Honors Math

About

This is a summery of **Honors Calculus** (117 + 118) (or as some people call it: "**Abstract Calculus**")
This course will focus on set theory and number theory (and logic)

* note this won't display properly on GitHub, I use a program called Typora with inline math enabled

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Notes

0. Shorthand

Ex. $\rightarrow$ example

Thm $\rightarrow$ Theorem (will be in block quotes)

Def $\rightarrow$ definition

s.t. $\rightarrow$ such that

, or ; or : $\rightarrow$ such that, so (general connector depending on context)

wLOG $\rightarrow$ without loss of generality (i.e. similar to previous case, but I'm too lazy to do it)

seq. $\rightarrow$ sequence

diff. $\rightarrow$ differentiable

RHS $\rightarrow$ right hand side

LHS $\rightarrow$ left hand side

Formal Logic

$\in \rightarrow$ element of

$\subset \ or \subseteq \quad \rightarrow$ subset of

$\cup \rightarrow$ union

$\cap \rightarrow$ intercept

$\wedge \rightarrow$ and

$\vee \rightarrow$ or

$\rightarrow \quad \rightarrow$ then

$\iff \rightarrow$ if and only if (this is the same as '=')

$=: \rightarrow$ is defined to be

$\forall \rightarrow$ for all

$\exists \rightarrow$ there exists

$\exists! \rightarrow$ there exists unique

$Q.E.D. \rightarrow$ and it is proved

$\therefore \quad \rightarrow$ therefore

$\because \quad \rightarrow$ because

Fields

$\mathbb{N} \rightarrow$ natural numbers (i.e. 1,2,3,4...) [* note 0 might be included in natural numbers depending on what text book is in use]

$W \rightarrow$ whole numbers ($\{0\} \cup \mathbb{N}$)

$\mathbb{Z} \rightarrow$ integers (i.e. ..-2,-1,0,1,2...)

$\mathbb{Q} \rightarrow$ rational numbers (anything that can be expressed as $\frac{p}{q}, where\ p \in \mathbb{Z}\ and\ q \in \mathbb{N}$)

$\mathbb{R} \rightarrow$ will be discussed later

1. Sets

a set is defined as a number (can be 0 but then that would be the trivial or stupid set) of elements belonging to the same group

> Ex.
>
> - {1,2,3} = {2,1,3}
> - {a,b}
> - {even, odd}
> - $\mathbb{N}$ = {1,2,3,4,5...}
> - {} (trivial set)

If an element 'x' is part of a set 'A', it is described as $x \in A$

If $A \cup B = A,\ B \subset A$. If for all elements of B are in A, then B is considered a subset of A

If elements of A are present in B, then those elements 'C' are said to be $A \cap B$ or $A \cap B = C$

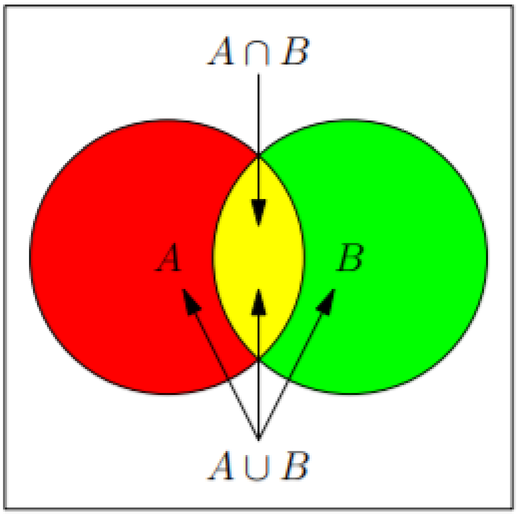

Figure 1.1: Venn Diagram

From Bowmen notes

* Most of the time, Russell's paradox will be introduced to scare any students still in the class.

Optional Exercise: look into Russell's paradox because it is quite interesting

2. Logic

Truth Values

There are 2 truth values, true (T or topology) and false (F or contradiction)

In math, there are established rules (which were set arbitrarily but is important for consistence around the world) for operations on truth values

Most of the time, a truth table will be used to illustrate how operators effect the truth value of a statement

Some common ones are:

$A \wedge B$; (A and B)

		A	A
		T	F
B	T	T	F
B	F	F	F

$A \lor B$; (A or B)

		A	A
		T	F
B	T	T	T
B	F	T	F

$A \to B$; (If A then B)

		A	A
		T	F
B	T	T	T
B	F	F	T

$A \iff B$; (A if and only if B or A = B)

* note this is a different way for displaying the truth table (important for chain multiple operators with more than 2 variables)

A	B	$A \iff B$	$A \to B \, and \, B \to A$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

To chain multiple operators together, you can use an extended truth table that considers all the variables (like the one above, but you must consider all variables and all the cases possible use a)

As seen from the table above, an iff is logically equivalent to a if b and b if a, therefore for proving an iff, the easiest way is to prove it using the A premise to get to B and then using the B premise to derive A

Contraction

The easiest way to prove something is to use *proof by contradiction*. Assume the opposite of the theorem or a fact that is true and find a contraction in reasoning

> Ex.

