The Design and Analysis of Algorithms

Lecture 22 Approximation Algorithm III

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Content

LP Rounding: Vertex Cover

Generalized Load Balancing

Weighted Vertex Cover

Def. Given a graph $G = (V, E)$, a vertex cover is a set $S \subseteq V$ such that each edge in E has at least one end in S.

- *Weighted Vertex Cover.* Given a graph G with vertex weights, find a vertex cover of minimum weight.

$$
\begin{aligned}
\min \sum_{i \in V} w_i x_i & & \\
s.t. x_i + x_j \geq 1 & \quad \forall (i, j) \in E, & \text{(ILP)} \\
x_i \in \{0, 1\} & \quad \forall i \in V. &
\end{aligned}
$$

Weighted Vertex Cover: LP Relaxation

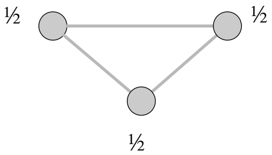
- The LP relaxation of weighted vertex cover:

$$
\begin{aligned}
&\min \sum_{i \in V} w_i x_i && && \\
&s.t. x_i + x_j \geq 1 && \forall (i,j) \in E, && \text{(LP)} \\
&x_i \geq 0, && \forall i \in V. &&
\end{aligned}
$$

Obs. Optimal value of (LP) is $\leq$ optimal value of (ILP).

Pf. LP has fewer constraints.

Note. LP is not equivalent to vertex cover.



Q. How can solving LP help us find a small vertex cover?

A. Solve LP and round fractional values.

Weighted Vertex Cover: LP Rounding Algorithm

Lemma 1

If x^* is optimal solution to (LP), then $S = \{i \in V : x_i^* \geq 1/2\}$ is a vertex cover whose weight is at most twice the min possible weight.

Pf. [S is a vertex cover]

Consider an edge $(i,j) \in E$.

Since $x_i^* + x_j^* \geq 1$, either $x_i^* \geq 1/2$ or $x_j^* \geq 1/2$,
$\Rightarrow (i,j)$ covered.

Pf. [S has desired cost]

Let S^* be optimal vertex cover. Then

$$\sum_{i \in S^*} w_i \geq \sum_{i \in V} w_i x_i^* \geq \sum_{i \in S} w_i x_i^* \geq \frac{1}{2} \sum_{i \in S} w_i. \ \square$$

Weighted Vertex Cover and Inapproximability

Theorem 2

The rounding algorithm is a 2-approximation algorithm.

Pf. Lemma + fact that LP can be solved in poly-time. □

Theorem 3 (Dinur-Safra, Ann. Math., 2004)

If P ≠ NP, then no ρ-approximation for weighted vertex cover for any $\rho < 1.3606$ (even if all weights are 1).

On the Hardness of Approximating Minimum Vertex Cover

Irit Dinur* Samuel Safra†

May 26, 2004

Abstract

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and borrow and extend ideas from several fields.

Weighted Vertex Cover: Inapproximability

Theorem 4 (Khot-Regev, J. Comput. Syst. Sci., 2008)

Based on Unique Game Conjecture, vertex cover is hard to approximate within $2 - \epsilon$ for any $\epsilon > 0$.

Vertex Cover Might be Hard to Approximate to within $2 - \varepsilon$

Subhash Khot * Oded Regev †

Abstract

Based on a conjecture regarding the power of unique 2-prover-1-round games presented in [Khot02], we show that vertex cover is hard to approximate within any constant factor better than 2. We actually show a stronger result, namely, based on the same conjecture, vertex cover on k-uniform hypergraphs is hard to approximate within any constant factor better than k.

Generalized Load Balancing

Input. Set of m machines M; set of n jobs J.

- For each job there is just a subset of machines to which it can be assigned.
- Job j must run contiguously on an authorized machine in $M_j \subseteq M$.
- Each machine can process at most one job at a time.

Def. Let $J(i)$ be the subset of jobs assigned to machine i.

The *load* of machine i is $L_i = \sum_{j \in J(i)} t_j$.

Def. The *makespan* is the maximum load on any machine $L = \max_i L_i$.

Def. *Generalized Load balancing.* Assign each job to a machine to minimize makespan.

Generalized Load Balancing: ILP

ILP formulation

x_{ij} is time of machine i spends processing job j.

$$
\begin{aligned}
& \min \ L \\
& s.t. \sum_i x_{ij} = t_j && \forall j \in J, \\
& \quad \sum_j x_{ij} \le L && \forall i \in M, \\
& \quad x_{ij} \in \{0, t_j\} && \forall j \in J \text{ and } i \in M_j, \\
& \quad x_{ij} = 0 && \forall j \in J \text{ and } i \notin M_j.
\end{aligned} \tag{ILP}
$$

Generalized Load Balancing: Relaxation

LP relaxation

$$
\begin{aligned}
&\min \ L \\
&s.t. \sum_i x_{ij} = t_j && \forall j \in J, \\
&\quad \sum_j x_{ij} \leq L && \forall i \in M, \\
&\quad x_{ij} \geq 0 && \forall j \in J \text{ and } i \in M_j, \\
&\quad x_{ij} = 0 && \forall j \in J \text{ and } i \notin M_j.
\end{aligned}
\qquad \text{(LP)}
$$

Generalized Load Balancing: Lower Bounds

Lemma 5

The optimal makespan $L^* \geq \max_j t_j$.

Pf. Some machine must process the most time-consuming job. □

Lemma 6

Let L be optimal value to the LP. Then, optimal makespan $L^* \geq L$.

