TTIC 31230, Fundamentals of Deep Learning

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Entropy and Compressibility

Rate-Distortion Autoencoders

The Fundamental Equation:

Conditional vs. Unconditional

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}}\, E_{(x,y)\sim \mathrm{Pop}} \; -\ln \; P(y|x)$$

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}}\, E_{y\sim \mathrm{Pop}} \; -\ln \; P(y)$$

This is a non-distinction: the issues in the to the conditional case are exactly the same as in the unconditional case.

The Fundamental Equation:

Distributions on Exponentially Large Sets

The structured case: $y \in \mathcal{Y}$ where $\mathcal{Y}$ is discrete but iteration over $\hat{y} \in \mathcal{Y}$ is infeasible.

Language modeling (unconditional) and machine translation (conditional) are distributions on exponentially large (even infinite) sets.

Friendly and Unfriendly Distributions

Autoregressive Language models (unconditional) and machine translation models (conditional) are friendly.

A model $P_\Phi(y)$ will be called friendly if we can efficiently sample from it and, for any given y, can efficiently compute $P(y)$.

Distributions which are not friendly in this sense will be called unfriendly.

Graphical Models are Generally Unfriendly:

Semantic Segmentation

SLIC superpixels, Achanta et al.

We want to assign each pixel one of k semantic classes such as "person", "car", "building", "sky", "other" (multiclass classification of each superpixel).

$\hat{y}[i]$ is the semantic class value of superpixel i.

Exponential Softmax

$$P_\Phi(\hat{y}|x) = \underset{\hat{y}}{\text{softmax}}\ s_\Phi(\hat{y}|x)$$

Let $\mathcal{C}$ be the semantic classes, $\mathcal{I}$ be superpixels, and $\mathcal{E}$ be edges.

We will compute

a unary potential tensor $s_i[c] = s_\Phi(c|x, i)$

a binary potential tensor $s_e[c, c'] = s_\Phi(c, c'|x, e)$

$$s_\Phi(\hat{y}|x) = \sum_{i \in \mathcal{I}} s_i[\hat{y}[i]] + \sum_{e \in \mathcal{E}} s_e[\hat{y}[e.i], \hat{y}[e.j]]$$

Exponential Softmax

$$P_\Phi(\hat{y}|x) = \operatorname*{softmax}_{\hat{y}} s_\Phi(\hat{y}|x)$$

Distributions defined by exponential softmax operations are not friendly in general — there is no efficient general sampling algorithm or efficient general method of computing Z.

Computing Z is easily shown to be #P hard.

Big Picture: Cross-Entropy as a Data Rate

For a distribution $P(y)$ on a discrete set $\mathcal{Y}$, the entropy $H(P)$, when measured using $\log_2$ rather than ln, gives the number of bits needed on average, when drawing from P, to represent the elements of $\mathcal{Y}$.

The cross-entropy $H(P, Q)$ give the number of bits used to code for items drawn from P but using the code defined by Q.

Cross-entropy gives the "data rate" when transmitting codes for items drawn from P but using the code defined by Q.

Entropy and Compressibility

Let S be a finite set.

Let z be a compression (or coding) function assigning a bit string $z(y)$ to each $y \in S$.

The compression function z is called *prefix-free* if for $y' \neq y$ we have that $z(y')$ is not a prefix of $z(y)$.

Prefix-Free Codes as Probabilities

A prefix-free code defines a binary branching tree — branch on the first code bit, then the second, and so on.

For a prefix-free code, only the leaves of this tree can be labeled with the elements of S.

The code defines a probability distribution on S by randomly selecting branches.

We have $P_z(y) = 2^{-|z(y)|}$.

Bits vs. Nats

We have that $|z(y)|$ is a number of bits.

We can define entropy in units of bits by

$$H_2(y) = E_y \ -\log_2 P(y) = H(y)/(\ln\ 2)$$

If y is uniformly distributed over 8 values then $H_2(y)$ is 3 bits.

