Treinamento de Velocidade em Equipes

Guilherme Zeus Moura
zeusdanmou@gmail.com

0 Pontuação e Instruções

A pontuação segue a seguinte regra:

- A k-ésima equipe a acertar a questão $n.i.$, na sua t-ésima tentativa, ganha

$$\left\lceil \frac{n+2}{kt} \right\rceil$$

pontos.

Example 0.1.
A equipe A acerta a questão 1.1, na primeira tentativa. A ganha 3 pontos.
A equipe B chuta, mas erra a questão 1.1. A pontuação não muda.
A equipe B acerta a questão 1.1, na sua segunda tentativa. B ganha $\left\lceil \frac{1+2}{2\cdot 2} \right\rceil = 1$ ponto.
A equipe A acerta a questão 3.2, na primeira tentativa. A ganha 5 pontos.
A equipe B acerta a questão 8.1, na primeira tentativa. B ganha 10 pontos.

0.1 Definições

Definition 0.1. O *valor esperado* de uma variável aleatória X é

$$\mathbb{E}(X) = \sum_x x \cdot \mathbb{P}(X = x).$$

Example 0.2. O valor esperado da variável aleatória D_6, determinada pela face superior um dado comum de seis faces é

$$\begin{aligned}\mathbb{E}(D_6) &= 1 \cdot \mathbb{P}(D_6 = 1) + 2 \cdot \mathbb{P}(D_6 = 2) + \cdots + 6 \cdot \mathbb{P}(D_6 = 6)\\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6}\\ &= \frac{7}{2}.\end{aligned}$$

1 Round 1

Problem 1.1. Let $\triangle\ ABC$ be an equilateral triangle with side length 1 and let Γ the circle tangent to AB and AC at B and C, respectively. Let P be on side AB and Q be on side AC so that $PQ//BC$, and the circle through A, P, and Q is tangent to Γ . If the area of $\triangle\ APQ$ can be written in the form $\frac{\sqrt{a}}{b}$ for positive integers a and b, where a is not divisible by the square of any prime, find $a+b$.

Problem 1.2. What is the smallest positive integer n such that $2016n$ is a perfect cube?

Problem 1.3. Chitoge is painting a cube; she can paint each face either black or white, but she wants no vertex of the cube to be touching three faces of the same color. In how many ways can Chitoge paint the cube? Two paintings of a cube are considered to be the same if you can rotate one cube so that it looks like the other cube.

2 Round 2

Problem 2.1. 32 teams, ranked 1 through 32, enter a basketball tournament that works as follows: the teams are randomly paired and in each pair, the team that loses is out of the competition. The remaining 16 teams are randomly paired, and so on, until there is a winner. A higher ranked team always wins against a lower-ranked team. If the probability that the team ranked 3 (the third-best team) is one of the last four teams remaining can be written in simplest form as $\dfrac{m}{n}$, compute $m+n$.

Problem 2.2. Let $ABCD$ be a square with side length 8. Let M be the midpoint of BC and let ω be the circle passing through M, A, and D. Let O be the center of ω, X be the intersection point (besides A) of ω with AB, and Y be the intersection point of OX and AM. If the length of OY can be written in simplest form as $\frac{m}{n}$, compute $m+n$.

Problem 2.3. For positive integers i and j, define $d(i,j)$ as follows: $d(1,j)=1, d(i,1)=1$ for all i and j, and for $i,j>1$, $d(i,j)=d(i-1,j)+d(i,j-1)+d(i-1,j-1)$. Compute the remainder when $d(3,2016)$ is divided by 1000.

3 Round 3

Problem 3.1. For odd positive integers n, define $f(n)$ to be the smallest odd integer greater than n that is not relatively prime to n. Compute the smallest n such that $f(f(n))$ is not divisible by 3.

Problem 3.2. Alice, Bob, Charlie, Diana, Emma, and Fred sit in a circle, in that order, and each roll a six-sided die. Each person looks at his or her own roll, and also looks at the roll of either the person to the right or to the left, deciding at random. Then, at the same time, Alice, Bob, Charlie, Diana, Emma and Fred each state the expected sum of the dice rolls based on the information they have. All six people say different numbers; in particular, Alice, Bob, Charlie, and Diana say 19, 22, 21, and 23, respectively. Compute the product of the dice rolls.

Problem 3.3. Let C be a right circular cone with apex A. Let P_1, P_2, P_3, P_4 and P_5 be points placed evenly along the circular base in that order, so that $P_1P_2P_3P_4P_5$ is a regular pentagon. Suppose that the shortest path from P_1 to P_3 along the curved surface of the cone passes through the midpoint of AP_2. Let h be the height of C, and r be the radius of the circular base of C. If $\left(\frac{h}{r}\right)^2$ can be written in simplest form as $\frac{a}{b}$, find $a+b$.

4 Round 4

Problem 4.1. Let $\triangle\ ABC$ be a triangle with integer side lengths such that $BC=2016$. Let G be the centroid of $\triangle\ ABC$ and I be the incenter of $\triangle\ ABC$. If the area of $\triangle\ BGC$ equals the area of $\triangle\ BIC$, find the largest possible length of AB.

Problem 4.2. A knight is placed at the origin of the Cartesian plane. Each turn, the knight moves in an chess L-shape (2 units parallel to one axis and 1 unit parallel to the other) to one of eight possible location, chosen at random. After 2016 such turns, what is the expected value of the square of the distance of the knight from the origin?

