Assignment

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Consider the experiment of throwing a die. If a multiple of 3 comes up, throw the die again. If any other number comes up, toss a coin. Find the conditional probability of the event the coin shows a tail, given that at least one die shows a 3.

Solution: Let, the states S_0 and S_1 describe the

S_0	$\Sigma(Y = k); k \in (3, 6)$
S_1	$\Sigma(Y = k); k \in (1, 2, 4, 5)$
S_2	Outcome of coin toss is heads
S_3	Outcome of coin toss is tails

outcomes of dice throws.
S_2 and S_3 describe the outcomes of coin toss. Conditional Probability is that "The coin shows tails" given that "at least one die shows a 3". Since, a Markov chain does not depend on the past outcomes,

$$p_{S_3|S_0} = \Pr\left(X_n = S_3|X_1 = S_0\right) \tag{1}$$

Transition Probability Matrix is given as,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{pmatrix} \tag{2}$$

The state vector is defined as,

$$\mathbf{Q_n} = \begin{pmatrix} p^n_{S_0} \\ p^n_{S_1} \\ p^n_{S_2} \\ p^n_{S_3} \end{pmatrix} \tag{3}$$

where $p^n_{S_i}$ is the probability of S_i state after n time interval from initial state.
The given conditon is "3 occurs at least once", and we let occurrence of 3 as the initial state, Since Markov chain does not depend on past outcomes. So, $\Pr\left(X = S_0\right) = 1$ and 0 for all other states.
And State vector is ,

$$\mathbf{Q_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

the state vector after one step in time can be written as,

$$\mathbf{Q_1} = \mathbf{PQ_0} \tag{5}$$
$$\mathbf{Q_2} = \mathbf{PQ_1} \tag{6}$$
$$\vdots \tag{7}$$
$$\mathbf{Q_n} = \mathbf{PQ_{n-1}} \tag{8}$$

we can see that the markov chain is reducible, a unique state vector exists and it is equal to limiting distribution.i.e, after a long time,

$$\mathbf{Q_n} = \mathbf{Q_{n-1}} \tag{9}$$

substituting the chain of state vectors, we get

$$\mathbf{Q_n} = \mathbf{P^n Q_0} \tag{10}$$

applying limits to find the stationary probability vector,

$$\lim_{n\to\infty} \mathbf{Q_n} = \mathbf{Q} = \mathbf{P^n Q_0} \tag{11}$$

We can find Q using eigenvector and eigenvalues. Now, to find the eigenvalues, let λ be the eigen value for the Transition Probability matrix.

$$\implies |\mathbf{P} - \lambda\mathbf{I}| = 0 \tag{12}$$

$$\begin{pmatrix} \frac{1}{3} - \lambda & 0 & 0 & 0 \\ \frac{2}{3} & 0 - \lambda & 0 & 0 \\ 0 & \frac{1}{2} & 1 - \lambda & 0 \\ 0 & \frac{1}{2} & 0 & 1 - \lambda \end{pmatrix} = 0 \tag{13}$$

$$\left(\frac{1}{3} - \lambda\right)(1 - \lambda)^2 (-\lambda) = 0 \tag{14}$$

$$\implies \lambda = \frac{1}{3}, 0, 1, 1 \tag{15}$$

Now, to find eigenvector,

$$\left(\mathbf{P} - \lambda\mathbf{I_4}\right)\mathbf{X} = \mathbf{0} \tag{16}$$

where X is the eigen vector coresponding to eigen value λ , using row reduction
$\implies \lambda = \frac{1}{3}$

$$\mathbf{X} = \begin{pmatrix} \frac{-2}{3} \\ \frac{-4}{3} \\ 1 \\ 1 \end{pmatrix} \tag{17}$$

$\implies \lambda = 0$

$$\mathbf{X} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \tag{18}$$

and for $\implies \lambda = 1$, using row echelon form,

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{19}$$

also

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{20}$$

then the eigenvector matrix is ,

$$\mathbf{S} = \begin{pmatrix} \frac{-2}{3} & 0 & 0 & 0 \\ \frac{-4}{3} & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{21}$$

Diagonal matrix $\mathbf{D}$ is,

$$\mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

It is said that,

$$\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} \tag{23}$$
$$\implies \mathbf{P}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{24}$$
$$\implies \lim_{n\to\infty} \mathbf{P}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{25}$$

finding as $n \to \infty$, value of $\mathbf{D}$

$$\lim_{n\to\infty} \mathbf{D}^n = \begin{pmatrix} \left(\frac{1}{3}\right)^n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1^n & 0 \\ 0 & 0 & 0 & 1^n \end{pmatrix} \tag{26}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{27}$$

and value of $\mathbf{S}^{-1}$ is

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{-3}{2} & 0 & 0 & 0 \\ 1 & \frac{-1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix} \tag{28}$$

by substititing the values,

$$\lim_{n\to\infty} \mathbf{P}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{29}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix} \tag{30}$$

By substituting the values of Q_0 and P_∞ in the above equation, We get the steady state probability vector as,

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{31}$$
$$= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{32}$$

So, the probability that "coin shows tails" given that "Die shows at least one 3" is,

$$p_{3|1} = \frac{1}{2} \tag{33}$$

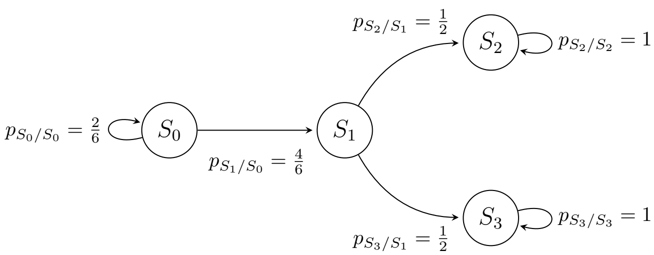

Fig. 1: State diagram generated using LatexTikZ