We have that $H_2(y)$ is a number of bits while $H(y)$ is a number of "nats".

The Source Coding (compression) Theorem

(1) There exists a prefix-free code z such that

$$|z(y)| <= (-\log_2 \mathrm{Pop}(y)) + 1$$

and hence

$$E_{y \sim \mathrm{Pop}} |z(y)| \leq H_2(\mathrm{Pop}) + 1$$

(2) For any prefix-free code z

$$E_{y \sim \mathrm{Pop}} \, |z(y)| \geq H_2(\mathrm{Pop})$$

Code Construction

We construct a code by iterating over $y \in S$ in order of decreasing probability (most likely first).

For each y select a code word $z(y)$ (a tree leaf) with length (depth)

$$|z(y)| = \lceil -\log_2 \mathrm{Pop}(y) \rceil$$

and where $z(y)$ is not an extension of (under) any previously selected code word.

Code Existence Proof

At any point before coding all elements of S we have

$$\sum_{y \in \text{Defined}} 2^{-|z(y)|} \leq \sum_{y \in \text{Defined}} \text{Pop}(y) < 1$$

Therefore there exists an infinite descent into the tree that misses all previous code words.

Hence there exists a code word $z(x)$ not under any previous code word with $|z(x)| = \lceil -\log_2 \text{Pop}(y) \rceil$.

Furthermore $z(x)$ is at least as long as all previous code words and hence $z(x)$ is not a prefix of any previously selected code word.

No Better Code Exists

Let z be an arbirtary coding.

$$\begin{aligned} E_y\,|z(y)| &= E_y\ -\log_2 P_z(y) \\ &= H_2(\mathrm{Pop}, P_z) \\ &= H_2(\mathrm{Pop}) + KL_2(\mathrm{Pop}, P_Z) \\ &\geq H_2(\mathrm{Pop}) \end{aligned}$$

Big Picture: Differential Entropy is Always Infinite

For a continuous set $\mathcal{Y}$ (such as the unit interval on real numbers) and for any probability density (smooth measure) on $\mathcal{Y}$ the probability of any single point $y \in \mathcal{Y}$ is zero.

This implies that the entropy is infinite — it takes an infinite number of bits to represent a random real number.

Cross-entropy between continuous densities is also always infinite and differential cross-entropy loss is not meaningful.

Differential Entropy

Consider a continuous density $p(x)$. For example

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{\frac{-x^2}{2\sigma^2}}$$

Differential entropy is often defined as

$$H(p) \doteq \int \left(\ln \frac{1}{p(x)}\right) p(x)dx$$

Finite Differential Entropy is Not Meaningful

$$H(\mathcal{N}(0,\sigma)) = +\int \left(\ln(\sqrt{2\pi}\sigma) + \frac{x^2}{2\sigma^2}\right) p(x)dx$$
$$= \ln(\sigma) + \ln(\sqrt{2\pi}) + \frac{1}{2}$$

But if we take $y \doteq x/2$ we get $H(y) = H(x) - \ln 2$.

Also for $\sigma << 1$, we get $H(p) < 0$

Hence differential entropy then depends on the choice of units — a distributions on lengths will have a different entropy when measuring in inches than when measuring in feet.

Differential Entropy is Always Infinite

Consider quantizing the the real numbers into bins.

A continuous probability densisty p assigns a probability $p(B)$ to each bin.

As the bin size decreases toward zero the entropy of the bin distribution increases toward ∞.

A meaningful convention is that $H(p) = +\infty$ for any continuous density p.

Differential KL-divergence is Meaningful

$$KL(p,q) = \int \left(\ln \frac{p(x)}{q(x)}\right) p(x)dx$$

This integral can be computed by dividing the real numbers into bins and computing the KL divergence between the distributions on bins.

The KL divergence between the bin distribution typically approaches a finite limit as the bin size goes to zero.

