Putnam (1985 – 2021)

	A1	A2	A3	A4	A5	A6	B1	B2	B3	B4	B5	B6
21												
20												
19												
18												
17												
16												
15												
14												
13												
12												
11												
10												
09												
08												
07												
06												
05												
04												
03												
02												
01												
00												
99												
98												
97												
96												
95												
94												
93												
92												
91												
90												
89												
88												
87												
86												
85												

Todo e qualquer feedback, especialmente sobre erros neste livreto (mesmo erros tipográficos pequenos), é apreciado. Você também pode contribuir enviando suas soluções (de preferência, formatadas em TEX).

Você pode enviar comentários e soluções para `zeusdanmou+tex@gmail.com`.

A versão mais atualizada desse arquivo (provavelmente) está disponível clicando aqui. Última atualização: December 22, 2021.

Session A

PROBLEM 1
A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

PROBLEM 2
For every positive real number x, let

$$g(x) = \lim_{r \to 0}((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}.$$

Find $\lim_{x\to\infty} \frac{g(x)}{x}$.

PROBLEM 3
Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

has an inscribed regular tetrahedron whose vertices have integer coordinates.

PROBLEM 4
Let

$$I(R) = \iint_{x^2+y^2 \le R^2} \left(\frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4} \right) dx\, dy.$$

Find

$$\lim_{R\to\infty} I(R),$$

or show that this limit does not exist.

PROBLEM 5
Let A be the set of all integers n such that $1 \le n \le 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let

$$S(j) = \sum_{n\in A} n^j.$$

Determine all values of j such that $S(j)$ is a multiple of 2021.

PROBLEM 6
Let $P(x)$ be a polynomial whose coefficients are all either 0 or 1. Suppose that $P(x)$ can be written as a product of two nonconstant polynomials with integer coefficients. Does it follow that $P(2)$ is a composite integer?

Session B

PROBLEM 1
Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?

PROBLEM 2
Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

PROBLEM 3
Let $h(x, y)$ be a real-valued function that is twice continuously differentiable throughout $\mathbb{R}^2$, and define

$$\rho(x, y) = y h_x - x h_y.$$

Prove or disprove: For any positive constants d and r with $d > r$, there is a circle $\mathcal{S}$ of radius r whose center is a distance d away from the origin such that the integral of ρ over the interior of $\mathcal{S}$ is zero.

PROBLEM 4
Let $F_0, F_1, \ldots$ be the sequence of Fibonacci numbers, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $m > 2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m - 1} k^k$ is divided by F_m. Prove that R_m is also a Fibonacci number.

PROBLEM 5
Say that an n-by-n matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1, 2, \ldots, n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \geq 1$.

PROBLEM 6
Given an ordered list of $3N$ real numbers, we can *trim* it to form a list of N numbers as follows: We divide the list into N groups of 3 consecutive numbers, and within each group, discard the highest and lowest numbers, keeping only the median.

Consider generating a random number X by the following procedure: Start with a list of 3^{2021} numbers, drawn independently and uniformly at random between 0 and 1. Then trim this list as defined above, leaving a list of 3^{2020} numbers. Then trim again repeatedly until just one number remains; let X be this number. Let μ be the expected value of $|X - \frac{1}{2}|$. Show that

$$\mu \geq \frac{1}{4} \left(\frac{2}{3}\right)^{2021}.$$

Session A

PROBLEM 1
How many positive integers N satisfy all of the following three conditions?

(i) N is divisible by 2020.

(ii) N has at most 2020 decimal digits.

(iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

PROBLEM 2
Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}.$$

PROBLEM 3
Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

PROBLEM 4
Consider a horizontal strip of $N+2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbors with equal probability, and color this neighboring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbors. Let $w(N)$ be the expected number of white squares remaining. Find

$$\lim_{N \to \infty} \frac{w(N)}{N}.$$

PROBLEM 5
Let a_n be the number of sets S of positive integers for which

$$\sum_{k \in S} F_k = n,$$

where the Fibonacci sequence $(F_k)_{k \geq 1}$ satisfies $F_{k+2} = F_{k+1} + F_k$ and begins $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$. Find the largest integer n such that $a_n = 2020$.

PROBLEM 6
For a positive integer N, let f_N[1] be the function defined by

$$f_N(x) = \sum_{n=0}^{N} \frac{N + 1/2 - n}{(N+1)(2n+1)} \sin((2n+1)x).$$

Determine the smallest constant M such that $f_N(x) \leq M$ for all N and all real x.

[1] Corrected from F_N in the source.

Session B

PROBLEM 1
For a positive integer n, define $d(n)$ to be the sum of the digits of n when written in binary (for example, $d(13) = 1+1+0+1 = 3$). Let

$$S = \sum_{k=1}^{2020} (-1)^{d(k)} k^3.$$

Determine S modulo 2020.

PROBLEM 2
Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

PROBLEM 3
Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \ldots$, a point x_{n+1} is chosen uniformly from the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ.

PROBLEM 4
Let n be a positive integer, and let V_n be the set of integer $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \cdots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\frac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$. Evaluate $M(2020)$.

PROBLEM 5
For $j \in \{1, 2, 3, 4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that

$$3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \neq 0.$$

PROBLEM 6
Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} (-1)^{\lfloor k(\sqrt{2}-1) \rfloor} \geq 0.$$

(As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Session A

PROBLEM 1
Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

PROBLEM 2
In the triangle $\triangle ABC$, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}(1/3)$. Find α.

PROBLEM 3
Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \cdots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \ldots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.$$

PROBLEM 4
Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x, y, z)$ over the surface of S equals 0. Must $f(x, y, z)$ be identically 0?

PROBLEM 5
Let p be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo p. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by

$$q(x) = \sum_{k=1}^{p-1} a_k x^k,$$

where

$$a_k = k^{(p-1)/2} \mod p.$$

Find the greatest nonnegative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

PROBLEM 6
Let g be a real-valued function that is continuous on the closed interval $[0, 1]$ and twice differentiable on the open interval $(0, 1)$. Suppose that for some real number $r > 1$,

$$\lim_{x \to 0^+} \frac{g(x)}{x^r} = 0.$$

Prove that either

$$\lim_{x \to 0^+} g'(x) = 0 \qquad \text{or} \qquad \limsup_{x \to 0^+} x^r |g''(x)| = \infty.$$

Session B

PROBLEM 1
Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

PROBLEM 2
For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)}.$$

Determine

$$\lim_{n\to\infty} \frac{a_n}{n^3}.$$

PROBLEM 3
Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

PROBLEM 4
Let $\mathcal{F}$ be the set of functions $f(x, y)$ that are twice continuously differentiable for $x \geq 1$, $y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):

$$x f_x + y f_y = xy \ln(xy),$$
$$x^2 f_{xx} + y^2 f_{yy} = xy.$$

For each $f \in \mathcal{F}$, let

$$m(f) = \min_{s\geq 1} \left(f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s)\right).$$

Determine $m(f)$, and show that it is independent of the choice of f.

