# Olimpíada Internacional de Matemática (1978 - 2020)

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Todo e qualquer feedback, especialmente sobre erros neste livreto (mesmo erros tipográficos pequenos), é apreciado. Você também pode contribuir enviando suas soluções (de preferência, formatadas em TEX).

Você pode enviar comentários e soluções para zeusdanmou+tex@gmail.com.

A versão mais atualizada desse arquivo (provavelmente) está disponível clicando aqui. Última atualização: 9 de janeiro de 2021.

## Dia I

**PROBLEMA 1**
Considere o quadrilátero convexo $ABCD$. O ponto $P$ está no interior do $ABCD$. Verificam-se as seguintes igualdades entre razões:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove que as três seguintes retas se intersectam num ponto: as bissetrizes internas dos ângulos $\angle ADP$ e $\angle PCB$ e a metriatriz do segmento $AB$.

**PROBLEMA 2**
Os números reais $a$, $b$, $c$, $d$ são tais que $a \geq b \geq c \geq d > 0$ e $a + b + c + d = 1$. Prove que

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1.$$

**PROBLEMA 3**
Temos $4n$ pedras com pesos $1, 2, 3, \cdots, 4n$. Cada pedra está colorida com uma de $n$ cores e há quatro pedras de cada cor. Mostre que podemos organizar as pedras em dois grupos de modo que as seguintes condições são ambas satisfeitas:

- Os pesos totais dos dois grupos são iguais.
- Cada grupo contém duas pedras de cada cor.

## Dia II

**PROBLEMA 4**
Seja $n > 1$ um inteiro. Na encosta de uma montanha existem $n^2$ estações, todas com diferentes altitudes. Duas companhias de teleféricos, $A$ e $B$, operam $k$ teleféricos cada uma. Cada teleférico faz a viagem de uma estação para uma de maior altitude (sem paragens intermediárias). Os $k$ teleféricos de $A$ partem de $k$ estações diferentes e terminam em $k$ estações diferentes; além disso, se um teleférico parte de uma estação de maior altitude do que a partida de outro, também termina numa estação de menor altitude do que a de chegada do outro. A companhia $B$ satisfaz as mesmas condições. Dizemos que duas estações estão *ligadas* por uma companhia se podemos começar na estação com menor altitude e chegar à de maior altitude usando um ou mais teleféricos dessa companhia (não são permitidos quaisquer outros movimentos entre estações).

Determine o menor inteiro positivo $k$ que garante que existam duas estações ligadas por ambas companhias.

**PROBLEMA 5**
Temos um baralho de $n > 1$ cartas, com um inteiro positivo escrito em cada carta. O baralho tem a propriedade de que a média aritmética dos números escritos em cada par de cartas é também a média geométrica dos números escritos nalguma coleção de uma ou mais cartas. Para que valores de $n$ podemos concluir que os números escritos nas cartas são todos iguais?

**PROBLEMA 6**
Prove que existe uma constante positiva $c$ para a qual a seguinte afirmação é verdadeira:

Considere um inteiro $n > 1$, e um conjunto $\mathcal{S}$ de $n$ pontos no plano tal que a distância entre quaisquer dois pontos diferentes de $\mathcal{S}$ é pelo menos 1. Então existe uma reta $\ell$ que separa $\mathcal{S}$ tal que a distância de qualquer ponto de $\mathcal{S}$ a $\ell$ é pelo menos $cn^{-1/3}$.

(Uma reta $\ell$ *separa* um conjunto de pontos $\mathcal{S}$ se existe algum segmento com extremos em dois pontos de $\mathcal{S}$ que interseta $\ell$.)

*Observação.* A resultados mais fracos obtidos substituindo $cn^{-1/3}$ por $cn^{-\alpha}$ podem ser atribuídos pontos dependendo do valor da constante $\alpha > 1/3$.

## Dia I

**PROBLEMA 1**
Determine todas as funções $f : \mathbb{Z} \to \mathbb{Z}$ tais que

$$f(2a) + 2f(b) = f(f(a + b))$$

para quaisquer $a$ e $b$ inteiros.

**PROBLEMA 2**
No triângulo $ABC$, o ponto $A_1$ está no lado $BC$ e o ponto $B_1$ está no lado $AC$. Sejam $P$ e $Q$ pontos nos segmentos $AA_1$ e $BB_1$, respectivamente, tal que $PQ$ é paralelo a $AB$. Seja $P_1$ um ponto na reta $PB_1$, tal que $B_1$ está estritamente entre $P$ e $P_1$ e $\angle PP_1C = \angle BAC$. Analogamente, seja $Q_1$ um ponto na reta $QA_1$, tal que $A_1$ está estritamente entre $Q$ e $Q_1$ e $\angle CQ_1Q = \angle CBA$. Prove que os pontos $P$, $Q$, $P_1$ e $Q_1$ são concíclicos.

**PROBLEMA 3**
Uma rede social possui 2019 usuários, alguns deles são amigos. Sempre que o usuário $A$ é amigo do usuário $B$, o usuário $B$ também é amigo do usuário $A$. Eventos do seguinte tipo podem acontecer repetidamente, um de cada vez:

Três usuários $A$, $B$ e $C$ tais que $A$ é amigo de $B$ e $A$ é amigo de $C$, mas $B$ e $C$ não são amigos, mudam seus estados de amizade de modo que $B$ e $C$ agora são amigos, mas $A$ deixa de ser amigo de $B$ e $A$ deixa de ser amigo de $C$. Todos os outros estados de amizade não são alterados.

Inicialmente, 1010 usuários possuem exatamente 1009 amigos cada e 1009 usuários possuem exatamente 1010 amigos cada. Prove que existe uma sequência de tais eventos tal que, após essa sequência, cada usuário é amigo de no máximo um outro usuário.

## Dia II

**PROBLEMA 4**
Encontre todos os pares $(k, n)$ de inteiros positivos tais que

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

**PROBLEMA 5**
O Banco de Bath emite moedas com um $H$ num lado e um $T$ no outro. Harry possui $n$ dessas moedas colocadas em linha, ordenadas da esquerda para a direita. Ele repetidamente realiza a seguinte operação: se há exatamente $k > 0$ moedas mostrando $H$, então ele vira a $k$-ésima moeda contada da esquerda para a direita; caso contrário, todas as moedas mostram $T$ e ele para. Por exemplo, se $n = 3$ o processo começando com a configuração $THT$ é $THT \to HHT \to HTT \to TTT$, que acaba depois de três operações.

(a) Mostre que, para qualquer configuração inicial, Harry para após um número finito de operações.

(b) Para cada configuração inicial $C$, seja $L(C)$ o número de operações antes de Harry parar. Por exemplo, $L(THT) = 3$ e $L(TTT) = 0$. Determine a média de $L(C)$ sobre todas as $2^n$ possíveis configurações iniciais $C$.

**PROBLEMA 6**
Seja $I$ o incentro do triângulo acutângulo $ABC$ com $AB \neq AC$. A circunferência inscrita (incírculo) $\omega$ de $ABC$ é tangente aos lados $BC$, $CA$ e $AB$ nos pontos $D$, $E$ e $F$, respectivamente. A reta que passa por $D$ perpendicular a $EF$ intersecta $\omega$ novamente em $R$. A reta $AR$ intersecta $\omega$ novamente em $P$. As circunferências circunscritas (circuncírculos) dos triângulos $PCE$ e $PBF$ se intersectam novamente no ponto $Q$. Prove que as retas $DI$ e $PQ$ se intersectam sobre a reta que passa por $A$ perpendicular a $AI$.

## Dia I

**PROBLEMA 1**
Seja $\Gamma$ o circuncírculo do triângulo acutângulo $ABC$. Os pontos $D$ e $E$ estão sobre os segmentos $AB$ e $AC$, respectivamente, de modo que $AD = AE$. As mediatrizes de $BD$ e $CE$ intersectam os arcos menores $AB$ e $AC$ de $\Gamma$ nos pontos $F$ e $G$, respectivamente. Prove que as retas $DE$ e $FG$ são paralelas (ou são a mesma reta).

**PROBLEMA 2**
Determine todos os inteiros $n \geq 3$ para os quais existem números reais $a_1, a_2, \ldots, a_{n+2}$, tais que $a_{n+1} = a_1$, $a_{n+2} = a_2$ e

$$a_i a_{i+1} + 1 = a_{i+2}$$

para $i = 1, 2, \ldots, n$.

**PROBLEMA 3**
Um triângulo *anti-Pascal* é uma disposição de números em forma de triângulo equilátero tal que, exceto para os números na última linha, cada número é o módulo da diferença entre os dois números imediatamente abaixo dele. Por exemplo, a seguinte disposição de números é um triângulo anti-Pascal com quatro linhas que contém todos os inteiros de 1 até 10.

4

2 6

5 7 1

8 3 10 9

Determine se existe um triângulo anti-Pascal com 2018 linhas que contenha todos os inteiros de 1 até $1+2+\cdots+2018$.

## Dia II

**PROBLEMA 4**
Um *local* é um ponto $(x, y)$ no plano tal que $x$ e $y$ são ambos inteiros positivos menores ou iguais a 20.

Inicialmente, cada um dos 400 locais está vazio. Ana e Beto colocam pedras alternadamente com Ana a iniciar. Na sua vez, Ana coloca uma nova pedra vermelha num local vazio tal que a distância entre quaisquer dois locais ocupados por pedras vermelhas seja diferente de $\sqrt{5}$. Na sua vez, Beto coloca uma nova pedra azul em qualquer local vazio. (Um local ocupado por uma pedra azul pode estar a qualquer distância de outro local ocupado.) Eles param quando um dos jogadores não pode colocar uma pedra.

Determine o maior $K$ tal que Ana pode garantir que ela coloca pelo menos $K$ pedras vermelhas, não importando como Beto coloca suas pedras azuis.

**PROBLEMA 5**
Sejam $a_1, a_2, \ldots$ uma sequência infinita de inteiros positivos. Suponha que existe um inteiro $N > 1$ tal que, para cada $n \geq N$, o número

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n+1}}{a_n} + \frac{a_n}{a_1}$$

é um inteiro. Prove que existe um inteiro positivo $M$ tal que $am = a_{m+1}$ para todo $m \geq M$.