> $\sqrt{2}\, is\ not\ rational$
>
> Start of proof by contraction:
>
> Suppose for contradiction, $\sqrt{2}\, is\ rational$, i.e. it can be expressed as $\frac{p}{q}, where\ p \in \mathbb{Z}\ and\ q \in \mathbb{N}.$
>
> $$\sqrt{2} = \frac{p}{q}$$
> $$\rightarrow \sqrt{2}^2 = \frac{p^2}{q^2}$$
> $$2q^2 = p^2$$
>
> * note even number are expressed as 2n and odd number can be expressed as 2n+1 for $n \in \mathbb{Z}$
>
> Therefore p must be even because only an even squared is even $((2n)^2 = 4n^2\ and\ (2n+1)^2 = 4n^2 + 2n + 1 \therefore odd)$

3. What is a number? (Algebraic Properties)

For real numbers, the following must be true

* assume $a, b, c \in \mathbb{R}$

1. Must be associative

> $a + (b + c) = (a + b) + c$

2. Must have an additive identity

> $a + 0 = 0 + a = a$

3. Must have an *additive inverse* -a s.t.

> $a + -a = -a + a = 0$

4. Follows additive commutativity

> $a + b = b + a$

5. Are associate

> $a(bc) = (ab)c$

6. There exists are multiplicative identity where it is not 0

> $a \times 1 = a$

7. It can distribute

> $a \times (b + c) = a \times b + a \times c$

8. Has an inverse that is not 0, i.e. $a^{-1} \neq 0$

> $a \times a^{-1} = 1$

9. Follows multiplicative commutativity

> $a \times b = b \times a$

10. Trichotomy Law

> must be one and only one of the following relations:

> $a < b, \ a = b, \ a > b$

11. Closed under addition

> $a > 0 \ and \ b > 0 \rightarrow a + b > 0$

12. Closed under multiplication

> $a > 0 \ and \ b > 0 \rightarrow a \times b > 0$

More rules to come

Also note the first lemma (lemma is like a theorem but arbitrary defined to be a small theorem)

* lemmas or thm can be assumed

Lemma 1.1 (Midpoint Lemma):

$$a < b \quad \Rightarrow \quad a < \frac{a+b}{2} < b.$$

Proof:

$$a < b \quad \Rightarrow \quad a + a \quad < \quad a + b < b + b$$

$$\Rightarrow \quad a = \frac{a+a}{2} \quad < \quad \frac{a+b}{2} < \frac{b+b}{2} = b.$$

4. Induction

Steps

1. Prove for any element in set (usually 1 or 0 if they are in the set)
2. Suppose true for n
3. Prove for n+1

Notice: This makes it so it holds true for the first tested element and for every subsequent element (like a domino effect)

> Ex.
>
> Gauss' claim:

$$1+2+...+n \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$Let\ S\ be\ set\ n$$
$$Step\ 1:\ Check\ 1 \in S$$
$$1 = \frac{1(1+1)}{2} = 1$$
$$Step\ 2:\ Suppose\ k \in S$$
$$\therefore \sum_{i=1}^{n} i = \frac{k(k+1)}{2}$$
$$Then\ prove\ k+1$$
$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} = 1+2+3+...+k+(k+1)$$
$$= \frac{k(k+1)}{2} + (k+1)$$
$$= (k+1)(\frac{k}{2}+1)$$
$$= \frac{(k+1)((k+1)+1)}{2}$$

> Hence $k+1 \in S$ or $k \in S \to k+1 \in S$
>
> * see Bowmen notes (starting from page 19) for more examples

5. Absolute Values and Binomial Thm

Absolute value is defined to be the following:

$|x| = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases}$

Properties of absolute values:

A1. $|x| \geq 0$

A2. $|x| = 0 \iff x = 0$

A3. $|x| = |-x|$

A4. $|xy| = |x||y|$

A5. $If c \geq 0,\ then$
$|x| \leq c \iff -c \leq x \leq x$

A6. $-|x| \leq x \leq |x|$

A7. $||x| - |y|| \leq |x \pm y| \leq |x| + |y|$ by triangle inequality

Binomials

$n! = 1 \times 2 \times ... \times (n-1) \times n$ if $n\in\N

$0! = 1$

and

$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)...(n-k+1)}{1 \times 2 \times ... \times k}$ * note equal to 1 if k=0

Helpful Thm:

- $\binom{n}{k} = \binom{n}{n-k}$

- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$
- $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n, \ \forall n \in W$ (can be proved via induction as exercise)

Binomial Thm:

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (can also be proved by induction, see Bowmen notes page 26)

$$= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

6. Intervals and bounds

Open and closed intervals are defined as the following

$$\begin{gathered} For\ a, b \in \mathbb{R}\ and\ a < b \\ [a, b] = \{x : a \leq x \leq b\}, \rightarrow closed \\ (a, b) = \{x : a < x < b\}, \rightarrow open \\ [a, b) = \{x : a \leq x < b\}, \rightarrow relatively\ open \\ (a, b] = \{x : a < x \leq b\}, \rightarrow relatively\ open \end{gathered}$$

not finite intervals

$$\begin{gathered} (-\infty, \infty) = \mathbb{R} \\ [a, \infty) = \{x : x \geq a\} \\ (-\infty, a) = \{x : x < a\} \end{gathered}$$

Upper/lower bounds

A real number b is an upper bound S if

$x \leq b\ for\ each\ x \in S$

If no b exists are a upper bound of S, we say it is unbounded above

* Lower bound is similar (wLOG)

Supremum and Infimum (Sup and Inf)

b is the sup of S if it is the least upper bound ($b = \sup\ S$)

If b is the sup of S and is in S, (i.e. $b = \sup\ S\ and\ x \in S$), it is said to be the maximum

* Infimum is similar (wLOG)

Max/min

Now what max and min functions are (they are self explanatory, otherwise, look it up in the python docs)

> 1. max f $= max\{f(x) | x \in A\}$ if right hand side exists
> 2. min f $= min\{f(x) | x \in A\}$ if right hand side exists

7. Real numbers

A real number is only defined if it satisfies all the rules laid out in chapter 3 and it follows the completeness axiom.

Completeness Axiom:

For every non-trivial subset of $\mathbb{R}$ with an upper bound has a lease upper bound in $\mathbb{R}$, (i.e. the $\sup$ exists in $\mathbb{R}$)

> - $\{\frac{p}{q} : p^2 \leq 2p^2, p \in \mathbb{Z}, q \in \mathbb{N}\}$
> - [0, 1] as sup at 1
> - [0,1) has sup at 1

> Lemma - Archimedean Property:
>
> No real number is an upper bound for $\mathbb{N}$
>
> Notes: $\mathbb{N} \subset \mathbb{R}$ (can be proved inductively)

8. Sequences and limits

A sequence of real numbers is a function (see chapter below) s.t. $a : \mathbb{N} \to \mathbb{R}$. Usually a sequence is written as a_n instead of $a(n)$. Sometimes a sequence can start at $k, s.t.\ k \in \mathbb{Z}, instead\ of\ k \in \mathbb{N}.$

A sequence is improper if $\exists\, k \in a_n\ if\ k\ is\ \pm\infty$

* also note an extended real number system is $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

Rules on extended real number system :

1. $c + \infty = \infty$
2. $c - \infty = -\infty$
3. $c \cdot \infty = (-c)(-\infty) = \infty$ if $c > 0$
4. $c \cdot \infty = (-c)(-\infty) = -\infty$ if $c < 0$
5. $\infty \cdot \infty == \infty + \infty = \infty$
6. $\infty \cdot (-\infty) = -\infty$

Limits (This is mega important)

a_n is a sequence. The limit of the sequence (a_n) is L, s.t. $L \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0, \exists\, N \in \mathbb{N}, s.t. |a_n - L| < \epsilon, \forall\, n > N$

Translation to English: The limit of the sequence as the index approaches infinity exists, if there exists an error (ϵ = epsilon that is greater than 0) small enough the element in sequence at index very very large (approaches ∞) minus the defined limit is less than that error. [Think of an error bar of value ϵ and $|a_n - L|$ being within the error bar if limit exists]

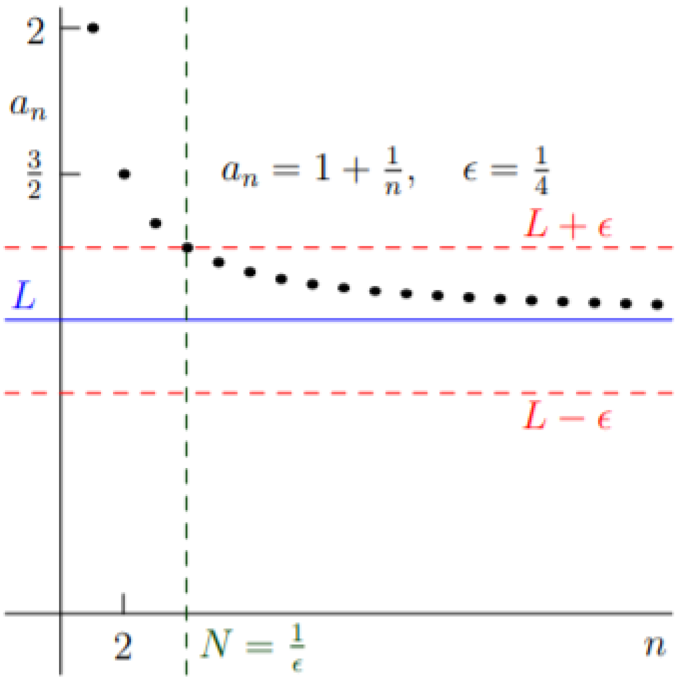

Figure 2.1: Limit of a sequence

If that limit exists, it is convergent, otherwise, it is divergent (improper limit of the sequence)