PROBLEM 5
Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.

PROBLEM 6
Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Two points in $\mathbb{Z}^n$ are called *neighbors* if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers $n \geq 1$ does there exist a set of points $S \subset \mathbb{Z}^n$ satisfying the following two conditions?

(1) If p is in S, then none of the neighbors of p is in S.

(2) If $p \in \mathbb{Z}^n$ is not in S, then exactly one of the neighbors of p is in S.

Session A

PROBLEM 1
Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

PROBLEM 2
Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

PROBLEM 3
Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.

PROBLEM 4
Let m and n be positive integers with $\gcd(m, n) = 1$, and let

$$a_k = \left\lfloor \frac{mk}{n} \right\rfloor - \left\lfloor \frac{m(k-1)}{n} \right\rfloor$$

for $k = 1, 2, \ldots, n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1} gh^{a_2} \cdots gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$. (As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

PROBLEM 5
Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

PROBLEM 6
Suppose that A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}$$

is a rational number.

Session B

PROBLEM 1
Let $\mathcal{P}$ be the set of vectors defined by

$$\mathcal{P} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \, 0 \le a \le 2, 0 \le b \le 100, \text{ and } a, b \in \mathbb{Z} \right\}.$$

Find all $\mathbf{v} \in \mathcal{P}$ such that the set $\mathcal{P} \setminus \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathcal{P}$ can be partitioned into two sets of equal size and equal sum.

PROBLEM 2
Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \colon |z| \le 1\}$.

PROBLEM 3
Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

PROBLEM 4
Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \ge 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

PROBLEM 5
Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.

PROBLEM 6
Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

Session A

PROBLEM 1
Let S be the smallest set of positive integers such that

(a) 2 is in S,

(b) n is in S whenever n^2 is in S, and

(c) $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S?

(The set S is “smallest” in the sense that S is contained in any other such set.)

PROBLEM 2
Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

PROBLEM 3
Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

PROBLEM 4
A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

PROBLEM 5
Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A, B, and C, take turns in the order $A, B, C, A, \ldots$ choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

PROBLEM 6
The 30 edges of a regular icosahedron are distinguished by labeling them $1, 2, \ldots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color? [Note: the top matter on each exam paper included the logo of the Mathematical Association of America, which is itself an icosahedron.]

Session B

PROBLEM 1
Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2, there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$.

PROBLEM 2
Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a+1) + (a+2) + \cdots + (a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

PROBLEM 3
Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

PROBLEM 4
Evaluate the sum

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left(3 \cdot \frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5} \right) \\
&= 3 \cdot \frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + 3 \cdot \frac{\ln 6}{6} - \frac{\ln 7}{7} \\
&\quad - \frac{\ln 8}{8} - \frac{\ln 9}{9} + 3 \cdot \frac{\ln 10}{10} - \cdots .
\end{aligned}$$

(As usual, $\ln x$ denotes the natural logarithm of x.)

PROBLEM 5
A line in the plane of a triangle T is called an *equalizer* if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

PROBLEM 6
Find the number of ordered 64-tuples $(x_0, x_1, \dots, x_{63})$ such that $x_0, x_1, \dots, x_{63}$ are distinct elements of $\{1, 2, \dots, 2017\}$ and

$$x_0 + x_1 + 2x_2 + 3x_3 + \cdots + 63x_{63}$$

Session A

PROBLEM 1
Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \left. \frac{d^j}{dx^j} p(x) \right|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

PROBLEM 2
Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n\to\infty} \frac{M(n)}{n}.$$

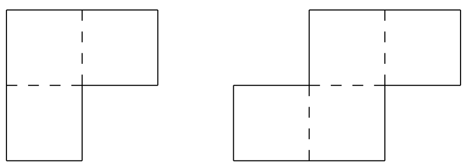
PROBLEM 3
Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$.) Find

$$\int_0^1 f(x)\,dx.$$

PROBLEM 4
Consider a $(2m-1) \times (2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into 1×1 squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?

PROBLEM 5
Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1} h^{n_1} g^{m_2} h^{n_2} \cdots g^{m_r} h^{n_r}$$

with $1 \leq r \leq |G|$ and $m_1, n_1, m_2, n_2, \ldots, m_r, n_r \in \{-1, 1\}$. (Here $|G|$ is the number of elements of G.)

PROBLEM 6
Find the smallest constant C such that for every real polynomial $P(x)$ of degree 3 that has a root in the interval $[0,1]$,

$$\int_0^1 |P(x)|\,dx \leq C \max_{x\in[0,1]} |P(x)|.$$

Session B

PROBLEM 1
Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

PROBLEM 2
Define a positive integer n to be *squarish* if either n is itself a perfect square or the distance from n to the nearest perfect square is a perfect square. For example, 2016 is squarish, because the nearest perfect square to 2016 is $45^2 = 2025$ and $2025 - 2016 = 9$ is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squarish.)

For a positive integer N, let $S(N)$ be the number of squarish integers between 1 and N, inclusive. Find positive constants α and β such that

$$\lim_{N\to\infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist.

PROBLEM 3
Suppose that S is a finite set of points in the plane such that the area of triangle $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

PROBLEM 4
Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

PROBLEM 5
Find all functions f from the interval $(1, \infty)$ to $(1, \infty)$ with the following property: if $x, y \in (1, \infty)$ and $x^2 \leq y \leq x^3$, then $(f(x))^2 \leq f(y) \leq (f(x))^3$.

PROBLEM 6
Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

Session A

PROBLEM 1
Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

PROBLEM 2
Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

PROBLEM 3
Compute

$$\log_2 \left(\prod_{a=1}^{2015} \prod_{b=1}^{2015} (1 + e^{2\pi i ab/2015}) \right)$$

Here i is the imaginary unit (that is, $i^2 = -1$).

PROBLEM 4
For each real number x, let

$$f(x) = \sum_{n \in S_x} \frac{1}{2^n},$$

where S_x is the set of positive integers n for which $\lfloor nx \rfloor$ is even. What is the largest real number L such that $f(x) \geq L$ for all $x \in [0, 1)$? (As usual, $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

PROBLEM 5
Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

PROBLEM 6
Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.

Session B

PROBLEM 1
Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

PROBLEM 2
Given a list of the positive integers $1, 2, 3, 4, \ldots$, take the first three numbers $1, 2, 3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4, 5, 7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced: $6, 16, 27, 36, \ldots$. Prove or disprove that there is some number in the sequence whose base 10 representation ends with 2015.

PROBLEM 3
Let S be the set of all 2×2 real matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

PROBLEM 4
Let T be the set of all triples (a, b, c) of positive integers for which there exist triangles with side lengths a, b, c. Express

$$\sum_{(a,b,c) \in T} \frac{2^a}{3^b 5^c}$$

as a rational number in lowest terms.

