Riemann Problem for Inviscid Burgers Equation

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1 Riemann Problem

The Riemann problem for inviscid Burgers equation is as below:

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, \\ u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x \geq 0. \end{cases} \end{cases} \tag{1}$$

We know that the analytical solution to equation 1 is

$$u(x,t) = \begin{cases} u_l & x < st, \\ u_r & x > st. \end{cases} \tag{2}$$

where $s = (u_l + u_r)/2$.

2 Different Schemes

2.1 Lax-Friedrichs Scheme

For non-linear system, Lax-Friedrichs method has the following scheme:

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{\triangle t}{2\triangle x}[f(u_{j+1}^n) - f(u_{j-1}^n)]. \tag{3}$$

For Burgers equation, we have:

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{\triangle t}{2\triangle x}[\frac{1}{2}(u_{j+1}^n)^2 - \frac{1}{2}(u_{j-1}^n)^2]. \tag{4}$$

2.2 Lax-Wendroff Scheme

For non-linear conservation equation, Lax-Wendroff scheme is as follow:

$$\begin{aligned} u_j^{n+1} = u_j^n - \frac{\triangle t}{2\triangle x}(f(u_{j+1}^n) - f(u_{j-1}^n)) + \frac{\triangle t^2}{2\triangle x^2} \\ [A_{j+\frac{1}{2}}(f(u_{j+1}^n) - f(u_j^n)) - A_{j-\frac{1}{2}}(f(u_j^n) - f(u_{j-1}^n))] \end{aligned} \tag{5}$$

Particularly, for Burgers equation, we have:

$$\begin{aligned} u_j^{n+1} = u_j^n - \frac{\triangle t}{2\triangle x}(\frac{1}{2}u_{j+1}^n - \frac{1}{2}(u_{j-1}^n)^2) + \frac{\triangle t^2}{2\triangle x^2} \\ [(\frac{1}{2}(u_{j+1}^n + u_j^n))(\frac{1}{2}(u_{j+1}^n)^2 - \frac{1}{2}(u_j^n)^2) - (\frac{1}{2}(u_j^n + u_{j-1}^n))(\frac{1}{2}(u_j^n)^2 - \frac{1}{2}(u_{j-1}^n)^2)]. \end{aligned} \tag{6}$$

2.3 Godunov Scheme

For inviscid burgers equation, Godunov scheme has terse conservation law form:

$$u_j^{n+1} = u_j^n - \frac{\triangle t}{\triangle x}[F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)], \tag{7}$$

where the definition of flux, F is 其 $F(u, v) = (u^\star)^2/2$， and the definition of $u^\star$ is:
when$u \geq v$

$$u^\star = \begin{cases} u, & \frac{u+v}{2} > 0, \\ v, & \frac{u+v}{2} \leq 0. \end{cases}$$

when$u < v$

$$u^\star = \begin{cases} u, & u > 0, \\ v, & v < 0, \\ 0, & u \leq 0 \leq v. \end{cases}$$

2.4 Beam-Warming Scheme

For typical governing equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{8}$$

set

$$a(u) = \frac{\partial f(u)}{\partial u}, \tag{9}$$

$$\Delta t/\Delta x = \lambda, \tag{10}$$

then Beam-Warming scheme is as below:
when $a(u^n) > 0$

$$\begin{aligned} u_i^{n+1} = u_i^n - \frac{\lambda}{2}\left((3f(u_i^n) - 4f(u_{i-1}^n) + f(u_{i-2}^n)\right) + \\ \frac{\lambda^2}{2}\left[a(u_{i-1/2}^n)\left((f(u_i^n) - f(u_{i-1}^n)\right) - a(u_{i-3/2}^n)\left(f(u_{i-1}^n) - f(u_{i-2}^n)\right)\right], \end{aligned} \tag{11}$$

when $a(u^n) < 0$

$$\begin{aligned} u_i^{n+1} = u_i^n + \frac{\lambda}{2}\left((3f(u_i^n) - 4f(u_{i+1}^n) + f(u_{i+2}^n)\right) + \\ \frac{\lambda^2}{2}\left[a(u_{i+3/2}^n)\left((f(u_{i+2}^n) - f(u_{i+1}^n)\right) - a(u_{i+1/2}^n)\left(f(u_{i+1}^n) - f(u_i^n)\right)\right]. \end{aligned} \tag{12}$$

For Burgers equation, set

$$f(u) = \frac{u^2}{2}, \tag{13}$$

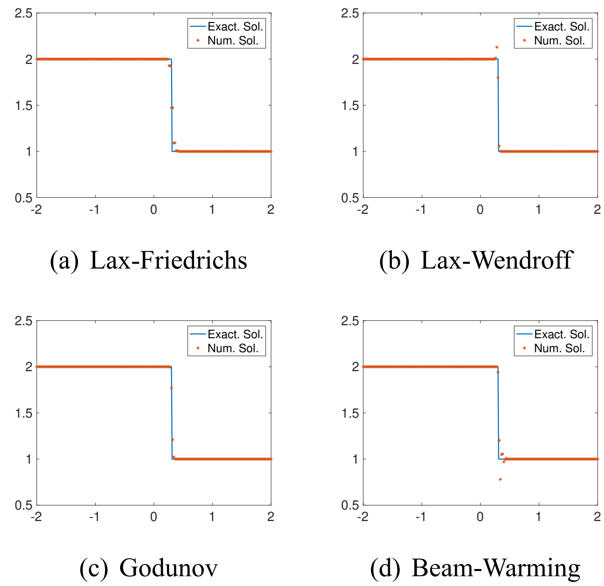
then we have the Beam-Warming scheme for Burgers equation.

