Dia 1

PROBLEMA 1
The positive integers $a_0, a_1, a_2, \ldots, a_{3030}$ satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \ldots, 3028.$$

Prove that at least one of the numbers $a_0, a_1, a_2, \ldots, a_{3030}$ is divisible by 2^{2020}.

PROBLEMA 2
Find all lists $(x_1, x_2, \ldots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:$x_1 \leq x_2 \leq \ldots \leq x_{2020}$;$x_{2020} \leq x_1 + 1$;there is a permutation $(y_1, y_2, \ldots, y_{2020})$ of $(x_1, x_2, \ldots, x_{2020})$ such that

$$\sum_{i=1}^{2020} ((x_i + 1)(y_i + 1))^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

A permutation of a list is a list of the same length, with the same entries, but the entries are allowed to be in any order. For example, $(2, 1, 2)$ is a permutation of $(1, 2, 2)$, and they are both permutations of $(2, 2, 1)$. Note that any list is a permutation of itself.

PROBLEMA 3
Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A$, $\angle C$, and $\angle E$ are concurrent.

Prove that the (interior) angle bisectors of $\angle B$, $\angle D$, and $\angle F$ must also be concurrent.Note that $\angle A = \angle FAB$. The other interior angles of the hexagon are similarly described.

Dia 2

PROBLEMA 4
A permutation of the integers $1, 2, \ldots, m$ is called fresh if there exists no positive integer $k < m$ such that the first k numbers in the permutation are $1, 2, \ldots, k$ in some order. Let f_m be the number of fresh permutations of the integers $1, 2, \ldots, m$.

Prove that $f_n \geq n \cdot f_{n-1}$ for all $n \geq 3$.For example, if $m = 4$, then the permutation $(3, 1, 4, 2)$ is fresh, whereas the permutation $(2, 3, 1, 4)$ is not.

PROBLEMA 5
Consider the triangle ABC with $\angle BCA > 90°$. The circumcircle Γ of ABC has radius R. There is a point P in the interior of the line segment AB such that $PB = PC$ and the length of PA is R. The perpendicular bisector of PB intersects Γ at the points D and E.

Prove P is the incentre of triangle CDE.

PROBLEMA 6
Let $m > 1$ be an integer. A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = a_2 = 1$, $a_3 = 4$, and for all $n \geq 4$,

$$a_n = m(a_{n-1} + a_{n-2}) - a_{n-3}.$$

Determine all integers m such that every term of the sequence is a square.

Dia 1

PROBLEMA 1
Encontre todas as triplas (a, b, c) de números reais tais que $ab + bc + ca = 1$ e

$$a^2b + c = b^2c + a = c^2a + b$$

.

PROBLEMA 2
Seja n um inteiro positivo. Dominós são colocados em um tabuleiro $2n \times 2n$ de forma que toda casa do tabuleiro é adjacente a exatamente uma casa coberta por um dominó. Para cada n, determine o maior número de dominós que podem ser colocados no tabuleiro dessa maneira.

(Um *dominó* é uma peça de tamanho 2×1 ou 1×2. Dominós são colocados no tabuleiro de maneira que cada dominó cobre exatamente 2 casas do tabuleiro, e dominós não podem se sobrepor. Duas casas do tabuleiro são chamadas *adjacentes* se elas são diferentes e têm um lado em comum.)

PROBLEMA 3
Seja ABC um triângulo tal que $\angle CAB > \angle ABC$, e seja I o seu incentro. Seja D o ponto do segmento BC tal que $\angle CAD = \angle ABC$. Seja ω a circunferência tangente a AC em A que também passa por I. Seja X o segundo ponto de interseção entre ω e o circuncírculo de ABC. Prove que as bissetrizes dos ângulos $\angle DAB$ e $\angle CXB$ se intersectam em um ponto na reta BC.

Dia 2

PROBLEMA 4
Seja ABC um triângulo com incentro I. A circunferência que passa por B e é tangente a AI no ponto I intersecta o lado AB novamente no ponto P. A circunferência que passa por C e é tangente a AI no ponto I intersecta o lado AC novamente em Q. Prove que PQ é tangente ao incírculo de ABC.

PROBLEMA 5
Seja $n \geq 2$ um inteiro, e sejam $a_1, a_2, \cdots a_n$ inteiros positivos. Mostre que existem inteiros positivos $b_1, b_2, \cdots, b_n$, que satizfazem as seguintes três condições:

1. $a_i \leq b_i$, para $i = 1, 2, 3, \ldots, n$;
2. os restos de $b_1, b_2, \ldots, b_n$ na divisão por n são distintos dois a dois;
3. $b_1 + b_2 + \cdots + b_n \leq n \cdot \left(\frac{n-1}{2} + \left\lceil \frac{a_1 + a_2 + \cdots + a_n}{n} \right\rceil \right)$

($\lceil x \rceil$ denota a parte inteira do número real x, isto é, o maior inteiro que é menor ou igual a x.)

