TTIC 31230, Fundamentals of Deep Learning

David McAllester, Winter 2019

Rate-Distortion Autoencoders (RDAs)

Noisy Channel RDAs

Gaussian Variational Autoencoders (Gaussian VAEs)

Rate-Distortion Autoencoders

Given a continuous signal y we can compress it into a (discrete) bit string $\tilde{z}_\Phi(y)$.

We let $y_\Phi(\tilde{z}_\Phi(y))$ be the decompression of $\tilde{z}_\Phi(y)$.

We can then define a rate-distortion loss.

$$\mathcal{L}(\Phi) = E_{y\sim\mathrm{Pop}}\ |\tilde{z}_\Phi(y)| + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}_\Phi(y)))$$

Common Distortion Functions

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}} \; E_{y \sim \mathrm{Pop}} \, |\tilde{z}_\Phi(y)| + \lambda \mathrm{Dist}(y, y_\Phi(y))$$

It is common to take

$$\mathrm{Dist}(y, y') = ||y - y'||^2 \qquad (L_2)$$

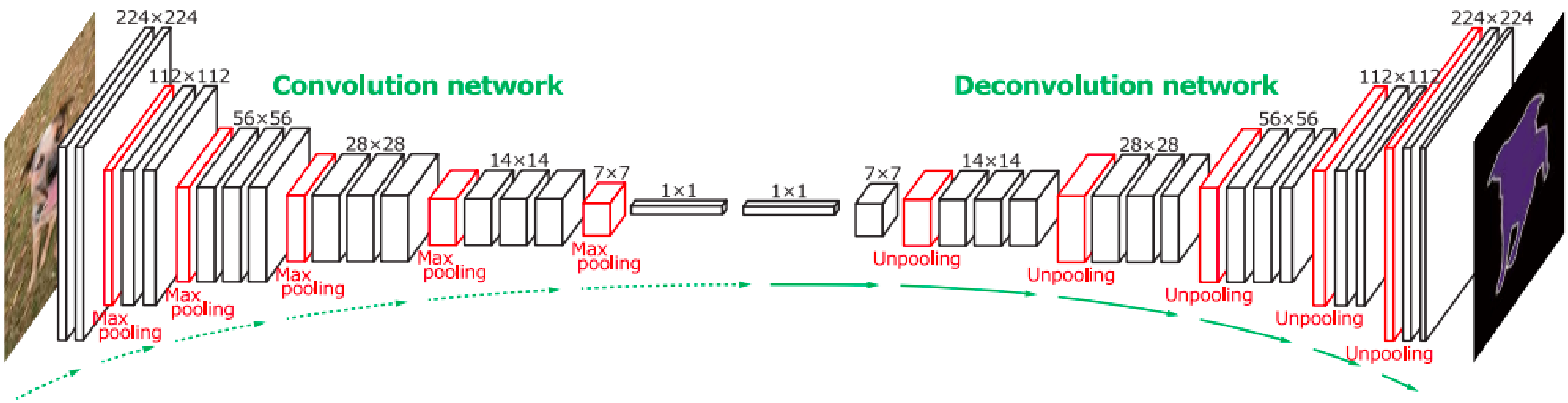
or

$$\mathrm{Dist}(y, y') = ||y - y'||_1 \qquad (L_1)$$

A Case Study in Image Compression

End-to-End Optimized Image Compression, Balle, Laparra, Simoncelli, ICLR 2017.

y $\qquad \tilde{z}_\Psi(y)$ $\qquad \tilde{z}$ $\qquad y'_\Phi(\tilde{z})$ $\qquad ||y - y'||^2$



JPEG at 4283 bytes or .121 bits per pixel

JPEG, 4283 bytes (0.121 bit/px), PSNR: 24.85 dB/29.23 dB, MS-SSIM: 0.8079

JPEG 2000 at 4004 bytes or .113 bits per pixel

JPEG 2000, 4004 bytes (0.113 bit/px), PSNR: 26.61 dB/33.88 dB, MS-SSIM: 0.8860

Deep Autoencoder at 3986 bytes or .113 bits per pixel

Proposed method, 3986 bytes (0.113 bit/px), PSNR: 27.01 dB/34.16 dB, MS-SSIM: 0.9039

A CNN Encoder

A three layer CNN is used as an encoder.