KL-Divergence can also be Infinite

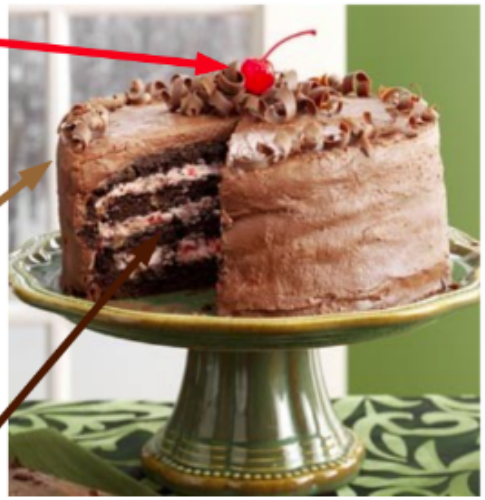
$$KL(p, q) = E_{x \sim p} \log \frac{p(x)}{q(x)}$$

In either the discrete or continuous case, if a set is assigned nonzero probability by p but zero probability by q then $KL(p, q) = +\infty$.

If every set assigned nonzero probability by p is also assigned nonzero probability by q then we say that p is absolutely continuous with respect to q.

Big Picture: Unsupervised Learning

- "Pure" Reinforcement Learning (cherry)
  - The machine predicts a scalar reward given once in a while.
  - **A few bits for some samples**
- Supervised Learning (icing)
  - The machine predicts a category or a few numbers for each input
  - Predicting human-supplied data
  - **10→10,000 bits per sample**
- Unsupervised/Predictive Learning (cake)
  - The machine predicts any part of its input for any observed part.
  - Predicts future frames in videos
  - **Millions of bits per sample**



- (Yes, I know, this picture is slightly offensive to RL folks. But I'll make it up)

Consider the power of BERT.

We would like to density estimation for images and sound waves.

The Big Picture: Rate-Distortion Loss

For density estimation of naturally occurring continuous distributions we consider a parameterized "rounding" operation mapping y to $\tilde{y}_\Phi(y)$ with $\tilde{y} \in \mathcal{Y}$ for $\mathcal{Y}$ discrete.

We then define rate-distortion loss

$$\mathcal{L}(\Phi) = E_{y \sim \text{Pop}} \left(-\ln P_\Phi(\tilde{y}_\Phi(y)) \right) + \lambda D(y, \tilde{y}_\Phi(y))$$

where $D(y, \tilde{y})$ is some "distortion function" measuring a distance between y and $\tilde{y}$.

The Rate-Distortion Tradeoff

$$\mathcal{L}(\Phi) = E_{y \sim \text{Pop}} \left(-\ln P_\Phi(\tilde{y}_\Phi(y)) \right) + \lambda D(y, \tilde{y}_\Phi(y))$$

The first term is just cross-entropy loss which we are calling the "rate". This terminology is explained below.

The meta-parameter λ controls the trade off between rate and distortion.

Common Distortion Functions

$$\Phi^* = E_{y \sim \text{Pop}} \left(-\ln P_\Phi(\tilde{y}_\Phi(y)) \right) + \lambda D(y, \tilde{y}_\Phi(y))$$

It is common to take

$$D(y, \tilde{y}) = ||y - \tilde{y}||^2 \qquad (L_2)$$

or

$$D(y, \tilde{y}) = ||y - \tilde{y}||_1 \qquad (L_1)$$

Rate-Distortion Autoencoders

Given a continuous signal y we can compress it into a (discrete) bit string $\tilde{z}_\Phi(y)$.

We let $\tilde{y}_\Phi(\tilde{z}_\Phi(y))$ be the decompression of $\tilde{z}_\Phi(y)$.

