Quantum Singular Value Transformation

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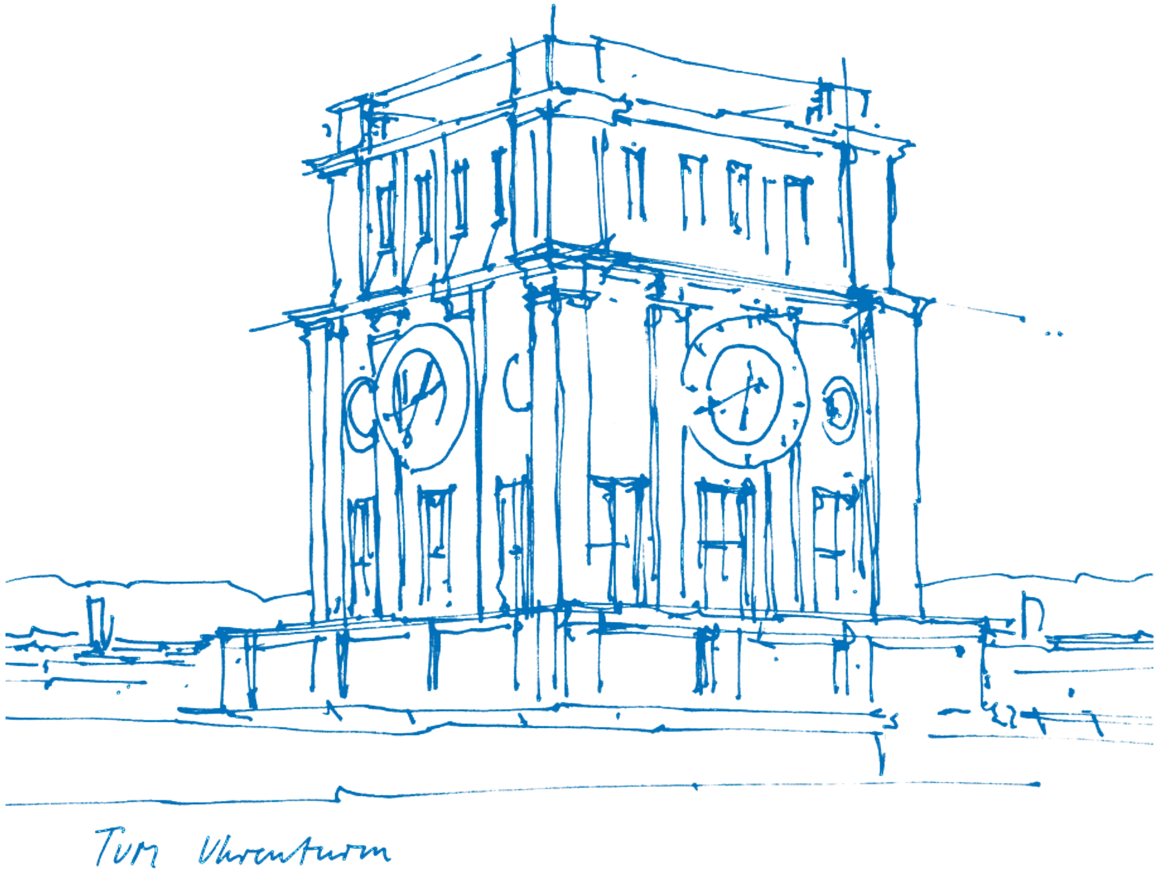


Overview

1. Quantum Algorithms: Why
2. Quantum signal processing (QSP)
3. Quantum eigenvalue transforms (QEVT)
4. Quantum singular value transformation (QSVT)
5. Hamiltonian Simulation
6. Conclusion
7. References

1. Quantum Algorithms: Why

- Significant speed ups

- Principles of quantum mechanics: superposition, entanglement, interference

- Factoring Large Numbers (Cryptography): Shor’s algorithm
—> threatens widely used encryption schemes

- Database search: Grover’s algorithm
—> unstructured database in $O(\sqrt{N})$

- Quantum singular value transformation
—> mathematical framework
—> systematic way to design quantum algorithms by manipulating the eigenvalue or singular values of a target operator

