Honors Math

About

This is a summery of **Honors Calculus** (117 + 118) (or as some people call it: "**Abstract Calculus**")
This course will focus on set theory and number theory (and logic)

* note this won't display properly on GitHub, I use a program called [Typora] with inline math enabled

Index

Notes

0. Shorthand

Ex. $\rightarrow$ example

Thm $\rightarrow$ Theorem (will be in block quotes)

s.t. $\rightarrow$ such that

, or ; $\rightarrow$ such that, so (general connector depending on context) wLOG $\rightarrow$ without loss of generality (i.e. similar to previous case, but I'm too lazy to do it)

Formal Logic

$\in\rightarrow$ element of

$\subset$ or $\subseteq$ $\rightarrow$ subset of

$\cup\rightarrow$ union

$\cap\rightarrow$ intercept

$\wedge\rightarrow$ and

$\vee\rightarrow$ or

$\rightarrow$ $\rightarrow$ then

$\iff\rightarrow$ if and only if (this is the same as '=')

$=:\rightarrow$ is defined to be

$\forall\rightarrow$ for all

$\exists\rightarrow$ there exists

$\exists!\rightarrow$ there exists unique

$Q.E.D.\rightarrow$ and it is proved

$\therefore$ $\rightarrow$ therefore

$\because$ $\rightarrow$ because

Fields

$\mathbb{N}\rightarrow$ natural numbers (i.e. 1,2,3,4...) [* note 0 might be included in natural numbers depending on what text book is in use]

$W\rightarrow$ whole numbers ($\{0\}\cup\mathbb{N}$)

$\mathbb{Z}\rightarrow$ integers (i.e. ..-2,-1,0,1,2...)

$\mathbb{Q}\rightarrow$ rational numbers (anything that can be expressed as $\frac{p}{q}, where\ p\in\mathbb{Z}\ and\ q\in\mathbb{N}$)

$\mathbb{R}\rightarrow$ will be discussed later

1. Sets

a set is defined as a number (can be 0 but then that would be the trivial or stupid set) of elements belonging to the same group

> Ex.
>
> - {1,2,3} = {2,1,3}
> - {a,b}
> - {even, odd}

> - $\mathbb{N}$ = {1,2,3,4,5...}
> - {} (trivial set)

If an element 'x' is part of a set 'A', it is described as $x \in A$

If $A \cup B = A,\ B \subset A$. If for all elements of B are in A, then B is considered a subset of A

If elements of A are present in B, then those elements 'C' are said to be $A \cap B$ or $A \cap B = C$

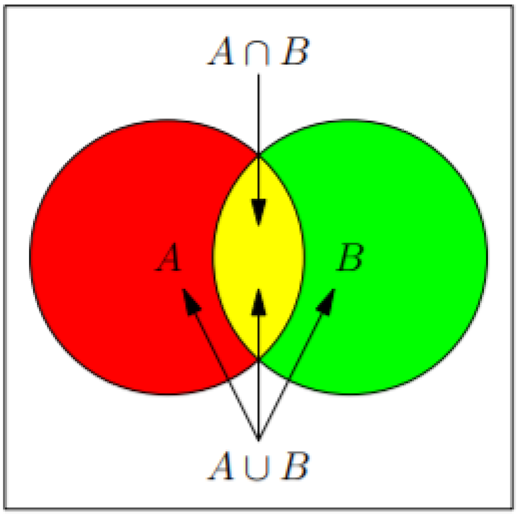

Figure 1.1: Venn Diagram

From Bowmen notes

* Most of the time, Russell's paradox will be introduced to scare any students still in the class.

