i} \leq n + i - 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

prove that

$$a_1 + \cdots + a_n \leq n^2.$$

**Problema 5.** Let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\geq 0}$.

**Problema 6.** Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) = 2(x^3 - mx + 1)P(x)$$

for all real number $x$.

# Combinatória

**Problema 1.** Let $n$ be an positive integer. Find the smallest integer $k$ with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \le a_i \le 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into $k$ groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

**Problema 2.** A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

(a) No line passes through any point of the configuration.

(b) No region contains points of both colors.

Find the least value of $k$ such that for any Colombian configuration of 4027 points, there is a good arrangement of $k$ lines.

**Problema 3.** A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in the lab by creating a copy $I'$ of each imon $I$. During this procedure, the two copies $I'$ and $J'$ become entangled if and only if the original imons $I$ and $J$ are entangled, and each copy $I'$ becomes entangled with its original imon $I$; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

**Problema 4.** Let $n$ be a positive integer, and let $A$ be a subset of $\{1, \cdots, n\}$. An $A$-partition of $n$ into $k$ parts is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \cdots, a_k$ belong to $A$ and are not necessarily distinct. The number of different parts in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \cdots, a_k\}$.

We say that an $A$-partition of $n$ into $k$ parts is optimal if there is no $A$-partition of $n$ into $r$ parts with $r < k$. Prove that any optimal $A$-partition of $n$ contains at most $\sqrt[3]{6n}$ different parts.

**Problema 5.** Let $r$ be a positive integer, and let $a_0, a_1, \cdots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers $m$ and $s$ there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}$$

Prove that the sequence is periodic, i.e. there exists some $p \ge 1$ such that $a_{n+p} = a_n$ for all $n \ge 0$.

**Problema 6.** In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The distance between two cities is defined to be the least possible numbers of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

**Problema 7.** Let $n \ge 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, ..., n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called beautiful if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled $a$ and $d$ does not intersect the chord joining the points labelled $b$ and $c$.

Let $M$ be the number of beautiful labelings, and let N be the number of ordered pairs $(x, y)$ of positive integers such that $x + y \le n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

**Problema 8.** Players $A$ and $B$ play a "paintful"game on the real line. Player $A$ has a pot of paint with four units of black ink. A quantity $p$ of this ink suffices to blacken a (closed) real interval of length $p$. In every round, player $A$ picks some positive integer $m$ and provides $1/2^m$ units of ink from the pot. Player $B$ then picks an integer $k$ and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player $A$ is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened.

Decide whether there exists a strategy for player $A$ to win in a finite number of moves.

# Geometria

**Problema 1.** Let $ABC$ be an acute triangle with orthocenter $H$, and let $W$ be a point on the side $BC$, lying strictly between $B$ and $C$. The points $M$ and $N$ are the feet of the altitudes from $B$ and $C$, respectively. Denote by $\omega_1$ is the circumcircle of $BWN$, and let $X$ be the point on $\omega_1$ such that $WX$ is a diameter of $\omega_1$. Analogously, denote by $\omega_2$ the circumcircle of triangle $CWM$, and let $Y$ be the point such that $WY$ is a diameter of $\omega_2$. Prove that $X, Y$ and $H$ are collinear.

**Problema 2.** Let $\omega$ be the circumcircle of a triangle $ABC$. Denote by $M$ and $N$ the midpoints of the sides $AB$ and $AC$, respectively, and denote by $T$ the midpoint of the arc $BC$ of $\omega$ not containing $A$. The circumcircles of the triangles $AMT$ and $ANT$ intersect the perpendicular bisectors of $AC$ and $AB$ at points $X$ and $Y$, respectively; assume that $X$ and $Y$ lie inside the triangle $ABC$. The lines $MN$ and $XY$ intersect at $K$. Prove that $KA = KT$.

**Problema 3.** In a triangle $ABC$, let $D$ and $E$ be the feet of the angle bisectors of angles $A$ and $B$, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$) . Let $\varphi$ be the non-obtuse angle of the rhombus. Prove that $\varphi \leq \max\{\angle BAC, \angle ABC\}$.

**Problema 4.** Let $ABC$ be a triangle with $\angle B > \angle C$. Let $P$ and $Q$ be two different points on line $AC$ such that $\angle PBA = \angle QBA = \angle ACB$ and $A$ is located between $P$ and $C$. Suppose that there exists an interior point $D$ of segment $BQ$ for which $PD = PB$. Let the ray $AD$ intersect the circle $ABC$ at $R \neq A$. Prove that $QB = QR$.

**Problema 5.** Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals $AD$,
$BE$, and $CF$ are concurrent.

**Problema 6.** Let the excircle of triangle $ABC$ opposite the vertex $A$ be tangent to the side $BC$ at the point $A_1$. Define the points $B_1$ on $CA$ and $C_1$ on $AB$ analogously, using the excircles opposite $B$ and $C$, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle $ABC$. Prove that triangle $ABC$ is right-angled.

# Teoria dos Números

**Problema 1.** Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers $m$ and $n$.

**Problema 2.** Assume that $k$ and $n$ are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

**Problema 3.** Prove that there exist infinitely many positive integers $n$ such that the largest prime divisor of $n^4+n^2+1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

**Problema 4.** Existe um inteiro positivo $N$ e uma sequência infinita de dígitos $a_1, a_2, \ldots$, todos não-nulos, tais que, para todo $k > N$, o número cuja representação decimal é $(a_k a_{k-1} \ldots a_1)$ é um quadrado perfeito?

**Problema 5.** Fix an integer $k > 2$. Two players, called Ana and Banana, play the following game of numbers. Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number $m$ just written on the blackboard and replaces it by some number $m'$ with $k \leq m' < m$ that is coprime to $m$. The first player who cannot move anymore loses.

An integer $n \geq k$ is called *good* if Banana has a winning strategy when the initial number is $n$, and *bad* otherwise.

Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides $n$ if and only if it divides $n'$. Prove that either both $n$ and $n'$ are good or both are bad.

**Problema 6.** Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying

$$f\left(\frac{f(x) + a}{b}\right) = f\left(\frac{x + a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$.

**Problema 7.** Let $\nu$ be an irrational positive number, and let $m$ be a positive integer. A pair of $(a, b)$ of positive integers is called good if

$$a \lceil b\nu \rceil - b \lfloor a\nu \rfloor = m.$$

A good pair $(a, b)$ is called excellent if neither of the pair $(a - b, b)$ and $(a, b - a)$ is good.

Prove that the number of excellent pairs is equal to the sum of the positive divisors of $m$.

# Álgebra

**Problema 1.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers $a, b, c$ that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

**Problema 2.** Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively.

(a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets $A, B, C$ such that the sets $A + B, B + C, C + A$ are disjoint?

(b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets $A, B, C$ such that the sets $A + B, B + C, C + A$ are disjoint?

Here, $X + Y$ denotes the set $\{x + y : x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and for $X, Y \subseteq \mathbb{Q}$.

