Finite Difference Methods for the Numerical Solution of Differential Equations

Zexuan Sun

Major: Information and Computational Science

Student ID: 2016301000038

1 Example 1: Dirichlet Boundary Conditions

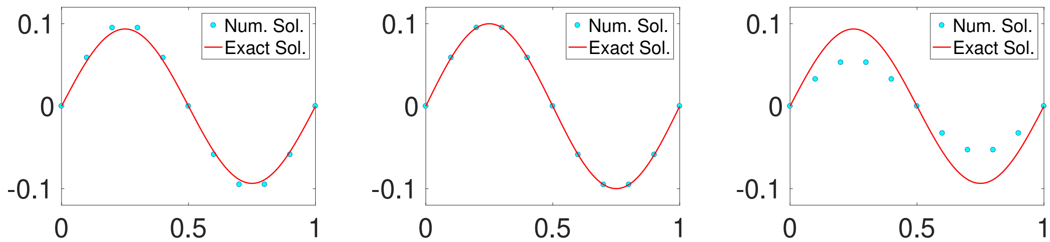
Give numerical solution to the heat conduction equation

$$\begin{cases} v_t = v_{xx}, & x \in (0,1), t > 0, \\ v(x,0) = sin2\pi x, & x \in [0,1], \\ v(0,x) = v(1,t) = 0, & t \geq 0 \end{cases}$$

test the impacts of different time step on the scheme, when $t = 0.06, 0.1, 0.9, 50.0$. PS: The analytical solution is $v(x,t) = exp(-(2*\pi)^2 t)sin2\pi x$.

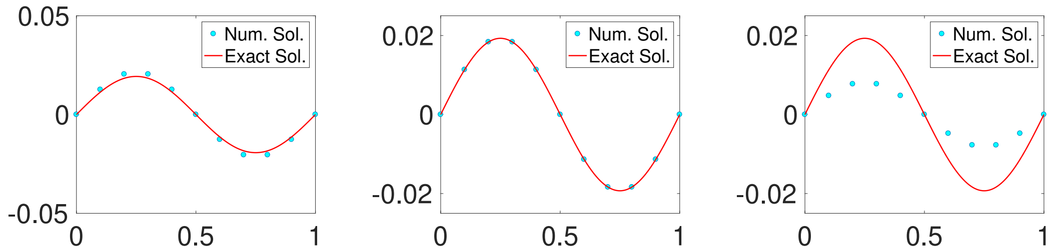
We approximate time and space derivatives using finite difference method. And then use Matlab to implement the scheme. We wrote the funciton FDM_eg1(DT,TEND), where DT is the time step, TEDN is the end time. We set the width of the grid as $1/10$ of the interval length. Thus, we need to caculate the values of 11 points. With the help of this function, we get the result when the end time $t = 0.06$ and time steps are $0.01/36, 0.01/6, 0.01$. And we drew the numerical solutions and the analytical solution together. The comparison charts are shown in Fig.1. We can seet that the numerical solution has the highest accuracy when $\Delta t = 0.01/6$. Either reduce Δt or enlarge it will decrease the accuracy of the numerical solutions. The decrease becomes more obvious as the end time t increases. Therefore, only proper time step is able to give good numerical solutions.

Similarly, when $t = 0.1, 0.9, 50.0$, we get the figures exhibited in Fig.2-4. We can see that when $t = 0.1, 0.9$ and $\Delta t = 0.01/6$, numerical solutions are quite precise. However, with the increase of the end time, either reduce or enlarge Δt, the accuracy decreases. When $t = 0.9$ and $\Delta t = 0.01$, the numerical solution has not held the shape of trigonometry. When $t = 50.0$, the numerical solutions have large error in all three situations, especially when $\Delta t = 0.01$.