**PROBLEMA 6**
Um quadrilátero convexo $ABCD$ satisfaz $AB \cdot CD = BC \cdot DA$. O ponto $X$ está no interior de $ABCD$ de modo que

$$\angle XAB = \angle XCD \quad \text{e} \quad \angle XBC = \angle XDA$$

Prove que $\angle BXA + \angle DXC = 180°$.

## Dia I

**PROBLEMA 1**
Para cada inteiro $a_0 < 1$, define-se a sequência $a_0, a_1, a_2, \ldots$ tal que, para cada $n \geq 0$:

$$a_{n+1} = \begin{cases} \sqrt{a_n}, & \text{se } \sqrt{a_n} \text{ é inteiro,} \\ a_n + 3, & \text{caso contrário.} \end{cases}$$

Determine todos os valores de $a_0$ para os quais existe um número $A$ tal que $a_n = A$ para infinitos valores de $n$.

**PROBLEMA 2**
Determine todas as funções $f : \mathbb{R} \to \mathbb{R}$ tais que

$$f(f(x)f(y)) + f(x + y) = f(xy)$$

para todos $x$ e $y$ reais.

**PROBLEMA 3**
Um coelho invisível e um caçador jogam da seguinte forma no plano euclidiano. O ponto de partida $A_0$ do coelho e o ponto de partida $B_0$ são iguais. Depois de $n-1$ rodadas do jogo, o coelho encontra-se no ponto $A_{n-1}$ e o caçador encontra-se no potno $B_{n-1}$. Na $n$-ésima jogada do jogo, ocorrem três coisas na seguinte ordem:

- O coelho move-se de forma invisível para um ponto $A_n$ tal que a distância entre $A_{n-1}$ e $A_n$ é exatamente 1.
- Um aparelho de localização informa um ponto $P_n$ ao caçador. A única informação garantida pelo aparelho ao caçador é que a distância entre $P_n$ e $A_n$ é menor ou igual a 1.
- O caçador move-se de forma visível para um ponto $B_n$ tal que a distância entre $B_{n-1}$ e $B_n$ é exatamente 1.

É sempre possível que, qualquer que seja a maneira em que se mova o coelho e quaisquer que sejam os pontos informados pelo aparelho de localização, o caçador possa escolher os seus movimentos de modo que depois de $10^9$ rodadas o caçador possa garantir que a distância entre ele e o coelho seja menor ou igual que 100?

## Dia II

**PROBLEMA 4**
Sejam $R$ e $S$ pontos distintos sobre a circunferência $\Omega$ tal que $RS$ não é um diâmetro. Seja $\ell$ a reta tangente a $\Omega$ em $R$. O ponto $T$ é tal que $S$ é o ponto médio do segmento $RT$. O ponto $J$ escolhe-se no menor arco $RS$ de $\Omega$ de maneira que $\Gamma$, a circunferência circunscrita ao triângulo $JST$, intersecta $\ell$ em dois pontos distintos. Seja $A$ o ponto comum de $\Gamma$ e $\ell$ mais próximo de $R$. A reta $AJ$ intersecta pela segunda vez $\Omega$ em $K$. Demonstre que a reta $KT$ é tangente a $\Gamma$.

**PROBLEMA 5**
Seja $N \geq 2$ um inteiro dado. Um conjunto de $N(N+1)$ jogadores de futebol, todos de diferentes alturas, colocam-se na fila. O treinador deseja retirar $N(N-1)$ jogadores desta fila, de modo que a fila que sobra formada pelos $2N$ jogadores restantes satisfaça as $N$ condições seguintes:

(1) Não resta ninguém entre os dois jogadores mais altos.

(2) Não resta ninguém entre o terceiro jogador mais alto e o quarto jogador mais alto.

$\vdots$

($N$) Não resta ninguém entre os dois jogadores mais baixos.

Demonstre que isto é sempre possível.

**PROBLEMA 6**
Um par ordenado $(x, y)$ de inteiros é *um ponto primitivo* se o máximo divisor comum entre $x$ e $y$ é 1. Dado um conjunto finito $S$ de pontos primitivos, demonstre que existem um inteiro positivo $n$ e inteiros $a_0, a_1, \ldots, a_n$ tais que, para cada $(x, y)$ de $S$, se verifica:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

## Dia I

**PROBLEMA 1**
O triângulo $BCF$ é retângulo em $B$. Seja $A$ o ponto da reta $CF$ tal que $FA = FB$ e que $F$ esteja entre $A$ e $C$. Escolhe-se o ponto $D$ de modo que $DA = DC$ e que $AC$ seja bissetriz de $\angle DAB$. Escolhe-se o ponto $E$ de modo que $EA = ED$ e que $AD$ seja a bissetriz de $\angle EAC$. Seja $M$ o ponto médio de $CF$. Seja $X$ o ponto tal que $AMXE$ seja um paralelogramo. Prove que $BD$, $FX$ e $ME$ são concorrentes.

**PROBLEMA 2**
Determine todos os inteiros positivos $n$ tais que pode-se preencher cada casa de um $n \times n$ com uma das letras $I$, $M$ e $O$ de tal forma que ambas as condições seguintes sejam satisfeitas:

- em cada linha e em cada coluna, exatamente um terço das casas tenha um $I$, um terço tenha um $M$ e um terço tenha um $O$;
- em cada diagonal formada por um número de casas que seja múltiplo de 3, exatamente um terço das casas tenha um $I$, um terço tenha um $M$ e um terço tenha um $O$.

*Observação.* As linhas e colunas de um tabuleiro $n \times n$ são numeradas de 1 e $n$. Assim, cada casa corresponde a um par de inteiros positivos $(i, j)$ com $1 \le i, j \le n$. Para $n > 1$, o tabuleiro tem $4n - 2$ *diagonais* de dois tipos. Uma diagonal do primeiro tipo é formada por todas as casas $(i, j)$ para as quais $i + j$ é igual a uma constante. Uma diagonal do segundo tipo é formada por todas as casas $(i, j)$ para as quais $i - j$ é igual a uma constante.

**PROBLEMA 3**
Seja $P = A_1A_2 \cdots A_k$ um polígono convexo no plano. Os vértices $A_1, A_2, \ldots, A_k$ têm coordenadas inteiras e pertencem a uma circunferência. Seja $S$ a área de $P$. Seja $n$ um inteiro positivo ímpar tal que os quadrados dos comprimentos dos lados de $P$ sejam todos números inteiros divisíveis por $n$. Demonstre que $2S$ é um inteiro divisível por $n$.

## Dia II

**PROBLEMA 4**
Um conjunto de números inteiros positivos é chamado *fragante* se contém pelo menos dois elementos e cada um de seus elementos tem algum fator primo em comum com pelo menos um dos elementos restantes. Seja $P(n) = n^2 + n + 1$. Determine o menor número inteiro positivo $b$ para o qual exista algum número inteiro não negativo $a$ tal que

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

seja fragante?

**PROBLEMA 5**
No quadro está a escrita a equação

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

que tem 2016 fatores lineares de cada lado. Determine o menor valor possível de $k$ para o qual é possível apagar exatamente $k$ destes 4032 fatores lineares, de modo que fique pelo menos um fator de cada lado e que a equação resultante não admita nenhuma raiz real.

**PROBLEMA 6**
Há $n \ge 2$ segmentos no plano tais que cada par de segmentos se intersecta num ponto interior a ambos e não há três segmentos que tenham um ponto em comum. Geoff deve escolher um dos extremos de cada segmento e colocar sobre ele um sapo, virado para o outro extremo. Depois Geoff baterá palmas $n - 1$ vezes. Cada vez que Geoff bater as mãos, cada sapo saltará imediatamente para a frente até o próximo ponto de intersecção sobre o seu segmento. Os sapos nunca mudam a direção dos seus saltos. Geoff deseja colocar os sapos de tal forma que dois sapos nunca ocupem ao mesmo tempo o mesmo ponto de intersecção.

(a) Prove que se $n$ é ímpar, Geoff sempre tem uma maneira de realizar o seu desejo.

(b) Prove que se $n$ é par, Geoff nunca realiza o seu desejo.

## Dia I

**PROBLEMA 1**
Dizemos que um conjunto finito $\mathcal{S}$ de pontos no plano é *balanceado* se, para quaisquer dois pontos distintos $a$ e $B$ em $\mathcal{S}$, existe um ponto $C$ em $\mathcal{S}$ tal que $AC = BC$. Dizemos que $\mathcal{S}$ é *livre de centro* se, para quaisquer três pontos distintos $A$, $B$ e $C$ em $\mathcal{S}$, não existe ponto $P$ em $\mathcal{S}$ tal que $PA = PB = PC$.

(a) Mostre que, para todos os inteiros $n \geq 3$, existe um conjunto balanceado com exatamente $n$ pontos.

(b) Determine todos os inteiros $n \geq 3$ para os quais existe um conjunto balanceado livre de centro com exatamente $n$ pontos.

**PROBLEMA 2**
Ache todos os ternos $(a, b, c)$ de inteiros positivos tais que cada um dos números

$$ab - c, \quad bc - a, \quad ca - b$$

é uma potência de 2.

**PROBLEMA 3**
Seja $ABC$ um triângulo acutângulo com $AB > AC$. Sejam $\Gamma$ o seu circuncírculo, $H$ o seu ortocentro, e $F$ o pé da perpendicular a partir de $A$. Seja $M$ o ponto médio de $BC$. Seja $Q$ o ponto de $\Gamma$ tal que $\angle HQA = 90°$, e seja $K$ o ponto de $\Gamma$ tal que $\angle HKQ = 90°$. Admita que os pontos $A$, $B$, $C$, $K$ e $Q$ são todos diferentes, e estão sobre $\Gamma$ nesta ordem.

Prove que os circuncírculos dos triângulos $KQH$ e $FKM$ são tangentes.

## Dia II

**PROBLEMA 4**
O triângulo $ABC$ tem circuncírculo $\Omega$ e circuncentro $O$. Uma circunferência $\Gamma$ de centro $A$ intersecta o segmento $BC$ nos pontos $D$ e $E$, de modo que $B$, $D$, $E$ e $C$ são todos diferentes e estão na reta $BC$, nesta oderm. Sejam $F$ e $G$ os pontos de interseção de $\Gamma$ e $\Omega$, tais que $A$, $F$, $B$, $C$ e $G$ estão em $\Omega$ nesta ordem. Seja $K$ o segundo ponto de interseção do circuncírculo do triângulo $BDF$ com o segmento $AB$. Seja $L$ o segundo ponto de interseção do circuncírculo do triângulo $CGE$ com o segmento $CA$.