PROBLEM 5
Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that

$$|i - j| = 1 \text{ implies } |\pi(i) - \pi(j)| \leq 2$$

for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

does not depend on n, and find its value.

PROBLEM 6
For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $[1, \sqrt{2k})$. Evaluate

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{A(k)}{k}.$$

Session A

PROBLEM 1
Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

PROBLEM 2
Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

PROBLEM 3
Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

PROBLEM 4
Suppose X is a random variable that takes on only nonnegative integer values, with $E[X] = 1$, $E\left[X^2\right] = 2$, and $E\left[X^3\right] = 5$. (Here $E[y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

PROBLEM 5
Let

$$P_n(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}.$$

Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers j and k with $j \neq k$.

PROBLEM 6
Let n be a positive integer. What is the largest k for which there exist $n \times n$ matrices $M_1, \ldots, M_k$ and $N_1, \ldots, N_k$ with real entries such that for all i and j, the matrix product $M_i N_j$ has a zero entry somewhere on its diagonal if and only if $i \neq j$?

Session B

PROBLEM 1
A *base* 10 *over-expansion* of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \ldots, 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

PROBLEM 2
Suppose that f is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

PROBLEM 3
Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m + n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

PROBLEM 4
Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

PROBLEM 5
In the 75th annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

(1) A player cannot choose an element that has been chosen by either player on any previous turn.

(2) A player can only choose an element that commutes with all previously chosen elements.

(3) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)

PROBLEM 6
Let $f : [0, 1] \to \mathbb{R}$ be a function for which there exists a constant $K > 0$ such that $|f(x) - f(y)| \leq K\,|x - y|$ for all $x, y \in [0, 1]$. Suppose also that for each rational number $r \in [0, 1]$, there exist integers a and b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \ldots, I_n$ such that f is a linear function on each I_i and $[0, 1] = \bigcup_{i=1}^{n} I_i$.

Session A

PROBLEM 1
Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

PROBLEM 2
Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minumum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

PROBLEM 3
Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

PROBLEM 4
A finite collection of digits 0 and 1 is written around a circle. An *arc* of length $L \geq 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's in w and the number of 1's in w, respectively. Assume that $|Z(w) - Z(w')| \leq 1$ for any two arcs w, w' of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k}\sum_{j=1}^{k} Z(w_j) \text{ and } N = \frac{1}{k}\sum_{j=1}^{k} N(w_j)$$

are both integers. Prove that there exists an arc w with $Z(w) = Z$ and $N(w) = N$.

PROBLEM 5
For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \leq i << j < k \leq m)$ is said to be *area definite* for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\Delta A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

PROBLEM 6
Define a function $w : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|, |b| \leq 2$, let $w(a, b)$ be as in the table shown; otherwise, let $w(a, b) = 0$.

$w(a,b)$		b				
		-2	-1	0	1	2
	-2	-1	-2	2	-2	-1
	-1	-2	4	-4	4	-2
a	0	2	-4	12	-4	2
	1	-2	4	-4	4	-2
	2	-1	-2	2	-2	-1

For every finite subset S of $\mathbb{Z} \times \mathbb{Z}$, define

$$A(S) = \sum_{(\mathbf{s}, \mathbf{s}') \in S \times S} w(\mathbf{s} - \mathbf{s}').$$

Prove that if S is any finite nonempty subset of $\mathbb{Z} \times \mathbb{Z}$, then $A(S) > 0$. (For example, if $S = \{(0,1), (0,2), (2,0), (3,1)\}$, then the terms in $A(S)$ are $12, 12, 12, 12, 4, 4, 0, 0, 0, 0, -1, -1, -2, -2, -4, -4$.)

Session B

PROBLEM 1
For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

PROBLEM 2
Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which:

(i) $f(x) \geq 0$ for all real x, and

(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

PROBLEM 3
Let $\mathcal{P}$ be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

(i) if $S, S' \in \mathcal{P}$, then $S \cup S' \in \mathcal{P}$ and $S \cap S' \in \mathcal{P}$, and

(ii) if $S \in \mathcal{P}$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in \mathcal{P}$ and T contains exactly one fewer element than S.

Suppose that $f : \mathcal{P} \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \text{ for all } S, S' \in \mathcal{P}.$$

Must there exist real numbers $f_1, \ldots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in \mathcal{P}$?

PROBLEM 4
For any continuous real-valued function f defined on the interval $[0,1]$, let

$$\mu(f) = \int_0^1 f(x)\,dx,\ \mathrm{Var}(f) = \int_0^1 (f(x) - \mu(f))^2\,dx,$$
$$M(f) = \max_{0 \leq x \leq 1} |f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0,1]$, then

$$\mathrm{Var}(fg) \leq 2\mathrm{Var}(f)M(g)^2 + 2\mathrm{Var}(g)M(f)^2.$$

PROBLEM 5
Let $X = \{1, 2, \ldots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \geq 0$ such that $f^{(j)}(x) \leq k$. [Here $f^{(j)}$ denotes the j^{th} iterate of f, so that $f^{(0)}(x) = x$ and $f^{(j+1)}(x) = f(f^{(j)}(x))$.]

PROBLEM 6
Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

Session A

PROBLEM 1
Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

PROBLEM 2
Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

PROBLEM 3
Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that

(i) $f(x) = \frac{2-x^2}{2} f\left(\frac{x^2}{2-x^2}\right)$ for every x in $[-1, 1]$,

(ii) $f(0) = 1$, and

(iii) $\lim_{x \to 1^-} \frac{f(x)}{\sqrt{1-x}}$ exists and is finite.

Prove that f is unique, and express $f(x)$ in closed form.

PROBLEM 4
Let q and r be integers with $q > 0$, and let A and B be intervals on the real line. Let T be the set of all $b + mq$ where b and m are integers with b in B, and let S be the set of all integers a in A such that ra is in T. Show that if the product of the lengths of A and B is less than q, then S is the intersection of A with some arithmetic progression.

PROBLEM 5
Let $\mathbb{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a fixed vector in $\mathbb{F}_p^n$, let M be an $n \times n$ matrix with entries of $\mathbb{F}_p$, and define $G : \mathbb{F}_p^n \to \mathbb{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the k-fold composition of G with itself, that is, $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p, n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1, 2, \ldots, p^n$ are distinct.

PROBLEM 6
Let $f(x, y)$ be a continuous, real-valued function on $\mathbb{R}^2$. Suppose that, for every rectangular region R of area 1, the double integral of $f(x, y)$ over R equals 0. Must $f(x, y)$ be identically 0?

Session B

PROBLEM 1
Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

PROBLEM 2
Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

PROBLEM 3
A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

PROBLEM 4
Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.)

