Comp 408 Advanced Topics in Artificial Intelligence

Lecture 4

n-gram Language Models

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Most of the Slides are by D. Jurafsky and J. M. Martin

Outline

- Language Models
- The Chain Rule
- Estimating n-gram Probabilities
- Evaluating n-gram Models
  - Perplexity
- Generalization and zeros
- Smoothing

Predicting words

- The water of Walden Pond is beautifully ...

blue
green
clear

*refrigerator
*that

Language Models

- A language model is a machine learning model that predicts upcoming words.
  - Can assign a probability to each potential next word.
  - Can assign a probability to a whole sentence.

Why word prediction?

It's a helpful part of language tasks

- Grammar or spell checking

  Their are two midterms ~~Their~~ There are two midterms

  Everything has improve Everything has ~~improve~~ improved

- Speech recognition

  I will be back soonish * I will be bassoon dish

Why word prediction?

It's how **large language models (LLMs)** work!

LLMs are **trained** to predict words

- Left-to-right (autoregressive) LMs learn to predict next word

LLMs **generate** text by predicting words

- By predicting the next word over and over again

Probabilistic Language Models

P(x) the probability of x

- Goal: assign a probability to a sentence

Why?

- Machine Translation:
  - P(**high** winds tonite) > P(**large** winds tonite)
- Spell Correction
  - The office is about fifteen **minuets** from my house
    - P(about fifteen **minutes** from) > P(about fifteen **minuets** from)
- Speech Recognition
  - P(I saw a van) >> P(eyes awe of an)
- + Summarization, question-answering, etc., etc.!!

Language Modeling (LM) more formally

- Goal: compute the probability of a sentence or sequence of words:

$$P(W) = P(w_1,w_2,w_3,w_4,w_5 \ldots w_n)$$

Joint probability

- Related task: probability of an upcoming word:

$$P(w_5|w_1,w_2,w_3,w_4)$$; probability of w_5 given w_1,w_2,w_3,w_4

Conditional probability

- A model that computes either of these:

$P(W)$ or $P(w_n|w_1,w_2 \ldots w_{n-1})$ is called a **language model**.

How to estimate these probabilities

- Could we just count and divide?

$$P(\text{blue} \mid \text{The water of Walden Pond is so beautifully}) = \frac{C(\text{The water of the Red sea is so beautifully blue})}{C(\text{The water of the Red sea is so beautifully})}$$

- No! Too many possible sentences!
- We'll never see enough data for estimating these

How to compute P(W) or $P(w_n|w_1, \ldots w_{n-1})$ with an N-gram Language Model

- How to compute the joint probability P(W):

  P(The, water, of, Walden, Pond , is, so, beautifully, blue)

- Intuition: let's rely on the Chain Rule of Probability

Reminder: The Chain Rule

- Recall the definition of conditional probabilities

  P(B|A) = P(A, B)/P(A) Rewriting: **P(A, B) = P(A)P(B|A)**

- 4 variables:

  P(A, B, C, D) = P(A)P(B|A)P(C|A, B)P(D|A, B, C)

- The Chain Rule in General

$$P(x_1,x_2,x_3,\ldots,x_n) = P(x_1)P(x_2|x_1)P(x_3|x_1,x_2)\ldots P(x_n|x_1,\ldots,x_{n-1})$$

The Chain Rule applied to compute joint probability of words in sentence

$$P(w_{1:n}) = P(w_1)P(w_2|w_1)P(w_3|w_{1:2})\ldots P(w_n|w_{1:n-1})$$

$$= \prod_{k=1}^{n} P(w_k|w_{1:k-1})$$

W1:n is w1…wn

W1:2 is W1 W2

P("The water of Walden Pond") =

P(The) × P(water|The) × P(of|The water)

× P(Walden|The water of) × P(Pond|The water of Walden)

Markov Assumption

- Simplifying assumption:

Andrei Markov

$$P(\texttt{blue}|\texttt{The water of Walden Pond is so beautifully})$$

$$\approx P(\texttt{blue}|\texttt{beautifully})$$

$$P(w_n|w_{1:n-1}) \approx P(w_n|w_{n-1})$$

Bigram Markov Assumption

$$P(w_{1:n}) \approx \prod_{k=1}^{n} P(w_k | w_{k-1})$$

- instead of:

$$\prod_{k=1}^{n} P(w_k | w_{1:k-1})$$

P(wk | w1 … wk-1)

We approximate each component in the product

$$P(w_n | w_{1:n-1}) \approx P(w_n | w_{n-N+1:n-1})$$

General N-gram model for each component

$$P(w_n|w_{1:n-1}) \approx P(w_n|w_{n-N+1:n-1})$$

N is the n-gram used, N = 2 for bigram, N = 3 for trigram , ...