**Problema 3.** Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

**Problema 4.** Let $f$ and $g$ be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes $p$ the polynomial $pf + g$ has a rational root. Prove that $f$ has a rational root.

**Problema 5.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$

and $f(-1) \neq 0$.

**Problema 6.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let $f^m$ be $f$ applied $m$ times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let $k_n$ be the smallest such $k$. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

**Problema 7.** We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a metapolynomial if, for some positive integers $m$ and $n$, it can be represented in the form

$$f(x_1, \cdots, x_k) = \max_{i=1,\cdots,m} \min_{j=1,\cdots,n} P_{i,j}(x_1, \cdots, x_k),$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

# Combinatória

**Problema 1.** Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers $x$ and $y$ such that $x > y$ and $x$ is to the left of $y$, and replaces the pair $(x, y)$ by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

**Problema 2.** Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding $n$?

**Problema 3.** In a $999 \times 999$ square table some cells are white and the remaining ones are red. Let $T$ be the number of triples $(C_1, C_2, C_3)$ of cells, the first two in the same row and the last two in the same column, with $C_1, C_3$ white and $C_2$ red. Find the maximum value $T$ can attain.

**Problema 4.** Guilherme and Zeus play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially Guilherme distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order Zeus, Guilherme, Zeus, Guilherme, ... by the following rules:

(a) On every of their moves, Zeus passes 1 coin from every box to an adjacent box.

(b) On every of their moves, Guilherme chooses several coins that were not involved in Zeus's previous move and are in different boxes. She passes every coin to and adjacent box.

Guilherme's goal is to ensure at least 1 coin in each box after every move of them, regardless of how Zeus plays and how many moves are made. Find the least $N$ that enables Guilherme to succeed.

**Problema 5.** The columns and the row of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square $(x, y)$ with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is $0, 1$ or $2$ respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most $d$ from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

**Problema 6.** The liar's guessing game is a game played between two players $A$ and $B$. The rules of the game depend on two positive integers $k$ and $n$ which are known to both players.

At the start of the game $A$ chooses integers $x$ and $N$ with $1 \leq x \leq N$. Player $A$ keeps $x$ secret, and truthfully tells $N$ to player $B$. Player $B$ now tries to obtain information about $x$ by asking player $A$ questions as follows: each question consists of $B$ specifying an arbitrary set $S$ of positive integers (possibly one specified in some previous question), and asking $A$ whether $x$ belongs to $S$. Player $B$ may ask as many questions as he wishes. After each question, player $A$ must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k + 1$ consecutive answers, at least one answer must be truthful.

After $B$ has asked as many questions as he wants, he must specify a set $X$ of at most $n$ positive integers. If $x$ belongs to $X$, then $B$ wins; otherwise, he loses. Prove that:

(a) If $n \geq 2^k$, then $B$ can guarantee a win.

(b) For all sufficiently large $k$, there exists an integer $n \geq (1.99)^k$ such that $B$ cannot guarantee a win.

**Problema 7.** There are given $2^{500}$ points on a circle labeled $1, 2, \ldots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of theses points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chord are equal.

# Geometria

**Problema 1.** Given triangle $ABC$ the point $J$ is the centre of the excircle opposite the vertex $A$. This excircle is tangent to the side $BC$ at $M$, and to the lines $AB$ and $AC$ at $K$ and $L$, respectively. The lines $LM$ and $BJ$ meet at $F$, and the lines $KM$ and $CJ$ meet at $G$. Let $S$ be the point of intersection of the lines $AF$ and $BC$, and let $T$ be the point of intersection of the lines $AG$ and $BC$. Prove that $M$ is the midpoint of $ST$.

**Problema 2.** Let $ABCD$ be a cyclic quadrilateral whose diagonals $AC$ and $BD$ meet at $E$. The extensions of the sides $AD$ and $BC$ beyond $A$ and $B$ meet at $F$. Let $G$ be the point such that $ECGD$ is a parallelogram, and let $H$ be the image of $E$ under reflection in $AD$. Prove that $D, H, F, G$ are concyclic.

**Problema 3.** In an acute triangle $ABC$ the points $D, E$ and $F$ are the feet of the altitudes through $A, B$ and $C$ respectively. The incenters of the triangles $AEF$ and $BDF$ are $I_1$ and $I_2$ respectively; the circumcenters of the triangles $ACI_1$ and $BCI_2$ are $O_1$ and $O_2$ respectively. Prove that $I_1I_2$ and $O_1O_2$ are parallel.

**Problema 4.** Let $ABC$ be a triangle with $AB \neq AC$ and circumcenter $O$. The bisector of $\angle BAC$ intersects $BC$ at $D$. Let $E$ be the reflection of $D$ with respect to the midpoint of $BC$. The lines through $D$ and $E$ perpendicular to $BC$ intersect the lines $AO$ and $AD$ at $X$ and $Y$ respectively. Prove that the quadrilateral $BXCY$ is cyclic.

**Problema 5.** Let $ABC$ be a triangle with $\angle BCA = 90°$, and let $D$ be the foot of the altitude from $C$. Let $X$ be a point in the interior of the segment $CD$. Let $K$ be the point on the segment $AX$ such that $BK = BC$. Similarly, let $L$ be the point on the segment $BX$ such that $AL = AC$. Let $M$ be the point of intersection of $AL$ and $BK$.

Show that $MK = ML$.

**Problema 6.** Let $ABC$ be a triangle with circumcenter $O$ and incenter $I$. The points $D, E$ and $F$ on the sides $BC, CA$ and $AB$ respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles $BFD$ and $CDE$ intersect at $P \neq D$.

Prove that $OP = OI$.

**Problema 7.** Let $ABCD$ be a convex quadrilateral with non-parallel sides $BC$ and $AD$. Assume that there is a point $E$ on the side $BC$ such that the quadrilaterals $ABED$ and $AECD$ are circumscribed.

Prove that there is a point $F$ on the side $AD$ such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if $AB$ is parallel to $CD$.

**Problema 8.** Let $ABC$ be a triangle with circumcircle $\omega$ and $\ell$ a line without common points with $\omega$. Denote by $P$ the foot of the perpendicular from the center of $\omega$ to $\ell$. The side-lines $BC, CA, AB$ intersect $\ell$ at the points $X, Y, Z$ different from $P$.

Prove that the circumcircles of the triangles $AXP$, $BYP$ and $CZP$ have a common point different from $P$ or are mutually tangent at $P$.

## Teoria dos Números

**Problema 1.** Call admissible a set $A$ of integers that has the following property:

If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer $k$.

Determine all pairs $m, n$ of nonzero integers such that the only admissible set containing both $m$ and $n$ is the set of all integers.

**Problema 2.** Find all triples $(x, y, z)$ of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

**Problema 3.** Determine all integers $m \geq 2$ such that every $n$ with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

**Problema 4.** An integer $a$ is called friendly if the equation $(m^2 + n)(n^2 + m) = a(m - n)^3$ has a solution over the positive integers.

(a) Prove that there are at least 500 friendly integers in the set $\{1, 2, \ldots, 2012\}$.