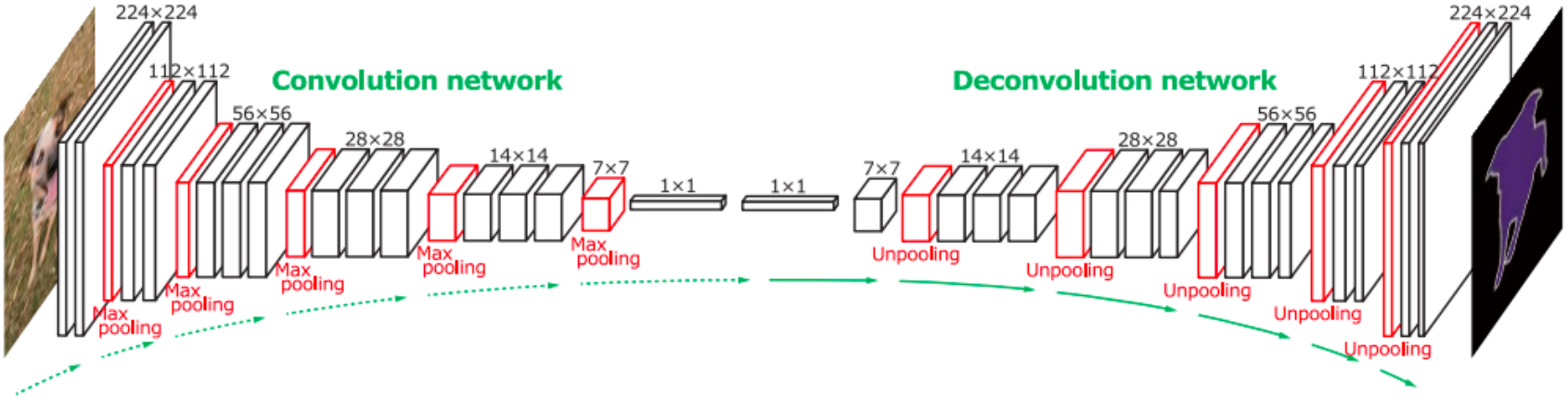
Rate-Disrtion loss can then be written as

$$\mathcal{L}(\Phi) = E_{y \sim \text{Pop}} \, |\tilde{z}_\Phi(y)| + \lambda D(y, \tilde{y}(\tilde{z}(y)))$$

A Case Study in Image Compression

End-to-End Optimized Image Compression, Balle, Laparra, Simoncelli, ICLR 2017.

y $\tilde{z}_\Psi(y)$ $\tilde{z}$ $\tilde{y}_\Phi(\tilde{z})$ $||y - \tilde{y}||^2$



The model described here has been simplified from the original.

JPEG at 4283 bytes or .121 bits per pixel

JPEG, 4283 bytes (0.121 bit/px), PSNR: 24.85 dB/29.23 dB, MS-SSIM: 0.8079

JPEG 2000 at 4004 bytes or .113 bits per pixel

JPEG 2000, 4004 bytes (0.113 bit/px), PSNR: 26.61 dB/33.88 dB, MS-SSIM: 0.8860

Deep Autoencoder at 3986 bytes or .113 bits per pixel

Proposed method, 3986 bytes (0.113 bit/px), PSNR: 27.01 dB/34.16 dB, MS-SSIM: 0.9039

The Encoder $z_\Phi(y)$

This paper uses a three layer CNN for the encoder.

The first layer is computed stride 4.

The two remaining layers are computed stride 2.

The number of numbers

The first layer is computed stride 4.