PROBLEMA 6
Alina traça 2019 cordas em uma circunferência. Os pontos extremos dessas cordas são todos distintos. Um ponto é considerado *marcado* se ele é de um dos seguintes tipos:

(a) um dos 4038 pontos extremos das cordas; ou

(b) um ponto de interseção de pelo menos duas das cordas.

Alina escreve um número em cada ponto marcado. Dos 4038 pontos do tipo (i), ela escreve o número 0 em 2019 destes pontos, e escreve o número 1 nos outros 2019 pontos. Em cada ponto do tipo (ii), Alina escreve um inteiro qualquer (não necessariamente positivo).

Em cada corda, Alina considera os segmentos que conectam 2 pontos marcados consecutivos (uma corda com k pontos marcados tem $k - 1$ desses segmentos). Em cada um desses segmentos ela escreve 2 números. Em amarelo, ela escreve a soma dos números escritos nos pontos extremos desse segmento. Em azul, ela escreve o valor absoluto da diferença dos números escritos nos pontos extremos dessesegmento.

Alina percebe que os $N + 1$ números escritos em amarelo são exatamente os números $0, 1, \ldots, N$. Mostre que pelo menos um dos números escritos em azul é um número múltiplo de 3.

(Uma corda é um segmento de reta que conecta 2 pontos distintos de uma circunferência.)

Dia 1

PROBLEMA 1
Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120°$, and let M be the midpoint of AB. Let P be a variable point on the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N.

Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P.

PROBLEMA 2
Consider the set

$$A = \left\{1 + \frac{1}{k} : k = 1, 2, 3, \ldots\right\}.$$

(a) Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

(b) For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different.

Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2, y \geq 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$.)

PROBLEMA 3
The n contestant of EGMO are named $C_1, \ldots, C_n$. After the competition they queue in front of the restaurant according to the following rules.

- The Jury chooses the initial order of the contestants in the queue.
- Every minute, the Jury chooses an integer i with $1 \leq i \leq n$.
  - If contestant C_i has at least i other contestants in front of her, she pays one euro to the Jury and moves forward in the queue by exactly i positions.
  - If contestant C_i has fewer than i other contestants in front of her, the restaurant opens and the process ends.

(a) Prove that the process cannot continue indefinitely, regardless of the Jury?s choices.

(b) Determine for every n the maximum number of euros that the Jury can collect by cunningly choosing the initial order and the sequence of moves.

Dia 2

PROBLEMA 4
A *domino* is a 1×2 or 2×1 tile.

Let $n \geq 3$ be an integer. Dominoes are placed on an $n \times n$ board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap.

The *value* of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called *balanced* if there exists some $k \geq 1$ such that each row and each column has a value of k.

Prove that a balanced configuration exists for every $n \geq 3$, and find the minimum number of dominoes needed in such a configuration.

PROBLEMA 5
Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q.

PROBLEMA 6 (a) Prove that for every real number t such that $0 < t < \frac{1}{2}$ there exists a positive integer n with the following property: for every set S of n positive integers there exist two different elements x and y of S, and a *non-negative* integer m (i.e. $m \geq 0$), such that

$$|x - my| \leq ty.$$

(b) Determine whether for every real number t such that $0 < t < \frac{1}{2}$ there exists an infinite set S of positive integers such that

$$|x - my| > ty$$

for every pair of different elements x and y of S and every *positive* integer m (i.e. $m > 0$).

Dia 1

PROBLEMA 1
Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90°$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$. Let the midpoint of BD be M and the midpoint of QR be N. Prove that the points M, N, A, and C lie on a circle.

PROBLEMA 2
Find the smallest positive integer k for which there exists a colouring of the positive integers $\mathbb{Z}_{>0}$ with k colours and a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

(i) For all positive integers m, n of the same colour, $f(m + n) = f(m) + f(n)$.

(ii) There are positive integers m, n such that $f(m + n) \neq f(m) + f(n)$.

In a colouring of $\mathbb{Z}_{>0}$ with k colours, every integer is coloured in exactly one of the k colours. In both (i) and (ii) the positive integers m, n are not necessarily distinct.

PROBLEMA 3
There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

Dia 2

PROBLEMA 4
Let $n \geq 1$ be an integer and let $t_1 < t_2 < \cdots < t_n$ be positive integers. In a group of $t_n + 1$ people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following two conditions to hold at the same time:

(i) The number of games played by each person is one of $t_1, t_2, \ldots, t_n$.