If $\lim_{n \to \infty} a_n = \infty$, if $\forall$ M>0, there is $N \in \mathbb{N}$ s.t. $a_n > M \, \forall \, n > N$ (wLOG for $-\infty$)

Properties of limits

Let a_n, b_n be sequences with limit L and M

1. If $c, d \in \mathbb{R}, and\ cL, dM$ are both finite of opposing signs, then $ca_n + db_n \to cL + dM$
2. If LM is not undefined, 0 $x \pm \infty, a_b b_n \to LM$
3. If $M \neq 0$, and both M, N are finite, then $\frac{a_n}{b_n} \to \frac{L}{M}$ (only for n large enough)
4. If $L = \pm\infty$, then $\frac{1}{a_n} \to 0$ (only for n large enough)

* look at page 36 of Bowmen notes to see proofs

> Lemma
>
> If $a_n \leq b_n \; \forall \; n \; and \; a_n \to L, b_n \to M. \to \; L \leq M$
>
> Squeeze thm (or sandwich thm if and only if you are hungry)
>
> If $a_n, c_n \to L \; and \; a_n \leq b_n \leq c_n \forall n. \to b_n \to L$
>
> A sequence is bounded if $\exists M \in \mathbb{R}, s.t. |a_n| \leq M \, \forall \, n$
>
> Helpful example:
>
> $|sin(x)| \leq 1$
>
> Why? Homework question

> Thm:
>
> Convergent sequence implies bounded but not the other way
>
> (convergent $\to$ bounded)
>
> (bounded $\nrightarrow$ convergent)

Monotone

A sequence is increasing if for a_n,

$$a_1 \leq a_2 \leq \ \ldots, i.e.\ a_n \leq a_{n+1}\ \forall n \in \mathbb{N}$$

A sequence is strictly increasing if for a_n,

$$a_1 < a_2 < \ \ldots, i.e.\ a_n < a_{n+1}\ \forall n \in \mathbb{N}$$

* likewise for decreasing

A sequence if **monotone** if it is either an increasing or a decreasing sequence (or both)

In monotone sequences, $convergent \iff bounded$

Subsequences

Given a seq. $\lim_{n\to\infty} a_n$ and a strictly increasing sequence of natural numbers $\lim_{n\to\infty} n_k$, we can form subsequence $\lim_{n\to\infty}\ a_{n_k}$ of $\lim_{n\to\infty} a_n$

> Ex.
>
> $\lim_{n\to\infty}(2k-1)^2 = \{1, 9, 25...\}$ is a subsequence of $\lim_{n\to\infty} n^2 = \{1, 4, 9, 16, \ldots\}$
>
> Thm.
>
> convergent $\iff$ all subsequence convergent
>
> Lemma:
>
> a) $0 \leq c < 1 \to c^n \leq c \leq 1$
>
> b) $c > 1 \to c^n \geq c > 1$
>
> $$\lim_{n\to\infty} c^n = \begin{cases} 0 & \text{if } \ 0 \leq c < 1, \\ 1 & \text{if } \ c = 1, \\ \nexists & \text{if } \ c > 1 \text{ (divergent; in fact, unbounded (exercise))} \end{cases}$$
>
> $\forall n \in \mathbb{N}$
>
> c) $0 \leq c < 1 \to c \leq c^{1/n} < 1$
>
> d) $c > 1 > 1 \to c \geq c^{1/n} > 1$
>
> **Ratio test for seq**
>
> $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = r\ where\ r \in [0,1)\ means\ that\ a_n\ is\ bounded\ and \to 0$

Bolzano-Weierstrass Thm:

A bounded sequence has a convergent subsequence (see page 49 for proof)

Cauchy Criterion

A sequence a_n is a Cauchy sequence (or just Cauchy) if $\forall \epsilon > 0\ \exists\ N\ s.t.$
$$m, n > N \to |a_m - a_n| < \epsilon$$

$\therefore$ a_n is convergent $\iff$ a_n is Cauchy

Revisit min/max

Let $f: I \to \mathbb{R}$ be a function and let x_0 be an accumulation point of I. x_0 is allowed to be $\pm\infty$. Let $\dot{N}(x_0) = \{J \setminus \{x_0\} \mid J \subseteq I$ is an interval with accumulation point $x_0\}$.

We put $\limsup_{x\to x_0} f(x) = \inf\{\sup_J f \mid J \in \dot{N}(x_0)\}$.

If x_0 is a boundary point or $\pm\infty$, then this definition coincides with

$$\limsup_{x\to x_0} f = \inf M(x)$$

where $M(x) = \sup_{[x,x_0)} f$ if x_0 is an upper bound (and $M(x) = \sup_{(x_0,x]} f$ if x_0 is a lower bound for I).

