

Assignment

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Consider the experiment of throwing a die. If a multiple of 3 comes up, throw the die again. If any other number comes up, toss a coin. Find the conditional probability of the event the coin shows a tail, given that at least one die shows a 3.

Solution: Let, the states S_0 and S_1 describe the

S_0	$\Sigma(Y = k); k \in (3, 6)$
S_1	$\Sigma(Y = k); k \in (1, 2, 4, 5)$
S_2	Outcome of coin toss is heads
S_3	Outcome of coin toss is tails

outcomes of dice throws.

S_2 and S_3 describe the outcomes of coin toss. Conditional Probability is that "The coin shows tails" given that "at least one die shows a 3". Since, a Markov chain does not depend on the past outcomes,

$$p_{S_3|S_0} = \Pr(X_n = S_3 | X_1 = S_0) \quad (1)$$

Transition Probability Matrix is given as,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{pmatrix} \quad (2)$$

The state vector is defined as,

$$\mathbf{Q}_n = \begin{pmatrix} p_{S_0}^n \\ p_{S_1}^n \\ p_{S_2}^n \\ p_{S_3}^n \end{pmatrix} \quad (3)$$

where $p_{S_i}^n$ is the probability of S_i state after n time interval from initial state.

The given condition is "3 occurs at least once", and we let occurrence of 3 as the initial state, Since Markov chain does not depend on past outcomes. So, $\Pr(X = S_0) = 1$ and 0 for all other states.

And State vector is ,

$$\mathbf{Q}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4)$$

the state vector after one step in time can be written as,

$$\mathbf{Q}_1 = \mathbf{P}\mathbf{Q}_0 \quad (5)$$

$$\mathbf{Q}_2 = \mathbf{P}\mathbf{Q}_1 \quad (6)$$

$$\vdots \quad (7)$$

$$\mathbf{Q}_n = \mathbf{P}\mathbf{Q}_{n-1} \quad (8)$$

we can see that the markov chain is reducible, a unique state vector exists and it is equal to limiting distribution.i.e, after a long time,

$$\mathbf{Q}_n = \mathbf{Q}_{n-1} \quad (9)$$

substituting the chain of state vectors, we get

$$\mathbf{Q}_n = \mathbf{P}^n \mathbf{Q}_0 \quad (10)$$

applying limits to find the stationary probability vector,

$$\lim_{n \rightarrow \infty} \mathbf{Q}_n = \mathbf{Q} = \mathbf{P}^n \mathbf{Q}_0 \quad (11)$$

We can find \mathbf{Q} using eigenvector and eigenvalues. Now, to find the eigenvalues, let λ be the eigen value for the Transition Probability matrix.

$$\Rightarrow |\mathbf{P} - \lambda \mathbf{I}| = 0 \quad (12)$$

$$\begin{pmatrix} \frac{1}{3} - \lambda & 0 & 0 & 0 \\ \frac{2}{3} & 0 - \lambda & 0 & 0 \\ 0 & \frac{1}{2} & 1 - \lambda & 0 \\ 0 & \frac{1}{2} & 0 & 1 - \lambda \end{pmatrix} = 0 \quad (13)$$

$$\left(\frac{1}{3} - \lambda\right) (1 - \lambda)^2 (-\lambda) = 0 \quad (14)$$

$$\Rightarrow \lambda = \frac{1}{3}, 0, 1, 1 \quad (15)$$

Now, to find eigenvector,

$$|\mathbf{P} - \lambda \mathbf{I}| \mathbf{X} = 0 \quad (16)$$

where X is the eigen vector corresponding to eigen value λ , using row reduction

$$\Rightarrow \lambda = \frac{1}{3}$$

$$\mathbf{X} = \begin{pmatrix} \frac{-2}{3} \\ \frac{-4}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \quad (17)$$

$$\Rightarrow \lambda = 0$$

$$\mathbf{X} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \quad (18)$$

$$\Rightarrow \lambda = 1$$

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (19)$$

also

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (20)$$

then the eigenvector matrix is ,

$$\mathbf{S} = \begin{pmatrix} \frac{-2}{3} & 0 & 0 & 0 \\ \frac{-4}{3} & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad (21)$$

Diagonal matrix \mathbf{D} is,

$$\mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (22)$$

It is said that,

$$\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} \quad (23)$$

$$\Rightarrow \mathbf{P}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \quad (24)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \mathbf{P}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \quad (25)$$

$$(26)$$

finding as $n \rightarrow \infty$, value of \mathbf{D}

$$\lim_{n \rightarrow \infty} \mathbf{D}^n = \begin{pmatrix} \left(\frac{1}{3}\right)^n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1^n & 0 \\ 0 & 0 & 0 & 1^n \end{pmatrix} \quad (27)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (28)$$

$$\lim_{n \rightarrow \infty} \mathbf{P}^n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \quad (29)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix} \quad (30)$$

By substituting the values of Q_0 and P_∞ in the above equation, We get the steady state probability vector as,

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (31)$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad (32)$$

So, the probability that "coin shows tails" given that "Die shows at least one 3" is,

$$p_{3|1} = \frac{1}{2} \quad (33)$$

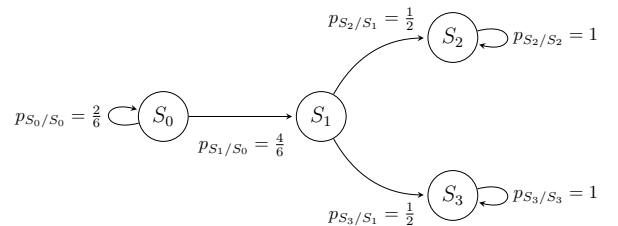


Fig. 1: State diagram generated using LatexTikZ