Pf. LP has fewer constraints than IP formulation. □

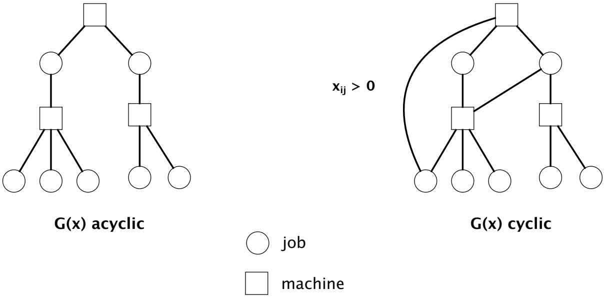
Generalized Load Balancing: Structure of LP Solution

Lemma 7

Let x be solution to LP. Let $G(x)$ be the graph with an edge between machine i and job j if $x_{ij} > 0$. Then there exists x such that $G(x)$ is acyclic.

- If LP solver doesn't return such an x, can transform x into another LP solution where $G(x)$ is acyclic.

Pf. (deferred)



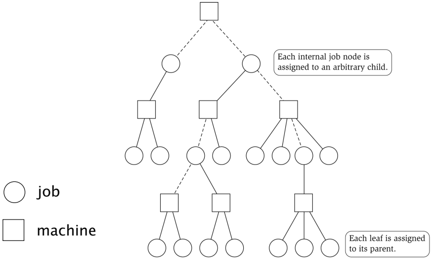
Generalized Load Balancing: Rounding

- *Rounded solution.* Find LP solution x where $G(x)$ is a forest.
- Root forest $G(x)$ at some arbitrary machine node r.
- If job j is a leaf node, assign j to its parent machine i.
- If job j is not a leaf node, assign j to any one of its children.

Lemma 8

Rounded solution only assigns jobs to authorized machines.

- If job j is assigned to machine i, then $x_{ij} > 0$. LP solution can only assign positive value to authorized machines. □



Generalized Load Balancing: Analysis

Lemma 9

If job j is a leaf node and machine $i = parent(j)$, then $x_{ij} = t_j$.

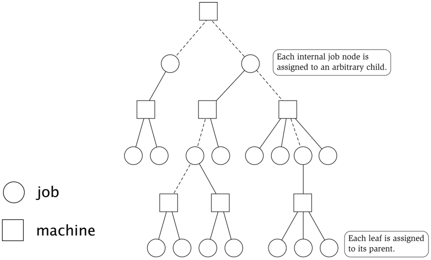
Pf. Since i is a leaf, $x_{ij} = 0$ for all $j \neq parent(i)$.

LP constraint $\sum_i x_{ij} = t_j$ guarantees $x_{ij} = t_j$. $\square$

Lemma 10

At most one non-leaf job is assigned to a machine.

Pf. The only possible non-leaf job assigned to i is $parent(i)$. $\square$



Generalized Load Balancing: Analysis

Theorem 11

Rounded solution is a 2-approximation.

Pf. Let $J(i)$ be the jobs assigned to machine i.

By lemma 10, the load L_i on machine i has two components:

- leaf nodes:

$$\sum_{j \in J(i),\ j \text{ is a leaf}} t_j = \sum_{j \in J(i),\ j \text{ is a leaf}} x_{ij} \leq \sum_{j \in J} x_{ij} \leq L \leq L^*$$

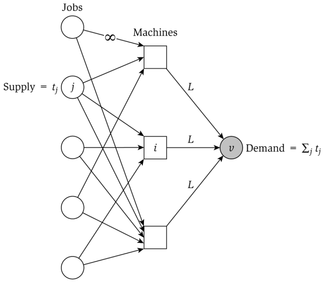
- parent: $t_{parent(i)} \leq L^*$.

Thus, the overall load $L_i \leq 2L^*$. □

Generalized Load Balancing: Flow Formulation

Flow formulation of LP.

$$
\begin{array}{ll}
\sum_i x_{ij} = t_j & \forall j \in J \\
\sum_j x_{ij} \leq L & \forall i \in M \\
x_{ij} \geq 0 & \forall j \in J \text{ and } i \in M_j \\
x_{ij} = 0 & \forall j \in J \text{ and } i \notin M_j
\end{array}
$$



Obs. Solution to feasible flow problem with value L are in 1-to-1 correspondence with LP solutions of value L.

Generalized Load Balancing: Structure of Solution

Lemma 7

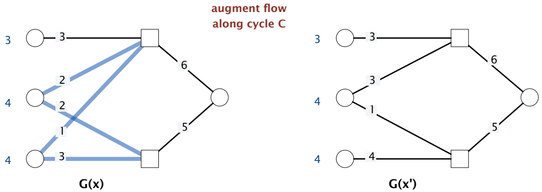
Let (x, L) be solution to LP. Let $G(x)$ be the graph with an edge between machine i and job j if $x_{ij} > 0$. We can find another solution (x', L) such that $G(x')$ is acyclic.

Pf. Let C be a cycle in $G(x)$.

Augment flow along the cycle C.

At least one edge from C is removed (and none are added).

Repeat until $G(x')$ is acyclic.



Conclusions

- *Running time.* The bottleneck operation in our 2-approximation algorithm is to solve LPs with $mn + 1$ variables.
- *Remark.* Can solve LP using flow techniques on a graph with $m + n + 1$ nodes: binary search to find L^*, and find feasible flow if it exists.
- *Extensions: unrelated parallel machines.* [Lenstra-Shmoys-Tardos, Math. Program., 1990]
  - Job j takes t_{ij} time if processed on machine i.
  - 2-approximation algorithm via LP rounding.
  - If $P \neq NP$, then no ρ-approximation exists for any $\rho < 3/2$.

Mathematical Programming 46 (1990) 259–271 259
North-Holland

APPROXIMATION ALGORITHMS FOR SCHEDULING UNRELATED PARALLEL MACHINES

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Homework

- Read Chapter 11 of the textbook.
- Exercises 8 & 9 in Chapter 11.