The Design and Analysis of Algorithms

Lecture 25 Local Search II

Zhenbo Wang

Department of Mathematical Sciences, Tsinghua University

Content

Nash Equilibrium

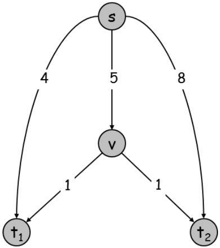
Algorithmic Game Theory: An Introduction

Multicast Routing

- *Multicast routing*. Given a directed graph $G = (V, E)$ with edge costs $c_e \geq 0$, a source node s, and k agents located at terminal nodes $t_1, \cdots, t_k$. Agent j must construct a path P_j from node s to its terminal t_j.

- *Fair share*. If x agents use edge e, they each pay c_e/x.

1	2	1 pays	2 pays
outer	outer	4	8
outer	middle	4	5 + 1
middle	outer	5 + 1	8
middle	middle	5/2 + 1	5/2 + 1



Multicast Routing

- *Best response dynamics*. Each agent is continually prepared to improve its solution in response to changes made by other agents.
- *Nash equilibrium*. Solution where no agent has an incentive to switch.
- *Fundamental question*. When do Nash equilibria exist?

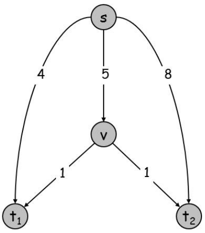
Ex.

Two agents start with outer paths.

Agent 1 has no incentive to switch paths (since $4 < 5 + 1$), but agent 2 does (since $8 > 5 + 1$).

Once this happens, agent 1 prefers middle path (since $4 > 5/2 + 1$).

Both agents using middle path is a Nash equilibrium.



Nash Equilibrium and Local Search

- *Local search algorithm*. Each agent is continually prepared to improve its solution in response to changes made by other agents.
- *Analogies*.

  Nash equilibrium: local optimum.

  Best response dynamics: local search algorithm.

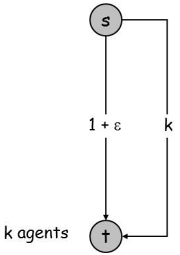
  Unilateral move by single agent: local neighborhood.
- *Contrast*. Best-response dynamics need not terminate since no single objective function is being optimized.

Socially Optimum

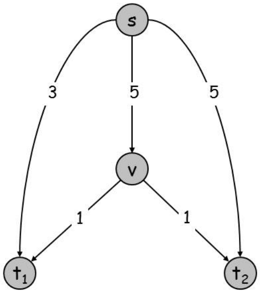
- *Social optimum*. Minimizes total cost to all agents.
- *Observation*.

  There can be many Nash equilibria.

  Even when its unique, it does not necessarily equal the social optimum.



Social optimum = $1 + \varepsilon$
Nash equilibrium A = $1 + \varepsilon$
Nash equilibrium B = k



Social optimum = 7
Unique Nash equilibrium = 8

Price of Stability

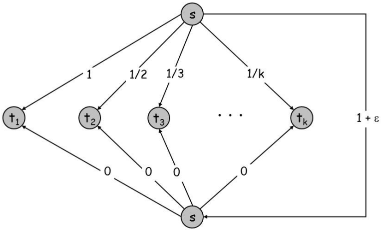
- *Price of stability*. Ratio of best Nash equilibrium to social optimum.
- What is price of stability of multicast routing?

Ex: Price of stability $= \Theta(\log k)$.

Social optimum. Everyone takes bottom paths.

Unique Nash equilibrium. Everyone takes top paths.

Price of stability. $H(k)/(1+\epsilon)$, $H(k) = 1 + 1/2 + \cdots + 1/k$.



Finding a Nash Equilibrium

Theorem 1

The following algorithm terminates with a Nash equilibrium.

$BEST - RESPONSE - DYNAMICS(G, c, k)$

for $j = 1$ to k **do**
    $P_j \leftarrow$ any path for agent j.
end for
while not a Nash equilibrium **do**
    $j \leftarrow$ some agent who can improve by switching paths.
    $P_j \leftarrow$ better path for agent i.
end while
return $(P_1, P_2, \cdots, P_k)$.

Finding a Nash Equilibrium

Pf. Consider a set of paths $P_1, P_2, \cdots, P_k$.

Let x_e denote the number of paths that use edge e.

Let $\Phi(P_1, P_2, \cdots, P_k) = \sum_{e \in E} c_e \cdot H(x_e)$ be a potential function, where $H(0) = 0, H(k) = \sum_{i=1}^{k} \frac{1}{i}$.

Since there are only finitely many sets of paths, it suffices to show that Φ strictly decreases in each step.

Consider agent j switching from path P_j to path P_j'.

