The Design and Analysis of Algorithms

Lecture 2 Stable Matching

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Content

Stable Matching Problem

Def. Given a perfect matching S, man m and woman w are unstable if:

m prefers w to his current partner.

w prefers m to her current partner.

Def. A *stable matching* is a perfect matching with no unstable pairs.

Stable matching problem

Given the preference lists of n men and n women, find a stable matching (if one exists).

Gale Shapley Algorithm

GALE-SHAPLEY (preference lists for men and women)

Initialize S to empty matching.
while some man m is unmatched and hasn't proposed to every woman **do**
    $w \leftarrow$ first woman on m's list to whom m has not yet proposed.
    if w is unmatched **then**
        Add pair $m - w$ to matching S.
    else if w prefers m to her current partner m' **then**
        Remove pair $m' - w$ from matching S.
        Add pair $m - w$ to matching S.
    else
        w rejects m.
    end if
end while
return stable matching S.

Proof of Correctness: Termination

Observation 1

Men propose to women in decreasing order of preference.

Observation 2

Once a woman is matched, she never becomes unmatched; she only “trades up”.

Claim 1

Algorithm terminates after at most n^2 iterations of while loop.

Pf. Each time through the while loop a man proposes to a new woman.

There are only n^2 possible proposals. $\square$

Proof of Correctness: Perfection

Claim 2

In Gale-Shapley matching, all men and women get matched.

Pf. [by contradiction]

Suppose that Zeus is not matched upon termination of *GS* algorithm.

Then some woman, say Amy, is not matched upon termination.

By Observation 2, Amy was never proposed to.

But, Zeus proposes to everyone, since he ends up unmatched. □

Proof of Correctness: Stability

Claim 3

In Gale-Shapley matching, there are no unstable pairs.

Pf. Suppose the *GS* matching S^* does not contain the pair $A - Z$.

Case 1: Z never proposed to A.

$\Rightarrow$ Z prefers his *GS* partner B to A.

$\Rightarrow$ $A - Z$ is stable.

Case 2: Z proposed to A.

$\Rightarrow$ A rejected Z (right away or later).

$\Rightarrow$ A prefers her *GS* partner Y to Z.

$\Rightarrow$ $A - Z$ is stable. $\square$

Summary

Stable matching problem

Given the preference lists of n men and n women, find a stable matching (if one exists).

Theorem 1 (Gale-Shapley 1962)

The Gale-Shapley algorithm guarantees to find a stable matching for any problem instance.

Understanding the Solution

- For a given problem instance, there may be several stable matchings.
- Two stable matching: $\{A - X, B - Y, C - Z\}$ and $\{A - Y, B - X, C - Z\}$.
- Do all executions of *GS* algorithm yield the same stable matching?
- If so, which one?

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Def. Woman w is a *valid partner* of man m if there exists some stable matching in which m and w are matched.

Def. *Man-optimal assignment*: Each man receives best valid partner.

Is it perfect?

Is it stable?

Claim 4

All executions of GS yield man-optimal assignment.

Corollary 2

Man-optimal assignment is a stable matching!

Man Optimality

Claim 5

GS matching S^* is man-optimal.

Pf. [by contradiction]

Suppose a man is matched with someone other than best valid partner.

$\Rightarrow$ Some man is rejected by valid partner during GS.

Let Y be first such man, and let A be the first valid woman that rejects him.

Let S be a stable matching where A and Y are matched.

When Y is rejected by A in GS, A forms engagement with a man, say Z.

$\Rightarrow$ A prefers Z to Y.

Man Optimality–Con't

Let B be partner of Z in S.

Z has not been rejected by any valid partner (including B) at the point when Y is rejected by A.

Thus, Z has not yet proposed to B when he proposes to A.

$\Rightarrow$ Z prefers A to B.

Thus $A - Z$ is unstable in S, a contradiction. $\square$

Woman Pessimality

Q. Does man-optimality come at the expense of the women?

A. Yes.

Def. *Woman-pessimal assignment*: Each woman receives worst valid partner.

Claim 6

GS finds woman-pessimal stable matching S^*.

Proof of the Claim

Pf. [by contradiction]

Suppose $A - Z$ matched in S^* but Z is not worst valid partner for A.

There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.

$\Rightarrow$ A prefers Z to Y.

Let B be the partner of Z in S. By man-optimality, A is the best valid partner for Z.

$\Rightarrow$ Z prefers A to B.

Thus $A - Z$ is unstable in S, a contradiction. $\square$

Deceit?

Q. Can there be an incentive to misrepresent your preference list?

Assume you know men's propose-and-reject algorithm.

