

# Honors Math

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## About

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This is a summary of **Honors Calculus** (117 + 118) (or as some people call it: "**Abstract Calculus**")

This course will focus on set theory and number theory (and logic)

\* note this won't display properly on GitHub, I use a program called [Typora](#) with inline math enabled

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## Notes

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### 0. Shorthand

Ex.  $\rightarrow$  example

Thm  $\rightarrow$  Theorem (will be in block quotes)

s.t.  $\rightarrow$  such that

, or ;  $\rightarrow$  such that, so (general connector depending on context)

wLOG  $\rightarrow$  without loss of generality (i.e. similar to previous case, but I'm too lazy to do it)

seq.  $\rightarrow$  sequence

## Formal Logic

$\in \rightarrow$  element of

$\subset$  or  $\subseteq \rightarrow$  subset of

$\cup \rightarrow$  union

$\cap \rightarrow$  intersection

$\wedge \rightarrow$  and

$\vee \rightarrow$  or

$\rightarrow \rightarrow$  then

$\iff \rightarrow$  if and only if (this is the same as '=')

$\Rightarrow \rightarrow$  is defined to be

$\forall \rightarrow$  for all

$\exists \rightarrow$  there exists

$\exists! \rightarrow$  there exists unique

$Q. E. D. \rightarrow$  and it is proved

$\therefore \rightarrow$  therefore

$\because \rightarrow$  because

## Fields

$\mathbb{N} \rightarrow$  natural numbers (i.e. 1,2,3,4...) [\* note 0 might be included in natural numbers depending on what text book is in use]

$\mathbb{W} \rightarrow$  whole numbers ( $\{0\} \cup \mathbb{N}$ )

$\mathbb{Z} \rightarrow$  integers (i.e. ...-2,-1,0,1,2...)

$\mathbb{Q} \rightarrow$  rational numbers (anything that can be expressed as  $\frac{p}{q}$ , where  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$ )

$\mathbb{R} \rightarrow$  will be discussed later

## 1. Sets

a set is defined as a number (can be 0 but then that would be the trivial or stupid set) of elements belonging to the same group

Ex.

- $\{1,2,3\} = \{2,1,3\}$
- $\{a,b\}$

- {even, odd}
- $\mathbb{N} = \{1,2,3,4,5...\}$
- $\{\}$  (trivial set)

If an element 'x' is part of a set 'A', it is described as  $x \in A$

If  $A \cup B = A$ ,  $B \subset A$ . If for all elements of B are in A, then B is considered a subset of A

If elements of A are present in B, then those elements 'C' are said to be  $A \cap B$  or  $A \cap B = C$

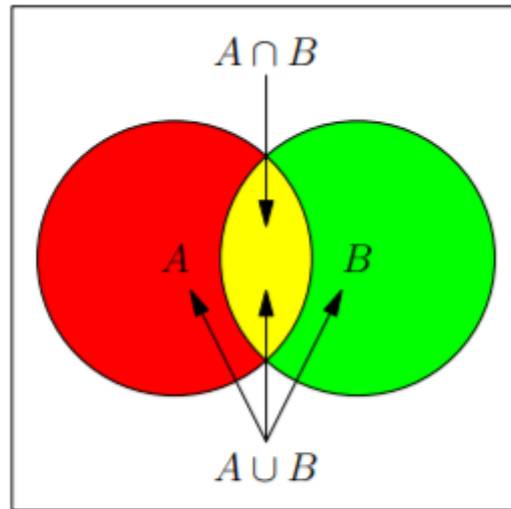


Figure 1.1: Venn Diagram

From Bowmen notes

\* Most of the time, Russell's paradox will be introduced to scare any students still in the class.