(b) Decide whether $a = 2$ is friendly.

**Problema 5.** For a nonnegative integer $n$ define $\operatorname{rad}(n) = 1$ if $n = 0$ or $n = 1$, and $\operatorname{rad}(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of $n$. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $\operatorname{rad}(f(n))$ divides $\operatorname{rad}(f(n^{\operatorname{rad}(n)}))$ for every nonnegative integer $n$.

**Problema 6.** Let $x$ and $y$ be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer $n$, prove that $x = 1$.

**Problema 7.** Find all positive integers $n$ for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

**Problema 8.** Prove that for every prime $p > 100$ and every integer $r$, there exist two integers $a$ and $b$ such that $p$ divides $a^2 + b^5 - r$.

# Álgebra

**Problema 1.** Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by $s_A$. Let $n_A$ denote the number of pairs $(i, j)$ with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides $s_A$. Find all sets $A$ of four distinct positive integers which achieve the largest possible value of $n_A$.

**Problema 2.** Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers, such that for every positive integer $n$ there exists an integer $a$ with

$$\sum_{j=1}^{2011} j x_j^n = a^{n+1} + 1$$

**Problema 3.** Determine all pairs $(f, g)$ of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers $x$ and $y$.

**Problema 4.** Determine all pairs $(f, g)$ of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer $n$. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

**Problema 5.** Prove that for every positive integer $n$, the set $\{2, 3, 4, \ldots, 3n+1\}$ can be partitioned into $n$ triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

**Problema 6.** Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq y f(x) + f(f(x))$$

for all real numbers $x$ and $y$. Prove that $f(x) = 0$ for all $x \leq 0$.

**Problema 7.** Let $a, b$ and $c$ be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

## Combinatória

**Problema 1.** Let $n > 0$ be an integer. We are given a balance and $n$ weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the $n$ weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.

Determine the number of ways in which this can be done.

**Problema 2.** Suppose that 1000 students are standing in a circle. Prove that there exists an integer $k$ with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

**Problema 3.** Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line $\ell$ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot $P$ until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, $Q$, takes over as the new pivot, and the line now rotates clockwise about $Q$, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point $P$ in $\mathcal{S}$ and a line $\ell$ going through $P$ such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

**Problema 4.** Determine the greatest positive integer $k$ that satisfies the following property: The set of positive integers can be partitioned into $k$ subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of $A_i$ whose sum is $n$.

**Problema 5.** Let $m$ be a positive integer, and consider a $m \times m$ checkerboard consisting of unit squares. At the centre of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in the opposite directions meet, they both turn $90°$ clockwise and continue moving with speed 1. When more than 2 ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard, or prove that such a moment does not necessarily exist.

**Problema 6.** Let $n$ be a positive integer, and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word, consisting of just letters $a$ and/or $b$. Suppose that the minimal period $N$ of $W$ is greater than $2^n$.

A finite nonempty word $U$ is said to appear in $W$ if there exist indices $k \leq \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word $U$ is called ubiquitous if the four words $Ua$, $Ub$, $aU$, and $bU$ all appear in $W$. Prove that there are at least $n$ ubiquitous finite nonempty words.

**Problema 7.** On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number $k$ of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of $k$?

# Geometria

**Problema 1.** Let $ABC$ be an acute triangle. Let $\omega$ be a circle whose centre $L$ lies on the side $BC$. Suppose that $\omega$ is tangent to $AB$ at $B'$ and $AC$ at $C'$. Suppose also that the circumcentre $O$ of triangle $ABC$ lies on the shorter arc $B'C'$ of $\omega$. Prove that the circumcircle of $ABC$ and $\omega$ meet at two points.

**Problema 2.** Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let $O_1$ and $r_1$ be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define $O_2, O_3, O_4$ and $r_2, r_3, r_4$ in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

**Problema 3.** Let $ABCD$ be a convex quadrilateral whose sides $AD$ and $BC$ are not parallel. Suppose that the circles with diameters $AB$ and $CD$ meet at points $E$ and $F$ inside the quadrilateral. Let $\omega_E$ be the circle through the feet of the perpendiculars from $E$ to the lines $AB, BC$ and $CD$. Let $\omega_F$ be the circle through the feet of the perpendiculars from $F$ to the lines $CD, DA$ and $AB$. Prove that the midpoint of the segment $EF$ lies on the line through the two intersections of $\omega_E$ and $\omega_F$.

**Problema 4.** Let $ABC$ be an acute triangle with circumcircle $\Omega$. Let $B_0$ be the midpoint of $AC$ and let $C_0$ be the midpoint of $AB$. Let $D$ be the foot of the altitude from $A$ and let $G$ be the centroid of the triangle $ABC$. Let $\omega$ be a circle through $B_0$ and $C_0$ that is tangent to the circle $\Omega$ at a point $X \neq A$. Prove that the points $D, G$ and $X$ are collinear.

**Problema 5.** Let $ABC$ be a triangle with incentre $I$ and circumcircle $\omega$. Let $D$ and $E$ be the second intersection points of $\omega$ with $AI$ and $BI$, respectively. The chord $DE$ meets $AC$ at a point $F$, and $BC$ at a point $G$. Let $P$ be the intersection point of the line through $F$ parallel to $AD$ and the line through $G$ parallel to $BE$. Suppose that the tangents to $\omega$ at $A$ and $B$ meet at a point $K$. Prove that the three lines $AE, BD$ and $KP$ are either parallel or concurrent.

**Problema 6.** Let $ABC$ be a triangle with $AB = AC$ and let $D$ be the midpoint of $AC$. The angle bisector of $\angle BAC$ intersects the circle through $D, B$ and $C$ at the point $E$ inside the triangle $ABC$. The line $BD$ intersects the circle through $A, E$ and $B$ in two points $B$ and $F$. The lines $AF$ and $BE$ meet at a point $I$, and the lines $CI$ and $BD$ meet at a point $K$. Show that $I$ is the incentre of triangle $KAB$.

**Problema 7.** Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle $\omega$ with centre $O$. Suppose that the circumcircle of triangle $ACE$ is concentric with $\omega$. Let $J$ be the foot of the perpendicular from $B$ to $CD$. Suppose that the perpendicular from $B$ to $DF$ intersects the line $EO$ at a point $K$. Let $L$ be the foot of the perpendicular from $K$ to $DE$. Prove that $DJ = DL$.

**Problema 8.** Let $ABC$ be an acute triangle with circumcircle $\Gamma$. Let $\ell$ be a tangent line to $\Gamma$, and let $\ell_a, \ell_b$ and $\ell_c$ be the lines obtained by reflecting $\ell$ in the lines $BC$,

$CA$ and $AB$, respectively. Show that the circumcircle of the triangle determined by the lines $\ell_a, \ell_b$ and $\ell_c$ is tangent to the circle $\Gamma$.

