# Improved Euler's method to solve 1st-order ODE numerically

In mathematics and computational science, Heun's method may refer to the improved or modified Euler's method (that is, the explicit trapezoidal rule]), or a similar two-stagage Runge–Kutta method.

Euler’s method is a numerical method for solving ordinary differential equations (ODEs) with a given initial value. It is named after the famous mathematician Leonhard Euler who introduced it in the 18th century. The method approximates the solution of an ODE by using small steps and computing the values of the function at each step.

y'=f(x,y), y(x0)=y0

the expensive part of the computation is the evaluation of . Therefore we want methods that give good results for a given number of such evaluations. This is what motivates us to look for numerical methods better than Euler’s. To clarify this point, suppose we want to approximate the value of ee by applying Euler’s method to the initial value problem

y'=y , y(0)=1 (with the solution y= e^x )

on [0,1], with h = 1/12 , 1/24, and 1/48, respectively. Since each step in Euler’s method requires one evaluation of f, the number of evaluations of f in each of these attempts is n=12, 24, and 48, respectively. In each case we accept yn as an approximation to e. The table below shows the results.

|  |  |  |
| --- | --- | --- |
| n | Euler | Exact |
| 12 | 2.613035290 | 2.718281828 |
| 24 | 2.663731258 | 2.718281828 |
| 48 | 2.690496599 | 2.718281828 |

The first column of the table indicates the number of evaluations of f required to obtain the approximation, and the last column contains the value of e rounded to ten significant figures.

In this module we’ll study the improved Euler method, which requires two evaluations of f at each step. We’ve used this method with h=1/6, 1/12, and 1/24. The required number of evaluations of f were 12, 24, and 48, as in the three applications of Euler’s method; however, you can see from the third column of the table below that the approximation to e obtained by the improved Euler method with only 12 evaluations of f is better than the approximation obtained by Euler’s method with 48 evaluations.

|  |  |  |  |
| --- | --- | --- | --- |
| n | Euler | Improved Euler | Exact |
| 12 | 2.613035290 | 2.707188994 | 2.718281828 |
| 24 | 2.663731258 | 2.715327371 | 2.718281828 |
| 48 | 2.690496599 | 2.717519565 | 2.718281828 |

The procedure for calculating the numerical solution to the initial value problem:

y'(t) = f(t,y(t)), y(t0) = y0,

by way of Heun's method, is to first calculate the intermediate value

~yi+1 and then the final approximation yi+1 at the next integration point.

~yi+1 = yi + hf( ti , yi)

yi+1 = yi + h/2[ f(ti, yi) + f(ti+1 , ~yi+1)],

where h is the step size and ti+1= ti + h

***Example***

y' = 2(y^2+1)/(x^2+4), y(0) = 1, with step size h =.1. With x0 = 0, y0 = 1 , we have m1 = f(0,1) = 1, M2 = f(.1,1.1) = 1.102244389, y1 = 1+.1(m1 + M2)/2= 1.105112219.

The true solution had (.1) = 1.105263158, so the error is .000150939, compared to .005263158 for the Euler method. Here is a table of the results of the first 10 steps for the Improved Euler and Euler methods with h = .1 , and their respective errors:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Xi | Improved Euler yi | Euler yi | (xi) | Improved Euler error | Euler error |
| 0.0 | 1.00000000 | 1.00000000 | 1.00000000 | 0.0 | 0.0 |
| 0.1 | 1.10511222 | 1.10000000 | 1.10526316 | 0.00015094 | 0.00526316 |
| 0.2 | 1.22185235 | 1.21022444 | 1.22222222 | 0.00036987 | 0.01199778 |
| 0.3 | 1.35225607 | 1.33223648 | 1.35294118 | 0.00068510 | 0.02070470 |
| 0.4 | 1.49886227 | 1.46792616 | 1.50000000 | 0.00113773 | 0.03207384 |
| 0.5 | 1.66487828 | 1.61959959 | 1.66666667 | 0.00178838 | 0.04706708 |
| 0.6 | 1.85441478 | 1.79009854 | 1.85714286 | 0.00272808 | 0.06704432 |
| 0.7 | 2.07282683 | 1.98296335 | 2.07692308 | 0.00409625 | 0.09395973 |
| 0.8 | 2.32722149 | 2.20265794 | 2.33333333 | 0.00611184 | 0.13067539 |
| 0.9 | 2.62723508 | 2.45488648 | 2.63636364 | 0.00912856 | 0.18147716 |
| 1.0 | 2.98626232 | 2.74704729 | 3.00000000 | 0.01373768 | 0.25295271 |

***Example***

***Find y(0.2) for y′= (x-y)/2 , y(0) = 1, with step length 0.1 using Improved Euler method***

Here, x0=0 ,y0=1 ,h=0.1 y′= (x-y)/2

∴f(x,y) = (x-y)/2

**Improved Euler method**

y(m+1) = y(m) + 1/2 h[ f(xm , ym) + f(xm + h , ym + hf(xm , ym)) ]

f(x0,y0) = f(0,1) = -0.5

f(x0+h , y0 + hf(x0,y0)) = f(0.1,0.95) = -0.425

y1 = y0 + 1/2 h[f(x0 , y0) + f(x0+h , y0 + hf(x0,y0))]

y1 = 1 + 0.1/2 ⋅ [-0.5-0.425] = 0.95375

∴y(0.1) = 0.95375

**Again taking (x1,y1) in place of (x0,y0) repeat the process**

f(x1,y1) = f(0.1 , 0.95375) = -0.42688

f(x1+h , y1 + hf(x1,y1)) = f(0.2,0.91106) = -0.35553

y2 = y1 + 1/2 h[f(x1,y1) + f(x1+h , y1 + hf(x1,y1))]

y2 = 0.95375 + 0.1/2 ⋅ [-0.42688-0.35553] = 0.91463

∴y(0.2) = 0.91463

# REFERENCES

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