Suponha que as retas $FK$ e $GL$ são diferentes e que se intersectam no ponto $X$. Prove que $X$ pertence a reta $AO$.

**PROBLEMA 5**
Seja $\mathbb{R}$ o conjunto dos números reais. Determine todas as funções $f : \mathbb{R} \to \mathbb{R}$ satisfazendo a equação

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

para todos os números reais $x$ e $y$.

**PROBLEMA 6**
A sequência $a_1, a_2, \ldots$ de inteiros satisfaz as condições seguintes:

(i) $1 \leq a_j \leq 2015$ para qualquer $j$ tal que $j \geq 1$,

(ii) $k + a_k \neq \ell + a_\ell$ para quaisquer $k, \ell$ tais que $1 \leq k < \ell$.

Prove que existem dois inteiros positivos $b$ e $N$ tais que

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leq 1007^2$$

para quaisquer inteiros $m$ e $n$ satisfazendo $n > m \geq N$.

## Dia I

**PROBLEMA 1**
Seja $a_0 < a_1 < a_2 < \cdots$ uma sequência infinita de inteiros positivos. Prove que existe um único inteiro $n \geq 1$ tal que

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

**PROBLEMA 2**
Seja $n \geq 2$ um inteiro. Considere um tabuleiro de xadrez $n \times n$ dividido em $n^2$ quadrados unitários. Uma configuração de $n$ torres neste tabuleiro é dita *pacífica* se, e somente se, cada linha e cada coluna contém exatamente uma torre. Encontre o maior inteiro positivo $k$ tal que, para qualquer configuração pacífica de $n$ torres, podemos encontrar um quadrado $k \times k$ sem torres em qualquer um de seus $k^2$ quadrados unitários.

**PROBLEMA 3**
Seja $ABCD$ um quadrilátero convexo com $\angle ABC = \angle CDA = 90°$. O ponto $H$ é o pé da perperndicular de $A$ sobre $BD$. Os pontos $S$ e $T$ são escolhidos sobre os lados $AB$ e $AD$, respectivamente, de modo que $H$ esteja no interior do triângulo $SCT$ e

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove que a reta $BD$ é tangente à circunferência circunscrita ao triângulo $TSH$.

## Dia II

**PROBLEMA 4**
Let $P$ and $Q$ be on segment $BC$ of an acute triangle $ABC$ such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let $M$ and $N$ be the points on $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$ and $Q$ is the midpoint of $AN$. Prove that the intersection of $BM$ and $CN$ is on the circumference of triangle $ABC$.

**PROBLEMA 5**
Para cada inteiro positivo $n$, o Banco da Cidade do Cabo emite moedas de valor $\frac{1}{n}$. Dada uma coleção finita de tais moedas (de valores não necessariamente distintos) com valor total de no máximo $99 + \frac{1}{2}$, prove que é possível particionar essa coleção em 100 ou menos grupos, cada um com valor total de no máximo 1.

**PROBLEMA 6**
A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions.

Prove that for all sufficiently large $n$, in any set of $n$ lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

*Observação.* Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant $c$.

## Dia I

**PROBLEMA 1**
Assume that $k$ and $n$ are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

**PROBLEMA 2**
A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

(i) No line passes through any point of the configuration.

(ii) No region contains points of both colors.

Find the least value of $k$ such that for any Colombian configuration of 4027 points, there is a good arrangement of $k$ lines.

**PROBLEMA 3**
Let the excircle of triangle $ABC$ opposite the vertex $A$ be tangent to the side $BC$ at the point $A_1$. Define the points $B_1$ on $CA$ and $C_1$ on $AB$ analogously, using the excircles opposite $B$ and $C$, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle $ABC$. Prove that triangle $ABC$ is right-angled.

## Dia II

**PROBLEMA 4**
Let $ABC$ be an acute triangle with orthocenter $H$, and let $W$ be a point on the side $BC$, lying strictly between $B$ and $C$. The points $M$ and $N$ are the feet of the altitudes from $B$ and $C$, respectively. Denote by $\omega_1$ is the circumcircle of $BWN$, and let $X$ be the point on $\omega_1$ such that $WX$ is a diameter of $\omega_1$. Analogously, denote by $\omega_2$ the circumcircle of triangle $CWM$, and let $Y$ be the point such that $WY$ is a diameter of $\omega_2$. Prove that $X, Y$ and $H$ are collinear.

**PROBLEMA 5**
Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;

(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

**PROBLEMA 6**
Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, ..., n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle.

A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled $a$ and $d$ does not intersect the chord joining the points labelled $b$ and $c$.

Let $M$ be the number of beautiful labelings, and let N be the number of ordered pairs $(x, y)$ of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

## Dia I

**PROBLEMA 1**
Dado um triângulo $ABC$, o ponto $J$ é o centro da circunferência ex-inscrita oposta ao vértice $A$. Esta circunferência ex-inscrita[1] é tangente ao lado $BC$ em $M$, e às retas $AB$ e $AC$ em $K$ e $L$, respectivamente. As retas $LM$ e $BJ$ intersectam-se em $F$, e as retas $KM$ e $CJ$ intersectam-se em $G$. Seja $S$ o ponto de interseção das retas $AF$ e $BC$, e seja $T$ o ponto de interseção das retas $AG$ e $BC$.

Prove que $M$ é o ponto médio de $ST$.

**PROBLEMA 2**
Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

**PROBLEMA 3**
The liar's guessing game is a game played between two players $A$ and $B$. The rules of the game depend on two positive integers $k$ and $n$ which are known to both players.

At the start of the game $A$ chooses integers $x$ and $N$ with $1 \leq x \leq N$. Player $A$ keeps $x$ secret, and truthfully tells $N$ to player $B$. Player $B$ now tries to obtain information about $x$ by asking player $A$ questions as follows: each question consists of $B$ specifying an arbitrary set $S$ of positive integers (possibly one specified in some previous question), and asking $A$ whether $x$ belongs to $S$. Player $B$ may ask as many questions as he wishes. After each question, player $A$ must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k + 1$ consecutive answers, at least one answer must be truthful.

After $B$ has asked as many questions as he wants, he must specify a set $X$ of at most $n$ positive integers. If $x$ belongs to $X$, then $B$ wins; otherwise, he loses.

Prove that:

(a) If $n \geq 2^k$, then $B$ can guarantee a win.

(b) For all sufficiently large $k$, there exists an integer $n \geq (1.99)^k$ such that $B$ cannot guarantee a win.

## Dia II

**PROBLEMA 4**
Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers $a, b, c$ that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

**PROBLEMA 5**
Let $ABC$ be a triangle with $\angle BCA = 90°$, and let $D$ be the foot of the altitude from $C$. Let $X$ be a point in the interior of the segment $CD$. Let $K$ be the point on the segment $AX$ such that $BK = BC$. Similarly, let $L$ be the point on the segment $BX$ such that $AL = AC$. Let $M$ be the point of intersection of $AL$ and $BK$.

Show that $MK = ML$.

**PROBLEMA 6**
Find all positive integers $n$ for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

[1] A circunferência ex-inscrita de $ABC$ oposta ao vértice $A$ é a circunferência tangente ao segmento $BC$, ao prolongamento do segmento $AB$ no sentido de $A$ para $B$ e ao prolongamento do segmento $AC$ no sentido de $A$ para $C$.

## Dia I

**PROBLEMA 1**
Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by $s_A$. Let $n_A$ denote the number of pairs $(i, j)$ with $1 \le i < j \le 4$ for which $a_i + a_j$ divides $s_A$. Find all sets $A$ of four distinct positive integers which achieve the largest possible value of $n_A$.

**PROBLEMA 2**
Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line $\ell$ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot $P$ until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, $Q$, takes over as the new pivot, and the line now rotates clockwise about $Q$, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point $P$ in $\mathcal{S}$ and a line $\ell$ going through $P$ such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

**PROBLEMA 3**
Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x + y) \le yf(x) + f(f(x))$$

for all real numbers $x$ and $y$. Prove that $f(x) = 0$ for all $x \le 0$.

## Dia II

**PROBLEMA 4**
Let $n > 0$ be an integer. We are given a balance and $n$ weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the $n$ weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.

Determine the number of ways in which this can be done.

**PROBLEMA 5**
Let $f$ be a function from the set of integers to the set of positive integers. Suppose that, for any two integers $m$ and $n$, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that, for all integers $m$ and $n$ with $f(m) \le f(n)$, the number $f(n)$ is divisible by $f(m)$.

**PROBLEMA 6**
Seja $ABC$ um triângulo acutângulo com circuncírculo $\Gamma$. Seja $\ell$ uma reta tangente a $\Gamma$, e sejam $\ell_a$, $\ell_b$ e $\ell_c$ as retas obtidas após refletir $\ell$ pelas retas $BC$, $CA$ e $AB$, respectivamente.

Mostre que o circuncírculo do triângulo determinado pelas retas $\ell_a, \ell_b$ e $\ell_c$ é tangente ao círculo $\Gamma$.

## Dia I

**PROBLEMA 1**
Encontre todas as funções $f : \mathbb{R} \to \mathbb{R}$ tais que

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

para quaisquer $x$ e $y$ reais.

**PROBLEMA 2**
Seja $ABC$ um triângulo, I o seu incentro e $\Gamma$ o seu circuncírculo. A reta $AI$ intersecta novamente $\Gamma$ no ponto $D$. Sejam $E$ um ponto no arco $BC$ que contém $D$ e $F$ um ponto do segmento $BC$, tal que

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$

Seja $G$ o ponto médio de $IF$, prove que as retas $EI$ e $DG$ se intersectam sobre $\Gamma$.

**PROBLEMA 3**
Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)\,(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

## Dia II

**PROBLEMA 4**
Let $P$ be a point interior to triangle $ABC$ (with $CA \neq CB$). The lines $AP$, $BP$ and $CP$ meet again its circumcircle $\Gamma$ at $K$, $L$, respectively $M$. The tangent line at $C$ to $\Gamma$ meets the line $AB$ at $S$. Show that from $SC = SP$ follows $MK = ML$.