We let $z_\Phi(y)$ be the final layer of this CNN.

Each continuous value in the final layer $z_\Phi(y)$ is then rounded to a (small) integer giving a discrete encoding $\tilde{z}(y)$.

Rate-Distortion Autoencoders

$$\Phi^* = \operatorname*{argmin}_{\Phi} \; E_{y \sim \mathrm{Pop}} \, |\tilde{z}_\Phi(y)| + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}_\Phi(y)))$$

Oops: Because of rounding, $\tilde{z}_\Phi(y)$ is discrete and the gradients are zero.

We will train using a differentiable approximation.

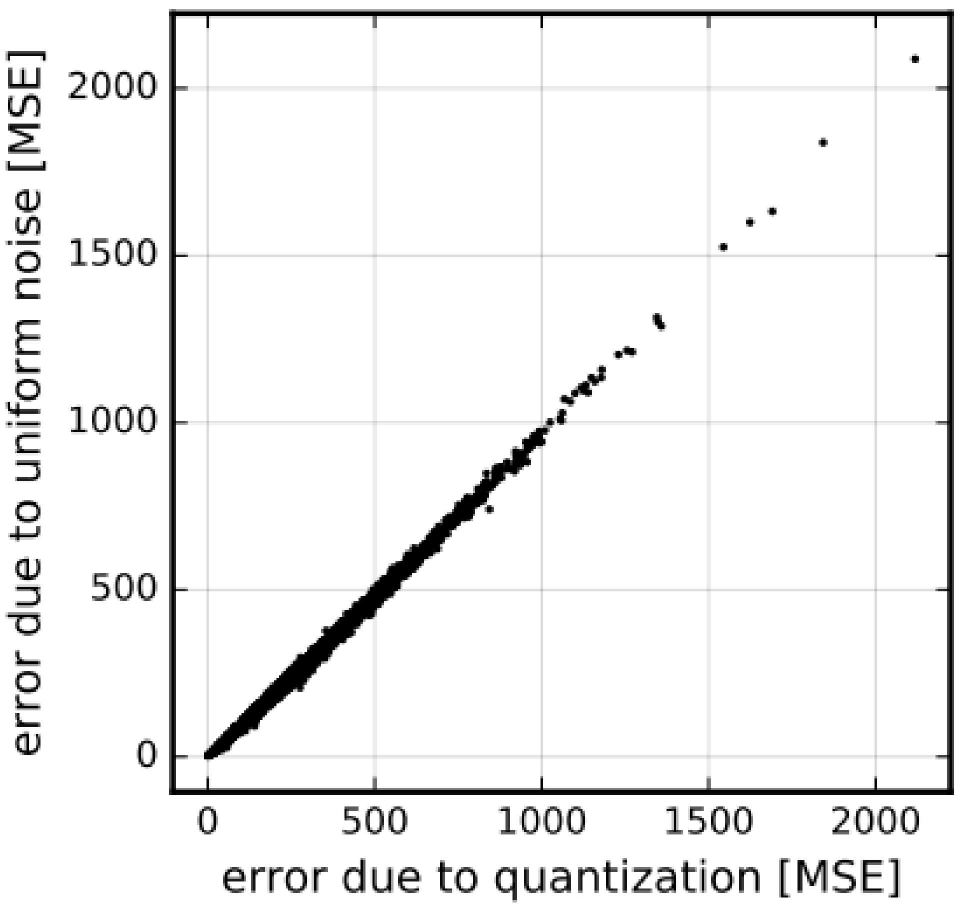
A Noise Model of Rounding

We can make distortion differentiable by modeling rounding as the addition of noise.

$$\begin{aligned}\mathcal{L}_{\mathrm{dist}}(\Phi) &= E_{y\sim\mathrm{Pop}}\,\mathrm{Dist}(y, y_\Phi(\tilde{z}_\Phi(y))) \\ &\approx E_{y,\epsilon}\,\mathrm{Dist}(y,\ y_\Phi(z_\Phi(y)+\epsilon))\end{aligned}$$

Here ϵ is a noise vector each component of which is drawn uniformly from $(-1/2, 1/2)$.