Optional Exercise: look into Russell's paradox because it is quite interesting

## 2. Logic

### Truth Values

There are 2 truth values, true (T or topology) and false (F or contradiction)

In math, there are established rules (which were set arbitrarily but is important for consistence around the world) for operations on truth values

Most of the time, a truth table will be used to illustrate how operators effect the truth value of a statement

Some common ones are:

$A \wedge B$ ; (A and B)

		A	A
		T	F
B	T	T	F
B	F	F	F

$A \vee B$ ; (A or B)

		<b>A</b>	<b>A</b>
		T	F
B	T	T	T
B	F	T	F

$A \rightarrow B$  ; (If A then B)

		<b>A</b>	<b>A</b>
		T	F
B	T	T	T
B	F	F	T

$A \iff B$  ; (A if and only if B or A = B)

\* note this is a different way for displaying the truth table (important for chain multiple operators with more than 2 variables)

<b>A</b>	<b>B</b>	$A \iff B$	$A \rightarrow B \text{ and } B \rightarrow A$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

To chain multiple operators together, you can use an extended truth table that considers all the variables (like the one above, but you must consider all variables and all the cases possible use a )

As seen from the table above, an iff is logically equivalent to a if b and b if a, therefore for proving an iff, the easiest way is to prove it using the A premise to get to B and then using the B premise to derive A

## Contraction

The easiest way to prove something is to use *proof by contradiction*. Assume the opposite of the theorem or a fact that is true and find a contraction in reasoning

Ex.

$\sqrt{2}$  is not rational

Start of proof by contraction:

Suppose for contradiction,  $\sqrt{2}$  is rational, i.e. it can be expressed as  $\frac{p}{q}$ , where  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$ .

$$\begin{aligned}\sqrt{2} &= \frac{p}{q} \\ \rightarrow \sqrt{2}^2 &= \frac{p^2}{q^2} \\ 2q^2 &= p^2\end{aligned}$$

\* note even number are expressed as  $2n$  and odd number can be expressed as  $2n+1$  for  $n \in \mathbb{Z}$

Therefore  $p$  must be even because only an even squared is even  
 $((2n)^2 = 4n^2 \text{ and } (2n+1)^2 = 4n^2 + 2n + 1 \therefore \text{odd})$

### 3. What is a number? (Algebraic Properties)

For real numbers, the following must be true

\* assume  $a, b, c \in \mathbb{R}$

1. Must be associative

$$a + (b + c) = (a + b) + c$$

2. Must have an additive identity

$$a + 0 = 0 + a = a$$

3. Must have an *additive inverse*  $-a$  s.t.

$$a + -a = -a + a = 0$$

4. Follows additive commutativity

$$a + b = b + a$$

5. Are associate

$$a(bc) = (ab)c$$

6. There exists a multiplicative identity where it is not 0

$$a \times 1 = a$$

7. It can distribute

$$a \times (b + c) = a \times b + a \times c$$

8. Has an inverse that is not 0, i.e.  $a^{-1} \neq 0$

$$a \times a^{-1} = 1$$

9. Follows multiplicative commutativity

$$a \times b = b \times a$$

10. Trichotomy Law

must be one and only one of the following relations:

$$a < b, a = b, a > b$$

11. Closed under addition

$$a > 0 \text{ and } b > 0 \rightarrow a + b > 0$$

12. Closed under multiplication

$$a > 0 \text{ and } b > 0 \rightarrow a \times b > 0$$

More rules to come

Also note the first lemma (lemma is like a theorem but arbitrary defined to be a small theorem)

\* lemmas or thm can be assumed

**Lemma 1.1** (Midpoint Lemma):

$$a < b \Rightarrow a < \frac{a+b}{2} < b.$$

Proof:

$$\begin{aligned} a < b &\Rightarrow a + a < a + b < b + b \\ \Rightarrow a = \frac{a+a}{2} &< \frac{a+b}{2} < \frac{b+b}{2} = b. \end{aligned}$$

## 4. Induction

Steps

1. Prove for any element in set (usually 1 or 0 if they are in the set)
2. Suppose true for n
3. Prove for n+1

**Notice:** This makes it so it holds true for the first tested element and for every subsequent element (like a domino effect)

Ex.