**PROBLEMA 5**
Each of the six boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$ initially contains one coin. The following operations are allowed:

**Type 1:** Choose a non-empty box $B_j$, $1 \leq j \leq 5$, remove one coin from $B_j$ and add two coins to $B_{j+1}$;

**Type 2:** Choose a non-empty box $B_k$, $1 \leq k \leq 4$, remove one coin from $B_k$ and swap the contents (maybe empty) of the boxes $B_{k+1}$ and $B_{k+2}$.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$ become empty, while box $B_6$ contains exactly $2010^{2010^{2010}}$ coins.

**PROBLEMA 6**
Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and $s$ be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \leq s$ and $N$, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N.$$

## Dia I

**PROBLEMA 1**
Let $n$ be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $1, 2, \ldots, n$ such that $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that $n$ does not divide $a_k(a_1 - 1)$.

**PROBLEMA 2**
Let $ABC$ be a triangle with circumcentre $O$. The points $P$ and $Q$ are interior points of the sides $CA$ and $AB$ respectively. Let $K, L$ and $M$ be the midpoints of the segments $BP, CQ$ and $PQ$. respectively, and let $\Gamma$ be the circle passing through $K, L$ and $M$. Suppose that the line $PQ$ is tangent to the circle $\Gamma$. Prove that $OP = OQ$.

**PROBLEMA 3**
Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1},\, s_{s_2},\, s_{s_3},\, \ldots \qquad \text{and} \qquad s_{s_1+1},\, s_{s_2+1},\, s_{s_3+1},\, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

## Dia II

**PROBLEMA 4**
Let $ABC$ be a triangle with $AB = AC$ . The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides $BC$ and $CA$ at $D$ and $E$ , respectively. Let $K$ be the incentre of triangle $ADC$. Suppose that $\angle BEK = 45^\circ$ . Find all possible values of $\angle CAB$.Jan Vonk, Belgium, Peter Vandendriessche, Belgium and Hojoo Lee, Korea

**PROBLEMA 5**
Determine todas as funções $f : \mathbb{Z}_+^* \to \mathbb{Z}_+^*$ tais que existe um triângulo não degenerado[2] cujos lados medem

$$a, \quad f(b) \quad \text{e} \quad f(b + f(a) - 1)$$

para todo $a, b \in \mathbb{Z}_+^* = \{1, 2, \ldots\}$.

**PROBLEMA 6**
Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let $M$ be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making $n$ jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in $M$.

[2] Um triângulo não degenerado é um triângulo cujos vértices não são colineares.

## Dia I

**PROBLEMA 1**
Seja $H$ o ortocentro do triângulo acutângulo $ABC$. O círculo $\Gamma_A$, centrado no ponto médio de $BC$ que passa por $H$ intersecta a reta $BC$ nos pontos $A_1$ e $A_2$. Da mesma maneira, defina os pontos $B_1$, $B_2$, $C_1$ e $C_2$.

Prove que os seis pontos $A_1$, $A_2$, $B_1$, $B_2$, $C_1$ e $C_2$ são concíclicos.

**PROBLEMA 2** (a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers $x$, $y$, $z$, each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers $x$, $y$, $z$, each different from 1, and satisfying $xyz = 1$.

**PROBLEMA 3**
Prove that there are infinitely many positive integers $n$ such that $n^2+1$ has a prime divisor greater than $2n+\sqrt{2n}$.

## Dia II

**PROBLEMA 4**
Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so $f$ is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers $w, x, y, z$, satisfying $wx = yz$.

**PROBLEMA 5**
Let $n$ and $k$ be positive integers with $k \geq n$ and $k-n$ an even number. Let $2n$ lamps labelled $1, 2, \ldots, 2n$ be given, each of which can be either on or off.

Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let $N$ be the number of such sequences consisting of $k$ steps and resulting in the state where lamps 1 through $n$ are all on, and lamps $n+1$ through $2n$ are all off.

Let $M$ be number of such sequences consisting of $k$ steps, resulting in the state where lamps 1 through $n$ are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

**PROBLEMA 6**
Seja $ABCD$ um quadrilátero convexo com $BA \neq BC$. Sejam $\omega_1$ e $\omega_2$ os incírculos de $ABC$ e $ADC$, respectivamente. Suponha que existe uma circunferência $\omega$ tangente à reta $BA$ de forma que $A$ está entre $B$ e o ponto de tangência, tangente à reta $BC$ de forma que $C$ está entre $B$ e o ponto de tangência, e que também seja tangente às retas $AD$ e $CD$. Prove que as tangentes externas comuns a $\omega_1$ e $\omega_2$ se intersectam sobre $\omega$.

## Dia I

**PROBLEMA 1**
Real numbers $a_1$, $a_2$, …, $a_n$ are given. For each $i$, $(1 \leq i \leq n)$, define

$$d_i = \max\{a_j \mid 1 \leq j \leq i\} - \min\{a_j \mid i \leq j \leq n\}$$

and let $d = \max\{d_i \mid 1 \leq i \leq n\}$.

(a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| \mid 1 \leq i \leq n\} \geq \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that the equality holds in eq. $(*)$.

**PROBLEMA 2**
Consider five points $A$, $B$, $C$, $D$ and $E$ such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let $\ell$ be a line passing through $A$. Suppose that $\ell$ intersects the interior of the segment $DC$ at $F$ and intersects line $BC$ at $G$. Suppose also that $EF = EG = EC$. Prove that $\ell$ is the bisector of angle $DAB$.

**PROBLEMA 3**
In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its size.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

## Dia II

**PROBLEMA 4**
In triangle $ABC$ the bisector of angle $BCA$ intersects the circumcircle again at $R$, the perpendicular bisector of $BC$ at $P$, and the perpendicular bisector of $AC$ at $Q$. The midpoint of $BC$ is $K$ and the midpoint of $AC$ is $L$. Prove that the triangles $RPK$ and $RQL$ have the same area.

**PROBLEMA 5**
Let $a$ and $b$ be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

**PROBLEMA 6**
Let $n$ be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$

as a set of $(n + 1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains $S$ but does not include $(0, 0, 0)$.

## Dia I

**PROBLEMA 1**
Let $ABC$ be triangle with incenter $I$. A point $P$ in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

**PROBLEMA 2**
Let $P$ be a regular 2006-gon. A diagonal is called good if its endpoints divide the boundary of $P$ into two parts, each composed of an odd number of sides of $P$. The sides of $P$ are also called good.

Suppose $P$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $P$. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

**PROBLEMA 3**
Determine the least real number $M$ such that the inequality

$$|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \leq M(a^2 + b^2 + c^2)^2$$

holds for all real numbers $a$, $b$ and $c$.

## Dia II

**PROBLEMA 4**
Determine all pairs $(x, y)$ of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

**PROBLEMA 5**
Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let $k$ be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where $P$ occurs $k$ times. Prove that there are at most $n$ integers $t$ such that $Q(t) = t$.

**PROBLEMA 6**
Assign to each side $b$ of a convex polygon $P$ the maximum area of a triangle that has $b$ as a side and is contained in $P$. Show that the sum of the areas assigned to the sides of $P$ is at least twice the area of $P$.

## Dia I

**PROBLEMA 1**
Six points are chosen on the sides of an equilateral triangle $ABC$: $A_1$, $A_2$ on $BC$, $B_1$, $B_2$ on $CA$ and $C_1$, $C_2$ on $AB$, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

Prove that the lines $A_1B_2$, $B_1C_2$ and $C_1A_2$ are concurrent.

**PROBLEMA 2**
Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer $n$ the numbers $a_1, a_2, \ldots, a_n$ leave $n$ different remainders upon division by $n$.

Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

**PROBLEMA 3**
Let $x, y, z$ be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

## Dia II

**PROBLEMA 4**
Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \ n \geq 1.$$

**PROBLEMA 5**
Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $BC$ not parallel with $DA$. Let two variable points $E$ and $F$ lie of the sides $BC$ and $DA$, respectively and satisfy $BE = DF$. The lines $AC$ and $BD$ meet at $P$, the lines $BD$ and $EF$ meet at $Q$, the lines $EF$ and $AC$ meet at $R$.

Prove that the circumcircles of the triangles $PQR$, as $E$ and $F$ vary, have a common point other than $P$.

**PROBLEMA 6**
In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

## Dia I

**PROBLEMA 1**

1. Let $ABC$ be an acute-angled triangle with $AB \neq AC$. The circle with diameter $BC$ intersects the sides $AB$ and $AC$ at $M$ and $N$ respectively. Denote by $O$ the midpoint of the side $BC$. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at $R$. Prove that the circumcircles of the triangles $BMR$ and $CNR$ have a common point lying on the side $BC$.

**PROBLEMA 2**

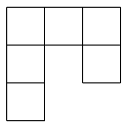
Encontre todos os polinômios $P(x) \in \mathbb{R}[x]$ que satisfaz a igualdade

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$

para todas os $a, b, c$ reais tais que $ab + bc + ca = 0$.

**PROBLEMA 3**

Define a *hook* to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

- the rectangle is covered without gaps and without overlaps
- no part of a hook covers area outside the rectangle.

## Dia II

**PROBLEMA 4**

Let $n \geq 3$ be an integer. Let $t_1$, $t_2$, ..., $t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that $t_i$, $t_j$, $t_k$ are side lengths of a triangle for all $i$, $j$, $k$ with $1 \leq i < j < k \leq n$.

**PROBLEMA 5**

In a convex quadrilateral $ABCD$, the diagonal $BD$ bisects neither the angle $ABC$ nor the angle $CDA$. The point $P$ lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

**PROBLEMA 6**

We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers $n$ such that $n$ has a multiple which is alternating.

## Dia I

**PROBLEMA 1**
Let $A$ be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1$, $t_2, \ldots, t_{100}$ in $S$ such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

**PROBLEMA 2**
Determine all pairs of positive integers $(a, b)$ such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

**PROBLEMA 3**
Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\frac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

## Dia II

**PROBLEMA 4**
Let $ABCD$ be a cyclic quadrilateral. Let $P$, $Q$, $R$ be the feet of the perpendiculars from $D$ to the lines $BC$, $CA$, $AB$, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with $AC$.

**PROBLEMA 5**
Let $n$ be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers.

Prove that

$$\left( \sum_{i,j=1}^{n} |x_i - x_j| \right)^2 \leq \frac{2(n^2 - 1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

**PROBLEMA 6**
Let $p$ be a prime number. Prove that there exists a prime number $q$ such that for every integer $n$, the number $n^p - p$ is not divisible by $q$.

