TTIC 31230 Fundamentals of Deep Learning

Problems for Rate Distortion Autoencoders.

Problem 1 The mutual information between two random variables x and y is defined by

$$I(x,y) = E_{x,y}\;\ln\frac{p(x,y)}{p(x)p(y)} = KL(p(x,y), p(x)p(y))$$

Mutual information has an interpretation as a channel capacity.

(a) Suppose that we draw a random bit $y \in \{0,1\}$ with $P(0) = P(1) = 1/2$ and send it across a noisy channel to a receiver who gets $y' = y + \epsilon$ where ϵ is random noise drawn from a Gaussian $\mathcal{N}(0,\sigma)$. Solve for the channel capacity $I(y,y')$ as a function of σ. This channel capacity, when measured in bits, has units of bits received per bit sent.

(b) Repeat (a) but for $y' = sy + \epsilon$. Here s can be interpreted as the signal strength and s/σ as a signal to noise ratio.

Problem 2. Consider a rate-distortion autoencoder.

$$\Phi^* = \operatorname*{argmin}_{\Phi}\; I_\Phi(y,z) + \lambda E_{y\sim\mathrm{Pop},\;z\sim p_\Phi(z|y)}\,\mathrm{Dist}(y, y_\Phi(z)).$$

Here $I_\Phi(y,z)$ is defined by the distribution where we draw y from Pop and z from $P_\Phi(z|y)$. The distribution $p_\Phi(z|y)$ is typically defined by $z = z_\Phi(y) + \epsilon$ for some form of random noise ϵ.

(a) Starting from the definition of $I_\Phi(y,z)$ given in problem 1, show

$$I_\Phi(y,z) = E_{y\sim\mathrm{Pop}} KL(p_\Phi(z|y), p_\Phi(z))$$

where $p_\Phi(z) = \sum_y \mathrm{Pop}(y) P_\Phi(z|y)$.

(b) Show the variational equation

$$I(y,z) = \inf_q\; E_{y\sim\mathrm{Pop}} KL(p_\Phi(z|y), q(z)).$$

Hint: It suffices to show

$$I(y,z) \leq\; E_{y\sim\mathrm{Pop}} KL(p_\Phi(z|y), q(z))$$

and that there exists a q achieving equality.

Problem 3. Consider a rate-distortion autoencoder

$$\Phi^*,\Psi^* = \operatorname*{argmin}_{\Phi,\Psi}\; E_{y\sim\mathrm{Pop}} KL(p_\Phi(z|y), p_\Psi(z)) + \lambda E_{y\sim\mathrm{Pop},\;z\sim p(z|y)}\,\mathrm{Dist}(y, y_\Phi(z)).$$

Define $p_\Phi(z|y)$ by $z = z_\Phi(y) + \epsilon$ with $z_\Phi[y] \in \mathbb{R}^d$ and ϵ drawn uniformly from $[0,1]^d$. In other words, we add noise drawn uniformly from $[0,1]$ to each component of $z_\Phi(y)$.

Define $p_\Psi(z)$ to be log-uniform in each dimension. More specifically $p_\Psi(z)$ is defined by drawing $s[i]$ uniformly from the interval $[1, s_{\max}]$ and then setting $z[i] = e^s$ so that $\ln z[i]$ is uniformly distributed over the interval $[0, s_{\max}]$. This gives

$$
\begin{aligned}
dz &= e^s ds &= z ds \\
dp &= \frac{1}{s_{\max}} ds \\
p_\Psi(z[i]) &= \frac{dp}{dz} &= \frac{1}{s_{\max} z[i]}
\end{aligned}
$$

Assume That we have that $z_\Phi(y) \in [0, e^{s_{\max}-1}]^d$ so that with probability 1 over the draw of ϵ $P_\Psi(z_\Phi(y) + \epsilon) > 0$.

(a) For $z \in [z_\Phi(y), z_\Phi(y) + 1]$ what is $p_\Phi(z|y)$?

(b) Solve for $KL(p_\Phi(z|y), p_\Psi(z))$ in terms of $z_\Phi(y)$ under the above specifications.

(b) Explain how these specifications model rounding down each number in $z_\Phi(y)$ to the nearest integer.