The Design and Analysis of Algorithms

Lecture 1 Introduction

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Content

What is an Algorithm?

Knuth An algorithm is a *finite*, *definite*, *effective* procedure, with some input and some output.

- Grade-school algorithm for computing addition.
- Multiplication by repeated addition: compute $x \times y$ by adding x to itself y times.
- Grade-school algorithm for multiplication.

$$
\begin{array}{cccccc}
 & & & 5 & 7 & 7 \\
 & & & 4 & 2 & 3 \\
\hline
 & & 1 & 7 & 3 & 1 \\
 & 1 & 1 & 5 & 4 & \\
2 & 3 & 0 & 8 & & \\
\hline
2 & 4 & 4 & 0 & 7 & 1
\end{array}
$$

Why Study Algorithms?

- Internet: Web search, packet routing, $\cdots$
- Biology: Human genome project, protein folding, $\cdots$
- Computers: Circuit layout, databases, networking, $\cdots$
- Computer graphics: Movies, video games, $\cdots$
- Security: Cell phones, e-commerce, $\cdots$
- Multimedia: MP3, JPG, DivX, face recognition, $\cdots$
- Social networks: Recommendations, advertisements, $\cdots$
- Artificial Intelligence: AlphaGo $\cdots$

Efficiency of Algorithms

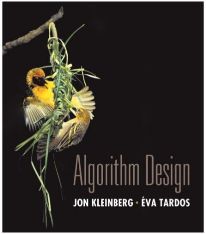
- The grade-school algorithm seems “better” than repeated addition.
- Assume both x and y have n digits and hence are between 10^{n-1} and 10^{n}.
- Repeated addition needs at least 10^{n-1} additions (each of which takes n single-digit addition).
- The grade-school algorithm uses only $O(n^2)$ times single-digit addition and multiplication.
- For slightly larger numbers even a supercomputer running the repeated addition algorithm is impractical.
- Fast Fourier Transform takes $O(n \log n)$ operations for multiplication.

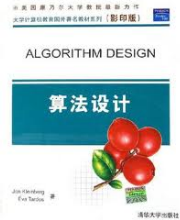
Contents of This Course

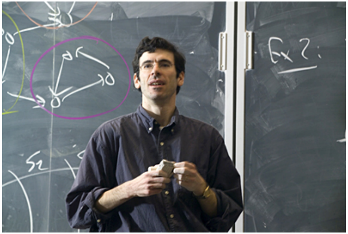
- Greedy algorithm.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- NP and computational intractability.
- Approximation algorithms.
- Local search.
- Randomized algorithms.

Textbook

- Kleinberg J and Tardos É. Algorithm Design. Prentice Hall, 2005 & Tsinghua Press, 2006.







Homework, Scores, Etc

- Homework: submit homework online.
- Score:

  Homework: 40 points;

  Final Exam: 60 points.
- Materials are available at `http://learn.tsinghua.edu.cn`.
- Contact: wangzhenbo@tsinghua.edu.cn, HeEr Office 212.
- TA:

Related Courses

- Three graduate courses:

  Computational Complexity (Spring): How hard is a problem? It is the fundamental of algorithm.

  Network Optimization (Spring). Provide a deep and broad understanding of network flow and related problems.

  The Design and Analyse of Algorithms (Autumn). Introduce algorithm and related problems from a general perspective.

- We focus on algorithms and techniques that are useful in practice.

- This course can be considered as an introductory course of *combinatorial optimization*.

Operations Research

- *Operations research*, is a discipline that deals with the application of advanced analytical methods to help make better decisions. – Wikipedia

  Operations Research is often considered to be a sub-field of mathematics.

- Major subdisciplines:

  Mathematical programming.

  Combinatorial optimization.

  Stochastic models and methods.

  Manufacturing, service sciences, and supply chain management $\cdots$

A First Problem: Stable Matching

- Matching med-school students to hospitals.

Goal Given a set of preferences among hospitals and med-school students, design a self-enforcing admissions process.

- *Unstable pair*: student x and hospital y are unstable if:

  x prefers y to its assigned hospital;

  y prefers x to one of its admitted students.

- *Stable assignment*: Assignment with no unstable pairs.

Stable Matching Problem

Goal Given a set of n men and a set of n women, find a "suitable" matching.

Participants rank members of opposite sex.

Each man lists women in order of preference from best to worst.

Each woman lists men in order of preference from best to worst.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Perfect Matching

Def. A matching S is a set of ordered pairs $m - w$ with $m \in M$ and $w \in W$, s.t.

Each man $m \in M$ appears in at most one pair of S.

Each woman $w \in W$ appears in at most one pair of S.

Def. A matching S is perfect if $|S| = |M| = |W| = n$.

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Figure 1: a perfect matching $S = \{X - C, Y - B, Z - A\}$

Unstable Pair

Def. Given a perfect matching S, man m and woman w are unstable if:

m prefers w to his current partner.

w prefers m to her current partner.

Key point An unstable pair m – w could each improve partner by joint action.

favorite ↓ ... least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ ... least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Figure 2: Bertha and Xavier are an unstable pair

Stable Matching Problem

Def. A *stable matching* is a perfect matching with no unstable pairs.

Stable matching problem

Given the preference lists of n men and n women, find a stable matching (if one exists).

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Figure 3: a stable matching $S = \{X - A, Y - B, Z - C\}$

Stable Roommate Problem

- $2n$ people; each person ranks others from 1 to $2n - 1$.
- Assign roommate pairs so that no unstable pairs.

	1^{st}	2^{nd}	3^{rd}
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

$A - B, C - D \Rightarrow B - C$ unstable.

$A - C, B - D \Rightarrow A - B$ unstable.

$A - D, B - C \Rightarrow A - C$ unstable.

- Stable matchings need not exist for stable roommate problem.

Gale Shapley Algorithm

GALE-SHAPLEY (preference lists for men and women)

Initialize S to empty matching.
while some man m is unmatched and hasn't proposed to every woman **do**
    $w \leftarrow$ first woman on m's list to whom m has not yet proposed.
    if w is unmatched **then**
        Add pair $m - w$ to matching S.
    else if w prefers m to her current partner m' **then**
        Remove pair $m' - w$ from matching S.
        Add pair $m - w$ to matching S.
    else
        w rejects m.
    end if
end while
return stable matching S.

Some Questions on Gale-Shapley Algorithm

Q. Does Gale-Shapley algorithm terminate in finite steps?

Q. Does Gale-Shapley algorithm find a perfect matching?

Q. Does Gale-Shapley algorithm find a stable matching?

Homework

- Read Chapter 1 of the textbook.
- Exercises 1 & 2 in Chapter 1.