Even if $x_0 \in I^\circ$, it is occasionally useful to define

$$\limsup_{x\to x_0^+} f = \limsup_{x\to x_0} f|_{(x_0,\infty)\cap I}$$

and

$$\limsup_{x\to x_0^-} f = \limsup_{x\to x_0} f|_{(-\infty,x_0)\cap I}$$

* This is not clear so watch this: https://www.youtube.com/watch?v=khypO8MQpdc

* Note: x_o is always a boundary point of $(x_o, \infty) \cap I \; and \; (-\infty, x_o) \cap I \; if \; x_o \in I^o$

9. Functions

A function $f : X \to Y$ is a rule that assigns each $x \in X$ to element $F(x) \in Y$.

The set of all X is called the domain of f, while the Y is called the codomain

The range or image of f is the set $f(X) = \{f(x)|x \in X\}$

* think of a rule as a machine that takes X and through a defined process turns that X or all X to Y

Note:

$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by f(x)= x^2 is different from$g : \mathbb{R} \to \mathbb{R}$ also defined by $x \mapsto x^2$ (Note $\mapsto$ is maps to and is used more often in linear algebra)

Trigonometry

> opp = opposite
>
> hyp = hypotonus
>
> adj = adjacent

$$sinx = \frac{opp}{hyp}$$
$$cosx = \frac{adj}{hyp}$$
$$tanx = \frac{opp}{adj} = \frac{sinx}{cosx}$$
$$cscx = \frac{1}{sinx}$$
$$\sec x = \frac{1}{cosx}$$
$$cotx = \frac{1}{tanx}$$
$$Pythagoras'\ Theorem$$
$$opp^2 + adj^2 = hyp^2$$
$$\rightarrow sin^2x + cos^2x = 1$$
$$1 + cot^2x = csc^2x$$
$$tan^2x + 1 = sec^2x$$
$$Also,\ |sinx| \leq 1\ \&\ |cosx| \leq 1$$

This can be demonstrated with a unit circle

* Note, degrees will not be used, use radians

π radians = 180^o

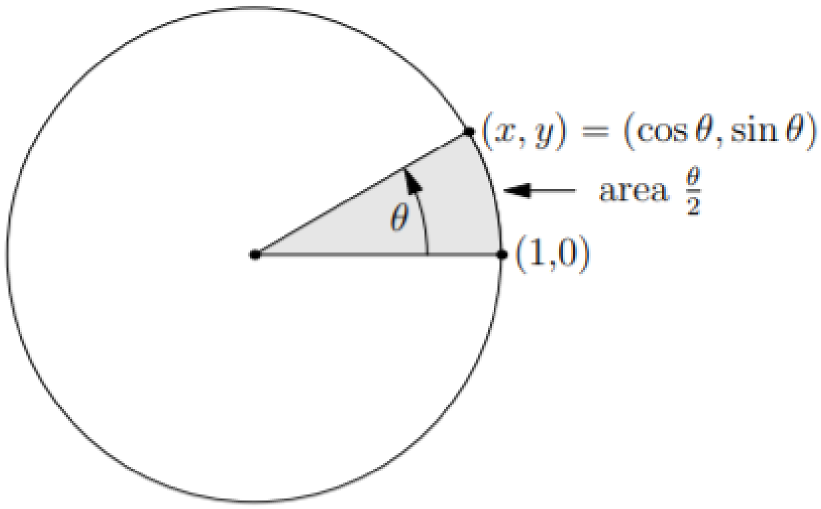

Figure 3.1: The unit circle

$cos\theta = sin(\frac{\pi}{2} - \theta),$

$sin\theta = cos(\frac{\pi}{2} - \theta),$

$$Supplementary\ Angle\ Identities:$$
$$sin(\pi - x) = sinx$$
$$cos(\pi - x) = -cosx$$
$$Symmetries:$$
$$sin(-x) = -sinx$$
$$cos(-x) = cosx$$
$$sin(x + 2\pi) = sinx$$
$$cos(x + 2\pi) = cosx$$

Special values:

$$sin(\pi/2) = cos0 = 1$$
$$sin(\pi/4) = cos(\pi/4) = 1/\sqrt{2}$$
$$sinx(\pi/6) = cos(\pi/3) = 1/2$$
$$sin(\pi/3) = cos(\pi/6) = \sqrt{3}/2$$
$$cos(A - B) = cosAcosB + sinAsinB$$

$$DoubleAngleFormulas:$$

$$sin2A = 2sinAcosA$$
$$cos2A = 2cos^2A - 1 = 1 - 2sin^2A$$
$$tan2A = \frac{2tanA}{1 - tan^2A}$$
$$sinx \leq x \leq tanx \; \forall x \in [0, \pi/2)$$
$$Also:$$
$$|sinx| \leq |x| \; \forall x \in \mathbb{R}$$

Let I be interval s.t. x_o is an accumulation point of f (see misc. notes below) (i.e. $x_o \in I \; or \; x_o \in \delta I$)

For any function with domain I, we say the real number ∞ is the limit of f as x approaches x_o