## Dia I

**PROBLEMA 1**
Let $n$ be a positive integer. Each point $(x, y)$ in the plane, where $x$ and $y$ are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point $(x, y)$ is red, then so are all points $(x', y')$ with $x' \leq x$ and $y' \leq y$. Let $A$ be the number of ways to choose $n$ blue points with distinct $x$-coordinates, and let $B$ be the number of ways to choose $n$ blue points with distinct $y$-coordinates. Prove that $A = B$.

**PROBLEMA 2**
The circle $S$ has centre $O$, and $BC$ is a diameter of $S$. Let $A$ be a point of $S$ such that $\angle AOB < 120°$. Let $D$ be the midpoint of the arc $AB$ which does not contain $C$. The line through $O$ parallel to $DA$ meets the line $AC$ at $I$. The perpendicular bisector of $OA$ meets $S$ at $E$ and at $F$. Prove that $I$ is the incentre of the triangle $CEF$.

**PROBLEMA 3**
Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers $a$ such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

## Dia II

**PROBLEMA 4**
Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \ldots + d_{k-1} d_k$ is always less than $n^2$, and determine when it is a divisor of $n^2$.

**PROBLEMA 5**
Ache todas as funções $f : \mathbb{R} \to \mathbb{R}$ satisfazendo

$$(f(x) + f(y))(f(w) + f(z)) = f(xw - yz) + f(xz + yw)$$

para todos os $x$, $y$, $w$ e $z$ reais.

**PROBLEMA 6**
Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

## Dia I

**PROBLEMA 1**
Consider an acute-angled triangle $ABC$. Let $P$ be the foot of the altitude of triangle $ABC$ issuing from the vertex $A$, and let $O$ be the circumcenter of triangle $ABC$. Assume that $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$.

**PROBLEMA 2**
Prove that for all positive real numbers $a, b, c$,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

**PROBLEMA 3**
Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

## Dia II

**PROBLEMA 4**
Let $n$ be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

**PROBLEMA 5**
Let $ABC$ be a triangle with $\angle BAC = 60^\circ$. Let $AP$ bisect $\angle BAC$ and let $BQ$ bisect $\angle ABC$, with $P$ on $BC$ and $Q$ on $AC$. If $AB + BP = AQ + QB$, what are the angles of the triangle?

**PROBLEMA 6**
Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

## Dia I

**PROBLEMA 1**
Two circles $G_1$ and $G_2$ intersect at two points $M$ and $N$. Let $AB$ be the line tangent to these circles at $A$ and $B$, respectively, so that $M$ lies closer to $AB$ than $N$. Let $CD$ be the line parallel to $AB$ and passing through the point $M$, with $C$ on $G_1$ and $D$ on $G_2$. Lines $AC$ and $BD$ meet at $E$; lines $AN$ and $CD$ meet at $P$; lines $BN$ and $CD$ meet at $Q$. Show that $EP = EQ$.

**PROBLEMA 2**
Let $a, b, c$ be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

**PROBLEMA 3**
Let $n \geq 2$ be a positive integer and $\lambda$ a positive real number. Initially there are $n$ fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points $A$ and $B$, with $A$ to the left of $B$, and letting the flea from $A$ jump over the flea from $B$ to the point $C$ so that $\frac{BC}{AB} = \lambda$.

Determine all values of $\lambda$ such that, for any point $M$ on the line and for any initial position of the $n$ fleas, there exists a sequence of moves that will take them all to the position right of $M$.

## Dia II

**PROBLEMA 4**
A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

How many ways are there to put the cards in the three boxes so that the trick works?

**PROBLEMA 5**
Does there exist a positive integer $n$ such that $n$ has exactly 2000 prime divisors and $n$ divides $2^n + 1$?

**PROBLEMA 6**
Let $AH_1, BH_2, CH_3$ be the altitudes of an acute angled triangle $ABC$. Its incircle touches the sides $BC, AC$ and $AB$ at $T_1, T_2$ and $T_3$ respectively. Consider the symmetric images of the lines $H_1H_2, H_2H_3$ and $H_3H_1$ with respect to the lines $T_1T_2, T_2T_3$ and $T_3T_1$. Prove that these images form a triangle whose vertices lie on the incircle of $ABC$.

## Dia I

**PROBLEMA 1**
A set $S$ of points from the space will be called completely symmetric if it has at least three elements and fulfills the condition that for every two distinct points $A$ and $B$ from $S$, the perpendicular bisector plane of the segment $AB$ is a plane of symmetry for $S$.

Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

**PROBLEMA 2**
Let $n \geq 2$ be a fixed integer. Find the least constant $C$ such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant $C$, characterize the instances of equality.

**PROBLEMA 3**
Let $n$ be an even positive integer. We say that two different cells of a $n \times n$ board are neighboring if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

## Dia II

**PROBLEMA 4**
Find all the pairs of positive integers $(x, p)$ such that p is a prime, $x \leq 2p$ and $x^{p-1}$ is a divisor of $(p-1)^x + 1$.

**PROBLEMA 5**
Duas circunferências $\Omega_1$ e $\Omega_2$ tangenciam internamente a circunferência $\Omega$ em $M$ e $N$ e o centro de $\Omega_2$ está sobre $\Omega_1$. A corda comum de $\Omega_1$ e $\Omega_2$ intersecta $\Omega$ em $A$ e $B$. $MA$ e $MB$ intersectam $\Omega_1$ em $C$ e $D$. Prove que $\Omega_2$ é tangente a $CD$.

**PROBLEMA 6**
Ache todas as funções $f : \mathbb{R} \to \mathbb{R}$ satisfazendo

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

para todos $x$ e $y$ reais.

## Dia I

**PROBLEMA 1**
A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides $AB$ and $CD$ meet at a unique point $P$ inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles $ABP$ and $CDP$ have equal areas.

**PROBLEMA 2**
Numa competição, existem $m$ atletas e $n$ jurados, com $n \geq 3$ um inteiro ímpar. Cada atleta é julgado por cada jurado como *bom* ou *ruim*. Suponha que cada par de jurados concorda com no máximo $k$ atletas. Prove que

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

**PROBLEMA 3**
For any positive integer $n$, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers $m$ for which there exists a positive integer $n$ such that $\frac{\tau(n^2)}{\tau(n)} = m$.

## Dia II

**PROBLEMA 4**
Determine all pairs $(x, y)$ of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

**PROBLEMA 5**
Let $I$ be the incenter of triangle $ABC$. Let $K, L$ and $M$ be the points of tangency of the incircle of $ABC$ with $AB, BC$ and $CA$, respectively. The line $t$ passes through $B$ and is parallel to $KL$. The lines $MK$ and $ML$ intersect $t$ at the points $R$ and $S$. Prove that $\angle RIS$ is acute.

**PROBLEMA 6**
Determine the least possible value of $f(1998)$, where $f : \mathbb{N} \to \mathbb{N}$ is a function such that for all $m, n \in \mathbb{N}$,

$$f\left(n^2 f(m)\right) = m\left(f(n)\right)^2.$$

## Dia I

**PROBLEMA 1**
In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers $m$ and $n$, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths $m$ and $n$, lie along edges of the squares. Let $S_1$ be the total area of the black part of the triangle and $S_2$ be the total area of the white part. Let $f(m, n) = |S_1 - S_2|$.

(a) Calculate $f(m, n)$ for all positive integers $m$ and $n$ which are either both even or both odd.

(b) Prove that $f(m, n) \leq \frac{1}{2}\max\{m, n\}$ for all $m$ and $n$.

(c) Show that there is no constant $C \in \mathbb{R}$ such that $f(m, n) < C$ for all $m$ and $n$.

**PROBLEMA 2**
It is known that $\angle BAC$ is the smallest angle in the triangle $ABC$. The points $B$ and $C$ divide the circumcircle of the triangle into two arcs. Let $U$ be an interior point of the arc between $B$ and $C$ which does not contain $A$. The perpendicular bisectors of $AB$ and $AC$ meet the line $AU$ at $V$ and $W$, respectively. The lines $BV$ and $CW$ meet at $T$.

Show that $AU = TB + TC$.

**PROBLEMA 3**
Let $x_1$, $x_2$, …, $x_n$ be real numbers satisfying the conditions:

- $|x_1 + x_2 + \cdots + x_n| = 1$
- $|x_i| \leq \dfrac{n+1}{2}$ for $i = 1, 2, \ldots, n$.

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

## Dia II

**PROBLEMA 4**
An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, \ldots, n$, the $i$-th row and the $i$-th column together contain all elements of $S$. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of $n$.

**PROBLEMA 5**
Find all pairs $(a, b)$ of positive integers that satisfy the equation: $a^{b^2} = b^a$.

**PROBLEMA 6**
Para cada inteiro positivo $n$, definimos $f(n)$ como o número de maneiras de representar $n$ como soma de potências de dois com expoentes não negativos. Representações que diferem somente na ordem das parcelas são consideradas a mesma. Por exemplo, $f(4) = 4$, pois o número 4 pode ser expresso das quatro seguintes maneiras: 4; $2+2$; $2+1+1$; $1+1+1+1$.

Prove que, para qualquer inteiro $n \geq 3$,

$$2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}.$$

## Dia I

**PROBLEMA 1**
We are given a positive integer $r$ and a rectangular board $ABCD$ with dimensions $AB = 20, BC = 12$. The rectangle is divided into a grid of $20 \times 12$ unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with $A$ as a vertex to the square with $B$ as a vertex.

(a) Show that the task cannot be done if $r$ is divisible by 2 or 3.

(b) Prove that the task is possible when $r = 73$.

(c) Can the task be done when $r = 97$?

**PROBLEMA 2**
Let $P$ be a point inside a triangle $ABC$ such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let $D$, $E$ be the incenters of triangles $APB$, $APC$, respectively. Show that the lines $AP$, $BD$, $CE$ meet at a point.

