Regularization of Cardinal B-Spline Curves

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1 Introduction

This note is about the calculation of the regularization of cardinal b-spline curves involved in the curve fitting problems. It seems convenient to use the vector/matrix representation other than the summation representation. The vector/matrix representation is also more clear, especially when wrap-around operation is needed. Alas, I have made a mistake when using summation representation.

2 The regularization terms

The curve is represented by cardinal b-spline of order k:

$$\boldsymbol{\mathcal{P}} = \sum_j Q_j(t)\boldsymbol{P}_j.$$

where $\boldsymbol{P}_j$ are the control points, and Q_j^k are the cardinal spline basis of order k. The support of Q_j^k is $[j-1, j-1+k)$.

For the simplicity, I consider the 1-dimensional case so I can use vector/matrix representation in a more clear way, and the order superscript k is dropped (order k is assumed unless specified):

$$\boldsymbol{\mathcal{P}} = \mathbf{P}^\intercal \mathbf{Q},$$

where $\mathbf{P} = [P_1, P_2, ..., P_m]^\intercal$ are control point vector, and $\mathbf{Q} = [Q_1(t), Q_2(t), ..., Q_m(t)]^\intercal$ are the cardinal basis vector. The vector/matrix expression for the curve, its first and second derivatives can be found in 3.4.

The regularization terms are F_1 and F_2:

$$\begin{aligned} F_1 &= \frac{1}{2}\int \|\boldsymbol{\mathcal{P}}'(t)\|^2 dt \\ &= \frac{1}{2}\int \|\mathbf{P}^\intercal \mathbf{M}_1 \mathbf{Q}^{k-1}\|^2 dt \\ &= \frac{1}{2}\mathbf{P}^\intercal \mathbf{M}_1 \Big(\int \mathbf{Q}\mathbf{Q}^\intercal dt\Big)\mathbf{M}_1^\intercal \mathbf{P}, \end{aligned}$$

and

$$\begin{aligned} F_2 &= \frac{1}{2}\int \|\boldsymbol{\mathcal{P}}''(t)\|^2 dt \\ &= \frac{1}{2}\int \|\mathbf{P}^\intercal \mathbf{M}_2 \mathbf{Q}^{k-2}\|^2 dt \\ &= \frac{1}{2}\mathbf{P}^\intercal \mathbf{M}_2 \Big(\int \mathbf{Q}\mathbf{Q}^\intercal dt\Big)\mathbf{M}_2^\intercal \mathbf{P}. \end{aligned}$$

Now the derivatives of F_1 and F_2 with respect to the control points $\mathbf{P}$ can be obtained easily. The extension to 2-dimensional control points and further discussion of the usage of B-spline in image processing can be found in chapter 3 of Blake and Isard's book "Active Contours".

3 Appendix

Here are some relations that would be used in the text.

3.1 Cardinal splines up to order 4

$$Q_1 = \begin{cases} 1 & 0 \le x < 1 \\ 0 & otherwise \end{cases},$$

$$Q_2 = \begin{cases} x & 0 \le x < 1 \\ -(x-1)+1 & 1 \le x < 2 \\ 0 & otherwise \end{cases},$$

$$Q_3 = \begin{cases} \frac{1}{2}x^2 & 0 \le x < 1 \\ -(x-1)^2 + (x-1) + \frac{1}{2} & 1 \le x < 2 \\ \frac{1}{2}(x-2)^2 - (x-2) + \frac{1}{2} & 2 \le x < 3 \\ 0 & otherwise \end{cases},$$

$$Q_4 = \begin{cases} \frac{1}{6}x^3 & 0 \le x < 1 \\ -\frac{1}{2}(x-1)^3 + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1) + \frac{1}{6} & 1 \le x < 2 \\ \frac{1}{2}(x-2)^3 - (x-2)^2 + \frac{2}{3} & 2 \le x < 3 \\ -\frac{1}{6}(x-3)^3 + \frac{1}{2}(x-3)^2 - \frac{1}{2}(x-3) + \frac{1}{6} & 3 \le x < 4 \\ 0 & otherwise \end{cases}.$$

3.2 Differentiation of cardinal spline basis

$$Q'_{i,k} = Q_{i,k-1} - Q_{i+1,k-1}.$$

3.3 Hodograph

The general way to construct the hodograph of a curve is to use the general differentiation method of the b-spline, c.f., page 115 of de Boor's book. Note that the evaluation range of the hodograph is the same as the original curve, e.g., $[t_k, \ldots, t_{m+1}]$, which is different from the range of the hodograph constructed by the method below. We can also make use of the properties of the cardinal splines: We compute the control points of the corresponding hodograph and construct the hodograph with cardinal spline of order $k-1$.