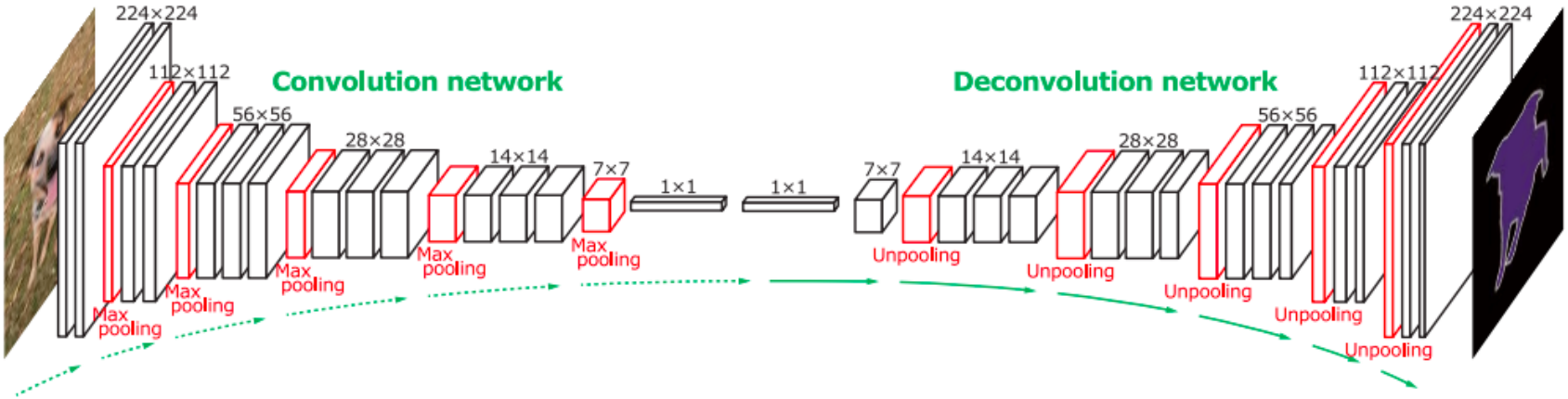
The next two layers are computed stride 2.

Final image dimension is reduced by a factor of 16 with 192 channels per pixel (192 channels is for color images).

$$192 < 16 \times 16 \times 3 = 768$$

The final values $z[x, y, i]$ are rounded to integers $\tilde{z}[x, y, i]$.

Increasing Spatial Dimension in Decoding



[Hyeonwoo Noh et al.]

In the ICLR 17 paper the deconvolution network has the shape as the input CNN but with independent parameters.

Increasing Spatial Dimensions in Deconvolution

Consider a stride 2 convolution

$$L_{\ell+1}[x, y, j] = \sigma\left(\sum_{\Delta x, \Delta y, i} W[\Delta x, \Delta y, i, j] L_\ell[2x + \Delta x, 2y + \Delta y, i]\right)$$

For deconvolution we use stride 1 with 4 times the features.

$$L'_\ell[x, y, i] = \sigma\left(\sum_{\Delta x, \Delta y, j} W[\Delta x, \Delta y, j, i] L'_{\ell+1}[x + \Delta x, y + \Delta y, j]\right)$$

The channels at each $L'_\ell[x, y]$ are divided among four higher resolution pixels.

This is done by a simple reshaping of $L'_\ell[x, y, i]$.

Rate-Distortion Autoencoders

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}} \; E_{y \sim \mathrm{Pop}} \, |\tilde{z}_\Phi(y)| + \lambda D(y, \tilde{y}_\Phi(z_\Phi(y)))$$

Oops: Because of rounding, $\tilde{z}_\Phi(y)$ is discrete and the gradients are zero.

Note, however that the rate-distortion loss is measurable.

We will approximate the gradient descent but still be able to measure loss.

Rate-Distortion Autoencoders

$$\Phi^* = \operatorname*{argmin}_{\Phi} \mathcal{L}_{\text{rate}}(\Phi) + \lambda \mathcal{L}_{\text{dist}}(\Phi)$$

$$\mathcal{L}_{\text{rate}}(\Phi) = E_{y \sim \text{Pop}} \, |\tilde{z}_\Phi(y)|$$

$$\mathcal{L}_{\text{dist}}(\Phi) = E_{y \sim \text{Pop}} \, D(y, \tilde{y}_\Phi(\tilde{z}_\Phi(y)))$$

We will consider differentiable approximations to both $\mathcal{L}_{\text{rate}}$ and $\mathcal{L}_{\text{dist}}$.