Simplest case of Markov model: Unigram model

$$P(w_1 w_2 \ldots w_n) \approx \prod_i P(w_i)$$

Simply estimate the probability of a whole sequence of word by the product of the probabilities of individual words (unigram)

Random sequence of words (independent words)

Some automatically generated sentences from a unigram model

```
fifth, an, of, futures, the, an, incorporated, a, a,
the, inflation, most, dollars, quarter, in, is, mass

thrift, did, eighty, said, hard, 'm, july, bullish

that, or, limited, the
```

That is the probability of the unigram model; words are independent in this model

Bigram model

- Condition on the previous word:

$$P(w_i \mid w_1 w_2 \ldots w_{i-1}) \approx P(w_i \mid w_{i-1})$$

texaco, rose, one, in, this, issue, is, pursuing, growth, in, a, boiler, house, said, mr., gurria, mexico, 's, motion, control, proposal, without, permission, from, five, hundred, fifty, five, yen

outside, new, car, parking, lot, of, the, agreement, reached

this, would, be, a, record, november

Problems with N-gram models

- N-grams can't handle **long-distance dependencies**:

The computer(s) which I had just put into the machine room on the fifth floor is(are) crashed.

- from the word floor alone can't predicate the word crashed
- Crashed referred to the word computer .

- N-grams don't do well at modeling new sequences with similar meanings

The solution: **Large language models**

- can handle much longer contexts
- because of using embedding spaces, can model synonymy better, and generate better novel strings

Why N-gram models?

A nice clear paradigm that lets us introduce many of the important issues for **large language models**

- training and test sets
- the perplexity metric
- sampling to generate sentences
- ideas like smoothing, interpolation and backoff

N-Grams Example

The big white cat

- Unigrams: P(cat)
- Bigrams: P(cat|white)
- Trigrams: P(cat|big white)
- Four-grams: P(cat|the big white)

- In general, we'll be dealing with

  P(Word| Some fixed prefix word)

- The intuition of the n-gram model is instead of computing the probability of a word given its entire word history, we can approximate the history by just the last few words.

Estimating bigram probabilities

- The Maximum Likelihood Estimate (MLE)

C : Count

$$P(w_n|w_{n-1}) = \frac{C(w_{n-1}w_n)}{\sum_w C(w_{n-1}w)}$$

$$P(w_n|w_{n-1}) = \frac{C(w_{n-1}w_n)}{C(w_{n-1})}$$

Example 1:

$$P(w_i \mid w_{i-1}) \underset{\text{MLE}}{=} \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

Mini Corpus

<s> I am Sam </s>

<s> Sam I am </s>

<s> I do not like green eggs and ham </s>

The calculation of some bigram probabilities from the corpus:

$$= \frac{C(<S>,I)}{C(<S>)}$$

<s> : special token start up sentence
<\s>: special token end of sentence

$$P(\texttt{I} \mid \texttt{<s>}) = \frac{2}{3} = .67 \qquad P(\texttt{Sam} \mid \texttt{<s>}) = \frac{1}{3} = .33 \qquad P(\texttt{am} \mid \texttt{I}) = \frac{2}{3} = .67$$

$$P(\texttt{</s>} \mid \texttt{Sam}) = \frac{1}{2} = 0.5 \qquad P(\texttt{Sam} \mid \texttt{am}) = \frac{1}{2} = .5 \qquad P(\texttt{do} \mid \texttt{I}) = \frac{1}{3} = .33$$

$$\frac{C(</S>, \text{Sam})}{C(Sam)}$$

Example 2
Berkeley Restaurant Project sentences

- can you tell me about any good cantonese restaurants close by
- mid priced thai food is what i'm looking for
- tell me about
- can you give me a listing of the kinds of food that are available
- i'm looking for a good place to eat breakfast
- when is caffe venezia open during the day