- open curve: Suppose that the curve is represented by k-order cardinal b-spline curve with m control points. The knot sequence is $1, \ldots, m+k$. The evaluation span is $[k, m+1]$. The curve is

$$\begin{aligned}\boldsymbol{\mathcal{P}} &= \mathbf{P}^\intercal \mathbf{Q}_m^k \\ &= \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \end{pmatrix} \left(Q_1^k, \ldots, Q_m^k\right).\end{aligned}$$

The hodograph is a curve represented by cardinal splines of order $k-1$:

$$\boldsymbol{\mathcal{P}'} = \mathbf{P}'^\intercal \mathbf{Q}_{m+1}^{k-1},$$

where

$$\mathbf{P}' = [P_1 - 0, P_2 - P_1, \ldots, P_m - P_{m-1}, -P_m]^\intercal$$

and

$$\mathbf{Q}_{m+1}^{k-1} = [Q_1^{k-1}, \ldots, Q_{m+1}^{k-1}]^\intercal.$$

Or in a more succint form:

$$\boldsymbol{\mathcal{P}'} = \mathbf{P}^\intercal \Big[\big(\boldsymbol{I_m}, \mathbf{0}\big) - \big(\mathbf{0}, \boldsymbol{I_m}\big) \Big] \mathbf{Q}_{m+1}^{k-1}.$$

The evaluation span is $[k-1, m+2]$.

- closed curve: The scenario is similar to that of open curve. Suppose that the curve is represented by k-order cardinal b-spline curve with m control points. Besides these m independent control points, there are $k-1$ wrapped-around control points. As a result, there are $m+k-1$ control points. The knot sequence is thus $1, \ldots, (m+k-1)+k$. The evaluation span is $[k, (m+k-1)+1]$. The curve is

$$\begin{aligned}\boldsymbol{\mathcal{P}} &= \mathbf{P}^\intercal \mathbf{Q}_{m+k-1}^k \\ &= \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \\ P_1 \\ \vdots \\ P_{k-1} \end{pmatrix} \left(Q_1^k, \ldots, Q_{m+k-1}^k\right) \\ &= \mathbf{P}^\intercal \left(\boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-1}}\right) \mathbf{Q}_{m+k-1}^k\end{aligned}$$

The corresponding hodograph is:

$$\boldsymbol{\mathcal{P}'} = \mathbf{P}'^\intercal \mathbf{Q}_{m+k-1}^{k-1},$$

where

$$\mathbf{P}' = [P_1 - P_m, P_2 - P_1, \ldots, P_m - P_{m-1}, P_1 - P_m, P_2 - P_1, \ldots, P_{k-1} - P_{k-2}]^\intercal$$

and

$$\mathbf{Q}_{m+k-1}^{k-1} = [Q_1^{k-1}, \ldots, Q_{m+k+1}^{k-1}]^\intercal.$$

Or in a more succint form:

$$\boldsymbol{\mathcal{P}'} = \mathbf{P}^\intercal \Big[\big(\boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-1}}\big) - \big(\boldsymbol{e_m}, \boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-2}}\big) \Big] \mathbf{Q}_{m+k-1}^{k-1}.$$

The evaluation span is $[k-1, (m+k-1)+1]$.