PROBLEM 5
Prove that, for any two bounded functions $g_1, g_2 : \mathbb{R} \to [1, \infty)$, there exist functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that, for every $x \in \mathbb{R}$,

$$\sup_{s \in \mathbb{R}} (g_1(s)^x g_2(s)) = \max_{t \in \mathbb{R}} (x h_1(t) + h_2(t)).$$

PROBLEM 6
Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

Session A

PROBLEM 1
of points with integer coordinates $P_0 = (0, 0), P_1, \ldots, P_n$ such that $n \geq 2$ and:

- the directed line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.;
- the lengths of these line segments are positive and strictly increasing.

[Picture omitted.] How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n of any growing spiral?

PROBLEM 2
real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$

converges, and evaluate S.

PROBLEM 3

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x \, dx}{\int_0^{\pi/2} x^r \cos x \, dx} = L.$$

PROBLEM 4
with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

PROBLEM 5
continuously differentiable functions with the following properties:

- $F(u, u) = 0$ for every $u \in \mathbb{R}$;
- for every $x \in \mathbb{R}$, $g(x) > 0$ and $x^2 g(x) \leq 1$;
- for every $(u, v) \in \mathbb{R}^2$, the vector $\nabla F(u, v)$ is either $\mathbf{0}$ or parallel to the vector $\langle g(u), -g(v) \rangle$.

Prove that there exists a constant C such that for every $n \geq 2$ and any $x_1, \ldots, x_{n+1} \in \mathbb{R}$, we have

$$\min_{i \neq j} |F(x_i, x_j)| \leq \frac{C}{n}.$$

PROBLEM 6

$$\{g_1 = e, g_2, \ldots, g_k\} \subsetneq G$$

be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1, g_2, \ldots, g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g \in G$. Prove that there exists a real number $b \in (0, 1)$ such that

$$\lim_{m\to\infty} \frac{1}{b^{2m}} \sum_{x \in G} \left(\mathrm{Prob}(g = x) - \frac{1}{n} \right)^2$$

is positive and finite.

Session B

PROBLEM 1
$\epsilon > 0$, there are positive integers m and n such that

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.$$

PROBLEM 2
numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

PROBLEM 3
interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

PROBLEM 4
multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

PROBLEM 5
a constant A such that for all n,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1 + (x - a_i)^2} \right)^2 dx \leq An.$$

Prove there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n} (1 + (a_i - a_j)^2) \geq Bn^3.$$

PROBLEM 6
Let p be an odd prime. Show that for at least $(p + 1)/2$ values of n in $\{0, 1, 2, \ldots, p - 1\}$,

$$\sum_{k=0}^{p-1} k!n^k \qquad \text{is not divisible by } p.$$

Session A

PROBLEM 1

Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

PROBLEM 2

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

PROBLEM 3

Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then h is identically zero.

PROBLEM 4

Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

PROBLEM 5

Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = 0$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in G$.

PROBLEM 6

Let $f\colon [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx$$

diverges.

Session B

PROBLEM 1
Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

PROBLEM 2
Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

PROBLEM 3
There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

PROBLEM 4
Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

PROBLEM 5
Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

PROBLEM 6
Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the kth power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

Session A

PROBLEM 1
Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane?

PROBLEM 2
Functions f,g,h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh+\frac{1}{gh}, \quad f(0)=1, \\ g' &= fg^2h+\frac{4}{fh}, \quad g(0)=1, \\ h' &= 3fgh^2+\frac{1}{fg}, \quad h(0)=1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

PROBLEM 3
Let d_n be the determinant of the $n\times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \dots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

PROBLEM 4
Let S be a set of rational numbers such that

(a) $0\in S$;

(b) If $x\in S$ then $x+1\in S$ and $x-1\in S$; and

(c) If $x\in S$ and $x\notin\{0,1\}$, then $\frac{1}{x(x-1)}\in S$.

Must S contain all rational numbers?

PROBLEM 5
Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?

PROBLEM 6
Let $f:[0,1]^2\to\mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a=\int_0^1 f(0,y)\,dy$, $b=\int_0^1 f(1,y)\,dy$, $c=\int_0^1 f(x,0)\,dx$, $d=\int_0^1 f(x,1)\,dx$. Prove or disprove: There must be a point (x_0,y_0) in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0)=b-a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0,y_0)=d-c.$$

Session B

PROBLEM 1
Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

PROBLEM 2
A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

PROBLEM 3
Call a subset S of $\{1, 2, \ldots, n\}$ *mediocre* if it has the following property: Whenever a and b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1, 2, \ldots, n\}$. [For instance, every subset of $\{1, 2, 3\}$ except $\{1, 3\}$ is mediocre, so $A(3) = 7$.] Find all positive integers n such that $A(n + 2) - 2A(n + 1) + A(n) = 1$.

PROBLEM 4
Say that a polynomial with real coefficients in two variables, x, y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.

PROBLEM 5
Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

PROBLEM 6
Prove that for every positive integer n, there is a sequence of integers $a_0, a_1, \ldots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k, or of the form $b \bmod c$ for some earlier positive terms b and c. [Here $b \bmod c$ denotes the remainder when b is divided by c, so $0 \le (b \bmod c) < c$.]

Session A

PROBLEM 1
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

PROBLEM 2
Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

PROBLEM 3
Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

PROBLEM 4
Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

PROBLEM 5
Let $n \ge 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1))$, $(f(2), g(2))$, $\ldots$, $(f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n - 1$.

PROBLEM 6
Prove that there exists a constant $c > 0$ such that in every nontrivial finite group G there exists a sequence of length at most $c \log |G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, $4, 4, 2$ is a subsequence of $2, 4, 6, 4, 2$, but $2, 2, 4$ is not.)

Session B

PROBLEM 1
What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

PROBLEM 2
Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate

$$\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n}.$$

PROBLEM 3
What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

PROBLEM 4
Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo p^3.

PROBLEM 5
Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.)

PROBLEM 6
Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \ldots, n\}$ is *k-limited* if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \ldots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

Session A

PROBLEM 1
Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

PROBLEM 2
Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$. (A set S in the plane is called *convex* if for any two points in S the line segment connecting them is contained in S.)

PROBLEM 3
Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k + 1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

PROBLEM 4
A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

PROBLEM 5
Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n + 1$.

PROBLEM 6
A *triangulation* $\mathcal{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side is a side of exactly one triangle in $\mathcal{T}$. Say that $\mathcal{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. For example, [figure omitted.] Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.

Session B

PROBLEM 1
Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n=1$. [Editor's note: one must assume f is nonconstant.]

PROBLEM 2
Suppose that $f:[0,1]\to\mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx=0$. Prove that for every $\alpha\in(0,1)$,

$$\left|\int_0^\alpha f(x)\,dx\right| \le \frac{1}{8}\max_{0\le x\le 1}|f'(x)|.$$

PROBLEM 3
Let $x_0=1$ and for $n\ge 0$, let $x_{n+1}=3x_n+\lfloor x_n\sqrt{5}\rfloor$. In particular, $x_1=5$, $x_2=26$, $x_3=136$, $x_4=712$. Find a closed-form expression for x_{2007}. ($\lfloor a\rfloor$ means the largest integer $\le a$.)