Noise vs. Rounding



A Differentiable Approximation of Code Length

$$\mathcal{L}_{\text{rate}}(\Phi) = E_{y\sim\text{Pop}}\;|\tilde{z}_\Phi(y)|$$

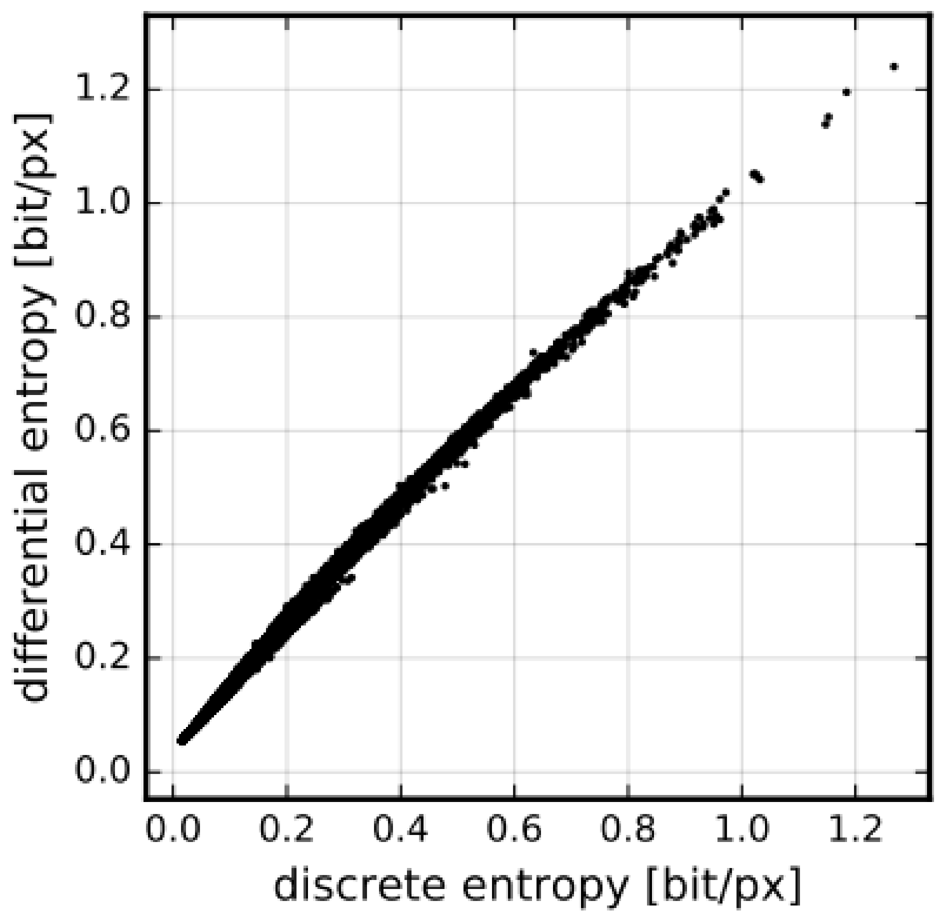
Recall that $\tilde{z}_\Phi(y)$ is a rounding of a continuous encoding $z_\Phi(y)$.

We approximate the code length after rounding using a differentiable function of the value before rounding.

$$|\tilde{z}_\Phi(y)| \approx \sum_i (\log_2 z_\Phi(y)[i])^+$$

This continuous value can be interpreted as a "differential entropy".