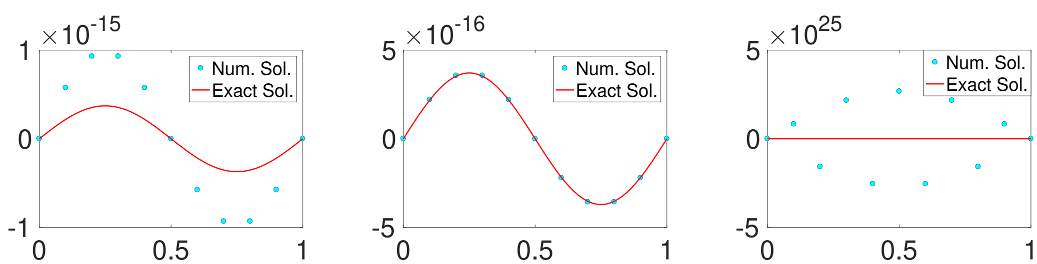
(a) $\Delta t = 0.01/36$ (b) $\Delta t = 0.01/6$ (c) $\Delta t = 0.01$

Figure 1: $t = 0.06$



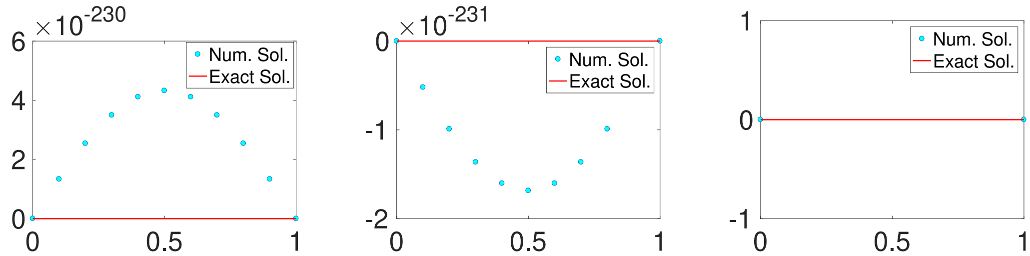
(a) $\Delta t = 0.01/36$ (b) $\Delta t = 0.01/6$ (c) $\Delta t = 0.01$

Figure 2: $t = 0.1$



(a) $\Delta t = 0.01/36$ (b) $\Delta t = 0.01/6$ (c) $\Delta t = 0.01$

Figure 3: $t = 0.9$



(a) $\Delta t = 0.01/36$ (b) $\Delta t = 0.01/6$ (c) $\Delta t = 0.01$

Figure 4: $t = 50.0$

2 Example 2: Neumann Boundary Conditions

Give numerical solution to the heat conduction equation

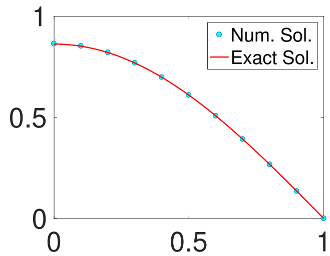
$$\begin{cases} v_t = v_{xx}, & x \in (0,1), t > 0, \\ v(x,0) = cos\frac{\pi x}{2}, & x \in [0,1], \\ \partial_x v(0,x) = v(1,t) = 0, & t \geq 0 \end{cases}$$

test the impacts of different time step on the scheme, when $t = 0.06, 0.1, 0.9, 50.0$. PS: The analytical solution is $v(x,t) = exp(-(\frac{\pi}{2})^2 t)cos\frac{\pi x}{2}$.

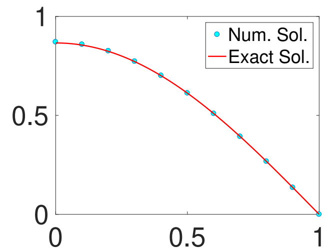
Alos, we make use of finite difference method. For the boundary conditions $\partial_x v(0,x) = 0$, we use central difference and set a ghost point to get two-order accuracy. To be specific, after we set the value of ghost point $u_{-1}^n = u_1^n$, we set

$$u_0^{n+1} = u_0^n + \frac{\triangle t}{\triangle x^2}(u_1^n - 2u_0^n + u_{-1}^n).$$

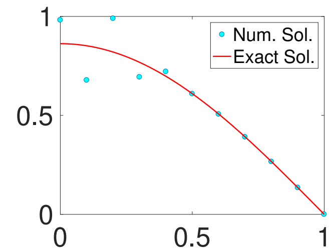
Based on this scheme, we write the function FDM_eg2 in Matlab. The parameters of this function and instructions are the same as FDM_eg1. We choose the same width of grid as example 1 and get the numerical solutions when $t = 0.06, 0.1, 0.9, 50.0$. The results are shown in Fig.5-8.