Optional Exercise: look into Russell's paradox because it is quite interesting

2. Logic

Truth Values

There are 2 truth values, true (T or topology) and false (F or contradiction)

In math, there are established rules (which were set arbitrarily but is important for consistence around the world) for operations on truth values

Most of the time, a truth table will be used to illustrate how operators effect the truth value of a statement

Some common ones are:

$A \wedge B$; (A and B)

		A	**A**
		T	F
B	T	T	F
B	F	F	F

$A \vee B$; (A or B)

		A	**A**
		T	F
B	T	T	T
B	F	T	F

$A \rightarrow B$; (If A then B)

		A	**A**
		T	F
B	T	T	T
B	F	F	T

$A \iff B$; (A if and only if B or A = B)

* note this is a different way for displaying the truth table (important for chain multiple operators with more than 2 variables)

A	**B**	$A \iff B$	$A \rightarrow B \, and \, B \rightarrow A$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

To chain multiple operators together, you can use an extended truth table that considers all the variables (like the one above, but you must consider all variables and all the cases possible use a)

As seen from the table above, an iff is logically equivalent to a if b and b if a, therefore for proving an iff, the easiest way is to prove it using the A premise to get to B and then using the B premise to derive A

Contraction

The easiest way to prove something is to use *proof by contradiction*. Assume the opposite of the theorem or a fact that is true and find a contraction in reasoning

> Ex.
>
> $\sqrt{2} \, is \, not \, rational$
>
> Start of proof by contraction:
>
> Suppose for contradiction, $\sqrt{2} \, is \, rational$, i.e. it can be expressed as $\frac{p}{q}, where \, p \in \mathbb{Z} \, and \, q \in \mathbb{N}.$

> $$\sqrt{2} = \frac{p}{q}$$
> $$\rightarrow \sqrt{2}^2 = \frac{p^2}{q^2}$$
> $$2q^2 = p^2$$
>
> * note even number are expressed as 2n and odd number can be expressed as 2n+1 for $n \in \mathbb{Z}$
>
> Therefore p must be even because only an even squared is even $((2n)^2 = 4n^2 \ and \ (2n+1)^2 = 4n^2 + 2n + 1 \therefore odd)$

3. What is a number? (Algebraic Properties)

For real numbers, the following must be true

* assume $a, b, c \in \mathbb{R}$

1. Must be associative

> $a + (b + c) = (a + b) + c$

2. Must have an additive identity

> $a + 0 = 0 + a = a$

3. Must have an *additive inverse* -a s.t.

> $a + -a = -a + a = 0$

4. Follows additive commutativity

> $a + b = b + a$

5. Are associate

> $a(bc) = (ab)c$

6. There exists are multiplicative identity where it is not 0

> $a \times 1 = a$

7. It can distribute

> $a \times (b + c) = a \times b + a \times c$

8. Has an inverse that is not 0, i.e. $a^{-1} \neq 0$

> $a \times a^{-1} = 1$

9. Follows multiplicative commutativity

> $a \times b = b \times a$

10. Trichotomy Law

> must be one and only one of the following relations:
>
> $a < b, \ a = b, \ a > b$

11. Closed under addition

> $a > 0 \ and \ b > 0 \rightarrow a + b > 0$

12. Closed under multiplication

> $a > 0 \; and \; b > 0 \rightarrow a \times b > 0$

More rules to come

Also note the first lemma (lemma is like a theorem but arbitrary defined to be a small theorem)

* lemmas or thm can be assumed

Lemma 1.1 (Midpoint Lemma):

$$a < b \quad \Rightarrow \quad a < \frac{a+b}{2} < b.$$

Proof:

$$a < b \quad \Rightarrow \quad a + a \quad < \quad a + b < b + b$$

$$\Rightarrow \quad a = \frac{a+a}{2} \quad < \quad \frac{a+b}{2} < \frac{b+b}{2} = b.$$

4. Induction

Steps

1. Prove for any element in set (usually 1 or 0 if they are in the set)
2. Suppose true for n
3. Prove for n+1

Notice: This makes it so it holds true for the first tested element and for every subsequent element (like a domino effect)

> Ex.
>
> Gauss' claim:
>
> $$1 + 2 + \ldots + n \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
> $$Let \; S \; be \; set \; n$$
> $$Step \; 1: \; Check \; 1 \in S$$
> $$1 = \frac{1(1+1)}{2} = 1$$
> $$Step \; 2: \; Suppose \; k \in S$$
> $$\therefore \sum_{i=1}^{n} i = \frac{k(k+1)}{2}$$
> $$Then \; prove \; k+1$$
> $$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} = 1 + 2 + 3 + \ldots + k + (k+1)$$
> $$= \frac{k(k+1)}{2} + (k+1)$$
> $$= (k+1)(\frac{k}{2} + 1)$$
> $$= \frac{(k+1)((k+1)+1)}{2}$$
>
> Hence $k+1 \in S$ or $k \in S \rightarrow k+1 \in S$
>
> * see Bowmen notes (starting from page 19) for more examples