Problem 4.3. Compute the sum of the two smallest positive integers b with the following property: there are at least ten integers $0 \le n < b$ such that n^2 and n end in the same digit in base b.

5 Round 5

Problem 5.1. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence where for all positive integers i, a_i is chosen to be a random positive integer between 1 and 2016, inclusive. Let S be the set of all positive integers k such that for all positive integers $j < k$, $a_j \neq a_k$. (So $1 \in S$; $2 \in S$ if and only if $a_1 \neq a_2$; $3 \in S$ if and only if $a_1 \neq a_3$ and $a_2 \neq a_3$; and so on.) In simplest form, let $\dfrac{p}{q}$ be the expected number of positive integers m such that m and $m+1$ are in S. Compute pq.

Problem 5.1 (versão em Português). Seja $a_1, a_2, a_3, \ldots$ uma sequência infinita em que, para todo inteiro positivo i, o número a_i é um inteiro positivo escolhido aleatóriamente (de maneira uniforme) entre 1 e 2016, inclusive. Seja S o conjunto de todos os inteiros positivos k com a seguinte propriedade: para todo inteiro positivo j, com $j < k$, vale que $a_j \neq a_k$. (Portanto, $1 \in S$; $2 \in S$ se e somente se $a_1 \neq a_2$; $3 \in S$ se e somente se $a_1 \neq a_3$ e $a_2 \neq a_3$.) Seja $\dfrac{p}{q}$ (com p e q coprimos) o valor experado da quantidade de inteiros positivos m tais que ambos m e $m+1$ estão em S. Calcule pq.

Problem 5.2. Let $k = 2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53$. Let S be the sum of $\frac{gcd(m,n)}{lcm(m,n)}$ over all ordered pairs of positive integers (m, n) where $mn = k$. If S can be written in simplest form as $\frac{r}{s}$, compute $r + s$.

Remark. $\gcd(m, n)$ denotes the greater common divisor ("maior divisor comum", in Portuguese) of m and n, and $\operatorname{lcm}(m, n)$ denotes the lower common multiple ("menor múltiplo comum", in Portuguese) of m and n.

Problem 5.3. Let D, E, and F respectively be the feet of the altitudes from A, B, and C of acute triangle $\triangle ABC$ such that $AF = 28, FB = 35$ and $BD = 45$. Let P be the point on segment BE such that $AP = 42$. Find the length of CP.

6 Round 6

Problem 6.1. Find the sum of the four smallest prime divisors of $2016^{239} - 1$.

Problem 6.2. In isosceles triangle ABC with base BC, let M be the midpoint of BC. Let P be the intersection of the circumcircle of $\triangle ACM$ with the circle with center B passing through M, such that $P \neq M$. If $\angle BPC = 135^o$, then $\frac{CP}{AP}$ can be written as $a + \sqrt{b}$ for positive integers a and b, where b is not divisible by the square of any prime. Find $a + b$.

Problem 6.3. The George Washington Bridge is 2016 meters long. Sally is standing on the George Washington Bridge, 1010 meters from its left end. Each step, she either moves 1 meter to the left or 1 meter to the right, each with probability $\dfrac{1}{2}$. What is the expected number of steps she will take to reach an end of the bridge?

7 Round 7

Problem 7.1. Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10, AD = 24, XA = 17$, and $XB = 21$, then the area of $\triangle DEY$ can be written in simplest form as $\frac{m}{n}$. Find $m + n$.

Problem 7.2. The Dinky is a train connecting Princeton to the outside world. It runs on an odd schedule: the train arrive once every one-hour block at some uniformly random time (once at a random time between 9am and 10am, once at a random time between 10am and 11am, and so on). One day, Emilia arrives at the station, at some uniformly random time, and does not know the time. She expects to wait for y minutes for the next train to arrive. After waiting for an hour, a train has still not come. She now expects to wait for z minutes. Find yz.

Problem 7.3. Compute the number of positive integers n between 2017 and 2017^2 such that $n^n \equiv 1 \pmod{2017}$. (2017 is prime.)

8 Round 8

Problem 8.1. Let $n = 2^8 \cdot 3^9 \cdot 5^{10} \cdot 7^{11}$. For k a positive integer, let $f(k)$ be the number of integers $0 \leq x < n$ such that $x^2 \equiv k^2 \pmod{n}$. Compute the number of positive integers k such that $k | f(k)$.

Problem 8.2. Let $\triangle$ ABC have side lengths $AB = 4, BC = 6, CA = 5$. Let M be the midpoint of BC and let P be the point on the circumcircle of $\triangle$ ABC such that $\angle MPA = 90^o$. Let D be the foot of the altitude from B to AC, and let E be the foot of the altitude from C to AB. Let PD and PE intersect line BC at X and Y , respectively. Compute the square of the area of $\triangle$ AXY .

Problem 8.3. Katie Ledecky and Michael Phelps each participate in 7 swimming events in the Olympics (and there is no event that they both participate in). Ledecky receives g_L gold, s_L silver, and b_L bronze medals, and Phelps receives g_P gold, s_P silver, and b_P bronze medals. Ledecky notices that she performed objectively better than Phelps: for all positive real numbers $w_b < w_s < w_g$, we have

$$w_g g_l + w_s s_L + w_b b_L > w_g g_P + w_s s_P + w_b b_P.$$

Compute the number of possible 6-tuples $(g_L, s_L, b_L, g_P, s_P, b_P)$.