# Teoria dos Números

**Problema 1.** For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly $d$ positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12)$ . Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

**Problema 2.** Consider a polynomial $P(x) = \prod_{j=1}^{9}(x + d_j)$, where $d_1, d_2, \ldots d_9$ are nine distinct integers. Prove that there exists an integer $N$, such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

**Problema 3.** Let $n \geq 1$ be an odd integer. Determine all functions $f$ from the set of integers to itself, such that for all integers $x$ and $y$ the difference $f(x) - f(y)$ divides $x^n - y^n$.

**Problema 4.** For each positive integer $k$, let $t(k)$ be the largest odd divisor of $k$. Determine all positive integers $a$ for which there exists a positive integer $n$, such that all the differences

$$t(n+a) - t(n); t(n+a+1) - t(n+1), \ldots, t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

**Problema 5.** Let $f$ be a function from the set of integers to the set of positive integers. Suppose that, for any two integers $m$ and $n$, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers $m$ and $n$ with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

**Problema 6.** Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients, such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer $n$ the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

**Problema 7.** Let $p$ be an odd prime number. For every integer $a$, define the number $S_a = \sum_{j=1}^{p-1} \frac{a^j}{j}$. Let $m, n \in \mathbb{Z}$, such that $S_3 + S_4 - 3S_2 = \frac{m}{n}$. Prove that $p$ divides $m$.

**Problema 8.** Let $k \in \mathbb{Z}^+$ and set $n = 2^k + 1$. Prove that $n$ is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, \ldots, g_{n-1}$, such that $n$ divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

# Álgebra

**Problema 1.** Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than $a$.

**Problema 2.** Let the real numbers $a, b, c, d$ satisfy the relations $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$. Prove that

$$36 \leq 4\left(a^3 + b^3 + c^3 + d^3\right) - \left(a^4 + b^4 + c^4 + d^4\right) \leq 48.$$

**Problema 3.** Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

**Problema 4.** A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1} x_k$ for all $k \geq 1$. Prove that $\forall n \geq 1$ $x_1 + x_2 + \ldots + x_n \geq 0$.

**Problema 5.** Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$ :

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

**Problema 6.** Suppose that $f$ and $g$ are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer $n$.

**Problema 7.** Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and $s$ be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \quad \text{for all} \quad n > s.$$

Prove there exist positive integers $\ell \leq s$ and $N$, such that

$$a_n = a_\ell + a_{n-\ell} \quad \text{for all} \quad n \geq N.$$

**Problema 8.** Given six positive numbers $a, b, c, d, e, f$ such that $a < b < c < d < e < f$. Let $a + c + e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)\left(S(bd+bf+df) + T(ac+ae+ce)\right)}.$$

# Combinatória

**Problema 1.** In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

**Problema 2.** On some planet, there are $2^N$ countries ($N \geq 4$). Each country has a flag $N$ units wide and one unit high composed of $N$ fields of size $1 \times 1$, each field being either yellow or blue. No two countries have the same flag. We say that a set of $N$ flags is diverse if these flags can be arranged into an $N \times N$ square so that all $N$ fields on its main diagonal will have the same color. Determine the smallest positive integer $M$ such that among any $M$ distinct flags, there exist $N$ flags forming a diverse set.

**Problema 3.** 2500 chess kings have to be placed on a $100 \times 100$ chessboard so that(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);(ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

**Problema 4.** Each of the six boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$ initially contains one coin. The following operations are allowed

Type 1) Choose a non-empty box $B_j$, $1 \leq j \leq 5$, remove one coin from $B_j$ and add two coins to $B_{j+1}$;

Type 2) Choose a non-empty box $B_k$, $1 \leq k \leq 4$, remove one coin from $B_k$ and swap the contents (maybe empty) of the boxes $B_{k+1}$ and $B_{k+2}$.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$ become empty, while box $B_6$ contains exactly $2010^{2010^{2010}}$ coins.

**Problema 5.** $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let $w_i$ and $l_i$ be respectively the number of wins and losses of the $i$-th player. Prove that

$$\sum_{i=1}^{n} (w_i - l_i)^3 \geq 0.$$

**Problema 6.** Given a positive integer $k$ and other two integers $b > w > 1$. There are two strings of pearls, a string of $b$ black pearls and a string of $w$ white pearls. The length of a string is the number of pearls on it. One cuts these strings in some steps by the following rules. In each step:(i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then $k$ first ones (if they consist of more than one pearl) are chosen; if there are less than $k$ strings longer than 1, then one chooses all of them.(ii) Next, one cuts each chosen string into two parts differing in length by at most one. (For instance, if there are strings of $5, 4, 4, 2$ black pearls, strings of $8, 4, 3$ white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4), (3, 2), (2, 2)$ and $(2, 2)$ respectively.) The process stops immediately after the step when a first isolated white pearl appears.

Prove that at this stage, there will still exist a string of at least two black pearls.

**Problema 7.** Let $P_1, \ldots, P_s$ be arithmetic progressions of integers, the following conditions being satisfied:(i) each integer belongs to at least one of them;(ii) each progression contains a number which does not belong to other progressions.

Denote by $n$ the least common multiple of the ratios of these progressions; let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ its prime factorization.

Prove that

$$s \geq 1 + \sum_{i=1}^{k} \alpha_i (p_i - 1).$$

## Geometria

**Problema 1.** Let $ABC$ be an acute triangle with $D, E, F$ the feet of the altitudes lying on $BC, CA, AB$ respectively. One of the intersection points of the line $EF$ and the circumcircle is $P$. The lines $BP$ and $DF$ meet at point $Q$. Prove that $AP = AQ$.

**Problema 2.** Let $P$ be a point interior to triangle $ABC$ (with $CA \neq CB$). The lines $AP$, $BP$ and $CP$ meet again its circumcircle $\Gamma$ at $K$, $L$, respectively $M$. The tangent line at $C$ to $\Gamma$ meets the line $AB$ at $S$. Show that from $SC = SP$ follows $MK = ML$.

**Problema 3.** Let $A_1A_2 \dots A_n$ be a convex polygon. Point $P$ inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

**Problema 4.** Given a triangle $ABC$, with $I$ as its incenter and $\Gamma$ as its circumcircle, $AI$ intersects $\Gamma$ again at $D$. Let $E$ be a point on the arc $BDC$, and $F$ a point on the segment $BC$, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If $G$ is the midpoint of $IF$, prove that the meeting point of the lines $EI$ and $DG$ lies on $\Gamma$.

**Problema 5.** Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let $M$ be the midpoint of $CE$, and let $O$ be the circumcenter of triangle $BCD$. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

**Problema 6.** The vertices $X, Y, Z$ of an equilateral triangle $XYZ$ lie respectively on the sides $BC, CA, AB$ of an acute-angled triangle $ABC$. Prove that the incenter of triangle $ABC$ lies inside triangle $XYZ$.