5. Absolute Values and Binomial Thm

Absolute value is defined to be the following:

$$|x| = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases}$$

Properties of absolute values:

A1. $|x| \geq 0$

A2. $|x| = 0 \iff x = 0$

A3. $|x| = |-x|$

A4. $|xy| = |x||y|$

A5. $If c \geq 0,\ then$
$|x| \leq c \iff -c \leq x \leq x$

A6. $-|x| \leq x \leq |x|$

A7. $||x| - |y|| \leq |x \pm y| \leq |x| + |y|$ by triangle inequality

Binomials

$n! = 1 \times 2 \times \ldots \times (n-1) \times n$ if $n\in\N

$0! = 1$

and

$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{1\times 2\times \ldots \times k}$ * note equal to 1 if k=0

Helpful Thm:

- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$
- $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n,\ \forall n \in W$ (can be proved via induction as exercise)

Binomial Thm:

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (can also be proved by induction, see Bowmen notes page 26)

$$= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

6. Intervals and bounds

Open and closed intervals are defined as the following

$$\begin{gathered} For\ a, b \in \mathbb{R}\ and\ a < b \\ [a,b] = \{x : a \leq x \leq b\}, \to closed \\ (a,b) = \{x : a < x < b\}, \to open \\ [a,b) = \{x : a \leq x < b\}, \to relatively\ open \\ (a,b] = \{x : a < x \leq b\}, \to relatively\ open \end{gathered}$$

not finite intervals

$$(-\infty, \infty) = \mathbb{R}$$
$$[a, \infty) = \{x : x \geq a\}$$
$$(-\infty, a) = \{x : x < a\}$$

Upper/lower bounds

A real number b is an upper bound S if

$x \leq b \ for \ each \ x \in S$

If no b exists are a upper bound of S, we say it is unbounded above

* Lower bound is similar (wLOG)

Supremum and Infimum (Sup and Inf)

b is the sup of S if it is the least upper bound ($b = \sup \ S$)

If b is the sup of S and is in S, (i.e. $b = \sup \ S \ and \ x \in S$), it is said to be the maximum

* Infimum is similar (wLOG)

7. Real numbers

A real number is only defined if it satisfies all the rules laid out in chapter 3 and it follows the completeness axiom.

Completeness Axiom:

For every non-trivial subset of $\mathbb{R}$ with an upper bound has a lease upper bound in $\mathbb{R}$, (i.e. the $\sup$ exists in $\mathbb{R}$)

> - $\{\frac{p}{q} : p^2 \leq 2p^2, p \in \mathbb{Z}, q \in \mathbb{N}\}$
> - [0, 1] as sup at 1
> - [0,1) has sup at 1

> Lemma - Archimedean Property:
>
> No real number is an upper bound for $\mathbb{N}$
>
> Notes: $\mathbb{N} \subset \mathbb{R}$ (can be proved inductively)

8. Sequences and limits

↑ Back to top

Resources

- school notes
- Bowmen notes

Note about this course and my experience

My school had an extraordinary honors calculus year. The main prof left by second semester and a German prof can in to sub, but, because of his Germanicness (he literally goes off of a German textbook), it was difficult to follow his sometimes rigorous and sometimes not rigorous (he called them trivial) proofs.

Also, at my university, compute 272 is trivially similar to 117.

Licence

The rules for copy and distributing this project licence are
outlined in the licence.txt file.

This project is under an MIT licence

Thanks

Dedicated to Dr. Terry Gannon who definitely did not abandon the class

Contributions

Contribute if and only if you know what you are doing (and optionally, it would help if you where Germanic or fluent in Germanics)

* If someone wants to convert this to Latex, go ahead and pull request (must have a index or contents page with links). I tried with pandoc, but it was not as complete as I thought so if anyone wants to do clean up, go ahead, also I can just export as pdf