Gauss' claim:

$$1 + 2 + \dots + n \equiv \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Let  $S$  be set  $n$

Step 1: Check  $1 \in S$

$$1 = \frac{1(1+1)}{2} = 1$$

Step 2: Suppose  $k \in S$

$$\therefore \sum_{i=1}^n i = \frac{k(k+1)}{2}$$

Then prove  $k+1$

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \frac{k(k+1)}{2} = 1 + 2 + 3 + \dots + k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

Hence  $k+1 \in S$  or  $k \in S \rightarrow k+1 \in S$

\* see Bowman notes (starting from page 19) for more examples

## 5. Absolute Values and Binomial Thm

Absolute value is defined to be the following:

$$|x| = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases}$$

Properties of absolute values:

A1.  $|x| \geq 0$

A2.  $|x| = 0 \iff x = 0$

A3.  $|x| = |-x|$

A4.  $|xy| = |x||y|$

A5. If  $c \geq 0$ , then

$$|x| \leq c \iff -c \leq x \leq c$$

A6.  $-|x| \leq x \leq |x|$

A7.  $||x| - |y|| \leq |x \pm y| \leq |x| + |y|$  by triangle inequality

### Binomials

$$n! = 1 \times 2 \times \dots \times (n-1) \times n \text{ if } n \in \mathbb{N}$$

$$0! = 1$$

and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{1 \times 2 \times \dots \times k} \quad \text{* note equal to 1 if } k=0$$

Helpful Thm:

- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$
- $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$
- $\sum_{k=0}^n \binom{n}{k} = 2^n, \forall n \in \mathbb{N}$  (can be proved via induction as exercise)

### Binomial Thm:

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \text{ (can also be proved by induction, see Bowmen notes page 26)}$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$

## 6. Intervals and bounds

Open and closed intervals are defined as the following

For  $a, b \in \mathbb{R}$  and  $a < b$

$$[a, b] = \{x : a \leq x \leq b\}, \rightarrow \text{closed}$$

$$(a, b) = \{x : a < x < b\}, \rightarrow \text{open}$$

$$[a, b) = \{x : a \leq x < b\}, \rightarrow \text{relatively open}$$

$$(a, b] = \{x : a < x \leq b\}, \rightarrow \text{relatively open}$$

not finite intervals

$$\begin{aligned}
 (-\infty, \infty) &= \mathbb{R} \\
 [a, \infty) &= \{x : x \geq a\} \\
 (-\infty, a) &= \{x : x < a\}
 \end{aligned}$$

### Upper/lower bounds

A real number  $b$  is an upper bound  $S$  if

$$x \leq b \text{ for each } x \in S$$

If no  $b$  exists as an upper bound of  $S$ , we say it is unbounded above

\* Lower bound is similar (wLOG)

### Supremum and Infimum (Sup and Inf)

$b$  is the sup of  $S$  if it is the least upper bound ( $b = \sup S$ )

If  $b$  is the sup of  $S$  and is in  $S$ , (i.e.  $b = \sup S$  and  $x \in S$ ), it is said to be the maximum

\* Infimum is similar (wLOG)

## 7. Real numbers

A real number is only defined if it satisfies all the rules laid out in [chapter 3](#) and it follows the completeness axiom.

### Completeness Axiom:

For every non-trivial subset of  $\mathbb{R}$  with an upper bound has a least upper bound in  $\mathbb{R}$ , (i.e. the sup exists in  $\mathbb{R}$ )

- $\{\frac{p}{q} : p^2 \leq 2q^2, p \in \mathbb{Z}, q \in \mathbb{N}\}$
- $[0, 1]$  as sup at 1
- $[0, 1)$  has sup at 1

Lemma - Archimedean Property:

No real number is an upper bound for  $\mathbb{N}$

Notes:  $\mathbb{N} \subset \mathbb{R}$  (can be proved inductively)

## 8. Sequences and limits

A sequence of real numbers is a function (see chapter below) s.t.  $a : \mathbb{N} \rightarrow \mathbb{R}$ . Usually a sequence is written as  $a_n$  instead of  $a(n)$ . Sometimes a sequence can start at  $k$ , s.t.  $k \in \mathbb{Z}$ , instead of  $k \in \mathbb{N}$ .