A Differentiable Approximation of $\mathcal{L}_{\text{rate}}$

$$\mathcal{L}_{\text{rate}}(\Phi) = E_{y \sim \text{Pop}} \, |\tilde{z}_\Phi(y)|$$

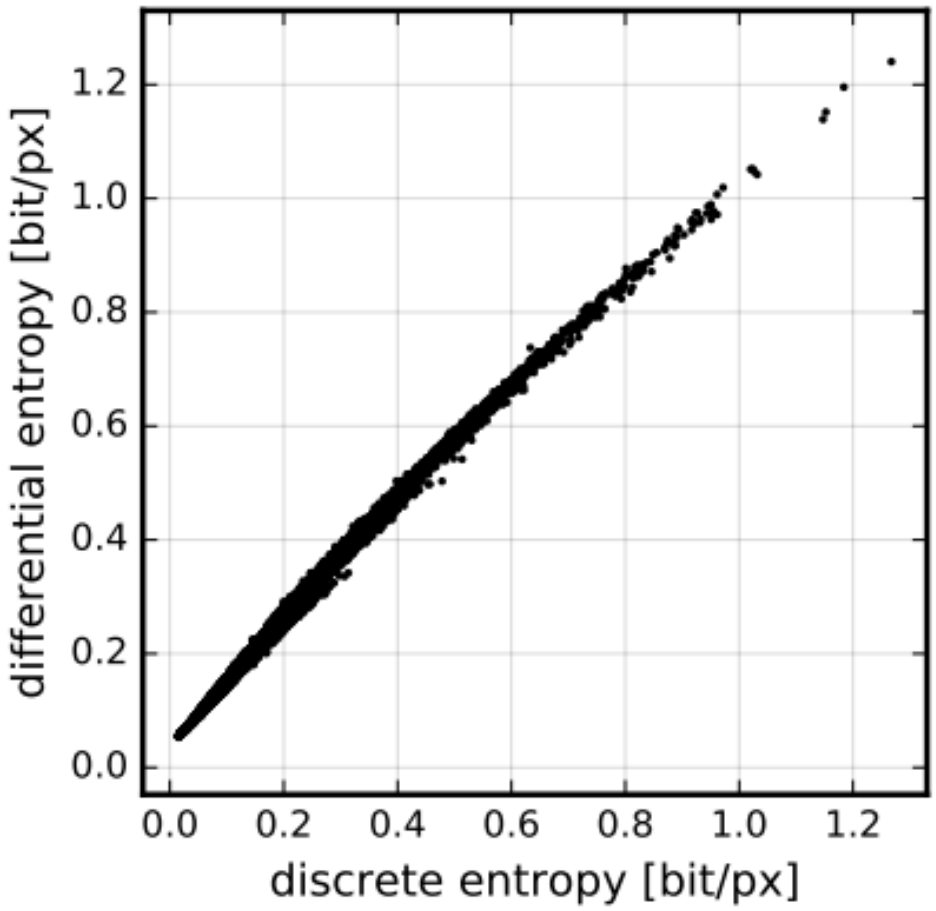
Recall that $\tilde{z}(y)$ is a rounding of a continuous tensor $z[x, y, i]$.

We can use the differentiable approximation

$$|\tilde{z}_\Phi(y)| \approx \sum_{x,y,i} \max(0, \log_2 z_\Phi[x, y, i])$$

This can be viewed as approximating a discrete entropy with differential entropy.