Raw bigram counts based on Example 2

- Out of 9222 sentences

	i	want	to	eat	chinese	food	lunch	spend
i	5	827	0	9	0	0	0	2
want	2	0	608	1	6	6	5	1
to	2	0	4	686	2	0	6	211
eat	0	0	2	0	16	2	42	0
chinese	1	0	0	0	0	82	1	0
food	15	0	15	0	1	4	0	0
lunch	2	0	0	0	0	1	0	0
spend	1	0	1	0	0	0	0	0

Raw bigram probabilities based on Example 2

- Normalize by unigram counts, e.g., P(want | i) = c (i, want) / c(i) = 827/2533= .33

- Result:

i	want	to	eat	chinese	food	lunch	spend
2533	927	2417	746	158	1093	341	278

	i	want	to	eat	chinese	food	lunch	spend
i	0.002	0.33	0	0.0036	0	0	0	0.00079
want	0.0022	0	0.66	0.0011	0.0065	0.0065	0.0054	0.0011
to	0.00083	0	0.0017	0.28	0.00083	0	0.0025	0.087
eat	0	0	0.0027	0	0.021	0.0027	0.056	0
chinese	0.0063	0	0	0	0	0.52	0.0063	0
food	0.014	0	0.014	0	0.00092	0.0037	0	0
lunch	0.0059	0	0	0	0	0.0029	0	0
spend	0.0036	0	0.0036	0	0	0	0	0

Bigram estimates of sentence probabilities

P (I | <s>) = 0.25	P(english | want) = 0.0011
P(food | english) =0.5	P(</s> | food) = 0.68
P(want | I) = 0.33	

P(<s> I want english food </s>) =

P(I |<s>)

× P(want|I)

× P(english|want)

× P(food|english)

× P(</s>|food)

= 0.25 x 0.33 x 0.0011 x 0.5 x 0.68

= .000031

What kinds of knowledge do N-grams represent?

- P(to|want) = .66
- P(want | spend) = 0
- P(of | to) = 0

Knowledge of grammar

("want" is followed by infinitive "to")

("of" is not a verb)

- P (dinner|lunch or) = .83
- P(dinner|for) ~ P(lunch|for)

Knowledge of meaning

The words "dinner" or "lunch" are semantically related.

- P(chinese|want) > P(english|want)

Knowledge about the world

Chinese food is very popular

Practical Issues

- We do everything in log space
  - Avoid underflow
  - (also adding is faster than multiplying)

$$\log(ab) = \log(a) + \log(b)$$

$$\log(p_1 \cdot p_2 \cdot p_3 \cdot p_4) = \log p_1 + \log p_2 + \log p_3 + \log p_4$$

If we need probabilities we can do one exp at the end

$$p_1 \times p_2 \times p_3 \times p_4 = \exp(\log p_1 + \log p_2 + \log p_3 + \log p_4)$$

Larger n-grams

- 4-grams, 5-grams
- Large datasets of large n-grams have been released
  - N-grams from Corpus of Contemporary American English (COCA) 1 billion words (Davies 2020)
  - Google Web 5-grams (Franz and Brants 2006) 1 trillion words)
  - Efficiency: quantize probabilities to 4-8 bits instead of 8-byte float

Newest model: infini-grams (∞-grams) (Liu et al 2024)

- No precomputing! Instead, store 5 trillion words of web text in **suffix arrays**. Can compute n-gram probabilities with any n!

H. W.

- Write out the equation for the trigram probability estimation and then write out all the non-zero trigram probability for the corpus of example 1.

How to evaluate N-gram models

- "Extrinsic (in-vivo) Evaluation"

To compare models A and B

1. Put each model in a real task
   - Machine Translation, speech recognition, etc.
2. Run the task, get a score for model A and for model B
   - How many words translated correctly
   - How many words transcribed correctly
3. Compare accuracy for A and B

Intrinsic (in-vitro) evaluation

- Extrinsic evaluation not always possible
  - Expensive, time-consuming
  - Doesn't always generalize to other applications
- Intrinsic evaluation: **perplexity**
  - Directly measures language model performance at predicting words.
  - Doesn't necessarily correspond with real application performance
  - But gives us a single general metric for language models
  - Useful for large language models (LLMs) as well as n-grams

Training sets and test sets

We train parameters of our model on a **training set**.

We test the model's performance on data we haven't seen.

- A **test set** is an unseen dataset; different from training set.
  - Intuition: we want to measure generalization to unseen data
- An **evaluation metric** (like **perplexity**) tells us how well our model does on the test set.