PROBLEM 4
Let n be a positive integer. Find the number of pairs P,Q of polynomials with real coefficients such that

$$(P(X))^2+(Q(X))^2=X^{2n}+1$$

and $\deg P>\deg Q$.

PROBLEM 5
Let k be a positive integer. Prove that there exist polynomials $P_0(n),P_1(n),\dots,P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor\frac{n}{k}\right\rfloor^k = P_0(n)+P_1(n)\left\lfloor\frac{n}{k}\right\rfloor+\cdots+P_{k-1}(n)\left\lfloor\frac{n}{k}\right\rfloor^{k-1}.$$

($\lfloor a\rfloor$ means the largest integer $\le a$.)

PROBLEM 6
For each positive integer n, let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k, $1\le k\le n$. Prove that for some constant C, independent of n,

$$n^{n^2/2-Cn}e^{-n^2/4}\le f(n)\le n^{n^2/2+Cn}e^{-n^2/4}.$$

Session A

PROBLEM 1
Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

PROBLEM 2
Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

PROBLEM 3
Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

PROBLEM 4
Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a *local maximum* at $k \in S$ if

(i) $\pi(k) > \pi(k + 1)$ for $k = 1$;

(ii) $\pi(k - 1) < \pi(k)$ and $\pi(k) > \pi(k + 1)$ for $1 < k < n$;

(iii) $\pi(k - 1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

PROBLEM 5
Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

PROBLEM 6
Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

Session B

PROBLEM 1
Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

PROBLEM 2
Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

PROBLEM 3
Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

PROBLEM 4
Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$. [Editorial note: the proposers probably intended to write $Z(V)$ instead of "the number of points in $V \cap Z$", but this changes nothing.]

PROBLEM 5
For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\, dx$ and $J(x) = \int_0^1 x \left(f(x)\right)^2\, dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

PROBLEM 6
Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$

Session A

PROBLEM 1
Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

PROBLEM 2
Let $\mathbf{S} = \{(a, b) | a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

(i) $p_i \in \mathbf{S}$,

(ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$,

(iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?

(An example of such a rook tour for $n = 5$ was depicted in the original.)

PROBLEM 3
Let $p(z)$ be a polynomial of degree n all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

PROBLEM 4
Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

PROBLEM 5
Evaluate $\int_0^1 \frac{\ln(x+1)}{x^2+1}\, dx$.

PROBLEM 6
Let n be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are the P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

Session B

PROBLEM 1
Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)

PROBLEM 2
Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

PROBLEM 3
Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

PROBLEM 4
For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

PROBLEM 5
Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) P(x_1, \ldots, x_n) = 0 \quad \text{(identically)}$$

and that

$$x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

PROBLEM 6
Let S_n denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1}.$$

Session A

PROBLEM 1

Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

PROBLEM 2

For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

PROBLEM 3

Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \ge 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

PROBLEM 4

Show that for any positive integer n there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$.

PROBLEM 5

An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

PROBLEM 6

Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \le x \le 1, 0 \le y \le 1$. Show that

$$\begin{aligned} &\int_0^1 \left(\int_0^1 f(x,y) dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y) dy \right)^2 dx \\ &\le \left(\int_0^1 \int_0^1 f(x,y) dx\, dy \right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2 \, dx\, dy. \end{aligned}$$

Session B

PROBLEM 1
Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,$$
$$\ldots,\ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

PROBLEM 2
Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

PROBLEM 3
Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$

has perimeter k units and area k square units for some real number k.

PROBLEM 4
Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots,$ then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

PROBLEM 5
Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

PROBLEM 6
Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding x. Let $\mathcal{B}$ denote the set of positive integers b that can be written in the form $b = a - a'$ with $a \in \mathcal{A}$ and $a' \in \mathcal{A}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then

$$\lim_{x \to \infty} N(x)/x = 0.$$

Session A

PROBLEM 1
Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

PROBLEM 2
Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$\begin{aligned}&(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \\ &\le [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.\end{aligned}$$

PROBLEM 3
Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

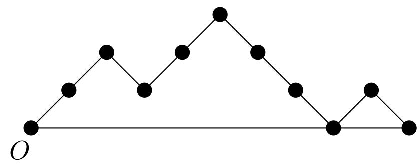
PROBLEM 4
Suppose that a, b, c, A, B, C are real numbers, $a \ne 0$ and $A \ne 0$, such that

$$|ax^2 + bx + c| \le |Ax^2 + Bx + C|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \le |B^2 - 4AC|.$$

PROBLEM 5
A Dyck n-path is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

PROBLEM 6
For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \ne s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n ?

Session B

PROBLEM 1
Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

PROBLEM 2
Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < 2/n$.

PROBLEM 3
Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

PROBLEM 4
Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

PROBLEM 5
Let A, B, and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distance from P to A, B, C, respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

PROBLEM 6
Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq \int_0^1 |f(x)| \, dx.$$

Session A

PROBLEM 1
Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

PROBLEM 2
Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

PROBLEM 3
Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

PROBLEM 4
In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

PROBLEM 5
Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \ldots \right\}.$$

PROBLEM 6
Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

Session B

PROBLEM 1
Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

PROBLEM 2
Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

PROBLEM 3
Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

PROBLEM 4
An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

PROBLEM 5
A palindrome in base b is a positive integer whose base-b digits read the same backwards and forwards; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b.

PROBLEM 6
Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients

Session A

PROBLEM 1
Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

PROBLEM 2
You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

PROBLEM 3
For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

PROBLEM 4
Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

PROBLEM 5
Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

PROBLEM 6
Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Session B

PROBLEM 1
Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

PROBLEM 2
Find all pairs of real numbers (x, y) satisfying the system of equations

$$\begin{aligned}\frac{1}{x} + \frac{1}{2y} &= (x^2+3y^2)(3x^2+y^2)\\ \frac{1}{x} - \frac{1}{2y} &= 2(y^4-x^4).\end{aligned}$$

PROBLEM 3
For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

PROBLEM 4
Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

PROBLEM 5
Let a and b be real numbers in the interval $(0, 1/2)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

PROBLEM 6
Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers

Session A

PROBLEM 1

Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

PROBLEM 2

Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

PROBLEM 3

The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

PROBLEM 4

Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

PROBLEM 5

Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

PROBLEM 6

Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

Session B

PROBLEM 1
Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

PROBLEM 2
Prove that the expression

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

PROBLEM 3
Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeroes (including multiplicities) of $\frac{d^k f}{dt^k}$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \text{ and } \lim_{k\to\infty} N_k = 2N.$$

[Editorial clarification: only zeroes in $[0, 1)$ should be counted.]