3.4 Differentiation of cardinal spline curves

The above discussion can be put in this summary:

$$\mathbf{P} = \begin{pmatrix} P_1, \\ P_2, \\ \vdots, \\ P_m \end{pmatrix}.$$

The curve itself:

$$\begin{aligned}
\boldsymbol{\mathcal{P}}_{open} &= \boldsymbol{\mathcal{P}}_o = \mathbf{P}^\intercal \mathbf{Q}_m^k, \\
\boldsymbol{\mathcal{P}}_{closed} &= \boldsymbol{\mathcal{P}}_c = \mathbf{P}^\intercal \left(\boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-1}} \right) \mathbf{Q}_{m+k-1}^k \\
&= \mathbf{P}^\intercal \mathbf{M}_{0c} \mathbf{Q}_{m+k-1}^k.
\end{aligned}$$

First derivative:

$$\begin{aligned}
\boldsymbol{\mathcal{P}}_o' &= \mathbf{P}^\intercal \Big[\left(\boldsymbol{I_m}, \mathbf{0} \right) - \left(\mathbf{0}, \boldsymbol{I_m} \right) \Big] \mathbf{Q}_{m+1}^{k-1} \\
&= \mathbf{P}^\intercal \mathbf{M}_{1o} \mathbf{Q}_{m+1}^{k-1}, \\
\boldsymbol{\mathcal{P}}_c' &= \mathbf{P}^\intercal \Big[\left(\boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-1}} \right) - \left(\boldsymbol{e_m}, \boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-2}} \right) \Big] \mathbf{Q}_{m+k-1}^{k-1} \\
&= \mathbf{P}^\intercal \mathbf{M}_{1c} \mathbf{Q}_{m+k-1}^{k-1}.
\end{aligned}$$

Second derivative :

$$\begin{aligned}
\boldsymbol{\mathcal{P}}_o'' &= \mathbf{P}^\intercal \Big\{ \Big[\left(\boldsymbol{I_m}, \mathbf{0}, \mathbf{0} \right) - \left(\mathbf{0}, \boldsymbol{I_m}, \mathbf{0} \right) \Big] - \Big[\left(\mathbf{0}, \boldsymbol{I_m}, \mathbf{0} \right) - \left(\mathbf{0}, \mathbf{0}, \boldsymbol{I_m} \right) \Big] \Big\} \mathbf{Q}_{m+2}^{k-2} \\
&= \mathbf{P}^\intercal \mathbf{M}_{2o} \mathbf{Q}_{m+2}^{k-2}, \\
\boldsymbol{\mathcal{P}}_c'' &= \mathbf{P}^\intercal \Big\{ \Big[\left(\boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-1}} \right) - \left(\boldsymbol{e_m}, \boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-2}} \right) \Big] \\
&\qquad - \Big[\left(\boldsymbol{e_m}, \boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-2}} \right) - \left(\boldsymbol{e_{m-1}}, \boldsymbol{e_m}, \boldsymbol{I_m}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{k-3}} \right) \Big] \\
&\qquad \Big\} \mathbf{Q}_{m+k-1}^{k-2} \\
&= \mathbf{P}^\intercal \mathbf{M}_{2c} \mathbf{Q}_{m+k-1}^{k-2}.
\end{aligned}$$

The evaluation range can be inferred by the superscript and subscript of $\mathbf{Q}_m^k$: $[k, m+k-k+1]$, where m is the number of control points, $m+k$ is the number of knots in the knot sequence, and k is order.

3.5 Others

Reorganization of vector of this form $[P_1, \ldots, P_m, 0]$:

$$\begin{pmatrix} P_1 \\ \vdots \\ P_m \\ 0 \end{pmatrix} = \begin{pmatrix} 1, 0, \ldots, 0 \\ 0, 1, \ldots, 0 \\ \vdots, \vdots, \ddots, \vdots \\ 0, 0, \ldots, 1 \\ 0, 0, \ldots, 0 \end{pmatrix} \begin{pmatrix} P_1 \\ \vdots \\ P_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{I_m} \\ \mathbf{0} \end{pmatrix} \mathbf{P},$$

and

$$\begin{pmatrix} P_1 \\ \vdots \\ P_m \\ P_1 \end{pmatrix} = \begin{pmatrix} 1, 0, \dots, 0 \\ 0, 1, \dots, 0 \\ \vdots, \vdots, \ddots, \vdots \\ 0, 0, \dots, 1 \\ 1, 0, \dots, 0 \end{pmatrix} \begin{pmatrix} P_1 \\ \vdots \\ P_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{I_m} \\ \boldsymbol{e_1^\intercal} \end{pmatrix} \mathbf{P}.$$