Choosing training and test sets

- If we're building an LM for a specific task
  - The test set should reflect the task language we want to use the model for
- If we're building a general-purpose model
  - We'll need lots of different kinds of training data
  - We don't want the training set or the test set to be just from one domain or author or language.

Training on the test set

We can't allow test sentences into the training set

- Or else the LM will assign that sentence an artificially high probability when we see it in the test set
- And hence assign the whole test set a falsely high probability.
- Making the LM look better than it really is

This is called "Training on the test set"

Bad science!

Dev sets

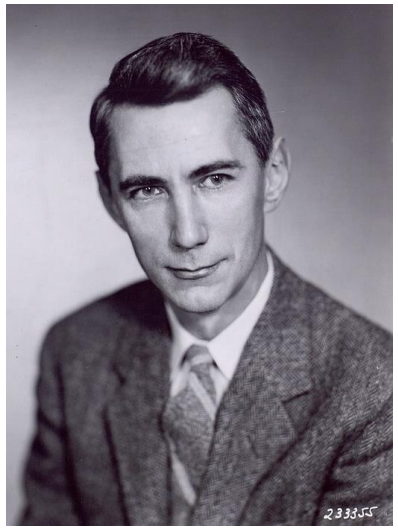
- If we test on the test set many times we might implicitly tune to its characteristics
  - Noticing which changes make the model better.
- So we run on the test set only once, or a few times
- That means we need a third dataset:
  - A **development test set** or, **devset**.
  - We test our LM on the devset until the very end
  - And then test our LM on the **test set** once

Intuition of perplexity as evaluation metric: How good is our language model?

Intuition: A good LM prefers "real" sentences

- Assign higher probability to “real” or “frequently observed” sentences
- Assigns lower probability to “word salad” or “rarely observed” sentences?

Intuition of perplexity 2:
Predicting upcoming words



Claude Shannon

The Shannon Game: **How well can we predict the next word?**

- Once upon a _____
- That is a picture of a _____
- For breakfast I ate my usual _____

time 0.9
dream 0.03
midnight 0.02
…
and 1e-100

Unigrams are terrible at this game (Why?)

A good LM is one that assigns a higher probability to the next word that actually occurs

Intuition of perplexity 3: The best language model is one that best predicts the entire unseen test set

- We said: a good LM is one that assigns a higher probability to the next word that actually occurs.
- Let's generalize to all the words!
  - The best LM assigns high probability to the entire test set.
- When comparing two LMs, A and B
  - We compute P_A(test set) and P_B(test set)
  - The better LM will give a higher probability to (=be less surprised by) the test set than the other LM.

Intuition of perplexity 4: Use perplexity instead of raw probability

- Probability depends on size of test set
  - Probability gets smaller the longer the text
  - Better: a metric that is **per-word**, normalized by length of sentence.

Example:

P(<s> Mona drinks tea <\s>) = 0.5

P((<s> Mona drinks tea in the garden <\s>) = 0.3

Intuition of perplexity 4 (cont.): Use perplexity instead of raw probability

- **Perplexity** is the inverse probability of the test set, normalized by the number of words

$$PP(W) = P(w_1 w_2 \ldots w_N)^{-\frac{1}{N}}$$

$$= \sqrt[N]{\frac{1}{P(w_1 w_2 \ldots w_N)}}$$

Example:

PP(<s> Mona drinks tea <\s>) = $(0.5)^{\frac{-1}{3}}$ = 1.3

P((<s> Mona drinks tea in the garden <\s>) = $(0.3)^{\frac{-1}{5}}$ = 1.3

Intuition of perplexity 5: the **inverse**

Perplexity is the **inverse** probability of the test set, normalized by the number of words

$$PP(W) = P(w_1 w_2 \ldots w_N)^{-\frac{1}{N}}$$

$$= \sqrt[N]{\frac{1}{P(w_1 w_2 \ldots w_N)}}$$

(The inverse comes from the original definition of perplexity from cross-entropy rate in information theory)

Probability range is [0,1], perplexity range is [1,∞]

Minimizing perplexity is the same as maximizing probability

Intuition of perplexity 6: N-grams

$$PP(W) = P(w_1 w_2 \ldots w_N)^{-\frac{1}{N}}$$

$$= \sqrt[N]{\frac{1}{P(w_1 w_2 \ldots w_N)}}$$

Chain rule:

$$\mathrm{PP}(W) = \sqrt[N]{\prod_{i=1}^{N} \frac{1}{P(w_i | w_1 \ldots w_{i-1})}}$$