**PROBLEMA 3**
Let $\mathbb{N}_0$ denote the set of nonnegative integers. Find all functions $f$ from $\mathbb{N}_0$ to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \text{for all } m, n \in \mathbb{N}_0.$$

## Dia II

**PROBLEMA 4**
The positive integers $a$ and $b$ are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

**PROBLEMA 5**
Let $ABCDEF$ be a convex hexagon such that $AB$ is parallel to $DE$, $BC$ is parallel to $EF$, and $CD$ is parallel to $FA$. Let $R_A, R_C, R_E$ denote the circumradii of triangles $FAB, BCD, DEF$, respectively, and let $P$ denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

**PROBLEMA 6**
Let $p, q, n$ be three positive integers with $p + q < n$. Let $(x_0, x_1, \cdots, x_n)$ be an $(n + 1)$-tuple of integers satisfying the following conditions:

- $x_0 = x_n = 0$, and
- For each $i$ with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

## Dia I

**PROBLEMA 1**
Sejam $A$, $B$, $C$, $D$ pontos distintos sobre uma reta, nesta ordem. As circunferências com diâmetros $AC$ e $BD$ se intersectam em $X$ e $Y$. A reta $XY$ intersecta $BC$ em $Z$. Seja $P$ um ponto sobre a reta $XY$ diferente de $Z$. A reta $CP$ intersecta a circunferência de diâmetro $AC$ em $C$ e $M$. A reta $BP$ intersecta a circufnerência de diâmetro $BD$ em $B$ e $N$. Prove que as retas $AM$, $DN$ e $XY$ são concorrentes.

**PROBLEMA 2**
Let $a$, $b$, $c$ be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3\left(b+c\right)}+\frac{1}{b^3\left(c+a\right)}+\frac{1}{c^3\left(a+b\right)}\geq\frac{3}{2}.$$

**PROBLEMA 3**
Determine all integers $n > 3$ for which there exist $n$ points $A_1, \cdots, A_n$ in the plane, no three collinear, and real numbers $r_1, \cdots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_iA_jA_k$ is $r_i + r_j + r_k$.

## Dia II

**PROBLEMA 4**
Find the maximum value of $x_0$ for which there exists a sequence $x_0, x_1 \cdots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that

$$x_{i-1}+\frac{2}{x_{i-1}}=2x_i+\frac{1}{x_i},$$

for all $i = 1, \cdots, 1995$.

**PROBLEMA 5**
Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose $G$ and $H$ are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$.

**PROBLEMA 6**
Seja $p$ um primo ímpar. Quantos subconjuntos de tamanho $p$ do conjunto $\{1, 2, \ldots, 2p\}$ possuem soma de seus elementos múltipla de $p$.

## Dia I

**PROBLEMA 1**
Let $m$ and $n$ be two positive integers. Let $a_1$, $a_2$, …, $a_m$ be $m$ different numbers from the set $\{1, 2, \ldots, n\}$ such that for any two indices $i$ and $j$ with $1 \leq i \leq j \leq m$ and $a_i + a_j \leq n$, there exists an index $k$ such that $a_i + a_j = a_k$. Show that

$$\frac{a_1 + a_2 + ... + a_m}{m} \geq \frac{n+1}{2}.$$

**PROBLEMA 2**
Let $ABC$ be an isosceles triangle with $AB = AC$. $M$ is the midpoint of $BC$ and $O$ is the point on the line $AM$ such that $OB$ is perpendicular to $AB$. $Q$ is an arbitrary point on $BC$ different from $B$ and $C$. $E$ lies on the line $AB$ and $F$ lies on the line $AC$ such that $E, Q, F$ are distinct and collinear. Prove that $OQ$ is perpendicular to $EF$ if and only if $QE = QF$.

**PROBLEMA 3**
For any positive integer $k$, let $f_k$ be the number of elements in the set $\{k + 1, k + 2, \ldots, 2k\}$ whose base 2 representation contains exactly three 1s.

(a) Prove that for any positive integer $m$, there exists at least one positive integer $k$ such that $f(k) = m$.

(b) Determine all positive integers $m$ for which there exists exactly one $k$ with $f(k) = m$.

## Dia II

**PROBLEMA 4**
Find all ordered pairs $(m, n)$ where $m$ and $n$ are positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.

**PROBLEMA 5**
Let $S$ be the set of all real numbers strictly greater than $-1$. Find all functions $f : S \to S$ satisfying the two conditions:

- $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y$ in $S$;
- $\frac{f(x)}{x}$ is strictly increasing on each of the two intervals $-1 < x < 0$ and $0 < x$.

**PROBLEMA 6**
Show that there exists a set $A$ of positive integers with the following property: for any infinite set $S$ of primes, there exist two positive integers $m$ in $A$ and $n$ not in $A$, each of which is a product of $k$ distinct elements of $S$ for some $k \geq 2$.

## Dia I

**PROBLEMA 1**
Let $n > 1$ be an integer and let $f(x) = x^n + 5 \cdot x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x) \cdot h(x)$.

**PROBLEMA 2**
Let $A$, $B$, $C$, $D$ be four points in the plane, with $C$ and $D$ on the same side of the line $AB$, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$.

Find the ratio $\frac{AB \cdot CD}{AC \cdot BD}$, and prove that the circumcircles of the triangles $ACD$ and $BCD$ are orthogonal.

**PROBLEMA 3**
On an infinite chessboard, a solitaire game is played as follows: at the start, we have $n^2$ pieces occupying a square of side $n$.

The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed.

For which $n$ can the game end with only one piece remaining on the board?

## Dia II

**PROBLEMA 4**
For three points $A, B, C$ in the plane, we define $m(ABC)$ to be the smallest length of the three heights of the triangle $ABC$, where in the case $A$, $B$, $C$ are collinear, we set $m(ABC) = 0$. Let $A$, $B$, $C$ be given points in the plane. Prove that for any point $X$ in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

**PROBLEMA 5**
Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine if there exists a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$ with the following properties:

- $f(1) = 2$;
- $f(f(n)) = f(n) + n, (n \in \mathbb{N})$.

**PROBLEMA 6**
Let $n > 1$ be an integer. In a circular arrangement of $n$ lamps $L_0, \ldots, L_{n-1}$, each of of which can either `ON` or `OFF`, we start with the situation where all lamps are `ON`, and then carry out a sequence of steps, $\text{Step}_0, \text{Step}_1, \ldots$.

If $L_{j-1}$ [3] is `ON` then $\text{Step}_j$ changes the state of $L_j$ (it goes from `ON` to `OFF` or from `OFF` to `ON`) but does not change the state of any of the other lamps. If $L_{j-1}$ is `OFF` then $\text{Step}_j$ does not change anything at all.

Show that:

(a) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again,

(b) If $n$ has the form $2^k$ then all the lamps are ON after $n^2 - 1$ steps,

(c) If $n$ has the form $2^k + 1$ then all lamps are ON after $n^2 - n + 1$ steps.

[3] $j$ is taken mod $n$

## Dia I

**PROBLEMA 1**
Find all integers $a, b, c$ with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

is a divisor of $abc - 1$.

**PROBLEMA 2**
Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2 + f(y)\right) = y + \left(f(x)\right)^2 \qquad \text{for all}\, x, y \in \mathbb{R}.$$

**PROBLEMA 3**
Consider 9 points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored.

Find the smallest value of $n$ such that whenever exactly $n$ edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

## Dia II

**PROBLEMA 4**
Seja $\Gamma$ uma circunferência, $\ell$ uma reta tangente à circunferência $\Gamma$ e $M$ um ponto em $\ell$. Ache o lugar geométrico de todos os pontos $P$ com a seguinte propriedade: existem dois pontos $Q$ e $R$ em $\ell$ tal que $M$ é ponto médio de $QR$ e $\Gamma$ é incírculo de $PQR$.

**PROBLEMA 5**
Let $S$ be a finite set of points in three-dimensional space. Let $S_x$, $S_y$, $S_z$ be the sets consisting of the orthogonal projections[4] of the points of $S$ onto the $yz$-plane, $zx$-plane, $xy$-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set $A$.

**PROBLEMA 6**
Para cada inteiro positivo $n$, $S(n)$ é definido como o maior inteiro tal que, para todo inteiro positivo $k \leq S(n)$, $n^2$ pode ser escrito como soma de $k$ quadrados positivos.

(a) Prove que $S(n) \leq n^2 - 14$ para cada $n \geq 4$.

(b) Ache um inteiro $n$ tal que $S(n) = n^2 - 14$.

(c) Prove que existem infinitos inteiros $n$ tal que $S(n) = n^2 - 14$.

[4] The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.

## Dia I

**PROBLEMA 1**
Given a triangle $ABC$, let $I$ be the center of its inscribed circle. The internal bisectors of the angles $A, B, C$ meet the opposite sides in $A', B', C'$ respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

**PROBLEMA 2**
Let $n > 6$ be an integer and $a_1, a_2, \cdots, a_k$ be all the natural numbers less than $n$ and relatively prime to $n$. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that $n$ must be either a prime number or a power of 2.

**PROBLEMA 3**
Let $S = \{1, 2, 3, \cdots, 280\}$. Find the smallest integer $n$ such that each $n$-element subset of $S$ contains five numbers which are pairwise relatively prime.

## Dia II

**PROBLEMA 4**
Suppose $G$ is a connected graph with $k$ edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

**PROBLEMA 5**
Let $ABC$ be a triangle and $P$ an interior point of $ABC$. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

**PROBLEMA 6**
An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be bounded if there is a constant $C$ such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j||i - j|^a \geq 1$$

for every pair of distinct nonnegative integers $i, j$.

## Dia I

**PROBLEMA 1**
Chords $AB$ and $CD$ of a circle intersect at a point $E$ inside the circle. Let $M$ be an interior point of the segment $EB$. The tangent line at $E$ to the circle through $D$, $E$, and $M$ intersects the lines $BC$ and $AC$ at $F$ and $G$, respectively.

If $\frac{AM}{AB} = t$, find $\frac{EG}{EF}$ in terms of $t$.

**PROBLEMA 2**
Let $n \geq 3$ and consider a set $E$ of $2n - 1$ distinct points on a circle. Suppose that exactly $k$ of these points are to be colored black. Such a coloring is good if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly $n$ points from $E$. Find the smallest value of $k$ so that every such coloring of $k$ points of $E$ is good.

**PROBLEMA 3**
Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

## Dia II

**PROBLEMA 4**
Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all $x$, $y$ in $\mathbb{Q}^+$.

**PROBLEMA 5**
Given an initial integer $n_0 > 1$, two players, $A$ and $B$, choose integers $n_1$, $n_2$, $n_3$, ... alternately according to the following rules:

(i) Knowing $n_{2k}$, $A$ chooses any integer $n_{2k+1}$ such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$

(ii) Knowing $n_{2k+1}$, $B$ chooses any integer $n_{2k+2}$ such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.

Player $A$ wins the game by choosing the number 1990; player $B$ wins by choosing the number 1. For which $n_0$ does:

(a) $A$ have a winning strategy?