**Problema 7.** Three circular arcs $\gamma_1, \gamma_2$, and $\gamma_3$ connect the points $A$ and $C$. These arcs lie in the same half-plane defined by line $AC$ in such a way that arc $\gamma_2$ lies between the arcs $\gamma_1$ and $\gamma_3$. Point $B$ lies on the segment $AC$. Let $h_1, h_2$, and $h_3$ be three rays starting at $B$, lying in the same half-plane, $h_2$ being between $h_1$ and $h_3$. For $i, j = 1, 2, 3$, denote by $V_{ij}$ the point of intersection of $h_i$ and $\gamma_j$ (see the Figure below). Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}$, $\widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}$, $\widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

# Teoria dos Números

**Problema 1.** Find the least positive integer $n$ for which there exists a set $\{s_1, s_2, \dots, s_n\}$ consisting of $n$ distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\cdots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

**Problema 2.** Find all pairs $(m, n)$ of nonnegative integers for which

$$m^2 + 2\cdot 3^n = m\left(2^{n+1}-1\right).$$

**Problema 3.** Find the smallest number $n$ such that there exist polynomials $f_1, f_2, \dots, f_n$ with rational coefficients satisfying

$$x^2+7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2.$$

**Problema 4.** Let $a, b$ be integers, and let $P(x) = ax^3 + bx$. For any positive integer $n$ we say that the pair $(a, b)$ is $n$-good if $n|P(m) - P(k)$ implies $n|m - k$ for all integers $m, k$. We say that $(a, b)$ is *very good* if $(a, b)$ is $n$-good for infinitely many positive integers $n$.(a) Find a pair $(a, b)$ which is 51-good, but not very good.(b) Show that all 2010-good pairs are very good.

**Problema 5.** Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m)+n)(g(n)+m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

**Problema 6.** The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been coloured with the following property being satisfied: for each $0 \le i, j \le 2^n - 1$, the $j$-th cell in the $i$-th row and the $(i + j)$-th cell in the $j$-th row have the same colour. (The indices of the cells in a row are considered modulo $2^n$.) Prove that the maximal possible number of colours is $2^n$.

# Álgebra

**Problema 1.** Find the largest possible integer $k$, such that the following statement is true:

Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are coloured, such that one is blue, one is red and one is white. Now, for every colour separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \leq b_2 \leq \ldots \leq b_{2009} & \text{the lengths of the blue sides} \\ & r_1 \leq r_2 \leq \ldots \leq r_{2009} & \text{the lengths of the red sides} \\ \text{and} & w_1 \leq w_2 \leq \ldots \leq w_{2009} & \text{the lengths of the white sides} \end{array}$$

Then there exist $k$ indices $j$ such that we can form a non-degenerated triangle with side lengths $b_j$, $r_j$, $w_j$.

**Problema 2.** Let $a$, $b$, $c$ be positive real numbers such that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=a+b+c$. Prove that:

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2} \leq \frac{3}{16}.$$

**Problema 3.** Determine all functions $f$ from the set of positive integers to the set of positive integers such that, for all positive integers $a$ and $b$, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b+f(a)-1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

**Problema 4.** Let $a$, $b$, $c$ be positive real numbers such that $ab+bc+ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3 \leq \sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)$$

**Problema 5.** Let $f$ be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers $x$ and $y$ such that

$$f\left(x-f(y)\right) > yf(x)+x$$

**Problema 6.** Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1},\, s_{s_2},\, s_{s_3},\, \ldots \qquad \text{and} \qquad s_{s_1+1},\, s_{s_2+1},\, s_{s_3+1},\, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

**Problema 7.** Find all functions $f$ from the set of real numbers into the set of real numbers which satisfy for all $x$, $y$ the identity

$$f\left(xf(x+y)\right) = f\left(yf(x)\right) + x^2$$

# Combinatória

**Problema 1.** Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

**Problema 2.** For any integer $n \geq 2$, let $N(n)$ be the maxima number of triples $(a_i, b_i, c_i)$, $i = 1, \ldots, N(n)$, consisting of nonnegative integers $a_i$, $b_i$ and $c_i$ such that the following two conditions are satisfied:

$a_i + b_i + c_i = n$ for all $i = 1, \ldots, N(n)$, If $i \neq j$ then $a_i \neq a_j$, $b_i \neq b_j$ and $c_i \neq c_j$

Determine $N(n)$ for all $n \geq 2$.

**Problema 3.** Let $n$ be a positive integer. Given a sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \ldots, n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \qquad \text{for each } i = 1, \ldots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \qquad \text{for each } i = 1, \ldots, n-1.$$

Prove that $a_n = b_n$.

**Problema 4.** For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of chessboard, in which each of the $2^m$ cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

**Problema 5.** Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighbouring buckets, empties them to the river and puts them back. Then the next round begins. The Stepmother goal's is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

**Problema 6.** On a $999 \times 999$ board a limp rook can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A non-intersecting route of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called cyclic, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

**Problema 7.** Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let $M$ be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making $n$ jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in $M$.

**Problema 8.** For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let $r$ be the rightmost digit of $n$.If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of $n$ by removing this rightmost digit 0.If $1 \leq r \leq 9$ we split the decimal representation of $n$ into a maximal right part $R$ that solely consists of digits not less than $r$ and into a left part $L$ that either is empty or ends with a digit strictly smaller than $r$. Then the decimal representation of $h(n)$ consists of the decimal representation of $L$, followed by two copies of the decimal representation of $R - 1$. For instance, for the number $17, 151, 345, 543$, we will have $L = 17, 151$, $R = 345, 543$ and $h(n) = 17, 151, 345, 542, 345, 542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of $h$ produces the integer 1 after finitely many steps.

# Geometria

**Problema 1.** Let $ABC$ be a triangle with $AB = AC$ . The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides $BC$ and $CA$ at $D$ and $E$ , respectively. Let $K$ be the incentre of triangle $ADC$. Suppose that $\angle BEK = 45°$ . Find all possible values of $\angle CAB$.Jan Vonk, Belgium, Peter Vandendriessche, Belgium and Hojoo Lee, Korea

**Problema 2.** Let $ABC$ be a triangle with circumcentre $O$. The points $P$ and $Q$ are interior points of the sides $CA$ and $AB$ respectively. Let $K, L$ and $M$ be the midpoints of the segments $BP, CQ$ and $PQ$. respectively, and let $\Gamma$ be the circle passing through $K, L$ and $M$. Suppose that the line $PQ$ is tangent to the circle $\Gamma$. Prove that $OP = OQ$.

**Problema 3.** Let $ABC$ be a triangle. The incircle of $ABC$ touches the sides $AB$ and $AC$ at the points $Z$ and $Y$, respectively. Let $G$ be the point where the lines $BY$ and $CZ$ meet, and let $R$ and $S$ be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram.

Prove that $GR = GS$.

**Problema 4.** Given a cyclic quadrilateral $ABCD$, let the diagonals $AC$ and $BD$ meet at $E$ and the lines $AD$ and $BC$ meet at $F$. The midpoints of $AB$ and $CD$ are $G$ and $H$, respectively. Show that $EF$ is tangent at $E$ to the circle through the points $E$, $G$ and $H$.

**Problema 5.** Let $P$ be a polygon that is convex and symmetric to some point $O$. Prove that for some parallelogram $R$ satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets $R$ and $P$, respectively.