(a) $\Delta t = 0.01/36$

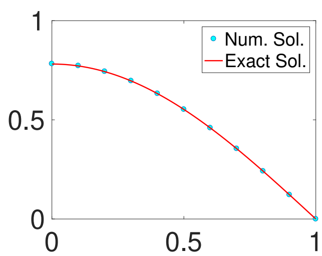


(b) $\Delta t = 0.01/6$

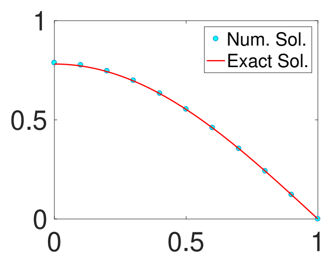


(c) $\Delta t = 0.01$

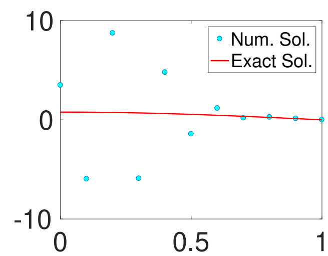
Figure 5: $t = 0.06$



(a) $\Delta t = 0.01/36$

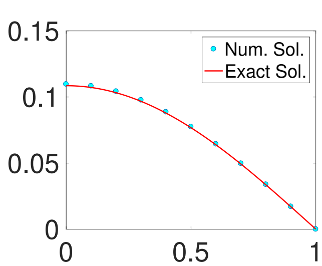


(b) $\Delta t = 0.01/6$

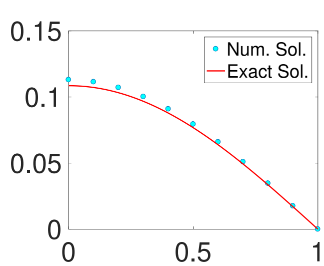


(c) $\Delta t = 0.01$

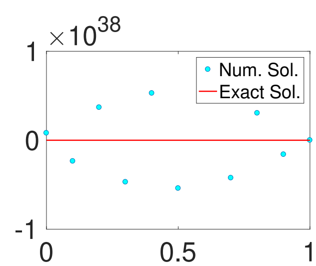
Figure 6: $t = 0.1$



(a) $\Delta t = 0.01/36$

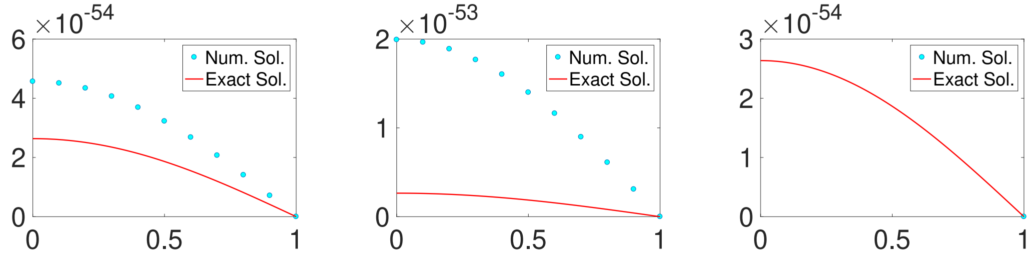


(b) $\Delta t = 0.01/6$



(c) $\Delta t = 0.01$

Figure 7: $t = 0.9$



(a) $\Delta t = 0.01/36$ (b) $\Delta t = 0.01/6$ (c) $\Delta t = 0.01$

Figure 8: $t = 50.0$

From the figures, we can notice that the accuracy of numerical solutions has not decreased as Δt decreases. By contrast, the accuracy increases as Δt decreases. What's more, as the end time increases, the accuracy decreases with the same time step. When $t = 50.0$, even using the smallest time step cannot give an accurate solution.

3 Conclusions and Analysis

From the two examples above, for Dirichlet boundary conditions, we need to choose proper time step. Actually, it is determined by the space stop we use. We can prove this theoretically. As for Neumann boundary conditions, the accuracy enhances as time step becomes smaller (at least it is ture according to the numerical experiments I have implemented). For both the two boundary conditions, the error of numerical solution increases as time goes by, so we are not able to give relative accurate solution when t is large.