A sequence is improper if  $\exists k \in a_n$  if  $k$  is  $\pm \infty$

\* also note an extended real number system is  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

### Rules on extended real number system :

1.  $c + \infty = \infty$
2.  $c - \infty = -\infty$
3.  $c \cdot \infty = (-c)(-\infty) = \infty$  if  $c > 0$
4.  $c \cdot \infty = (-c)(-\infty) = -\infty$  if  $c < 0$
5.  $\infty \cdot \infty = \infty + \infty = \infty$
6.  $\infty \cdot (-\infty) = -\infty$



## Limits (This is mega important)

$a_n$  is a sequence. The limit of the sequence  $(a_n)$  is  $L$ , s.t.  $L \in \mathbb{R}$  and  $\lim_{n \rightarrow \infty} a_n = L$  if for every  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$ , s.t.  $|a_n - L| < \epsilon, \forall n > N$

Translation to English: The limit of the sequence as the index approaches infinity exists, if there exists an error ( $\epsilon$  = epsilon that is greater than 0) small enough the element in sequence at index very very large (approaches  $\infty$ ) minus the defined limit is less than that error. [Think of an error bar of value  $\epsilon$  and  $|a_n - L|$  being within the error bar if limit exists]

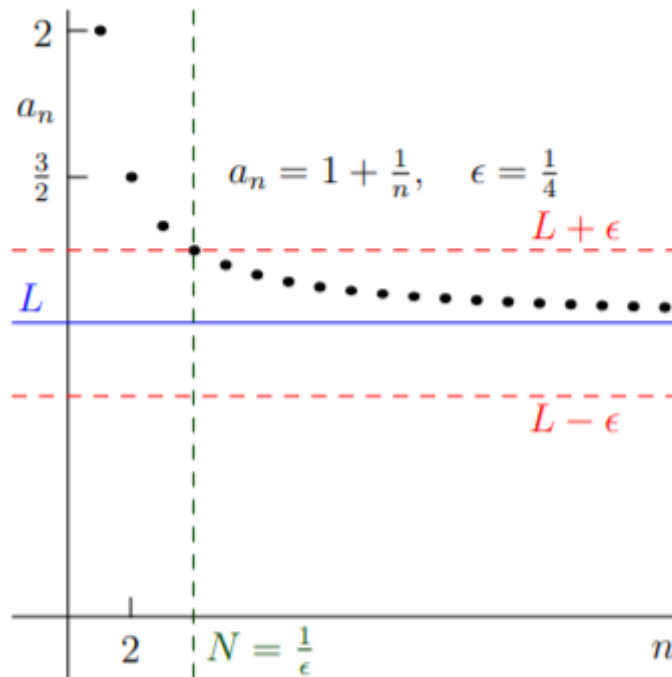


Figure 2.1: Limit of a sequence

If that limit exists, it is convergent, otherwise, it is divergent (improper limit of the sequence)

If  $\lim_{n \rightarrow \infty} a_n = \infty$ , if  $\forall M > 0$ , there is  $N \in \mathbb{N}$  s.t.  $a_n > M \forall n > N$  (wLOG for  $-\infty$ )

### Properties of limits

Let  $a_n, b_n$  be sequences with limit  $L$  and  $M$

1. If  $c, d \in \mathbb{R}$ , and  $cL, dM$  are both finite of opposing signs, then  $ca_n + db_n \rightarrow cL + dM$
2. If  $LM$  is not undefined,  $0 \neq \pm \infty$ ,  $a_n b_n \rightarrow LM$
3. If  $M \neq 0$ , and both  $M, N$  are finite, then  $\frac{a_n}{b_n} \rightarrow \frac{L}{M}$  (only for  $n$  large enough)
4. If  $L = \pm \infty$ , then  $\frac{1}{a_n} \rightarrow 0$  (only for  $n$  large enough)

\* look at page 36 of Bowman notes to see proofs

Lemma

If  $a_n \leq b_n \forall n$  and  $a_n \rightarrow L, b_n \rightarrow M \rightarrow L \leq M$

Squeeze thm (or sandwich thm if and only if you are hungry)

If  $a_n, c_n \rightarrow L$  and  $a_n \leq b_n \leq c_n \forall n \rightarrow b_n \rightarrow L$

A sequence is bounded if  $\exists M \in \mathbb{R}$ , s.t.  $|a_n| \leq M \forall n$

Helpful example:

$$|\sin(x)| \leq 1$$

## Why? Homework question

Now what max and min functions are (they are self explanatory, otherwise, look it up in the python docs)