Bigrams:

$$\mathrm{PP}(W) = \sqrt[N]{\prod_{i=1}^{N} \frac{1}{P(w_i | w_{i-1})}}$$

Holding test set constant:
Lower perplexity = better language model

- Training 38 million words, test 1.5 million words, Wall Street Journal corpus-WSJ

N-gram Order	Unigram	Bigram	Trigram
Perplexity	962	170	109

Visualizing Bigrams the Shannon Way

- Choose a random bigram (<s>, w)
- according to its probability p(w|<s>)
- Now choose a random bigram (w, x) according to its probability p(x|w)
- And so on until we choose </s>
- Then string the words together

<s> I
I want
want to
to eat
eat Chinese
Chinese food
food </s>

I want to eat Chinese food

The perils of overfitting

N-grams only work well for word prediction if the test corpus looks like the training corpus

- But even when we try to pick a good training corpus, the test set will surprise us!
- We need to train robust models that generalize!

One kind of generalization: **Zeros**

- Things that don't ever occur in the training set
  - But occur in the test set

Zeros

- Training set:
  … ate lunch
  … ate dinner
  … ate a
  … ate the

  P(“breakfast” | ate) = 0

- Test set
  … ate lunch
  … ate breakfast

Zero probability bigrams

Bigrams with zero probability

- Will hurt our performance for texts where those words appear!
- And mean that we will assign 0 probability to the test set!

And hence we cannot compute perplexity (can’t divide by 0)!

The problem of sparsity

Maximum likelihood estimation has a problem

- Sparsity! Our finite training corpus won't have some perfectly fine sequences
  - Perhaps it has "ruby" and "slippers"
  - But happens not to have "ruby slippers"

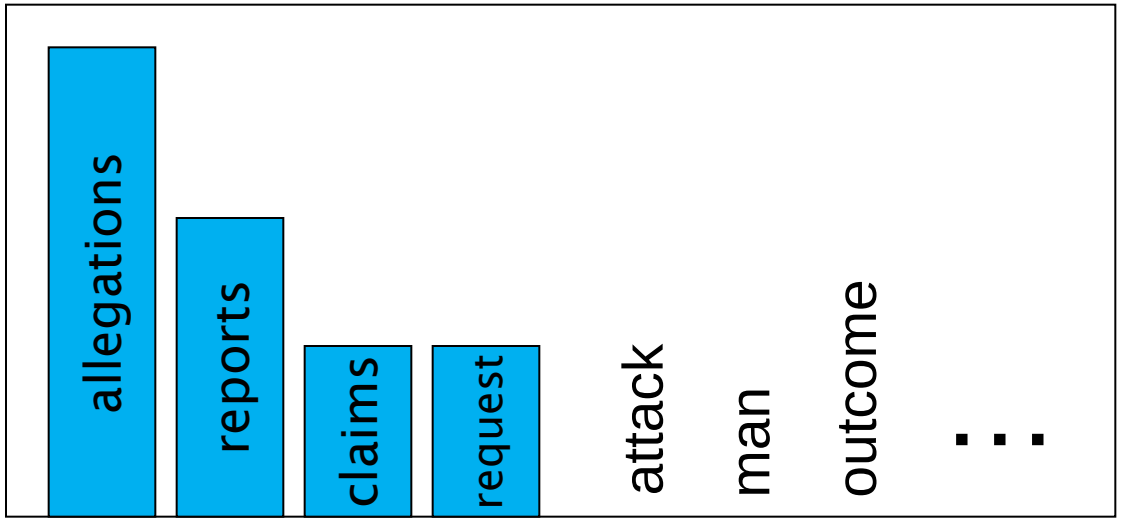
This sparsity can take many forms

- Verbs have direct objects; those can be sparse too!