PROBLEM 4
Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

PROBLEM 5
Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a - 1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

PROBLEM 6
Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm1, \pm1, \ldots, \pm1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an

Session A

PROBLEM 1
Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

PROBLEM 2
Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

PROBLEM 3
Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

PROBLEM 4
Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)}.$$

PROBLEM 5
Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^{1} |p(x)| \, dx.$$

PROBLEM 6
The sequence $(a_n)_{n \geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

Session B

PROBLEM 1

Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0} |EF|$.

PROBLEM 2

Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

PROBLEM 3

Let $A = \{(x, y) : 0 \le x, y < 1\}$. For $(x, y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \le \frac{m}{n} \le 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y)\to(1,1),(x,y)\in A} (1 - xy^2)(1 - x^2y)S(x, y).$$

PROBLEM 4

Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \le f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

PROBLEM 5

For an integer $n \ge 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

PROBLEM 6

Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$

Session A

PROBLEM 1
A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

PROBLEM 2
Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

PROBLEM 3
Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

PROBLEM 4
Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

PROBLEM 5
Let $\mathcal{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$E \subseteq \cup_{j=1}^{n} 3D_j.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

PROBLEM 6
Let A, B, C denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

Session B

PROBLEM 1
Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x>0$.

PROBLEM 2
Given a point (a,b) with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at (a,b), one on the x-axis, and one on the line $y=x$. You may assume that a triangle of minimum perimeter exists.

PROBLEM 3
let H be the unit hemisphere $\{(x,y,z): x^2+y^2+z^2=1, z\geq 0\}$, C the unit circle $\{(x,y,0): x^2+y^2=1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A\sin\alpha+B\cos\beta$, where A,B,α,β are real numbers.

PROBLEM 4
Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1}(-1)^{\lfloor i/m\rfloor+\lfloor i/n\rfloor}=0.$$

PROBLEM 5
Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N=1111\cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

PROBLEM 6
Prove that, for any integers a,b,c, there exists a positive integer n such that $\sqrt{n^3+an^2+bn+c}$ is not an integer.

Session A

PROBLEM 1
A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

PROBLEM 2
Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

PROBLEM 3
Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots\right)$$
$$\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right) dx.$$

PROBLEM 4
Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

PROBLEM 5
Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1+1/a_2+\ldots+1/a_n = 1$. Determine whether N_{10} is even or odd.

PROBLEM 6
For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

Session B

PROBLEM 1
Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

PROBLEM 2
Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

PROBLEM 3
For each positive integer n, write the sum $\sum_{m=1}^{n} 1/m$ in the form p_n/q_n, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

PROBLEM 4
Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all [integers] $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

PROBLEM 5
Prove that for $n \geq 2$,

$$\overbrace{2^{2^{\cdots^{2}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdots^{2}}}}^{n-1 \text{ terms}} \pmod{n}.$$

PROBLEM 6
The dissection of the 3–4–5 triangle shown below (into four congruent right triangles similar to the original) has diameter $5/2$. Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)

Session A

PROBLEM 1
Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

PROBLEM 2
Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

PROBLEM 3
Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

PROBLEM 4
Let S be the set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

1. $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
2. $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$;
3. (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

Prove that there exists a one-to-one function g from A to R such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$. Note: R is the set of real numbers.

PROBLEM 5
If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

PROBLEM 6
Let $c > 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : R \to R$ such that $f(x) = f(x^2 + c)$ for all $x \in R$. Note that R denotes the set of real numbers.

Session B

PROBLEM 1
Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

PROBLEM 2
Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

PROBLEM 3
Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

PROBLEM 4
For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

PROBLEM 5
Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

PROBLEM 6
Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0).$$

Session A

PROBLEM 1
multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

PROBLEM 2
improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}}-\sqrt{\sqrt{x}-\sqrt{x-b}}\right)dx$$

converge?

PROBLEM 3
necessarily distinct) decimal digits. The number $e_1e_2\dots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i is $d_1d_2\dots d_9$ by the corresponding digit e_i $(1 \le i \le 9)$ is divisible by 7. The number $f_1f_2\dots f_9$ is related to $e_1e_2\dots e_9$ is the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1d_2\dots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

PROBLEM 4
labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \dots, x_n$ satisfy

$$\sum_{i=1}^k x_i \le k-1 \qquad \text{for} \quad k=1,2,\dots,n.$$

PROBLEM 5
(real-valued) functions of a single variable f which satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots & \qquad\qquad\qquad\qquad\qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \dots, x_n$ necessarily linearly dependent?

PROBLEM 6
1,2,3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \le b \le c$. Show that for some $n \ge 1995$, it is at least four times as likely that both $b = a+1$ and $c = a+2$ as that $a = b = c$.

Session B

PROBLEM 1
let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

PROBLEM 2
rolls without slipping on the curve $y = c\sin\left(\frac{x}{a}\right)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

PROBLEM 3
associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det\begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

PROBLEM 4

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \frac{1}{2207 - \dots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.

PROBLEM 5
and 6 beans. The two players move alternately. A move consists of taking **either**

a) one bean from a heap, provided at least two beans are left behind in that heap, **or**

b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

PROBLEM 6

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \dots\}.$$

Prove that $\{1, 2, 3, \dots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha), S(\beta)$ and $S(\gamma)$. [As

Session A

PROBLEM 1
Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

PROBLEM 2
Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

PROBLEM 3
Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color whch are at least a distance $2 - \sqrt{2}$ apart.

PROBLEM 4
Let A and B be 2×2 matrices with integer entries such that $A, A + B, A + 2B, A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

PROBLEM 5
Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}},$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

PROBLEM 6
Let $f_1, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathcal{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}\},$$

$e_i = 0$ or 1 for $1 \leq i \leq 10$. (f_i^0 is the identity function and $f_i^1 = f_i$.) Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathcal{F}$ map A to itself.

Session B

PROBLEM 1
Find all positive integers n that are within 250 of exactly 15 perfect squares.

PROBLEM 2
For which real numbers c is there a straight line that intersects the curve

$$x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

PROBLEM 3
Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

PROBLEM 4
For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n\to\infty} d_n = \infty$.

PROBLEM 5
For any real number α, define the function $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

PROBLEM 6
For any integer n, set

$$n_a = 101a - 100 \cdot 2^a.$$

Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a, b\} = \{c, d\}$.

Session A

PROBLEM 1
The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the y-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.]

PROBLEM 2
Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

PROBLEM 3
Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

PROBLEM 4
Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

PROBLEM 5
Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx +$$
$$\int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx +$$
$$\int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx$$

is a rational number.

PROBLEM 6
The infinite sequence of 2's and 3's

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3,$$
$$3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

Session B

PROBLEM 1

Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

PROBLEM 2

Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

PROBLEM 3

Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

PROBLEM 4

The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x, $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y)\, dy = g(x)$$

and

$$\int_0^1 g(y)K(x, y)\, dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

PROBLEM 5

Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

PROBLEM 6

Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three

Session A

PROBLEM 1
defined on the integers that satisfies the following conditions.

(i) $f(f(n)) = n$, for all integers n;

(ii) $f(f(n+2)+2) = n$ for all integers n;

(iii) $f(0) = 1$.