Thm:

Convergent sequence implies bounded but not the other way

(convergent  $\rightarrow$  bounded)

(bounded  $\nrightarrow$  convergent)

## Monotone

A sequence is increasing if for  $a_n$ ,

$$a_1 \leq a_2 \leq \dots, i.e. a_n \leq a_{n+1} \forall n \in \mathbb{N}$$

A sequence is strictly increasing if for  $a_n$ ,

$$a_1 < a_2 < \dots, i.e. a_n < a_{n+1} \forall n \in \mathbb{N}$$

\* likewise for decreasing

A sequence is **monotone** if it is either an increasing or a decreasing sequence (or both)

In monotone sequences, *convergent*  $\iff$  *bounded*

## Subsequences

Given a seq.  $\lim_{n \rightarrow \infty} a_n$  and a strictly increasing sequence of natural numbers  $\lim_{n \rightarrow \infty} n_k$ , we can form subsequence  $\lim_{n \rightarrow \infty} a_{n_k}$  of  $\lim_{n \rightarrow \infty} a_n$

Ex.

$$\lim_{n \rightarrow \infty} (2k-1)^2 = \{1, 9, 25, \dots\} \text{ is a subsequence of } \lim_{n \rightarrow \infty} n^2 = \{1, 4, 9, 16, \dots\}$$

Thm.

convergent  $\iff$  all subsequence convergent

Lemma:

$$a) 0 \leq c < 1 \rightarrow c^n \leq c \leq 1$$

$$b) c > 1 \rightarrow c^n \geq c > 1$$

$$\lim_{n \rightarrow \infty} c^n = \begin{cases} 0 & \text{if } 0 \leq c < 1, \\ 1 & \text{if } c = 1, \\ \nexists & \text{if } c > 1 \text{ (divergent; in fact, unbounded (exercise))} \end{cases}$$

$$\forall n \in \mathbb{N}$$

$$c) 0 \leq c < 1 \rightarrow c \leq c^{1/n} < 1$$

$$d) c > 1 > 1 \rightarrow c \geq c^{1/n} > 1$$

## Ratio test for seq

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r \text{ where } r \in [0, 1) \text{ means that } a_n \text{ is bounded and } \rightarrow 0$$

## Bolzano-Weierstrass Thm:

A bounded sequence has a convergent subsequence (see page 49 for proof)

### Cauchy Criterion

A sequence  $a_n$  is a Cauchy sequence (or just Cauchy) if  $\forall \epsilon > 0 \exists N$  s.t.

$$m, n > N \rightarrow |a_m - a_n| < \epsilon$$

$\therefore a_n$  is convergent  $\iff a_n$  is Cauchy

## 9. Functions

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## Misc. Notes:

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Consider a statement of the form:  $\forall x \in D$ , if  $P(x)$  then  $Q(x)$ .

1. Its **contrapositive** is the statement:  $\forall x \in D$ , if  $\sim Q(x)$  then  $\sim P(x)$ .
2. Its **converse** is the statement:  $\forall x \in D$ , if  $Q(x)$  then  $P(x)$ .
3. Its **inverse** is the statement:  $\forall x \in D$ , if  $\sim P(x)$  then  $\sim Q(x)$ .

## Resources

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- school notes
- [Bowmen notes](#)

## Note about this course and my experience

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My school had an extraordinary honors calculus year. The main prof left by second semester and a German prof came in to sub, but, because of his Germanicness (he literally goes off of a German textbook), it was difficult to follow his sometimes rigorous and sometimes not rigorous (he called them trivial) proofs.

Also, at my university, compute 272 is trivially similar to 117.

## Licence

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The rules for copy and distributing this project licence are outlined in the licence.txt file.

This project is under an MIT licence

## Thanks

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Dedicated to Dr. Terry Gannon who definitely did not abandon the class

## Contributions

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Contribute if and only if you know what you are doing (and optionally, it would help if you where Germanic or fluent in Germanics)

\* If someone wants to convert this to Latex, go ahead and pull request (must have a index or contents page with links). I tried with pandoc, but it was not as complete as I thought so if anyone wants to do clean up, go ahead, also I can just export as pdf