Agent j switches because

$$\sum_{f \in P_j' - P_j} \frac{c_f}{x_f + 1} < \sum_{e \in P_j - P_j'} \frac{c_e}{x_e}.$$

Finding a Nash Equilibrium–Con't

Φ increases by

$$\sum_{f \in P_j' - P_j} c_f[H(x_f + 1) - H(x_f)] = \sum_{f \in P_j' - P_j} \frac{c_f}{x_f + 1}.$$

Φ decreases by

$$\sum_{e \in P_j - P_j'} c_e[H(x_e) - H(x_e - 1)] = \sum_{e \in P_j - P_j'} \frac{c_e}{x_e}.$$

Thus, net change in Φ is negative. $\square$

Bounding the Price of Stability

Lemma 2

Let $C(P_1, \cdots, P_k)$ denote the total cost of selecting paths $P_1, \cdots, P_k$. For any set of paths $P_1, P_2, \cdots, P_k$, we have

$$C(P_1, \cdots, P_k) \leq \Phi(P_1, \cdots, P_k) \leq H(k)C(P_1, \cdots, P_k).$$

Pf. Let x_e denote the number of paths containing edge e.

Let E^+ denote set of edges that belong to at least one of the paths.

Then,

$$C(P_1, \cdots, P_k) = \sum_{e \in E^+} c_e \leq \sum_{e \in E^+} c_e H(x_e)$$

$$\leq \sum_{e \in E^+} c_e H(k) = H(k)C(P_1, \cdots, P_k). \square$$

Bounding the Price of Stability

Theorem 3

There is a Nash equilibrium for which the total cost to all agents exceeds that of the social optimum by at most a factor of $H(k)$.

Pf. Let $(P_1^*, \cdots, P_k^*)$ denote a set of socially optimal paths.

Run best-response dynamics algorithm starting from P^*.

Since Φ is monotone decreasing,
$\Phi(P_1, \cdots, P_k) \leq \Phi(P_1^*, \cdots, P_k^*)$.

$$\begin{aligned} C(P_1, \cdots, P_k) \leq \Phi(P_1, \cdots, P_k) &\leq \Phi(P_1^*, \cdots, P_k^*) \\ &\leq H(k) \cdot C(P_1^*, \cdots, P_k^*). \square \end{aligned}$$

Summary

- *Existence*. Nash equilibria always exist for k-agent multicast routing with fair sharing.
- *Price of stability*. Best Nash equilibrium is never more than a factor of $H(k)$ worse than the social optimum.
- *Fundamental open problem*. Find any Nash equilibria in poly-time.

Game Theory

- *Game theory* aims to model situations in which multiple participants interact or affect each other's outcomes.

  Non-cooperative game.

  Cooperative game.

- *Nash equilibrium*. Solution where no agent has an incentive to switch.

  Pure strategy Nash equilibrium.

  Mixed strategy Nash equilibrium.

Prisoner's Dilemma

Prisoner's dilemma

Two prisoners are on trial for a crime and each one faces a choice of confessing of the crime or remaining silent.

P1 \ P2	Confess	Silent
Confess	4 4	5 1
Silent	1 5	2 2

- The only stable solution in this game is that both prisoners confess.

Algorithmic Game Theory

- Recently there has been a lot of interesting problems at the intersection of game theory, economics, and computer science, largely motivated by the emergence of the Internet.
- *Algorithmic Game* uses the ideas from game theory and algorithms to illuminate these problems.
  1. Existence of Nash equilibrium; price equilibrium for market; cost sharing in cooperative game.
  2. Hardness of finding Nash equilibrium: *PPAD*.
  3. *Price of Anarchy* (*PoA*): the ratio of worst NE to the social optimum; *Price of stability* (*POS*).
  4. Mechanism design.

N. Nisan, T. Roughgarden, É. Tardos, V.V. Vazirani.
Algorithmic Game Theory, Cambridge University Press, 2007.

Scheduling Game

Koutsoupias and Papadimitriou's Model

There are n jobs and m machines. Each job seeks to minimize the total number of jobs that select its machine. The objective of social optimum is to minimize the number of jobs on the most crowded machine.

- *PoA* for pure strategy Nash equilibrium: $2 - \frac{2}{m+1}$.
- *PoA* for mixed strategy Nash equilibrium: $\Theta(\log m / \log \log m)$.

E. Koutsoupias and C. H. Papadimitriou, Worst-case equilibria. STACS, 404–413, 1999.

Wardrop model

Selfish Routing

Given a directed graph $G = (V, E)$ with edge cost (time) function $c_e(x)$, some source nodes and some terminal nodes, each agent controls a small amount of flow, and wants to construct a path from its source node to its terminal with the minimal cost. The objective of social optimum is to minimize the total cost of all agents.

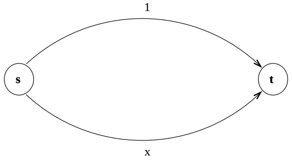


Figure 1: Pigou's example

Results

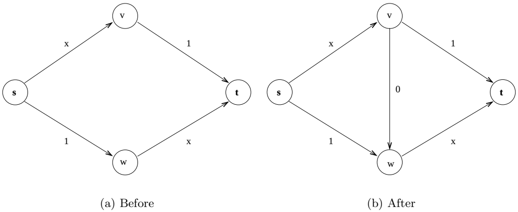


Figure 2: Braess's Paradox

- Assume the cost functions are nonnegative, continuous and nondecreasing.
- Pure Nash equilibrium exists.
- If the cost functions are linear, the *PoA* is $4/3$.

T. Roughgarden and É. Tardos. How bad is selfish routing? FOCS, 93–102, 2000 (JACM 2002).

Homework

- Read Chapter 12 of the textbook.
- Exercise 4 in Chapter 12.