Differential Entropy vs. Discrete Entropy



Details

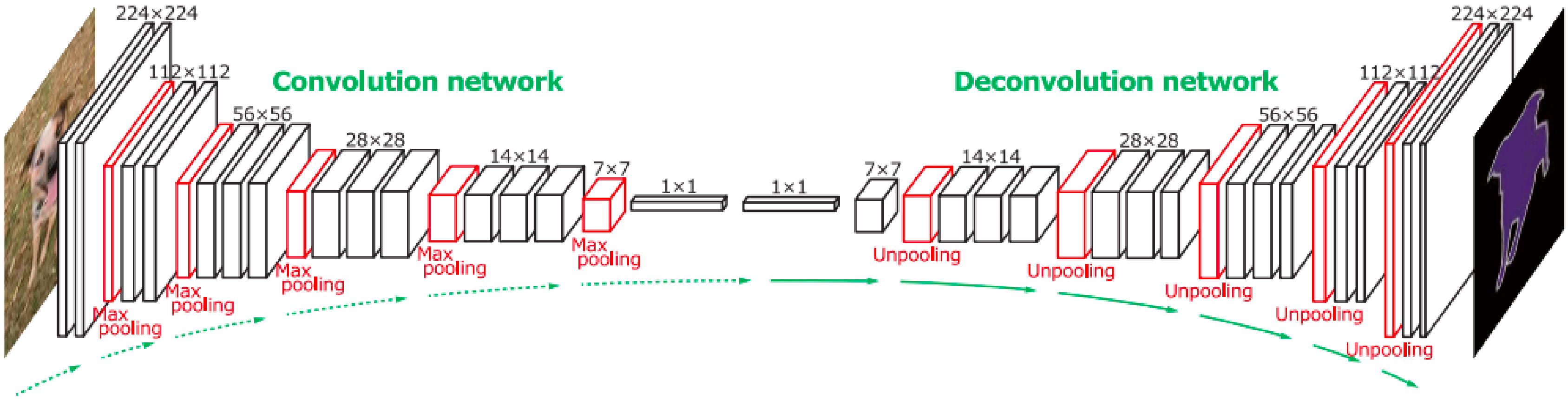
The first layer is computed stride 4.

The next two layers are computed stride 2.

Final image dimension is reduced by a factor of 16 with 192 channels per pixel (192 channels is for color images).

$$192 < 16 \times 16 \times 3 = 768$$

Increasing Spatial Dimension in Decoding



[Hyeonwoo Noh et al.]

In the ICLR 17 paper the deconvolution network has the shape as the input CNN but with independent parameters.

Increasing Spatial Dimension in Decoding

Consider a stride 2 convolution

$$L_{\ell+1}[x, y, j] = \sigma\left(\sum_{\Delta x, \Delta y, i} W[\Delta x, \Delta y, i, j] L_\ell[2x + \Delta x, 2y + \Delta y, i]\right)$$

For deconvolution we use stride 1 with 4 times the features.

$$L'_\ell[x, y, i] = \sigma\left(\sum_{\Delta x, \Delta y, j} W[\Delta x, \Delta y, j, i] L'_{\ell+1}[x + \Delta x, y + \Delta y, j]\right)$$

The channels at each $L'_\ell[x, y]$ are divided among four higher resolution pixels.

This is done by a simple reshaping of $L'_\ell[x, y, i]$.

Noisy-Channel RDAs (TZ)

Consider the differentiable loss used in training.

$$\Phi^* = \operatorname*{argmin}_{\Phi} \; E_{y\sim\mathrm{Pop}} -\ln p(z_\Phi(y)) + \lambda E_\epsilon \; \mathrm{Dist}(y, y_\Phi(z_\Phi(y)+\epsilon))$$

Intuitively, $-\ln p(z_\Phi(y))$ is a proxy for the number of bits used in the (intuitively rounded) encoding $z_\Phi(y) + \epsilon$.