Count(w | denied the)

Counts:
3 allegations
2 reports
1 claims
1 request
0 attack
0 outcome

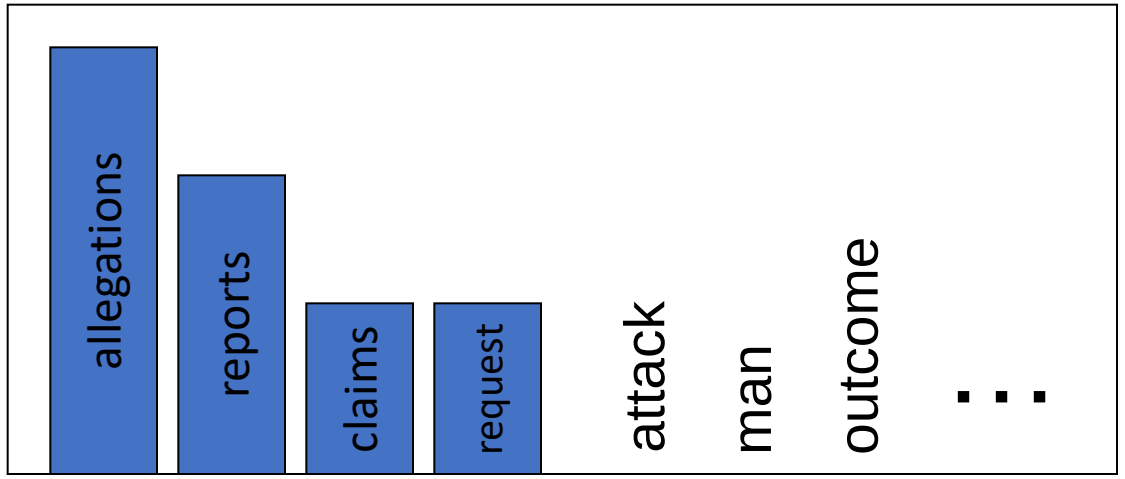
7 total



The intuition of smoothing (from Dan Klein)

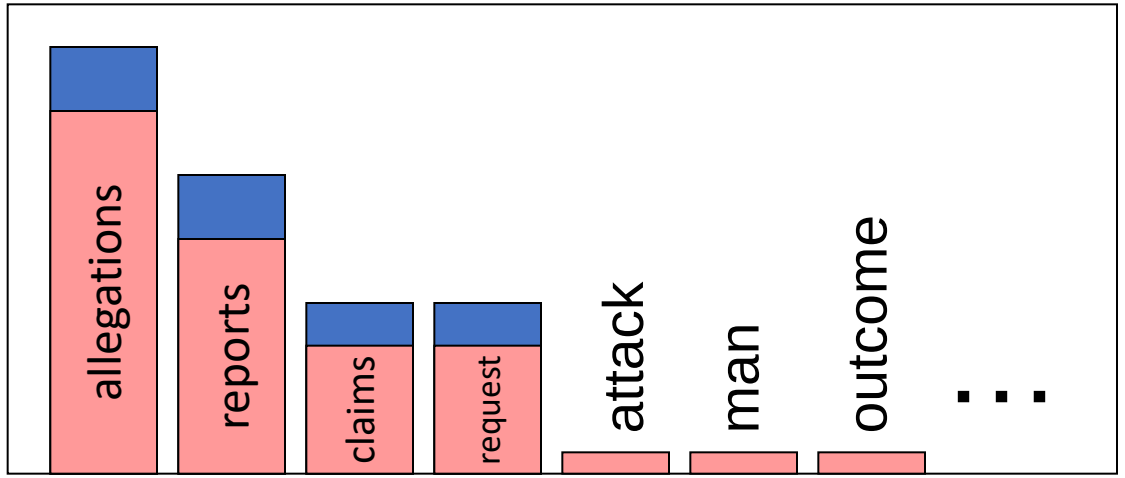
- When we have sparse statistics:

  Count(w | denied the)
  3 allegations
  2 reports
  1 claims
  1 request
  7 total



- Steal probability mass to generalize better

  Count (w | denied the)
  2.5 allegations
  1.5 reports
  0.5 claims
  0.5 request
  2 other
  7 total



Add-one estimation

- Also called Laplace smoothing
- Pretend we saw each word one more time than we did
- Just add one to all the counts!

- MLE estimate:

$$P_{\text{MLE}}(w_n|w_{n-1}) = \frac{C(w_{n-1}w_n)}{C(w_{n-1})}$$

- Add-1 estimate:

$$P_{\text{Laplace}}(w_n|w_{n-1}) = \frac{C(w_{n-1}w_n)+1}{\sum_w (C(w_{n-1}w)+1)} = \frac{C(w_{n-1}w_n)+1}{C(w_{n-1})+V}$$

Maximum Likelihood Estimates

- The maximum likelihood estimate
  - of some parameter of a model M from a training set T
  - maximizes the likelihood of the training set T given the model M
- Suppose the word “bagel” occurs 400 times in a corpus of a million words
- What is the probability that a random word from some other text will be “bagel”?
- MLE estimate is 400/1,000,000 = .0004
- This may be a bad estimate for some other corpus
  - But it is the **estimate** that makes it **most likely** that “bagel” will occur 400 times in a million-word corpus.