$\lim_{x \to x_o} f(x) = \infty$ if for every M>0, there is $\delta > 0$ s.t. $\forall x \neq x_o \in I \; with \; |x - x_o| < \delta, f(x) > M$, likewise for $-\infty$

Thm. Equivalence of Function and Sequence Limits:

$\lim_{x \to a} f(x) = L \iff f$ is defined near a and every sequence point in x_n in the domain of f with $x_n \neq a$, but $\lim_{n \to \infty} x_n = a$, satisfies $\lim_{n \to \infty} f(x_n) = L$

* See page 68 for proof

Corollary:

Assume $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

1. $\lim_{x \to a}(f(x) + g(x)) = L + M$
2. $\lim_{x \to a} f(x)g(x) = LM$
3. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

Cauchy Criterion for Function

$\lim_{x \to a} f(x)$ exists $\iff$ for every $\epsilon > 0, \exists \delta > 0$ s.t. x, y $\in$ $(a - \delta, a) \cup (a, a + \delta)$, their function values satisfy $|f(x) - f(y)| < \epsilon$

10. Continuity:

Simply, a function is continuous if when you draw it out, your pen/pencil does not leave the paper (sorry I have not defined what pen/pencil and paper is but that is for another course (possibly Bio 399))

Proper def:

Let D $\to$ R. A point c is an interior point of D if it belongs to some open interval (a, b) entirely contained in D: c $\in$ (a, b) $\subset$ D

> i.e. 1/10, 1/2, 3/4 are interior points of [0, 1], but 0 and 1 are not
>
> However, all points in (0, 1) are interior points of (0, 1)

A point is continuous at interior point a in domain if

$$\lim_{x \to a} f(x) = f(a)$$

> f is continuous at a $\iff$ for every $\epsilon > 0, \exists \delta > 0 \ s.t.$
> $$|x - a| < \delta \to |f(x) - f(a)| < \epsilon$$
>
> Corollary:
>
> a. If f and g are continuous at a, then f+g and fg are continuous at a and f/g continuous at a if $g(a) \neq 0$
>
> b. A rational function is continuous at all points of its domain
>
> c. If g is continuous at a and f is continuous at g(a). Then f ∘ g (i.e. f(g)) is continuous at a.

11. Inverse functions

Take $f : X \to Y$, then f is **invertible** if there is a $g : Y \to X$ s.t. f(g(y))=y $\forall \ y \in Y$ and g(f(x))=x $\forall \ x \in X$. The function g is unique if it exists.

The inverse of f is invertible and it is written as f^{-1}, but this is a bit ambiguous in some cases, so be careful.

> Take $f : X \to Y$
>
> 1. f is **injective** (or 1-1 or one-one) if $x = x_o$ then $f(x) = f(x_o)$
> 2. f is called **surjective** (onto) if f(X)=y (i.e. $y \in Y$ there is $x \in X \ s.t. \ f(x) = y$
> 3. f is bijective (one-to-one) if both injective and surjective
>
> Notes:
>
> - f is injective, if for every $y \in Y$ the equation $f(x) = y$ has at most one solution (but may have none)
>   - ie $\forall x, x_o : f(x_o) = f(x) \to x = x_o$
> - f is surjective, if for every $y \in Y$ the equation $f(x) = y$ has at least one solution (which may not necessarily be unique)
>   - ie $\forall y : \exists x : f(x) = y$
> - f is bijective, if for every $y \in Y$ the equation $f(x) = y$ has a unique solution
>   - ie $\forall y : \exists! x : f(x) = y$

Therefore, f is invertible $\iff$ it is **bijective**

Let f be continuous. If f is injective, f is strictly monotone on I; J = f(I) is an interval of the same type and $f^{-1} : J \to I$ is continuous

12. 1-sided limits:

$\lim_{x \to a^+} f(x) = L$ if for each $\epsilon > 0, \exists \ \delta > 0, s.t.$

$$0 < x - a < \delta \to |f(x) - L| < \epsilon$$

Likewise for $x \to a^-$

If the limit exists above, then it is said to be continuous from the right

13. Intermediate Value Theorem (IVT)

If a is continuous on [a,b] and f(a)<0<f(b)

then $\exists$ a number $c \in (a, b)\ s.t.\ f(c) = 0$

14. Differentiation

Differentiation is a measure of the rate of change (or slope, but that's a bad word in AP cal) of a function

The idea is given by a rate of change formula:

$$v(t) = \frac{\Delta x}{\Delta t}$$

The exact velocity is when $\Delta t = t - t_o$ is taken when the difference is smaller and smaller (ie $\lim_{t \to t_o}$) and the same for Δx

$$\therefore\ v(t_o) = \lim_{t \to t_o} \frac{x(t) - x(t_o)}{t - t_o}$$

> Proper def:
>
> Let I be interval s.t. f is defined on that interval (that means is it must be continuous near x_o). f is differentiable at $x_o \in I$ if
>
> $\lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} = f'(x) = f'$
>
> OR
>
> f'(x)= $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is finite and exists
>
> OR
>
> $\exists$ c=f'(x) s.t. f(x+h)=f(x)+ch+r(h) and $\lim_{h \to 0} \frac{r(h)}{h} = 0$
>
> * notion can also be like the following:
>
> - $\frac{df}{dx}(x_o)$
> - $\frac{d}{dx}|_{x_o}$