Differential Entropy vs. Discrete Entropy

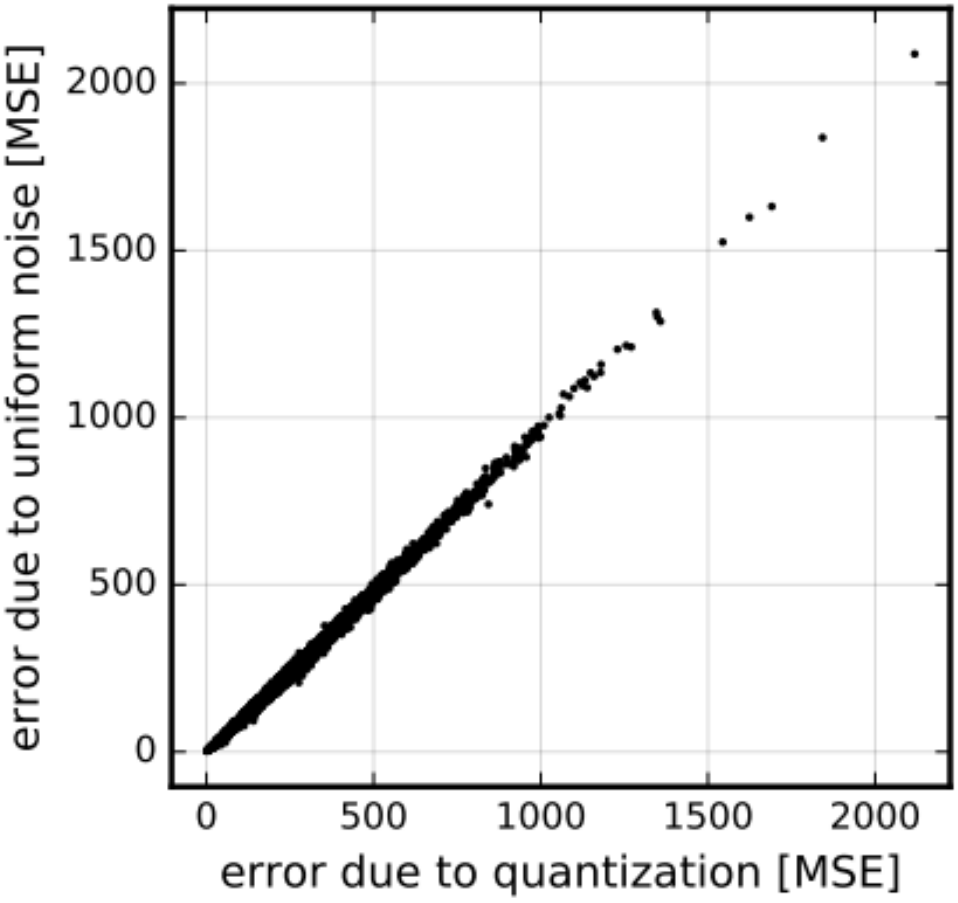


A Differentiable Approximation of $\mathcal{L}_{\text{dist}}$

$$\begin{aligned}\mathcal{L}_{\text{dist}}(\Phi) &= E_{y\sim\text{Pop}}\, D(y, \tilde{y}_\Phi(\tilde{z}_\Phi(y))) \\ &\approx E_{y,\epsilon}\, D(y,\ \tilde{y}(z_\Phi(y)+\epsilon))\end{aligned}$$

Here ϵ is a noise tensor with $\epsilon[x, y, i]$ drawn uniformly form $(-1/2, 1/2)$.

Noise vs. Rounding



Varying the Level Of Compression

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}} \; E_{y \sim \mathrm{Pop}} |\tilde{z}_\Phi(y)| + \frac{1}{2}\lambda ||y - \tilde{y}_\Phi(\tilde{z}_\Phi(y))||^2$$

Different levels of compression correspond to different values of λ.

In all levels of compression we replace 768 numbers by 192 numbers.

Higher levels of compression result in smaller integer values in the 192 numbers.

Conditional Rate-Distortion Autoencoders

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}} \quad E_{(x,y)\sim \mathrm{Pop}} \quad |\tilde{z}_\Phi(y|x)| + \lambda D(y \mid \tilde{y}_\Phi(x, \tilde{z}_\Phi(y|x)))$$

Colorization

$$\Phi^* = \underset{\Phi}{\text{argmin}}\, E_{(x,y)\sim \text{Pop}}\, |\tilde{z}_\Phi(y|x)| + \frac{1}{2}\lambda||y - \tilde{y}_\Phi(x, \tilde{z}_\Phi(y|x))||^2$$

If the image can be segmented based on x then $\tilde{z}_\Phi(y|x)$ can be a specification of color of each segment — this would be very compact.

END