**Problema 6.** Let the sides $AD$ and $BC$ of the quadrilateral $ABCD$ (such that $AB$ is not parallel to $CD$) intersect at point $P$. Points $O_1$ and $O_2$ are circumcenters and points $H_1$ and $H_2$ are orthocenters of triangles $ABP$ and $CDP$, respectively. Denote the midpoints of segments $O_1H_1$ and $O_2H_2$ by $E_1$ and $E_2$, respectively. Prove that the perpendicular from $E_1$ on $CD$, the perpendicular from $E_2$ on $AB$ and the lines $H_1H_2$ are concurrent.

**Problema 7.** Let $ABC$ be a triangle with incenter $I$ and let $X$, $Y$ and $Z$ be the incenters of the triangles $BIC$, $CIA$ and $AIB$, respectively. Let the triangle $XYZ$ be equilateral. Prove that $ABC$ is equilateral too.

**Problema 8.** Let $ABCD$ be a circumscribed quadrilateral. Let $g$ be a line through $A$ which meets the segment $BC$ in $M$ and the line $CD$ in $N$. Denote by $I_1$, $I_2$ and $I_3$ the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on $g$.

# Teoria dos Números

**Problema 1.** Seja $n$ um inteiro positivo e $a_1, a_2, \ldots, a_k$ $(k \geq 2)$ elementos distintos do conjunto $1, 2, \ldots, n$ tal que $n$ divide $a_i(a_{i+1} - 1)$ para $i = 1, 2, \ldots k - 1$. Prove que $n$ não divide $a_k(a_1 - 1)$.

**Problema 2.** A positive integer $N$ is called balanced, if $N = 1$ or if $N$ can be written as a product of an even number of not necessarily distinct primes. Given positive integers $a$ and $b$, consider the polynomial $P$ defined by $P(x) = (x + a)(x + b)$.

(a) Prove that there exist distinct positive integers $a$ and $b$ such that all the number $P(1)$, $P(2)$, $\ldots$, $P(50)$ are balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers $n$, then $a = b$.

**Problema 3.** Let $f$ be a non-constant function from the set of positive integers into the set of positive integer, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers $a$, $b$. Prove that there exist infinitely many primes $p$ such that $p$ divides $f(c)$ for some positive integer $c$.

**Problema 4.** Find all positive integers $n$ such that there exists a sequence of positive integers $a_1$, $a_2$, $\ldots$, $a_n$ satisfying:

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every $k$ with $2 \leq k \leq n - 1$.

**Problema 5.** Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function $T$ from the set of integers into the set of integers such that the number of integers $x$ with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where $T^n$ denotes the $n$-fold application of $T$.

**Problema 6.** Let $k$ be a positive integer. Show that if there exists a sequence $a_0, a_1, \ldots$ of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \text{ for all } n \geq 1,$$

then $k - 2$ is divisible by 3.

**Problema 7.** Let $a$ and $b$ be distinct integers greater than 1. Prove that there exists a positive integer $n$ such that $(a^n - 1)(b^n - 1)$ is not a perfect square.

# Álgebra

**Problema 1.** Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so $f$ is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers $w, x, y, z$, satisfying $wx = yz$.

**Problema 2.** (a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers $x$, $y$, $z$, each different from 1, and satisfying $xyz = 1$.(b) Prove that equality holds above for infinitely many triples of rational numbers $x$, $y$, $z$, each different from 1, and satisfying $xyz = 1$.

**Problema 3.** Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair $(f, g)$ of functions from $S$ into $S$ is a Spanish Couple on $S$, if they satisfy the following conditions:

(i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all $x$, $y \in S$ with $x < y$;

(ii) The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple:

(a) on the set $S = \mathbb{N}$ of positive integers;

(b) on the set $S = \{a - \frac{1}{b} : a, b \in \mathbb{N}\}$.

**Problema 4.** For an integer $m$, denote by $t(m)$ the unique number in $\{1, 2, 3\}$ such that $m + t(m)$ is a multiple of 3. A function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(-1) = 0$, $f(0) = 1$, $f(1) = -1$ and $f(2^n + m) = f(2^n - t(m)) - f(m)$ for all integers $m$, $n \geq 0$ with $2^n > m$. Prove that $f(3p) \geq 0$ holds for all integers $p \geq 0$.

**Problema 5.** Let $a$, $b$, $c$, $d$ be positive real numbers such that $abcd = 1$ and $a + b + c + d > \dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d} + \dfrac{d}{a}$. Prove that

$$a + b + c + d < \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}$$

**Problema 6.** Let $f : \mathbb{R} \to \mathbb{N}$ be a function which satisfies $f\left(x + \dfrac{1}{f(y)}\right) = f\left(y + \dfrac{1}{f(x)}\right)$ for all $x$, $y \in \mathbb{R}$. Prove that there is a positive integer which is not a value of $f$.

**Problema 7.** Prove that for any four positive real numbers $a$, $b$, $c$, $d$ the inequality

$$\frac{(a-b)(a-c)}{a+b+c} + \frac{(b-c)(b-d)}{b+c+d} + \frac{(c-d)(c-a)}{c+d+a} + \frac{(d-a)(d-b)}{d+a+b} \geq 0$$

holds. Determine all cases of equality.

# Combinatória

**Problema 1.** In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a box. Two boxes intersect if they have a common point in their interior or on their boundary. Find the largest $n$ for which there exist $n$ boxes $B_1$, ..., $B_n$ such that $B_i$ and $B_j$ intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

**Problema 2.** Let $n \in \mathbb{N}$ and $A_n$ set of all permutations $(a_1, \dots, a_n)$ of the set $\{1, 2, \dots, n\}$ for which

$$k | 2(a_1 + \cdots + a_k), \text{ for all } 1 \le k \le n.$$

Find the number of elements of the set $A_n$.

**Problema 3.** In the coordinate plane consider the set $S$ of all points with integer coordinates. For a positive integer $k$, two distinct points $a$, $B \in S$ will be called $k$-friends if there is a point $C \in S$ such that the area of the triangle $ABC$ is equal to $k$. A set $T \subset S$ will be called $k$-clique if every two points in $T$ are $k$-friends. Find the least positive integer $k$ for which there exits a $k$-clique with more than 200 elements.

**Problema 4.** Let $n$ and $k$ be positive integers with $k \ge n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let $N$ be the number of such sequences consisting of $k$ steps and resulting in the state where lamps 1 through $n$ are all on, and lamps $n + 1$ through $2n$ are all off.

Let $M$ be number of such sequences consisting of $k$ steps, resulting in the state where lamps 1 through $n$ are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

**Problema 5.** Let $S = \{x_1, x_2, \dots, x_{k+l}\}$ be a $(k + l)$-element set of real numbers contained in the interval $[0, 1]$; $k$ and $l$ are positive integers. A $k$-element subset $A \subset S$ is called nice if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{l} \sum_{x_j \in S \setminus A} x_j \right| \le \frac{k + l}{2kl}$$

Prove that the number of nice subsets is at least $\dfrac{2}{k + l}\dbinom{k + l}{k}$.