Berkeley Restaurant Corpus: Laplace smoothed bigram counts

$$C(w_{n-1}w_n) + 1$$

	i	want	to	eat	chinese	food	lunch	spend
i	6	828	1	10	1	1	1	3
want	3	1	609	2	7	7	6	2
to	3	1	5	687	3	1	7	212
eat	1	1	3	1	17	3	43	1
chinese	2	1	1	1	1	83	2	1
food	16	1	16	1	2	5	1	1
lunch	3	1	1	1	1	2	1	1
spend	2	1	2	1	1	1	1	1

Laplace-smoothed bigrams

$$P_{\text{Laplace}}(w_n|w_{n-1}) = \frac{C(w_{n-1}w_n)+1}{\sum_w (C(w_{n-1}w)+1)} = \frac{C(w_{n-1}w_n)+1}{C(w_{n-1})+V}$$

V = 1446

	i	want	to	eat	chinese	food	lunch	spend
i	0.0015	0.21	0.00025	0.0025	0.00025	0.00025	0.00025	0.00075
want	0.0013	0.00042	0.26	0.00084	0.0029	0.0029	0.0025	0.00084
to	0.00078	0.00026	0.0013	0.18	0.00078	0.00026	0.0018	0.055
eat	0.00046	0.00046	0.0014	0.00046	0.0078	0.0014	0.02	0.00046
chinese	0.0012	0.00062	0.00062	0.00062	0.00062	0.052	0.0012	0.00062
food	0.0063	0.00039	0.0063	0.00039	0.00079	0.002	0.00039	0.00039
lunch	0.0017	0.00056	0.00056	0.00056	0.00056	0.0011	0.00056	0.00056
spend	0.0012	0.00058	0.0012	0.00058	0.00058	0.00058	0.00058	0.00058

Compare with raw bigram counts

original

	i	want	to	eat	chinese	food	lunch	spend
i	5	827	0	9	0	0	0	2
want	2	0	608	1	6	6	5	1
to	2	0	4	686	2	0	6	211
eat	0	0	2	0	16	2	42	0
chinese	1	0	0	0	0	82	1	0
food	15	0	15	0	1	4	0	0
lunch	2	0	0	0	0	1	0	0
spend	1	0	1	0	0	0	0	0

add-1 smoothed

	i	want	to	eat	chinese	food	lunch	spend
i	3.8	527	0.64	6.4	0.64	0.64	0.64	1.9
want	1.2	0.39	238	0.78	2.7	2.7	2.3	0.78
to	1.9	0.63	3.1	430	1.9	0.63	4.4	133
eat	0.34	0.34	1	0.34	5.8	1	15	0.34
chinese	0.2	0.098	0.098	0.098	0.098	8.2	0.2	0.098
food	6.9	0.43	6.9	0.43	0.86	2.2	0.43	0.43
lunch	0.57	0.19	0.19	0.19	0.19	0.38	0.19	0.19
spend	0.32	0.16	0.32	0.16	0.16	0.16	0.16	0.16

Backoff and Interpolation

- Sometimes it helps to use **less** context
  - Condition on less context for contexts you know less about
- **Backoff:**
  - use trigram if you have good evidence,
  - otherwise bigram, otherwise unigram
- **Interpolation:**
  - mix unigram, bigram, trigram

- Interpolation works better

Linear Interpolation

- Simple interpolation

$$\begin{aligned}\hat{P}(w_n|w_{n-2}w_{n-1}) &= \lambda_1 P(w_n|w_{n-2}w_{n-1}) \\ &\quad +\lambda_2 P(w_n|w_{n-1}) \\ &\quad +\lambda_3 P(w_n)\end{aligned}$$

$$\sum_i \lambda_i = 1$$

- Lambdas conditional on context:

$$\begin{aligned}\hat{P}(w_n|w_{n-2}w_{n-1}) &= \lambda_1(w_{n-2}^{n-1})P(w_n|w_{n-2}w_{n-1}) \\ &\quad +\lambda_2(w_{n-2}^{n-1})P(w_n|w_{n-1}) \\ &\quad +\lambda_3(w_{n-2}^{n-1})P(w_n)\end{aligned}$$