PROBLEM 2
power series about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

PROBLEM 3
of positive integers, with n relatively prime to m, which satisfy

$$(x^2 + y^2)^m = (xy)^n.$$

PROBLEM 4
defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0), k = 1, 2, 3, \ldots$.

PROBLEM 5
number of 1's in the binary representation of n is even (or odd), respectively. Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j},$$

for $0 \le j \le m-1$.

PROBLEM 6
What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

Session B

PROBLEM 1
the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?

PROBLEM 2
the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.)

PROBLEM 3
$(a_n(x,y))_{n\geq 0}$ is defined as follows:

$$\begin{aligned} a_0(x,y) &= x, \\ a_{n+1}(x,y) &= \frac{(a_n(x,y))^2 + y^2}{2}, \qquad \text{for } n \geq 0. \end{aligned}$$

Find the area of the region

$$\{(x,y) | (a_n(x,y))_{n\geq 0} \text{ converges}\}.$$

PROBLEM 4
having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

PROBLEM 5

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

Is the set $\left\{\frac{D_n}{n!}\right\}_{n\geq 2}$ bounded?

PROBLEM 6 (i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;

(ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;

(iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;

(iv) if $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

Session A

PROBLEM 1
A 2×3 rectangle has vertices as $(0,0),(2,0),(0,3)$, and $(2,3)$. It rotates $90°$ clockwise about the point $(2,0)$. It then rotates $90°$ clockwise about the point $(5,0)$, then $90°$ clockwise about the point $(7,0)$, and finally, $90°$ clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

PROBLEM 2
Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

PROBLEM 3
Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

1. $p(r_i) = 0, \qquad i = 1, 2, \ldots, n$, and
2. $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0 \qquad i = 1, 2, \ldots, n-1$,

where $p'(x)$ denotes the derivative of $p(x)$.

PROBLEM 4
Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that

1. the c_i have no limit point in the finite plane,
2. the sum of the areas of the D_i is finite, and
3. every line in the plane intersects at least one of the D_i?

PROBLEM 5
Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 \leq y \leq 1$.

PROBLEM 6
Let $A(n)$ denote the number of sums of positive integers

$$a_1 + a_2 + \cdots + a_r$$

which add up to n with

$$a_1 > a_2 + a_3, a_2 > a_3 + a_4, \ldots,$$
$$a_{r-2} > a_{r-1} + a_r, a_{r-1} > a_r.$$

Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ which add up to n, with

1. $b_1 \geq b_2 \geq \cdots \geq b_s$,
2. each b_i is in the sequence $1, 2, 4, \ldots, g_j, \ldots$ defined by $g_1 = 1$, $g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and
3. if $b_1 = g_k$ then every element in $\{1, 2, 4, \ldots, g_k\}$ appears at least once as a b_i.

Prove that $A(n) = B(n)$ for each $n \geq 1$.

(For example, $A(7) = 5$ because the relevant sums are $7, 6+1, 5+2, 4+3, 4+2+1$, and $B(7) = 5$ because the relevant sums are $4+2+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$.)

Session B

PROBLEM 1
For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

PROBLEM 2
Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

PROBLEM 3
Does there exist a real number L such that, if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

PROBLEM 4
Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

PROBLEM 5
Let p be an odd prime and let $\mathbb{Z}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : y \in \mathbb{Z}_p\}?$$

PROBLEM 6
Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x} \leq a \frac{\sinh ux}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0 < |u| \leq c$ and for all x, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

Session A

PROBLEM 1
Let
$$T_0 = 2, T_1 = 3, T_2 = 6,$$
and for $n \geq 3$,
$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$
The first few terms are
$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$
Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

PROBLEM 2
Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \ldots)$?

PROBLEM 3
Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to 5/2.

PROBLEM 4
Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

PROBLEM 5
If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size such that $\mathbf{ABAB} = \mathbf{0}$, does it follow that $\mathbf{BABA} = \mathbf{0}$?

PROBLEM 6
If X is a finite set, let X denote the number of elements in X. Call an ordered pair (S, T) of subsets of $\{1, 2, \ldots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \ldots, 10\}$ are there? Prove your answer.

Session B

PROBLEM 1
Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\,dt + 1990.$$

PROBLEM 2
Prove that for $|x| < 1$, $|z| > 1$,

$$1 + \sum_{j=1}^{\infty} (1 + x^j) P_j = 0,$$

where P_j is

$$\frac{(1-z)(1-zx)(1-zx^2)\cdots(1-zx^{j-1})}{(z-x)(z-x^2)(z-x^3)\cdots(z-x^j)}.$$

PROBLEM 3
Let S be a set of 2×2 integer matrices whose entries a_{ij} (1) are all squares of integers and, (2) satisfy $a_{ij} \leq 200$. Show that if S has more than 50387 ($= 15^4 - 15^2 - 15 + 2$) elements, then it has two elements that commute.

PROBLEM 4
Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \ldots, g_{2n}$$

such that

(1) every element of G occurs exactly twice, and

(2) g_{i+1} equals $g_i a$ or $g_i b$ for $i = 1, 2, \ldots, 2n$. (Interpret g_{2n+1} as g_1.)

PROBLEM 5
Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

PROBLEM 6
Let S be a nonempty closed bounded convex set in the plane. Let K be a line and t a positive number. Let L_1 and L_2 be support lines for S parallel to K_1, and let $\overline{L}$ be the line parallel to K and midway between L_1 and L_2. Let $B_S(K,t)$ be the band of points whose distance from $\overline{L}$ is at most $(t/2)w$, where w is the distance between L_1 and L_2. What is the smallest t such that

$$S \cap \bigcap_K B_S(K,t) \neq \emptyset$$

for all S? (K runs over all lines in the plane.)

Session A

PROBLEM 1
How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

PROBLEM 2
Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\,dy\,dx$ where a and b are positive.

PROBLEM 3
Prove that if
$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$
then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

PROBLEM 4
If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

PROBLEM 5
Let m be a positive integer and let $\mathcal{G}$ be a regular $(2m+1)$-gon inscribed in the unit circle. Show that there is a positive constant A, independent of m, with the following property. For any points p inside $\mathcal{G}$ there are two distinct vertices v_1 and v_2 of $\mathcal{G}$ such that
$$\left|\, |p - v_1| - |p - v_2| \,\right| < \frac{1}{m} - \frac{A}{m^3}.$$
Here $|s - t|$ denotes the distance between the points s and t.

PROBLEM 6
Let $\alpha = 1 + a_1x + a_2x^2 + \cdots$ be a formal power series with coefficients in the field of two elements. Let
$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block} \\ 0 & \text{otherwise.} \end{cases}$$
(For example, $a_{36} = 1$ because $36 = 100100_2$ and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + x\alpha + 1 = 0$.