**Problema 6.** For $n \ge 2$, let $S_1$, $S_2$, ..., $S_{2^n}$ be $2^n$ subsets of $A = \{1, 2, 3, \dots, 2^{n+1}\}$ that satisfy the following property: There do not exist indices $a$ and $b$ with $a < b$ and elements $x$, $y$, $z \in A$ with $x < y < z$ and $y$, $z \in S_a$, and $x$, $z \in S_b$. Prove that at least one of the sets $S_1$, $S_2$, ..., $S_{2^n}$ contains no more than $4n$ elements.

## Geometria

**Problema 1.** An acute-angled triangle $ABC$ has orthocentre $H$. The circle passing through $H$ with centre the midpoint of $BC$ intersects the line $BC$ at $A_1$ and $A_2$. Similarly, the circle passing through $H$ with centre the midpoint of $CA$ intersects the line $CA$ at $B_1$ and $B_2$, and the circle passing through $H$ with centre the midpoint of $AB$ intersects the line $AB$ at $C_1$ and $C_2$. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

**Problema 2.** Given trapezoid $ABCD$ with parallel sides $AB$ and $CD$, assume that there exist points $E$ on line $BC$ outside segment $BC$, and $F$ inside segment $AD$ such that $\angle DAE = \angle CBF$. Denote by $I$ the point of intersection of $CD$ and $EF$, and by $J$ the point of intersection of $AB$ and $EF$. Let $K$ be the midpoint of segment $EF$, assume it does not lie on line $AB$. Prove that $I$ belongs to the circumcircle of $ABK$ if and only if $K$ belongs to the circumcircle of $CDJ$.

**Problema 3.** Let $ABCD$ be a convex quadrilateral and let $P$ and $Q$ be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point $E$ on the line segment $PQ$ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

**Problema 4.** In an acute triangle $ABC$ segments $BE$ and $CF$ are altitudes. Two circles passing through the point $A$ and $F$ and tangent to the line $BC$ at the points $P$ and $Q$ so that $B$ lies between $C$ and $Q$. Prove that lines $PE$ and $QF$ intersect on the circumcircle of triangle $AEF$.

**Problema 5.** Let $k$ and $n$ be integers with $0 \le k \le n-2$. Consider a set $L$ of $n$ lines in the plane such that no two of them are parallel and no three have a common point. Denote by $I$ the set of intersections of lines in $L$. Let $O$ be a point in the plane not lying on any line of $L$. A point $X \in I$ is colored red if the open line segment $OX$ intersects at most $k$ lines in $L$. Prove that $I$ contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

**Problema 6.** There is given a convex quadrilateral $ABCD$. Prove that there exists a point $P$ inside the quadrilateral such that $\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90°$
if and only if the diagonals $AC$ and $BD$ are perpendicular.

**Problema 7.** Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles $ABC$ and $ADC$ by $\omega_1$ and $\omega_2$ respectively. Suppose that there exists a circle $\omega$ tangent to ray $BA$ beyond $A$ and to the ray $BC$ beyond $C$, which is also tangent to the lines $AD$ and $CD$. Prove that the common external tangents to $\omega_1$ and $\omega_2$ intersect on $\omega$.

# Teoria dos Números

**Problema 1.** Let $n$ be a positive integer and let $p$ be a prime number. Prove that if $a$, $b$, $c$ are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa$$

then $a = b = c$.

**Problema 2.** Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices $i$ and $j$ such that $a_i + a_j$ does not divide any of the numbers $3a_1, 3a_2, \ldots, 3a_n$.

**Problema 3.** Let $a_0, a_1, a_2, \ldots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \geq 2^n$ for all $n \geq 0$.

**Problema 4.** Let $n$ be a positive integer. Show that the numbers

$$\binom{2^n-1}{0}, \binom{2^n-1}{1}, \binom{2^n-1}{2}, \ldots, \binom{2^n-1}{2^{n-1}-1}$$

are congruent modulo $2^n$ to 1, 3, 5, $\ldots$, $2^n - 1$ in some order.

**Problema 5.** For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of $n$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties:

$d\left(f(x)\right) = x$ for all $x \in \mathbb{N}.f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x$, $y \in \mathbb{N}$.

**Problema 6.** Prove that there are infinitely many positive integers $n$ such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

# Álgebra

**Problema 1.** Real numbers $a_1$, $a_2$, …, $a_n$ are given. For each $i$, $(1 \le i \le n)$, define

$$d_i = \max\{a_j \mid 1 \le j \le i\} - \min\{a_j \mid i \le j \le n\}$$

and let $d = \max\{d_i \mid 1 \le i \le n\}$.

(a) Prove that, for any real numbers $x_1 \le x_2 \le \dots \le x_n$,

$$\max\{|x_i - a_i| \mid 1 \le i \le n\} \ge \frac{d}{2}.$$

(b) Show that there are real numbers $x_1 \le x_2 \le \dots \le x_n$ such that the equality holds in the equation above.

**Problema 2.** Consider those functions $f : \mathbb{N} \mapsto \mathbb{N}$ which satisfy the condition

$$f(m+n) \ge f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

**Problema 3.** Let $n$ be a positive integer, and let $x$ and $y$ be a positive real number such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right) \cdot \left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)\cdot(1-y)}.$$

**Problema 4.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f\left(x + f\left(y\right)\right) = f\left(x+y\right) + f\left(y\right)$ for all pairs of positive reals $x$ and $y$. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

**Problema 5.** Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \le 2 \cdot a(m) + 2 \cdot a(n) \text{ for all } m, n \ge 1,$$

and $a\left(2^k\right) \le \frac{1}{(k+1)^c}$ for all $k \ge 0$. Prove that the sequence $a(n)$ is bounded.

**Problema 6.** Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \ldots + a_{100}^2 = 1$. Prove that

$$a_1^2 \cdot a_2 + a_2^2 \cdot a_3 + \ldots + a_{100}^2 \cdot a_1 < \frac{12}{25}.$$

**Problema 7.** Let $n$ be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains $S$ but does not include $(0, 0, 0)$.

# Combinatória

**Problema 1.** Let $n > 1$ be an integer. Find all sequences $a_1, a_2, \ldots a_{n^2+n}$ satisfying the following conditions:

(a) $a_i \in \{0, 1\}$ for all $1 \leq i \leq n^2 + n$;

(b) $a_{i+1} + a_{i+2} + \ldots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \ldots + a_{i+2n}$ for all $0 \leq i \leq n^2 - n$.

**Problema 2.** A rectangle $D$ is partitioned in several ($\geq 2$) rectangles with sides parallel to those of $D$. Given that any line parallel to one of the sides of $D$, and having common points with the interior of $D$, also has common interior points with the interior of at least one rectangle of the partition; prove that there is at least one rectangle of the partition having no common points with $D$'s boundary.

