对称锥优化问题

对称锥优化问题的标准形式

原始问题 $\mathcal{P}$ 的标准形式，

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathcal{K} \end{aligned} \tag{1}$$

对偶问题 $\mathcal{D}$ 的标准形式，

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & c - A^{\mathrm{T}} y \in \mathcal{K} \end{aligned} \tag{2}$$

算例1

$$\begin{aligned} \min \quad & x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 4 \\ & 2x_1 - x_2 + x_3 = 2 \end{aligned} \tag{3}$$

最优解为 $\mathbf{x}^* = \left(\frac{21}{11}, \frac{43}{22}, \frac{3}{22}\right)^T$， 目标函数值为 $f_{\min} = 3.9773$

上式转化成如下形式，

$$\begin{aligned} \min \quad & \mathbf{x}^T P_0 \mathbf{x} + 2q_0 \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = b \end{aligned} \tag{4}$$

进一步转化可得，

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & A\mathbf{x} = b \\ & \left\| P_0^{1/2} x + P_0^{-1/2} q_0 \right\| \leq t \end{aligned} \tag{5}$$

式中，

$$\begin{aligned} & A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\ & P_0 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad q_0 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \end{aligned} \tag{6}$$

对系数矩阵 P_0 进行开平方操作`(matlab sqrtm function)`可得，

$$P_0^{1/2} = \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ -0.4472 & 1.3416 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

作进一步处理，

$$s=c-A^{\mathrm{T}}y=\begin{bmatrix} t \\ P_0^{1/2}x+P_0^{-1/2}q_0 \end{bmatrix}\in\mathcal{K}^4 \tag{8}$$

其中，$y=(t,x_1,x_2,x_3)^T$，

$$\begin{aligned} c_f&=\begin{bmatrix}4\\2\end{bmatrix}, \qquad A_f^{\mathrm{T}}=\begin{bmatrix}0&1&1&1\\0&2&-1&1\end{bmatrix} \\ c_q&=\begin{bmatrix}0\\P_0^{-1/2}q_0\end{bmatrix}, \qquad A_q^{\mathrm{T}}=\begin{bmatrix}-1&\mathbf{0}_{1\times3}\\ \mathbf{0}_{3\times1}&-P_0^{1/2}\end{bmatrix}\end{aligned} \tag{9}$$

至此，算例1完全化成sedumi对偶问题的标准输入形式。

算例2

$$\begin{aligned} \min\quad & t \\ \textbf{s.t.}\quad & Ax\le b \\ & x^{\mathrm{T}}P_0x+2q_0x+r_0\le t \\ & x^{\mathrm{T}}P_1x+2q_1x+r_1\le 0 \\ & x^{\mathrm{T}}P_2x+2q_2x+r_2\le 0 \end{aligned} \tag{10}$$

其中，

$$\begin{aligned} A&=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\0&0&-1.5&1\end{bmatrix}, \qquad b=\begin{bmatrix}2.4\\-2.4\\2.4\end{bmatrix} \\ P_0&=\begin{bmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{bmatrix}, \qquad q_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \qquad r_0=0 \\ P_1&=\begin{bmatrix}\frac{1}{4}&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \qquad q_1=\begin{bmatrix}-\frac{1}{4}\\0\\0\\0\end{bmatrix}, \qquad r_1=-\frac{3}{4} \\ P_2&=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&5&3\\0&0&3&5\end{bmatrix}, \qquad q_2=\begin{bmatrix}0\\0\\-22\\-26\end{bmatrix}, \qquad r_2=140 \end{aligned} \tag{11}$$

上述问题做进一步处理得，

$$\begin{aligned} \min\quad & t \\ \textbf{s.t.}\quad & b-Ax\ge 0 \\ & t\ge\left\|P_0^{1/2}x+P_0^{-1/2}q_0\right\| \\ & \left(q_1^{\mathrm{T}}P_1^{-1}q_1-r_1\right)^{1/2}\ge\left\|P_1^{1/2}x+P_1^{-1/2}q_0\right\| \\ & \left(q_2^{\mathrm{T}}P_2^{-1}q_2-r_2\right)^{1/2}\ge\left\|P_2^{1/2}x+P_2^{-1/2}q_0\right\| \end{aligned} \tag{12}$$

令 $y=(t,x_1,x_2,x_3,x_4)^{\mathrm{T}}$，上式等价于，

$$\begin{aligned} \max\quad & b^{\mathrm{T}}y \\ \textbf{s.t.}\quad & c_f-A_f^{\mathrm{T}}y\in\mathcal{K}_l^2 \end{aligned}$$

$$
\begin{aligned}
c_{q_0} - A_{q_0}^{\mathrm{T}} y &= \begin{bmatrix} t \\ P_0^{1/2} x + P_0^{-1/2} q_0 \end{bmatrix} \in \mathcal{K}_{q_0}^5 \\
c_{q_1} - A_{q_1}^{\mathrm{T}} y &= \begin{bmatrix} \left(q_1^{\mathrm{T}} P_1^{-1} q_1 - r_1\right)^{1/2} \\ P_1^{1/2} x + P_1^{-1/2} q_1 \end{bmatrix} \in \mathcal{K}_{q_1}^5 \\
c_{q_2} - A_{q_2}^{\mathrm{T}} y &= \begin{bmatrix} \left(q_2^{\mathrm{T}} P_2^{-1} q_2 - r_2\right)^{1/2} \\ P_2^{1/2} x + P_2^{-1/2} q_2 \end{bmatrix} \in \mathcal{K}_{q_2}^5
\end{aligned}
\tag{13}
$$

其中，

$$
b = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}
$$