(b) $B$ have a winning strategy?

(c) Neither player have a winning strategy?

**PROBLEMA 6**
Prove that there exists a convex 1990-gon with the following two properties:

- All angles are equal.
- The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, \ldots, 1990^2$ in some order.

## Dia I

**PROBLEMA 1**
Prove that in the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets $A_i, \{i = 1, 2, \ldots, 117\}$ such that

- each $A_i$ contains 17 elements;
- the sum of all the elements in each $A_i$ is the same.

**PROBLEMA 2**
$ABC$ is a triangle, the bisector of angle $A$ meets the circumcircle of triangle $ABC$ in $A_1$, points $B_1$ and $C_1$ are defined similarly. Let $AA_1$ meet the lines that bisect the two external angles at $B$ and $C$ in $A_0$. Define $B_0$ and $C_0$ similarly. Prove that the area of triangle $A_0B_0C_0 = 2\cdot$ area of hexagon $AC_1BA_1CB_1 \geq 4\cdot$ area of triangle $ABC$.

**PROBLEMA 3**
Sejam $n$ e $k$ inteiros positivos e seja $S$ um conjunto de $n$ pontos no plano tal que não há três pontos de $S$ colineares e, para cada ponto $P$ em $S$, existem pelo menos $k$ pontos de $S$ que equidistam de $P$. Prove que

$$k < \frac{1}{2} + \sqrt{2n}.$$

## Dia II

**PROBLEMA 4**
Let $ABCD$ be a convex quadrilateral such that the sides $AB, AD, BC$ satisfy $AB = AD + BC$. There exists a point $P$ inside the quadrilateral at a distance $h$ from the line $CD$ such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

**PROBLEMA 5**
Prove that for each positive integer $n$ there exist $n$ consecutive positive integers none of which is an integral power of a prime number.

**PROBLEMA 6**
A permutation $\{x_1, x_2, \ldots, x_{2n}\}$ of the set $\{1, 2, \ldots, 2n\}$ where $n$ is a positive integer, is said to have property $T$ if $|x_i - x_{i+1}| = n$ for at least one $i$ in $\{1, 2, \ldots, 2n-1\}$. Show that, for each $n$, there are more permutations with property $T$ than without.

## Dia I

**PROBLEMA 1**
Consider 2 concentric circle radii $R$ and $r$ $(R > r)$ with centre $O$. Fix $P$ on the small circle and consider the variable chord $PA$ of the small circle. Points $B$ and $C$ lie on the large circle; $B, P, C$ are collinear and $BC$ is perpendicular to $AP$.

(a) For which values of $\angle OPA$ is the sum $BC^2 + CA^2 + AB^2$ extremal?

(b) What are the possible positions of the midpoints $U$ of $BA$ and $V$ of $AC$ as $\angle OPA$ varies?

**PROBLEMA 2**
Let $n$ be an even positive integer. Let $A_1, A_2, \ldots, A_{n+1}$ be sets having $n$ elements each such that any two of them have exactly one element in common while every element of their union belongs to at least two of the given sets. For which $n$ can one assign to every element of the union one of the numbers 0 and 1 in such a manner that each of the sets has exactly $\frac{n}{2}$ zeros?

**PROBLEMA 3**
A function $f$ defined on the positive integers (and taking positive integers values) is given by:

$$f(1) = 1, f(3) = 3$$
$$f(2 \cdot n) = f(n)$$
$$f(4 \cdot n + 1) = 2 \cdot f(2 \cdot n + 1) - f(n)$$
$$f(4 \cdot n + 3) = 3 \cdot f(2 \cdot n + 1) - 2 \cdot f(n),$$

for all positive integers $n$.

Determine with proof the number of positive integers $\leq 1988$ for which $f(n) = n$.

## Dia II

**PROBLEMA 4**
Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose length is 1988.

**PROBLEMA 5**
In a right-angled triangle $ABC$ let $AD$ be the altitude drawn to the hypotenuse and let the straight line joining the incentres of the triangles $ABD, ACD$ intersect the sides $AB, AC$ at the points $K, L$ respectively. If $E$ and $E_1$ dnote the areas of triangles $ABC$ and $AKL$ respectively, show that

$$\frac{E}{E_1} \geq 2.$$

**PROBLEMA 6**
Sejam $a$ e $b$ inteiros positivos tais que $ab + 1$ divide $a^2 + b^2$. Mostre que

$$\frac{a^2 + b^2}{ab + 1}$$

é um quadrado perfeito.

## Dia I

**PROBLEMA 1**
Seja $p_n(k)$ o número de permutações do conjunto $\{1, 2, 3, \ldots, n\}$ que possuem exatamente $k$ pontos fixos. Prove que

$$\sum_{k=0}^{n} kp_n(k) = n!.$$

**PROBLEMA 2**
In an acute-angled triangle $ABC$ the interior bisector of angle $A$ meets $BC$ at $L$ and meets the circumcircle of $ABC$ again at $N$. From $L$ perpendiculars are drawn to $AB$ and $AC$, with feet $K$ and $M$ respectively. Prove that the quadrilateral $AKNM$ and the triangle $ABC$ have equal areas.

**PROBLEMA 3**
Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i| \leq k - 1$ for all $i$, and $|a_1x_1 + a_2x_2 + \ldots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n-1}$.

## Dia II

**PROBLEMA 4**
Prove that there is no function $f$ from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for all $n$.

**PROBLEMA 5**
Let $n \geq 3$ be an integer. Prove that there is a set of $n$ points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

**PROBLEMA 6**
Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers $k$ such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers $k$ such that $0 \leq k \leq n - 2$.

## Dia I

**PROBLEMA 1**
Let $d$ be any positive integer not equal to $2, 5$ or $13$. Show that one can find distinct $a, b$ in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

**PROBLEMA 2**
Given a point $P_0$ in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \geq 4$. Construct a set of points $P_1, P_2, P_3, \ldots$ such that $P_{k+1}$ is the image of $P_k$ under a rotation center $A_{k+1}$ through an angle $120^o$ clockwise for $k = 0, 1, 2, \ldots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

**PROBLEMA 3**
To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers $x, y, z$ respectively, and $y < 0$, then the following operation is allowed: $x, y, z$ are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative.

Determine whether this procedure necessarily comes to an end after a finite number of steps.

## Dia II

**PROBLEMA 4**
Let $A, B$ be adjacent vertices of a regular $n$-gon ($n \geq 5$) with center $O$. A triangle $XYZ$, which is congruent to and initially coincides with $OAB$, moves in the plane in such a way that $Y$ and $Z$ each trace out the whole boundary of the polygon, with $X$ remaining inside the polygon. Find the locus of $X$.

**PROBLEMA 5**
Find all functions $f$ defined on the non-negative reals and taking non-negative real values such that: $f(2) = 0, f(x) \neq 0$ for $0 \leq x < 2$, and $f(xf(y))f(y) = f(x + y)$ for all $x, y$.

**PROBLEMA 6**
Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line $L$ parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on $L$ is not greater than 1?

## Dia I

**PROBLEMA 1**
A circle has center on the side $AB$ of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

**PROBLEMA 2**
Let $n$ and $k$ be relatively prime positive integers with $k < n$. Each number in the set $M = \{1, 2, 3, \ldots, n-1\}$ is colored either blue or white. For each $i$ in $M$, both $i$ and $n - i$ have the same color. For each $i \neq k$ in $M$ both $i$ and $|i - k|$ have the same color. Prove that all numbers in $M$ must have the same color.

**PROBLEMA 3**
For any polynomial $P(x) = a_0 + a_1 x + \ldots + a_k x^k$ with integer coefficients, the number of odd coefficients is denoted by $o(P)$. For $i - 0, 1, 2, \ldots$ let $Q_i(x) = (1+x)^i$. Prove that if $i_1, i_2, \ldots, i_n$ are integers satisfying $0 \leq i_1 < i_2 < \ldots < i_n$, then:

$$o(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) \geq o(Q_{i_1}).$$

## Dia II

**PROBLEMA 4**
Given a set $M$ of 1985 distinct positive integers, none of which has a prime divisor greater than 23, prove that $M$ contains a subset of 4 elements whose product is the 4th power of an integer.

**PROBLEMA 5**
A circle with center $O$ passes through the vertices $A$ and $C$ of the triangle $ABC$ and intersects the segments $AB$ and $BC$ again at distinct points $K$ and $N$ respectively. Let $M$ be the point of intersection of the circumcircles of triangles $ABC$ and $KBN$ (apart from $B$). Prove that $\angle OMB = 90°$.

**PROBLEMA 6**
For every real number $x_1$, construct the sequence $x_1, x_2, \ldots$ by setting:

$$x_{n+1} = x_n(x_n + \frac{1}{n}).$$

Prove that there exists exactly one value of $x_1$ which gives $0 < x_n < x_{n+1} < 1$ for all $n$.

## Dia I

**PROBLEMA 1**
Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where $x, y$ and $z$ are non-negative real numbers satisfying $x + y + z = 1$.

**PROBLEMA 2**
Find one pair of positive integers $a, b$ such that $ab(a+b)$ is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by $7^7$.

**PROBLEMA 3**
Given points $O$ and $A$ in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point $X$ in the plane, the circle $C(X)$ has center $O$ and radius $OX + \frac{\angle AOX}{OX}$, where $\angle AOX$ is measured in radians in the range $[0, 2\pi)$. Prove that we can find a point $X$, not on $OA$, such that its color appears on the circumference of the circle $C(X)$.

## Dia II

**PROBLEMA 4**
Let $ABCD$ be a convex quadrilateral with the line $CD$ being tangent to the circle on diameter $AB$. Prove that the line $AB$ is tangent to the circle on diameter $CD$ if and only if the lines $BC$ and $AD$ are parallel.

**PROBLEMA 5**
Let $d$ be the sum of the lengths of all the diagonals of a plane convex polygon with $n$ vertices (where $n > 3$). Let $p$ be its perimeter. Prove that:

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right] \cdot \left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding $x$.

**PROBLEMA 6**
Let $a, b, c, d$ be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers $k$ and $m$, then $a = 1$.

## Dia I

**PROBLEMA 1**
Find all functions $f$ defined on the set of positive reals which take positive real values and satisfy: $f(xf(y)) = yf(x)$ for all $x, y$; and $f(x) \to 0$ as $x \to \infty$.