**Problema 3.** Find all positive integers $n$ for which the numbers in the set $S = \{1, 2, \ldots, n\}$ can be colored red and blue, with the following condition being satisfied: The set $S \times S \times S$ contains exactly 2007 ordered triples $(x, y, z)$ such that:

(a) the numbers $x$, $y$, $z$ are of the same color, and

(b) the number $x + y + z$ is divisible by $n$.

**Problema 4.** Let $A_0 = (a_1, \ldots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \ldots, x_k)$ we construct a new sequence $A_{k+1}$ in the following way.

(i) We choose a partition $\{1, \ldots, n\} = I \cup J$, where $I$ and $J$ are two disjoint sets, such that the expression

$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$

attains the smallest value. (We allow $I$ or $J$ to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.

(ii) We set $A_{k+1} = (y_1, \ldots, y_n)$ where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some $k$, the sequence $A_k$ contains an element $x$ such that $|x| \geq \frac{n}{2}$.

**Problema 5.** In the Cartesian coordinate plane define the strips $S_n = \{(x, y) | n \leq x < n + 1\}$, $n \in \mathbb{Z}$ and color each strip black or white. Prove that any rectangle which is not a square can be placed in the plane so that its vertices have the same color.IMO Shortlist 2007 Problem C5 as it appears in the official booklet:

In the Cartesian coordinate plane define the strips $S_n = \{(x, y) | n \leq x < n + 1\}$ for every integer $n$. Assume each strip $S_n$ is colored either red or blue, and let $a$ and $b$ be two distinct positive integers. Prove that there exists a rectangle with side length $a$ and $b$ such that its vertices have the same color.

(Edited by Orlando Döhring)

**Problema 6.** In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its size.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

**Problema 7.** Let $\alpha < \frac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers $n$ and $p > \alpha \cdot 2^n$ for which one can select $2 \cdot p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \emptyset$ for all $1 \leq i, j \leq p$

**Problema 8.** Given is a convex polygon $P$ with $n$ vertices. Triangle whose vertices lie on vertices of $P$ is called good if all its sides are equal in length. Prove that there are at most $\frac{2n}{3}$ good triangles.

# Geometria

**Problema 1.** No triângulo $ABC$, a bissetriz do ângulo $\angle BCA$ intersecta o circumcírculo de novo em $R$, intersecta a mediatriz de $BC$ em $P$, e intersecta a mediatriz de $AC$ em $Q$. O ponto médio de $BC$ é $K$ e o ponto médio de $AC$ é $L$. Prove que os triângulos $RPK$ and $RQL$ têm a mesma área.

**Problema 2.** Denote by $M$ midpoint of side $BC$ in an isosceles triangle $\triangle ABC$ with $AC = AB$. Take a point $X$ on a smaller arc $MA$ of circumcircle of triangle $\triangle ABM$. Denote by $T$ point inside of angle $BMA$ such that $\angle TMX = 90$ and $TX = BX$.

Prove that $\angle MTB - \angle CTM$ does not depend on choice of $X$.

**Problema 3.** The diagonals of a trapezoid $ABCD$ intersect at point $P$. Point $Q$ lies between the parallel lines $BC$ and $AD$ such that $\angle AQD = \angle CQB$, and line $CD$ separates points $P$ and $Q$. Prove that $\angle BQP = \angle DAQ$.

**Problema 4.** Consider five points $A$, $B$, $C$, $D$ and $E$ such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let $\ell$ be a line passing through $A$. Suppose that $\ell$ intersects the interior of the segment $DC$ at $F$ and intersects line $BC$ at $G$. Suppose also that $EF = EG = EC$. Prove that $\ell$ is the bisector of angle $DAB$.

**Problema 5.** Let $ABC$ be a fixed triangle, and let $A_1$, $B_1$, $C_1$ be the midpoints of sides $BC$, $CA$, $AB$, respectively. Let $P$ be a variable point on the circumcircle. Let lines $PA_1$, $PB_1$, $PC_1$ meet the circumcircle again at $A'$, $B'$, $C'$, respectively. Assume that the points $A$, $B$, $C$, $A'$, $B'$, $C'$ are distinct, and lines $AA'$, $BB'$, $CC'$ form a triangle. Prove that the area of this triangle does not depend on $P$.

**Problema 6.** Determine the smallest positive real number $k$ with the following property. Let $ABCD$ be a convex quadrilateral, and let points $A_1$, $B_1$, $C_1$, and $D_1$ lie on sides $AB$, $BC$, $CD$, and $DA$, respectively. Consider the areas of triangles $AA_1D_1$, $BB_1A_1$, $CC_1B_1$ and $DD_1C_1$; let $S$ be the sum of the two smallest ones, and let $S_1$ be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \geq S$.

**Problema 7.** Given an acute triangle $ABC$ with $\angle B > \angle C$. Point $I$ is the incenter, and $R$ the circumradius. Point $D$ is the foot of the altitude from vertex $A$. Point $K$ lies on line $AD$ such that $AK = 2R$, and $D$ separates $A$ and $K$. Lines $DI$ and $KI$ meet sides $AC$ and $BC$ at $E, F$ respectively. Let $IE = IF$.

Prove that $\angle B \leq 3\angle C$.

**Problema 8.** Point $P$ lies on side $AB$ of a convex quadrilateral $ABCD$. Let $\omega$ be the incircle of triangle $CPD$, and let $I$ be its incenter. Suppose that $\omega$ is tangent to the incircles of triangles $APD$ and $BPC$ at points $K$ and $L$, respectively. Let lines $AC$ and $BD$ meet at $E$, and let lines $AK$ and $BL$ meet at $F$. Prove that points $E$, $I$, and $F$ are collinear.

# Teoria dos Números

**Problema 1.** Find all pairs of natural numbers $(a, b)$ such that $7^a - 3^b$ divides $a^4 + b^2$.

**Problema 2.** Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer $a_k$ such that $b - a_k^n$ is divisible by $k$. Prove that $b = A^n$ for some integer $A$.

**Problema 3.** Let $X$ be a set of 10,000 integers, none of them is divisible by 47. Prove that there exists a 2007-element subset $Y$ of $X$ such that $a - b + c - d + e$ is not divisible by 47 for any $a, b, c, d, e \in Y$.

**Problema 4.** For every integer $k \geq 2$, prove that $2^{3k}$ divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$

but $2^{3k+1}$ does not.

**Problema 5.** Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime $p$, the number $f(m+n)$ is divisible by $p$ if and only if $f(m) + f(n)$ is divisible by $p$.

**Problema 6.** Let $k$ be a positive integer. Prove that the number $(4 \cdot k^2 - 1)^2$ has a positive divisor of the form $8kn - 1$ if and only if $k$ is even.

**Problema 7.** For a prime $p$ and a given integer $n$ let $\nu_p(n)$ denote the exponent of $p$ in the prime factorisation of $n!$. Given $d \in \mathbb{N}$ and $\{p_1, p_2, \ldots, p_k\}$ a set of $k$ primes, show that there are infinitely many positive integers $n$ such that $d \mid \nu_{p_i}(n)$ for all $1 \leq i \leq k$.