(ii) For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

PROBLEMA 5
Let $n \geq 2$ be an integer. An n-tuple $(a_1, a_2, \ldots, a_n)$ of not necessarily different positive integers is *expensive* if there exists a positive integer k such that

$$(a_1 + a_2)(a_2 + a_3) \ldots (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}.$$

a) Find all integers $n \geq 2$ for which there exists an expensive n-tuple.

b) Prove that for every odd positive integer m there exists an integer $n \geq 2$ such that m belongs to an expensive n-tuple.

There are exactly n factors in the product on the left hand side.

PROBLEMA 6
Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcentre O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3 and O_1, O_2, O_3, respectively. Show that the circumcircles of triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

The centroid of a triangle is the intersection point of the three medians. A median is a line connecting a vertex of the triangle to the midpoint of the opposite side.

Dia 1

PROBLEMA 1
Let n be an odd positive integer, and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Show that

$$\min_{i=1,\ldots,n}\left(x_i^2+x_{i+1}^2\right) \leq \max_{j=1,\ldots,n}\left(2x_jx_{j+1}\right),$$

where $x_{n+1} = x_1$.

PROBLEMA 2
Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X. Let C_1, D_1 and M be the midpoints of segments CX, DX and CD, respecctively. Lines AD_1 and BC_1 intersect at Y, and line MY intersects diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E, F and X.

PROBLEMA 3
Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are *related* to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are coloured blue, such that every cell is related to at lest two blue cells. Determine the minimum number of blue cells.

Dia 2

PROBLEMA 4
Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1, and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

PROBLEMA 5
Let k and n be integers such that $k \geq 2$ and $k \leq n \leq 2k-1$. Place rectangular tiles, each of size $1 \times k$, or $k \times 1$ on a $n \times n$ chessboard so that each tile covers exactly k cells and no two tiles overlap. Do this until no further tile can be placed in this way. For each such k and n, determine the minimum number of tiles that such an arrangement may contain.

PROBLEMA 6
Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2+1, n^2+2, \ldots, n^2+2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m+1, 7m+2, 7m+5, 7m+6$ and no elements of S of the form $7m+3$ and $7m+4$, where m is an integer.

Dia 1

PROBLEMA 1
Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45°$, show that CF is tangent to ω.

PROBLEMA 2
A *domino* is a 2×1 or 1×2 tile. Determine in how many ways exactly n^2 dominoes can be placed without overlapping on a $2n \times 2n$ chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column.

PROBLEMA 3
Let n, m be integers greater than 1, and let $a_1, a_2, \ldots, a_m$ be positive integers not greater than n^m. Prove that there exist positive integers $b_1, b_2, \ldots, b_m$ not greater than n, such that

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) < n,$$

where $\gcd(x_1, x_2, \ldots, x_m)$ denotes the greatest common divisor of $x_1, x_2, \ldots, x_m$.

Dia 2

PROBLEMA 4
Determine whether there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers which satisfies the equality

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1} + a_n}$$

for every positive integer n.

PROBLEMA 5
Let m, n be positive integers with $m > 1$. Anastasia partitions the integers $1, 2, \ldots, 2m$ into m pairs. Boris then chooses one integer from each pair and finds the sum of these chosen integers. Prove that Anastasia can select the pairs so that Boris cannot make his sum equal to n.

PROBLEMA 6
Let H be the orthocentre and G be the centroid of acute-angled triangle ABC with $AB \neq AC$. The line AG intersects the circumcircle of ABC at A and P. Let P' be the reflection of P in the line BC. Prove that $\angle CAB = 60$ if and only if $HG = GP'$.

Dia 1

PROBLEMA 1
Determine all real constants t such that whenever a, b and c are the lengths of sides of a triangle, then so are $a^2 + bct$, $b^2 + cat$, $c^2 + abt$.

PROBLEMA 2
Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Let the lines CD and BE meet at F. Prove that the incentre I of triangle ABC, the orthocentre H of triangle DEF and the midpoint M of the arc BAC of the circumcircle of triangle ABC are collinear.

PROBLEMA 3
We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2 + b^2)$ for any positive integers a, b satisfying $a + b = n$.

Dia 2

PROBLEMA 4
Determine all positive integers $n \geq 2$ for which there exist integers $x_1, x_2, \ldots, x_{n-1}$ satisfying the condition that if $0 < i < n, 0 < j < n, i \neq j$ and n divides $2i + j$, then $x_i < x_j$.