3 Numerical Experiments

To better compare the performance of these four schemes, we implemented these schemes to Burgers equation with particular initial value. The example we used are equation 1 and intial value is $ul = 2, ur = 1$. The time step dt is set as 0.01 and space step dx is 0.04. The interval we carried out experiments is [-2,2].

First we compared the numerical solutions derived from the four schemes with analytical solution. The result is shown in Fig.1. We can see that all four schemes perform well. But Lax-Wendroff and Beam- Warming schemes have some points, which do not locate on the curve of analytical solution.

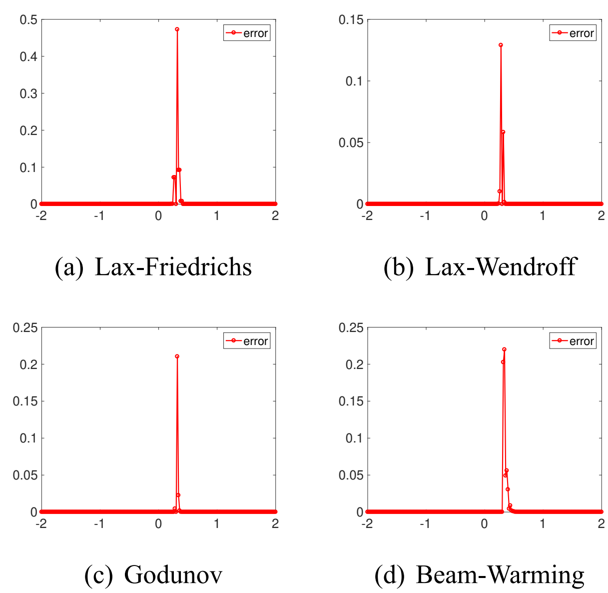
Then we calculate the errors of the four schemes. The result is shown in Fig.2. From the maximum error value, we can say that Lax-Wendroff scheme is the best. Godunov and Beam-Warming schemes performed relatively the same. And Lax-Friedrichs scheme is the worst.



(a) Lax-Friedrichs (b) Lax-Wendroff

(c) Godunov (d) Beam-Warming

Figure 1: Comparison of numerical solutions and analytical solution when $t = 0.2s$



(a) Lax-Friedrichs (b) Lax-Wendroff

(c) Godunov (d) Beam-Warming

Figure 2: The figure of error when $t = 0.2s$

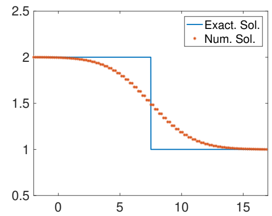
We then give some parameters of the error of the four schemes in Table.1 to better compare the four schemes.

Table 1: The parameters of the error of the four schemes when $t = 0.2s$

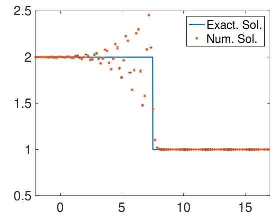
Scheme	Index of Maximum Error	Maximum Error	Average Absolute Error
Lax-Friedrichs	0.32	0.4726	0.0041
Lax-Wendroff	0.28	0.129	9.93×10^{-4}
Godunov	0.32	0.2103	0.0012
Beam-Warming	0.34	0.2198	0.0029

From the table, we can notice that Lax-Wendroff scheme has the smallest average absolute error and smallest maximum error. Needless to say, it performed the best. The average absolute error and maximum error of Godunov scheme are smaller than those of Beam-Warming scheme, thus it is better than Beam-Warming scheme. The average absolute error and maximum error of Lax-Friedrichs are the largest, therefore, it is the worst scheme.

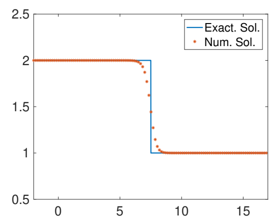
Next, we investigated the performances of the four schemes when t is relatively large. We set t as 5, and the interval as [-2,17]. Currently, the break occurs at $x = 7.5$. Similarly, we compared the numerical solutions derived from the four schemes with analytical solution. And the result is shown in Fig.3.



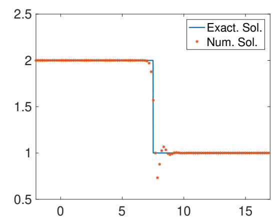
(a) Lax-Friedrichs



(b) Lax-Wendroff



(c) Godunov



(d) Beam-Warming

Figure 3: Comparison of numerical solutions and analytical solution when $t = 5s$

We can notice that Lax-Friedrichs and Lax-Wendroff schemes are not good choices right now, but Godunov and Beam-Warming shemes still perform well. Finally, we give the parameters of error of the four schemes in Table.2.

Table 2: The parameters of the error of the four schemes when $t = 5s$

Scheme	Index of Maximum Error	Maximum Error	Average Absolute Error
Lax-Friedrichs	7.52	0.4849	0.1181
Lax-Wendroff	6.67	0.5205	0.0469
Godunov	7.52	0.4437	0.0166
Beam-Warming	7.52	0.5697	0.0116

We can see that the errors of the four schemes increase compared with those when $t = 0.2s$. From the figure, it is easy to notice that Lax-Wendroff scheme cannot fit the break right now. The maximum error values of the four schemes are relative approximate. But from the average absolute error, obviously, Godunov and Beam-Warming schemes are better than Lax-Friedrichs and Lax-Wendroff schemes.

4 Conclusions and Analysis

From the results of numerical experiments, we can see that even though Lax-Wendroff scheme performed the best when t is small, as time increases, its accuracy decreases. But Godunov and Beam-Warming schemes still have relative high accuracy as time increases and are able to fit the break quite well. Lax-Friedrichs and Lax-Wendroff scheme cannot do a good job in this situation. Thus, we would better choose Godunov and Beam-Warming schemes to solve the Riemann problem for inviscid Burgers equation.