> Useful:
>
> 1. Any constant function is differentiable everywhere it is defined, and its derivative is 0
> 2. Any linear function is differentiable everywhere: f(x)=mx+b, then f'(x_o) = m
> 3. f(x) = $x^n \to$ f'(x)= $n(x)^{n-1}$
> 4. if f=|x|, then f' is defined everywhere except x=0
> 5. In the previous function, if the domain is bounded by either $x \leq 0\ or\ x \geq 0$, then it is defined
>
> Rule on diff.
>
> - f+g diff on x $\to (f+g)'(x) = f'(x) + g('x')$
> - fg diff at x $\to (fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (Product or Leibniz Rule)
> - g(x)$\neq 0 \forall x \in I, \to \frac{f}{g}$ diff. and $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ (Quotient Rule)
> - f(g(x)) = h(x) $\to h' = f'(g)g'$ (h must be defined at point and exist at point and same for g and f. Also f and g must be differentiable)
>
> To see proof, goto page 85

* Note, any polynomial function is diff. everywhere

For any natural number n we say f is n-times differentiable at x if:

1. $f^{(n-1)}$ is defined on a relative open interval containing x.
2. $f^{(n-1)}$ is differentiable at x

A function is called smooth if it is n times differentiable everywhere in its domain for every n

Therefore, know that all polynomial function (and trig. functions) are smooth everywhere

Any rational function is also diff. everywhere in its domain

Important Oversights

If f^2 is diff at x, f might not be diff. at x (see f=|x|)

> Trig diffs:
>
> 1. f(x0=sinx $\to f'(0) = \lim_{x\to 0} \frac{sinx}{x} = 1$ (see l'Hopital)
> 2. f(x)=sinx $\to$ f' = cosx
> 3. f(x)=cosx $\to$ f'=-sinx
> 4. f'(tanx) $= (secx)^2$
> 5. f'(secx)=secx tanx
> 6. f'(csc x) = -csc x cot x
> 7. f'(cotx)=$-(cscx)^2$
>
> Inverses
>
> $$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}, x \neq \pm 1$$
>
> $$\frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}, x \neq \pm 1$$
>
> $$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$
>
> $$\frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+x^2}$$
>
> $$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}}, x \neq \pm 1, 0$$
>
> $$\frac{d}{dx}\left(\csc^{-1}x\right) = \frac{-1}{|x|\sqrt{x^2-1}}, x \neq \pm 1, 0$$

15. Extrema

Let f be diff on $x \in I \to$ f has a

- local max at x if for $some\ \delta > 0, s.t. \forall x_o \in I \cap (x_o - \delta, x_o + \delta) : f(x_o) \leq f(x)$
- local min at x if for $some\ \delta > 0, s.t. \forall x_o \in I \cap (x_o - \delta, x_o + \delta) : f(x_o) \geq f(x)$

A local extremum exists at x if it is a local max or min at x

Global extremum is f(x)=sup{f(x)} or inf{f(x)}, therefore, global extremum may not be the same as local extremum

If f'=0, then a local extremum exists or is said to be a critical point

16. Rolle's theorem

if f is continuous on [a, b] and diff on at least (a,b), s.t. f(a)=f(b), then, $\exists c \in (a, b), s.t.\ f'(c) = 0$

This means that if the points a, b are equal on f, then in-between a and b, there is a least one point such that the derivate is 0

17. MVT

if f is continuous on [a, b] and diff on at least (a,b), then,
$\exists c \in (a, b), s.t.\ f(b) - f(a) = f'(c)(b - a)$

To see prove, go to page 11 on chpt 2 on class notes or pg 94 on Bowmen

> Corollary:
>
> zero derivative means constant
>
> Let f be continuous on I and diff on at least the interior of I
>
> - f is monotone increasing $\iff$ f' ≥ 0 on I
> - f is monotone decreasing $\iff$ f' ≤ 0 on I

18. First and second derivative test

First Derivative Test

Let I be an interval and f continuous on I and a $c \in I$, a critical point

- If there is a relative open subinterval ($J \subset I\ and\ f' \leq 0\ on\ J \cap (-\infty, c)\ and f' \leq 0\ on\ J \cap (c, \infty), then\ f$ has a local min at c
- Likewise for local max, but flip inequalities

Second Derivative Test

Let I be an interval and f continuous on I and twice diff at x

- if f'' < 0 has local max at x
- if f'' > 0 has local min at x
- if f'' = 0 unknown

19. L'Hôpital's Rule

L'Hôpital's rule is useful for computing division where the top and bottom might be $\pm\infty$ or 0