PROBLEMA 5
Let n be a positive integer. We have n boxes where each box contains a non-negative number of pebbles. In each move we are allowed to take two pebbles from a box we choose, throw away one of the pebbles and put the other pebble in another box we choose. An initial configuration of pebbles is called *solvable* if it is possible to reach a configuration with no empty box, in a finite (possibly zero) number of moves. Determine all initial configurations of pebbles which are not solvable, but become solvable when an additional pebble is added to a box, no matter which box is chosen.

PROBLEMA 6
Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f\left(y^2 + 2xf(y) + f(x)^2\right) = (y + f(x))(x + f(y))$$

for all real numbers x and y.

Dia 1

PROBLEMA 1
The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled.

PROBLEMA 2
Determine all integers m for which the $m \times m$ square can be dissected into five rectangles, the side lengths of which are the integers $1, 2, 3, \ldots, 10$ in some order.

PROBLEMA 3
Let n be a positive integer.

(a) Prove that there exists a set S of $6n$ pairwise different positive integers, such that the least common multiple of any two elements of S is no larger than $32n^2$.

(b) Prove that every set T of $6n$ pairwise different positive integers contains two elements the least common multiple of which is larger than $9n^2$.

Dia 2

PROBLEMA 4
Find all positive integers a and b for which there are three consecutive integers at which the polynomial

$$P(n) = \frac{n^5 + a}{b}$$

takes integer values.

PROBLEMA 5
Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q.

Prove that $\angle ACP = \angle QCB$.

PROBLEMA 6
Snow White and the Seven Dwarves are living in their house in the forest. On each of 16 consecutive days, some of the dwarves worked in the diamond mine while the remaining dwarves collected berries in the forest. No dwarf performed both types of work on the same day. On any two different (not necessarily consecutive) days, at least three dwarves each performed both types of work. Further, on the first day, all seven dwarves worked in the diamond mine.

Prove that, on one of these 16 days, all seven dwarves were collecting berries.

Dia 1

PROBLEMA 1
Let ABC be a triangle with circumcentre O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By interior we mean, for example, that the point D lies on the line BC and D is between B and C on that line.)

Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular.

PROBLEMA 2
Let n be a positive integer. Find the greatest possible integer m, in terms of n, with the following property: a table with m rows and n columns can be filled with real numbers in such a manner that for any two different rows $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ the following holds:

$$\max\left(\left|a_1 - b_1\right|, \left|a_2 - b_2\right|, \ldots, \left|a_n - b_n\right|\right) = 1$$

PROBLEMA 3
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(yf\left(x+y\right) + f\left(x\right)\right) = 4x + 2yf\left(x+y\right)$$

for all $x, y \in \mathbb{R}$.

PROBLEMA 4
A set A of integers is called *sum-full* if $A \subseteq A + A$, i.e. each element $a \in A$ is the sum of some pair of (not necessarily different) elements $b, c \in A$. A set A of integers is said to be *zero-sum-free* if 0 is the only integer that cannot be expressed as the sum of the elements of a finite nonempty subset of A.

Does there exist a sum-full zero-sum-free set of integers?

Dia 2

PROBLEMA 5
The numbers p and q are prime and satisfy

$$\frac{p}{p+1} + \frac{q+1}{q} = \frac{2n}{n+2}$$

for some positive integer n. Find all possible values of $q - p$.

PROBLEMA 6
There are infinitely many people registered on the social network Mugbook. Some pairs of (different) users are registered as friends, but each person has only finitely many friends. Every user has at least one friend. (Friendship is symmetric; that is, if A is a friend of B, then B is a friend of A.)

Each person is required to designate one of their friends as their best friend. If A designates B as her best friend, then (unfortunately) it does not follow that B necessarily designates A as her best friend. Someone designated as a best friend is called a 1-best friend. More generally, if $n > 1$ is a positive integer, then a user is an n-best friend provided that they have been designated the best friend of someone who is an $(n-1)$-best friend. Someone who is a k-best friend for every positive integer k is called popular.

(a) Prove that every popular person is the best friend of a popular person.

(b) Show that if people can have infinitely many friends, then it is possible that a popular person is not the best friend of a popular person.

PROBLEMA 7
Let ABC be an acute-angled triangle with circumcircle Γ and orthocentre H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM.

Show that the lines KH, EM and BC are concurrent. (The orthocentre of a triangle is the point on all three of its altitudes.)

PROBLEMA 8
A *word* is a finite sequence of letters from some alphabet. A word is *repetitive* if it is a concatenation of at least two identical subwords (for example, *ababab* and *abcabc* are repetitive, but *ababa* and *aabb* are not). Prove that if a word has the property that swapping any two adjacent letters makes the word repetitive, then all its letters are identical. (Note that one may swap two adjacent identical letters, leaving a word unchanged.)