2. Quantum Signal Processing (QSP)

GOAL: transform a $\longmapsto$ P(a) polynomially

- idea of interleaving two kinds of single-qubit rotation

signal rotation operator for $a \in [-1, 1]$

$$W(a) = \begin{bmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{bmatrix}$$

x rotation by angle −2*arccos(a)

signal-processing rotation operator

$$S(\phi) = e^{i\phi Z} = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$$

z rotation by angle −2φ

QSP for $\vec{\phi} = (\phi_0, \ldots, \phi_d) \in \mathbb{R}^{d+1}$

$$U_{\vec{\phi}} = e^{i\phi_0 Z} \prod_{k=1}^{d} W(a) e^{i\phi_k Z} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix}$$

(i) $deg(P) \leq d,\ deg(Q) \leq d-1$
(ii) P *has parity* $d \bmod 2$ *and* Q *has parity* $(d-1) \bmod 2$
(iii) $|P|^2 + (1-a^2)|Q|^2 = 1$

3. Quantum Eigenvalue Transforms (QEVT)

GOAL: transform Hamiltonian $\mathscr{H} \longmapsto P(\mathscr{H}) = \sum_\lambda Poly(\lambda)\big|\lambda\rangle\langle\lambda|$ polynomially

Block Encoding

- encode $\mathscr{H}$ into a larger unitary matrix U

$$U = \begin{array}{cc} & \begin{array}{cc} \Pi & \end{array} \\ \Pi & \begin{bmatrix} H & \cdot \\ \cdot & \cdot \end{bmatrix} \end{array}$$

- with Π := |0><0| -> access $\mathscr{H}$ in $\mathscr{H}$ = ΠUΠ

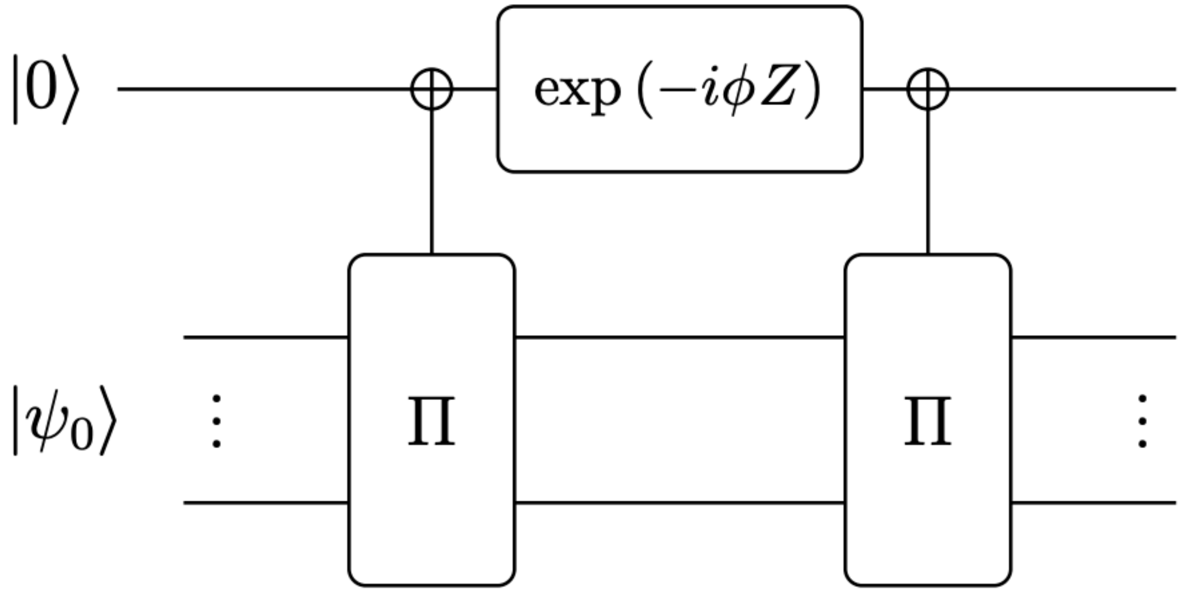
3. Quantum Eigenvalue Transforms (QEVT)

GOAL: transform Hamiltonian $\mathscr{H} \longmapsto P(\mathscr{H}) = \sum_\lambda Poly(\lambda)|\lambda\rangle\langle\lambda|$ polynomially

