Prove Beta-Binomial conjugation

⇒要prove之前要先介紹gamma function

$\Gamma(x) = \int_0^\infty p^{x-1} e^{-p} dp$ 該function有以下性質：

① $\Gamma(x) = (x-1)\Gamma(x-1)$

$$\int_0^\infty p^{x-1} e^{-p} dp = -p^{x-1} e^{-p} \Big|_0^\infty + (x-1)\int_0^\infty p^{x-2} e^{-p} dp$$

$$= (x-1)\Gamma(x-1)$$

② $\int_0^\infty p^{a-1}(1-p)^{b-1} dp = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

$$\int_0^\infty \beta(p,a,b)\,dp = 1 \Rightarrow \int_0^\infty p^{a-1}(1-p)^{b-1} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} dp$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^\infty p^{a-1}(1-p)^{b-1} dp = 1$$

$$\Rightarrow \int_0^\infty p^{a-1}(1-p)^{b-1} dp = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Prior 為 beta distribution，其分布為以下

$$p^{a-1}(1-p)^{b-1} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

令 a、b 為先前的knowledge，p為成功的

機率：N為測試次數、m為成功、n為失敗

$$P(\theta|\text{event}) = \frac{\text{likelihood} \times \text{prior}}{\text{marginal}} = \frac{C^N_m p^m (1-p)^{N-m} p^{a-1}(1-p)^{b-1} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}{\int_0^1 C^N_m \theta^m (1-\theta)^{N-m} \theta^{a-1}(1-\theta)^{b-1} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} d\theta}$$

$$= \frac{p^{m+a-1}(1-p)^{N-m+b-1}}{\int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1} d\theta}$$

$$\because \int_0^1 \beta(\theta | m+a, N-m+b) d\theta$$

$$= \int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1} \frac{\Gamma(a+N+b)}{\Gamma(m+a)\Gamma(N-m+b)} d\theta$$

$$= 1$$

$$\Rightarrow \int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1} d\theta = \frac{\Gamma(m+a)\Gamma(N-m+b)}{\Gamma(a+N+b)}$$

$$P(\theta|\text{event}) = \frac{p^{m+a-1}(1-p)^{N-m+b-1}}{\int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1} d\theta}$$

$$= p^{m+a-1}(1-p)^{N-m+b-1} \frac{\Gamma(a+N+b)}{\Gamma(m+a)\Gamma(N-m+b)}$$

$$= \beta(p \mid a+m, b+N-m)$$

由此可以 prove Beta-Binomial conjugation