By the channel capcacity theorem the number of bits that $z_\Phi(y) + \epsilon$ can carry about y is the mutual information between y and $z_\Phi(y) + \epsilon$.

$$I(y, z_\Phi(y) + \epsilon)$$

Noisy-Channel RDAs

We now consider $p_\Phi(z|y)$ as a generalization of $z_\Phi(y) + \epsilon$.

The channel capacity theorem motivates

$$p_\Phi(z) = E_{y\sim\mathrm{Pop}}\, p_\Phi(z|y)$$
$$I(y,z) = H(z) - H(z|y) = E_{y\sim\mathrm{Pop}}\, KL(p_\Phi(z|y), p_\Phi(z))$$

$$\begin{aligned}
\Phi^* &= \operatorname*{argmin}_\Phi \left(I(y,z) + \lambda E_{y\sim\mathrm{Pop},z\sim p_\Phi(z|y)}\, \mathrm{Dist}(y, y_\Phi(z)) \right) \\
&= \operatorname*{argmin}_\Phi E_{y\sim\mathrm{Pop}} \left(\begin{array}{c} KL(p_\Phi(z|y), p_\Phi(z)) \\ \\ +\lambda E_{z\sim p_\Phi(z|y)}\, \mathrm{Dist}(y, y_\Phi(z)) \end{array} \right)
\end{aligned}$$

A Variational Upper Bound

Unfortunately we cannot compute $p_\Phi(z) = E_{y\sim\text{Pop}}\; p_\Phi(z|y)$.

We now replace $p_\Phi(z)$ by a friendly (variational) model $p_\Psi(z)$.

$$
\begin{aligned}
& I_\Phi(y,z) \\
= \;& E_{y\sim\text{Pop}}\; KL(p_\Phi(z|y), p_\Phi(z)) \\
= \;& E_{y,z\sim P_\Phi(z|y)}\; \left(\ln \frac{p_\Phi(z|y)}{p_\Psi(z)} + \ln \frac{p_\Psi(z)}{p_\Phi(z)}\right) \\
= \;& E_{y\sim\text{Pop}}\; KL(p_\Phi(z|y), p_\Psi(z)) + \left(E_{y\sim\text{Pop},\; z\sim p_\Phi(z|y)}\; \ln \frac{p_\Psi(z)}{p_\Phi(z)}\right)
\end{aligned}
$$

A Variational Upper Bound

$$I(y, z)$$

$$= E_{y\sim\mathrm{Pop}}\; KL(p_\Phi(z|y), p_\Psi(z)) + \left(E_{y\sim\mathrm{Pop},\; z\sim p_\Phi(z|y)}\; \ln \frac{p_\Psi(z)}{p_\Phi(z)}\right)$$

$$= E_y\; KL(p_\Phi(z|y), p_\Psi(z)) + E_{z\sim p_\Phi(z)}\; \ln \frac{p_\Psi(z)}{p_\Phi(z)}$$

$$= E_y\; KL(p_\Phi(z|y), p_\Psi(z)) - KL(p_\Phi(z), p_\Psi(z))$$

$$\leq E_{y\sim\mathrm{Pop}}\; KL(p_\Phi(z|y), p_\Psi(z))$$

The Noisy-Channel RDA

$$\Phi^*, \Psi^* = \operatorname*{argmin}_{\Phi,\Psi} \; E_{y\sim \mathrm{Pop}} \left(\begin{array}{c} KL(p_\Phi(z|y), p_\Psi(z)) \\ \\ +\lambda \; E_{z\sim p_\Phi(z|y)} \; \mathrm{Dist}(y,\; y_\Phi(z)) \end{array} \right)$$