Projector-controlled phase-shift operation φ

applies a phase factor of exp(i2ϕ) to the subspace defined by Π

$$\Pi_\phi := e^{i2\phi\Pi} = e^{i\phi(2\Pi - I)}$$



3. Quantum Eigenvalue Transform (QEVT)

GOAL: transform Hamiltonian $\mathscr{H} \longmapsto$ P($\mathscr{H}$) = $\sum_\lambda Poly(\lambda)|\lambda\rangle\langle\lambda|$ polynomially

Quantum Eigenvalue Transform

$$U_{\vec{\phi}}^{(QEVT)} = \begin{cases} \left[\prod_{k=1}^{d/2} \Pi_{\phi_{2k-1}} U^\dagger \Pi_{\phi_{2k}} U\right] & \text{for } d \text{ even} \\ \Pi_{\phi_1} U \left[\prod_{k=1}^{(d-1)/2} \Pi_{\phi_{2k}} U^\dagger \Pi_{\phi_{2k+1}} U\right] & \text{for } d \text{ odd} \end{cases} = \begin{bmatrix} \mathrm{Poly}(A) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

where

$$\mathrm{Poly}(\mathcal{H}) = \sum_\lambda \mathrm{Poly}(\lambda)|\lambda\rangle\langle\lambda|$$

Hamiltonian matrix: $H = H^\dagger$

$$U_{\vec{\phi}}^{(QEVT)} = \Pi_{\phi_0} \prod_{k=1}^{d} U \Pi_{\phi_k}$$

4. Quantum Sigular Value Transformation (QSVT)

GOAL: transform $A \mapsto P(A) = \sum_{k=1}^{r} Poly(\sigma_k)\left|w_k\right\rangle\left\langle v_k\right|$ polynomially

Block Encoding

$$U = \begin{array}{cc} & \Pi \\ \tilde{\Pi} & \begin{bmatrix} A & \cdot \\ \cdot & \cdot \end{bmatrix} \end{array}$$

with $\Pi := \sum_k \left|v_k\right\rangle\left\langle v_k\right|$

and $\widetilde{\Pi} := \sum_k \left|w_k\right\rangle\left\langle w_k\right|$

Projector-controlled phase-shift operation φ

- alternate between $\widetilde{\Pi}$ and Π projector

Quantum Singular Value Transformation operation sequence, for $\vec{\phi} = (\phi_0, \dots, \phi_d) \in \mathbb{R}^{d+1}$

$$U_{\vec{\phi}} = \begin{cases} \left[\prod_{k=1}^{d/2} \Pi_{\phi_{2k-1}} U(A)^\dagger \widetilde{\Pi}_{\phi_{2k}} U(A)\right] \Pi_{\phi_{d+1}} & \text{for d even} \\ \widetilde{\Pi}_{\phi_1} \left[\prod_{k=1}^{(d-1)/2} \Pi_{\phi_{2k}} U(A)^\dagger \widetilde{\Pi}_{\phi_{2k+1}} U(A)\right] \Pi_{\phi_{d+1}} & \text{for d odd} \end{cases} = \begin{bmatrix} Poly(A) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

5. Hamiltonian Simulation

1. Function f representing our problem:

$$f(x) = e^{ixt} = \cos(xt) - isin(xt)$$

2. Block encoding $\mathscr{H}$:

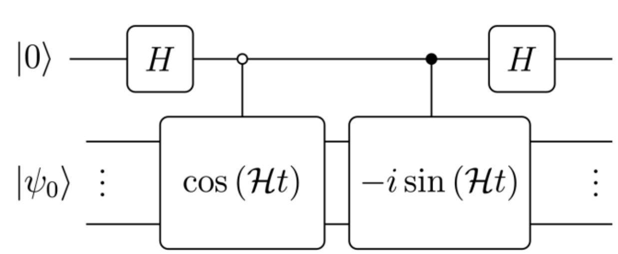
$$U = \begin{bmatrix} \mathcal{H} & \sqrt{I - \mathcal{H}^2} \\ \sqrt{I - \mathcal{H}^2} & -\mathcal{H} \end{bmatrix}$$

3. Polynomial approximation p for f:

$$\cos(xt) = J_0(t) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(t) T_{2k}(x)$$

$$\sin(xt) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(t) T_{2k+1}(x)$$

4. Constructing a circuit that applies p for $\mathscr{H}$:



6. Conclusion

Unifying mathematical framework for designing quantum algorithms:

- A systematic approach to design algorithms
- Applications in Hamiltonian simulation, Grover’s algorithm, and beyond
- A pathway to unlocking the full potential of quantum computing

7. References

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