Session B

PROBLEM 1
A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

PROBLEM 2
Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

PROBLEM 3
Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\,dx$$

(sometimes called the nth moment of f).

a) Express μ_n in terms of μ_0.

b) Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

PROBLEM 4
Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

PROBLEM 5
Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2, and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of T, and O is the center of the circle. Determine the least upper bound of $\frac{s_1 - s_2}{d}$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained.

PROBLEM 6
Let $(x_1, x_2, \ldots x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt$, where P is a polynomial of degree n,

Session A

PROBLEM 1
Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

PROBLEM 2
A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

PROBLEM 3
Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges.

PROBLEM 4 (a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

(b) What if "three" is replaced by "nine"?

PROBLEM 5
Prove that there exists a *unique* function f from the set R^+ of positive real numbers to R^+ such that

$$f(f(x)) = 6x - f(x)$$

and

$$f(x) > 0$$

for all $x > 0$.

PROBLEM 6
If a linear transformation A on an n-dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

Session B

PROBLEM 1
A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

PROBLEM 2
Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

PROBLEM 3
For every n in the set $\mathrm{N} = \{1, 2, \dots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r_n \leq g$ for all $n \in \mathrm{N}$.

PROBLEM 4
Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

PROBLEM 5
For positive integers n, let M_n be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of M_n. (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$M_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$
$$M_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

PROBLEM 6
Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t, the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \dots\}$.)

Session A

PROBLEM 1
Curves A, B, C and D are defined in the plane as follows:

$$\begin{aligned}
A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \\
B &= \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\}, \\
C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \\
D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}.
\end{aligned}$$

Prove that $A \cap B = C \cap D$.

PROBLEM 2
The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

PROBLEM 3
For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

PROBLEM 4
Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y-x, z-x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1,0,0) = 4$, $P(0,1,0) = 5$, and $P(0,0,1) = 6$. Also let A, B, C be complex numbers with $P(A,B,C) = 0$ and $|B - A| = 10$. Find $|C - A|$.

PROBLEM 5
Let

$$\vec{G}(x,y) = \left(\frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}, 0\right).$$

Prove or disprove that there is a vector-valued function

$$\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$

with the following properties:

(i) M, N, P have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;

(ii) $\operatorname{Curl} \vec{F} = \vec{0}$ for all $(x,y,z) \neq (0,0,0)$;

(iii) $\vec{F}(x,y,0) = \vec{G}(x,y)$.

PROBLEM 6
For each positive integer n, let $a(n)$ be the number of zeroes in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

Session B

PROBLEM 1
Evaluate
$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}.$$

PROBLEM 2
Let r, s and t be integers with $0 \le r$, $0 \le s$ and $r+s \le t$. Prove that
$$\frac{\binom{s}{0}}{\binom{t}{r}}+\frac{\binom{s}{1}}{\binom{t}{r+1}}+\cdots+\frac{\binom{s}{s}}{\binom{t}{r+s}}=\frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

PROBLEM 3
Let F be a field in which $1+1 \neq 0$. Show that the set of solutions to the equation $x^2+y^2=1$ with x and y in F is given by $(x,y)=(1,0)$ and
$$(x,y)=\left(\frac{r^2-1}{r^2+1},\frac{2r}{r^2+1}\right)$$
where r runs through the elements of F such that $r^2 \neq -1$.

PROBLEM 4
Let $(x_1,y_1)=(0.8,0.6)$ and let $x_{n+1}=x_n\cos y_n - y_n \sin y_n$ and $y_{n+1}=x_n \sin y_n + y_n \cos y_n$ for $n=1,2,3,\ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

PROBLEM 5
Let O_n be the n-dimensional vector $(0,0,\cdots,0)$. Let M be a $2n\times n$ matrix of complex numbers such that whenever $(z_1,z_2,\ldots,z_{2n})M=O_n$, with complex z_i, not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers $r_1,r_2,\ldots,r_{2n}$, there are complex numbers $w_1,w_2,\ldots,w_n$ such that
$$\operatorname{re}\left[M\begin{pmatrix} w_1\\ \vdots \\ w_n\end{pmatrix}\right]=\begin{pmatrix} r_1\\ \vdots \\ r_n\end{pmatrix}.$$

(Note: if C is a matrix of complex numbers, $\operatorname{re}(C)$ is the matrix whose entries are the real parts of the entries of C.)

PROBLEM 6
Let F be the field of p^2 elements, where p is an odd prime. Suppose S is a set of $(p^2-1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even.

Session A

PROBLEM 1
Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

PROBLEM 2
What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

PROBLEM 3
Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2+n+1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$.

PROBLEM 4
A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

PROBLEM 5
Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on $\mathbb{R}^n$. Suppose further that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that there is a function $g(x)$ on $\mathbb{R}^n$ such that $f_i + \partial g/\partial x_i$ is linear for all i, $1 \leq i \leq n$. (A linear function is one of the form

$$a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.)$$

PROBLEM 6
Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose that there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and n (but independent of $a_1, a_2, \ldots, a_n$).

Session B

PROBLEM 1
Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?

PROBLEM 2
Prove that there are only a finite number of possibilities for the ordered triple $T = (x - y, y - z, z - x)$, where x, y, z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy,$$

and list all such triples T.

PROBLEM 3
Let Γ consist of all polynomials in x with integer coefficients. For f and g in Γ and m a positive integer, let $f \equiv g \pmod{m}$ mean that every coefficient of $f - g$ is an integral multiple of m. Let n and p be positive integers with p prime. Given that f, g, h, r and s are in Γ with $rf + sg \equiv 1 \pmod{p}$ and $fg \equiv h \pmod{p}$, prove that there exist F and G in Γ with $F \equiv f \pmod{p}$, $G \equiv g \pmod{p}$, and $FG \equiv h \pmod{p^n}$.

PROBLEM 4
For a positive real number r, let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0.

PROBLEM 5
Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z), q(x, y, z)$, $r(x, y, z)$ be polynomials with real coefficients satisfying

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x, \pm y, \pm z$, where the number of minus signs is 0 or 2.

PROBLEM 6
Suppose A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^T and CD^T are symmetric and $AD^T - BC^T = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^T is its transpose. Prove that $A^T D - C^T B = I$.

Session A

PROBLEM 1
Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

PROBLEM 2
Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

PROBLEM 3
Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m,$$
$$a_m(j+1) = (a_m(j))^2 + 2a_m(j), \qquad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

PROBLEM 4
Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

PROBLEM 5
Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1 \leq m \leq 10$ is $I_m \neq 0$?

PROBLEM 6
If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial with real coefficients a_i, then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $F(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

(i) $g(0) = 1$, and

(ii) $\Gamma(f(x)^n) = \Gamma(g(x)^n)$

for every integer $n \geq 1$.

Session B

PROBLEM 1
Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

PROBLEM 2
Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

PROBLEM 3
Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

PROBLEM 4
Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

PROBLEM 5
Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\, dt$. You may assume that $\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$.

PROBLEM 6
Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.