Gaussian Noisy-Channel RDA

$$\Phi^*, \Psi^* = \operatorname*{argmin}_{\Phi,\Psi} \; E_{y\sim \text{Pop}} \left(\begin{array}{c} KL(p_\Phi(z|y), p_\Psi(z)) \\ \\ +\lambda \; E_{z\sim p_\Phi(z|y)} \; \text{Dist}(y,\; y_\Phi(z)) \end{array} \right)$$

$$p_\Phi(z[i] \mid y) \;=\; \mathcal{N}(z_\Phi(y)[i], \sigma_\Phi(y)[i]))$$

$$p_\Psi(z[i]) \;=\; \mathcal{N}(\mu_\Psi[i], \sigma_\Psi[i])$$

$$\text{Dist}(y, y') \;=\; ||y - y'||^2$$

Closed Form KL-Divergence

$$KL(p_\Phi(z|y), p_\Psi(z))$$

$$= \sum_i \frac{\sigma_\Phi(y)[i]^2 + (z_\Phi(y)[i] - \mu_\Psi[i])^2}{2\sigma_\Psi[i]^2} + \ln \frac{\sigma_\Psi[i]}{\sigma_\Phi(y)[i]} - \frac{1}{2}$$

Standardizing $p_\Psi(z)$

The KL-divergence term is

$$\sum_i \frac{\sigma_\Phi(y)[i]^2 + (z_\Phi(y)[i] - \mu_\Psi[i])^2}{2\sigma_\Psi[i]^2} + \ln\frac{\sigma_\Psi[i]}{\sigma_\Phi(y)[i]} - \frac{1}{2}$$

We can adjust Φ to Φ' such that

$$\begin{aligned} z_{\Phi'}(y)[i] &= z_\Phi(y)[i]/\sigma_\Psi[i] + \mu_\Psi[i] \\ \sigma_{\Phi'}(y)[i] &= \sigma_\Phi(y)[i]/\sigma_\Psi[i] \end{aligned}$$

We then get $KL(p_{\Phi'}(z|y), \mathcal{N}(0,I)) = KL(p_\Phi(z|y), p_\Psi(z))$.

Standardizing $p_\Psi(z)$

Without loss of generality the Gaussian noisy channel RDA becomes.

$$\Phi^* = \operatorname*{argmin}_{\Phi} \; E_{y\sim \mathrm{Pop}} \left(\begin{array}{c} KL(p_\Phi(z|y), \mathcal{N}(0, I)) \\ \\ +\lambda E_{z\sim p_\Phi(z|y)} \; \mathrm{Dist}(y, y_\Phi(z)) \end{array} \right)$$

Reparameterization Trick for Optimizing Distortion

$$p_\Phi(z[i]|y) = \mathcal{N}(z_\Phi(y)[i], \sigma_\Phi[i])$$

$$E_{z \sim p_\Phi(z|y)} \; ||y - y_\Phi(z)||^2$$

$$= \; E_{\epsilon \sim \mathcal{N}(0,I)} \; z[i] = z_\Phi(y)[i] + \sigma_\Phi(y)[i]\epsilon[i]; \quad ||y - y_\Phi(z)||^2$$

Sampling

Sample $z \sim \mathcal{N}(0, I)$ and compute $y_\Phi(z)$

[Alec Radford]

Summary: Rate-Distortion

RDA: y continuous, $\tilde{z}$ a bit string,

$$\Phi^* = \operatorname*{argmin}_{\Phi} E_{y\sim\mathrm{Pop}}\, |\tilde{z}_\Phi(y)| + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}_\Phi(y)))$$

Gaussian RDA: $z = z_\Phi(y) + \sigma_\Phi(y) \odot \epsilon, \qquad \epsilon \sim \mathcal{N}(0, I)$

$$\Phi^* = \operatorname*{argmin}_{\Phi} E_{y\sim\mathrm{Pop}} \left(\begin{array}{c} KL(p_\Phi(z|y), \mathcal{N}(0,I)) \\ \\ +\lambda E_{z\sim p_\Phi(z|y)}\, \mathrm{Dist}(y, y_\Phi(z)) \end{array}\right)$$

Issue: Do we expect compression to yield useful features?

END