**PROBLEMA 2**
Seja $A$ um dos dois pontos de intersecção de duas circunferências $C_1$ e $C_2$, com raios distintos e centros $O_1$ e $O_2$, respectivamente. Uma das tangentes comuns das circunferências tangencia $C_1$ em $P_1$ e $C_2$ em $P_2$, enquanto a outra tangente comum tangencia $C_1$ em $Q_1$ e $C_2$ em $Q_2$. Seja $M_1$ o ponto médio de $P_1Q_1$ e $M_2$ o ponto médio de $P_2Q_2$.

Prove que $\angle O_1AO_2 = \angle M_1AM_2$.

**PROBLEMA 3**
Let $a, b$ and $c$ be positive integers, no two of which have a common divisor greater than 1. Show that $2abc-ab-bc-ca$ is the largest integer which cannot be expressed in the form $xbc+yca+zab$, where $x, y, z$ are non-negative integers.

## Dia II

**PROBLEMA 4**
Let $ABC$ be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments $AB$, $BC$, and $CA$ (including $A$, $B$, and $C$). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

**PROBLEMA 5**
Is it possible to choose 1983 distinct positive integers, all less than or equal to $10^5$, no three of which are consecutive terms of an arithmetic progression?

**PROBLEMA 6**
Let $a$, $b$ and $c$ be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality occurs.

### Dia I

**PROBLEMA 1**
The function $f(n)$ is defined on the positive integers and takes non-negative integer values. $f(2) = 0, f(3) > 0, f(9999) = 3333$ and for all $m, n$ :

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1.$$

Determine $f(1982)$.

**PROBLEMA 2**
Um triângulo escaleno $A_1A_2A_3$ tem lados $a_1$, $a_2$, $a_3$, com o lado $a_i$ oposto ao vértice $A_i$. Seja $M_i$ o ponto médio do lado $a_i$, e seja $T_i$ o ponto onde o incírculo de $A_1A_2A_3$ tangencia o lado $a_i$. Seja $S_i$ a reflexão de $T_i$ pela bissetriz interna do ângulo $\angle A_i$.

Prove que as retas $M_1S_1$, $M_2S_2$ e $M_3S_3$ concorrem.

**PROBLEMA 3**
Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq x_1 \geq x_2 \geq \ldots$.

(a) Prove that for every such sequence there is an $n \geq 1$ such that:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence such that for all $n$:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4.$$

### Dia II

**PROBLEMA 4**
Prove that if $n$ is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers $x, y$, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

**PROBLEMA 5**
The diagonals $AC$ and $CE$ of the regular hexagon $ABCDEF$ are divided by inner points $M$ and $N$ respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine $r$ if $B, M$ and $N$ are collinear.

**PROBLEMA 6**
Let $S$ be a square with sides length 100. Let $L$ be a path within $S$ which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ with $A_0 = A_n$. Suppose that for every point $P$ on the boundary of $S$ there is a point of $L$ at a distance from $P$ no greater than $\frac{1}{2}$. Prove that there are two points $X$ and $Y$ of $L$ such that the distance between $X$ and $Y$ is not greater than 1 and the length of the part of $L$ which lies between $X$ and $Y$ is not smaller than 198.

## Dia I

**PROBLEMA 1**
Consider a variable point $P$ inside a given triangle $ABC$. Let $D$, $E$, $F$ be the feet of the perpendiculars from the point $P$ to the lines $BC$, $CA$, $AB$, respectively. Find all points $P$ which minimize the sum

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}.$$

**PROBLEMA 2**
Take $r$ such that $1 \leq r \leq n$, and consider all subsets of $r$ elements of the set $\{1, 2, \ldots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that:

$$F(n, r) = \frac{n+1}{r+1}.$$

**PROBLEMA 3**
Determine the maximum value of $m^2 + n^2$, where $m$ and $n$ are integers in the range $1, 2, \ldots, 1981$ satisfying $(n^2 - mn - m^2)^2 = 1$.

## Dia II

**PROBLEMA 4** (a) For which $n > 2$ is there a set of $n$ consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?

(b) For which $n > 2$ is there exactly one set having this property?

**PROBLEMA 5**
Três circunferências de raios iguais possuem um ponto $O$ em comum e estão no interior de um triângulo dado. Cada circunferência tangencia um par de lados do triângulo.

Prove que o ponto $O$ está na reta que liga o incentro e o circuncentro do triângulo.

**PROBLEMA 6**
The function $f(x, y)$ satisfies: $f(0, y) = y + 1, f(x + 1, 0) = f(x, 1), f(x + 1, y + 1) = f(x, f(x + 1, y))$ for all non-negative integers $x, y$. Find $f(4, 1981)$.

## Dia I

**PROBLEMA 1**
Let $\alpha, \beta$ and $\gamma$ denote the angles of the triangle $ABC$. The perpendicular bisector of $AB$ intersects $BC$ at the point $X$, the perpendicular bisector of $AC$ intersects it at $Y$. Prove that $\tan(\beta) \cdot \tan(\gamma) = 3$ implies $BC = XY$. Show that this condition is not necessary, and give a necessary and sufficient condition for $BC = XY$.

**PROBLEMA 2**
Define the numbers $a_0, a_1, \ldots, a_n$ in the following way:

$$a_0 = \frac{1}{2}, \quad a_{k+1} = a_k + \frac{a_k^2}{n} \quad (n > 1, k = 0, 1, \ldots, n-1).$$

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

**PROBLEMA 3**
Prove that the equation

$$x^n + 1 = y^{n+1},$$

where $n$ is a positive integer not smaller then 2, has no positive integer solutions in $x$ and $y$ for which $x$ and $n+1$ are relatively prime.

## Dia II

**PROBLEMA 4**
Determine all positive integers $n$ such that the following statement holds: If a convex polygon with with $2n$ sides $A_1 A_2 \ldots A_{2n}$ is inscribed in a circle and $n-1$ of its $n$ pairs of opposite sides are parallel, which means if the pairs of opposite sides

$$(A_1A_2, A_{n+1}A_{n+2}), (A_2A_3, A_{n+2}A_{n+3}), \ldots, (A_{n-1}A_n, A_{2n-1}A_{2n})$$

are parallel, then the sides

$$A_nA_{n+1}, A_{2n}A_1$$

are parallel as well.

**PROBLEMA 5**
In a rectangular coordinate system we call a horizontal line parallel to the $x$ -axis triangular if it intersects the curve with equation

$$y = x^4 + px^3 + qx^2 + rx + s$$

in the points $A, B, C$ and $D$ (from left to right) such that the segments $AB, AC$ and $AD$ are the sides of a triangle. Prove that the lines parallel to the $x$ - axis intersecting the curve in four distinct points are all triangular or none of them is triangular.

**PROBLEMA 6**
Find the digits left and right of the decimal point in the decimal form of the number

$$(\sqrt{2} + \sqrt{3})^{1980}.$$

## Dia I

**PROBLEMA 1**
If $p$ and $q$ are natural numbers so that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319},$$

prove that $p$ is divisible with 1979.

**PROBLEMA 2**
We consider a prism which has the upper and inferior basis the pentagons: $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$. Each of the sides of the two pentagons and the segments $A_iB_j$ with $i, j = 1, \ldots, 5$ is colored in red or blue. In every triangle which has all sides colored there exists one red side and one blue side. Prove that all the 10 sides of the two basis are colored in the same color.

**PROBLEMA 3**
Two circles in a plane intersect. $A$ is one of the points of intersection. Starting simultaneously from $A$ two points move with constant speed, each travelling along its own circle in the same sense. The two points return to $A$ simultaneously after one revolution. Prove that there is a fixed point $P$ in the plane such that the two points are always equidistant from $P$.

## Dia II

**PROBLEMA 4**
We consider a point $P$ in a plane $p$ and a point $Q \notin p$. Determine all the points $R$ from $p$ for which

$$\frac{QP + PR}{QR}$$

is maximum.

**PROBLEMA 5**
Determine all real numbers a for which there exists positive reals $x_1, \ldots, x_5$ which satisfy the relations $\sum_{k=1}^{5} kx_k = a$, $\sum_{k=1}^{5} k^3 x_k = a^2$, $\sum_{k=1}^{5} k^5 x_k = a^3$.

**PROBLEMA 6**
Let $A$ and $E$ be opposite vertices of an octagon. A frog starts at vertex $A$. From any vertex except $E$ it jumps to one of the two adjacent vertices. When it reaches $E$ it stops. Let $a_n$ be the number of distinct paths of exactly $n$ jumps ending at $E$. Prove that:

$$a_{2n-1} = 0, \quad a_{2n} = \frac{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}}{\sqrt{2}}.$$

## Dia I

**PROBLEMA 1**
Let $m$ and $n$ be positive integers such that $1 \leq m < n$. In their decimal representations, the last three digits of $1978^m$ are equal, respectively, to the last three digits of $1978^n$. Find $m$ and $n$ such that $m + n$ has its least value.

**PROBLEMA 2**
We consider a fixed point $P$ in the interior of a fixed sphere. We construct three segments $PA, PB, PC$, perpendicular two by two, with the vertexes $A, B, C$ on the sphere. We consider the vertex $Q$ which is opposite to $P$ in the parallelepiped (with right angles) with $PA, PB, PC$ as edges. Find the locus of the point $Q$ when $A, B, C$ take all the positions compatible with our problem.

**PROBLEMA 3**
Let $0 < f(1) < f(2) < f(3) < \ldots$ a sequence with all its terms positive. The $n - th$ positive integer which doesn't belong to the sequence is $f(f(n)) + 1$. Find $f(240)$.

## Dia II

**PROBLEMA 4**
Seja $ABC$ um triângulo com $AB = AC$. Um círculo que é internamente tangente ao circuncírculo de $ABc$ também é tangente aos lados $AB$ e $AC$ nos pontos $P$ e $Q$, respectivamente.

Prove que o ponto médio de $PQ$ é o incentro de $ABC$.

**PROBLEMA 5**
Let $f$ be an injective function from $1, 2, 3, \ldots$ in itself. Prove that for any $n$ we have: $\sum_{k=1}^{n} f(k)k^{-2} \geq \sum_{k=1}^{n} k^{-1}$.

**PROBLEMA 6**
An international society has its members from six different countries. The list of members contain 1978 names, numbered $1, 2, \ldots, 1978$.

Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.