Conditions for applying l'Hôpital's rule:

- If numerator and dementor both approach 0
- If numerator and dementor both approach $\pm\infty$

> Thm:
>
> If it passes the previous checks, you can
>
> $\lim_{x \to a} \frac{f}{g} = L = \lim_{x \to a} \frac{f'}{g'}$

* See page 100 for proof

20. Convex and Concave

A inflection point occurs when f'' = 0

- f is convex $\iff$ f' is increasing on I (or f'' > 0)
- f is concave $\iff$ f' is decreasing on I (or f'' < 0)

Notes:

Suppose f'(c) = 0 at some $c \in I$

- f'' $\geq 0 \forall x \in I$ then f has a global min at c
- f'' $\leq 0 \forall x \in I$ then f has a global max at c

Suppose f is continuous on I. Then f is one-to-one on I $\iff$ f is strictly monotonic on I

More Inverses

f^{-1}'(y)$= \frac{1}{f'(f^{-1}(y))}$

The Horse Race Thm

Let I = [a, b] and f, g continuous on I, diff on (a, b)

If a. f(a)$\geq$ f(b) and b. $f' \geq g'$ on (a,b), then c. f(b)$\geq$ g(a)

if f' > g' on (a,b) then f(b)>g(b)

21. Exponentials and logs

> Properties of logs
>
> 1. log(xy) = log x + log y
> 2. log(x/y) = log x - log y
> 3. log(x^r) = r log x

22. Logarithmic differentiation

> f'(e^x) = e^x
>
> f'(lnx) = f'(log x) = $\frac{1}{x}$
>
> f'(x^x) = $x^x(ln(x)+1)$

[↑ Back to top]

Misc. Notes:

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Useful def:

Let $S \subseteq \mathbb{R}$ be a subset.

1. A point $x \in \mathbb{R}$ is an **accumulation point** of S, if there is a sequence $a_n \in S$ such that $a_n \neq x \forall n$ and $a_n \to x$ as $n \to \infty$. If a_n is a sequence, an accumulation point of the sequence is an accumulation point of $\{a_n \mid n \in \mathbb{N}\}$.
2. A point $x \in \mathbb{R}$ is a **boundary point** of S, if for all $\varepsilon > 0$, the sets $(x-\varepsilon, x+\varepsilon) \cap S$ and $(x-\varepsilon, x+\varepsilon) \cap (\mathbb{R} \setminus S)$ are nonempty.
3. An **interior** o **inner** point of S is an element $x \in S$ such that there is $\varepsilon > 0$ for which $(x-\varepsilon, x+\varepsilon) \subseteq S$.
4. The set of all interior points is denoted $\mathring{S}$, and called the **interior** of S; set of all boundary points is denoted ∂S and called the **boundary** of S.
5. S is called **closed** if it contains all its accumulation points.
6. S is called **open** if it is equal to its interior $\mathring{S}$.
7. A subset $T \subseteq S$ is called **relative open**, if it is the intersection of an open set with S.
8. A subset $A \subseteq \mathbb{R}$ is called **discrete**, if for every $a \in A$ there is $\varepsilon > 0$ (maybe depending on a) such that $(a-\varepsilon, a+\varepsilon) \cap A = \{a\}$.

To clarify the "relative open" business: if c is an interior point of I, all this says is that there is $\delta > 0$ such that $J = (c-\delta, c+\delta) \subseteq I$ and, in the first scenario, $f' \leq 0$ on $(c-\delta, c)$ and $f' \geq 0$ on $(c, c+\delta)$.

In the second scenario, $f' \geq 0$ on $(c-\delta, c)$ and $f' \leq 0$ on $(c, c+\delta)$.

$\mathbb{N} \setminus \{3\} = \{1, 2, 4, 5, \dots\}$

When in doubt, write ""***clearly***..." even in multiple choice, and especially for T or F questions.

hand wavy stuff

> - $\frac{x}{\infty} \to 0$ if $x \neq \pm\infty$
> - $\frac{x}{0} \to \pm\infty$ if $x \neq 0$ and depending on x > 0 or x < 0
> - $\frac{\pm\infty}{x} \to \pm\infty$ if $x \neq \pm\infty$

Resources

- school notes
- [Bowmen notes]()

Note about this course and my experience

My school had an extraordinary honors calculus year. The main prof left by second semester and a German prof can in to sub, but, because of his Germanicness (he literally goes off of a German textbook), it was difficult to follow his sometimes rigorous and sometimes not rigorous (he called them trivial) proofs.

Also, at my university, compute 272 is trivially similar to 117.

Licence

Thanks

Dedicated to Dr. Terry Gannon who definitely did not abandon the class

Contributions

Contribute if and only if you know what you are doing (and optionally, it would help if you where Germanic or fluent in Germanics)

* If someone wants to convert this to Latex, go ahead and pull request (must have a index or contents page with links). I tried with pandoc, but it was not as complete as I thought so if anyone wants to do clean up, go ahead, also I can just export as pdf