Assume preference lists of all other participants are known.

Fact No, for any man; yes, for some women.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

- If Amy lies "I prefer Zeus to Xavier", GS will return $\{A - Y, B - X, C - Z\}$!

2012 Nobel Prize in Economics

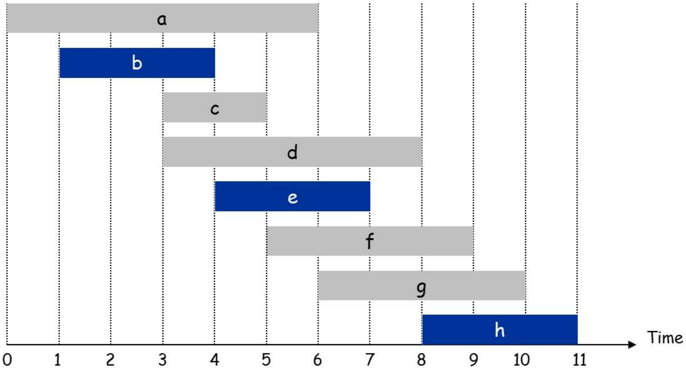
- Lloyd Shapley. Stable matching theory and Gale-Shapley algorithm.
- Alvin Roth. Applied Gale-Shapley to matching new doctors with hospitals, students with schools, and organ donors with patients.

Figure 1: Lloyd Shapley, Alvin Roth and Nobel Prize

Interval Scheduling

Input. Set of jobs with start times and finish times.

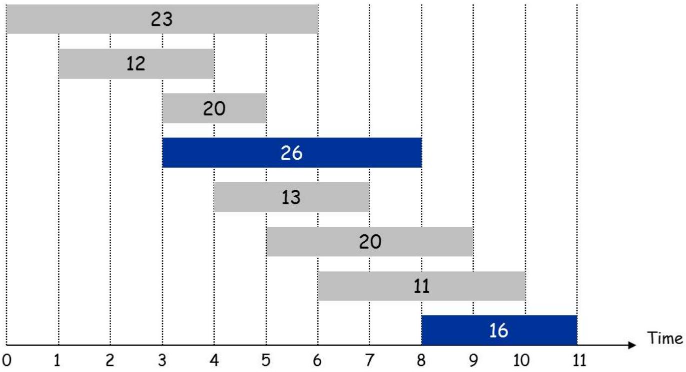
Goal. Find maximum cardinality subset of mutually compatible jobs (don't overlap).



Weighted Interval Scheduling

Input. Set of jobs with start times and finish times, and weights.

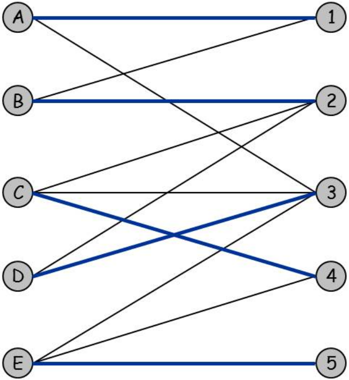
Goal. Find maximum weight subset of mutually compatible jobs.



Bipartite Matching

Problem. Given a bipartite graph $G = (L \cup R, E)$, find a max cardinality matching.

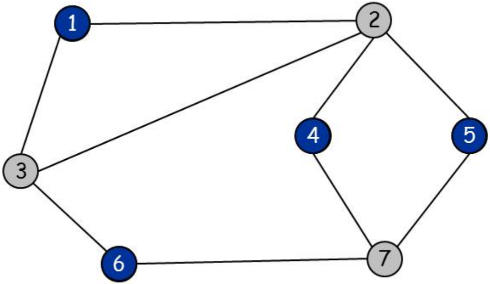
Def. A subset of edges $M \subseteq E$ is a *matching* if each node appears in exactly one edge in M.



Independent Set

Problem. Given a graph $G = (V, E)$, find a max cardinality independent set.

Def. A subset $S \subseteq V$ *independent* if for every $(u, v) \in E$, either $u \notin S$ or $v \notin S$ (or both).

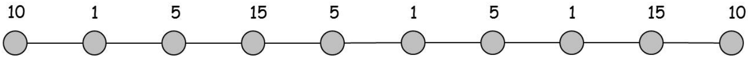


Competitive Facility Location

Input. Graph with weight on each node.

Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a *maximum weight* subset of nodes.



Five Representative Problems

- *Interval scheduling*: $O(n \log n)$ greedy algorithm.
- *Weighted interval scheduling*: $O(n \log n)$ dynamic programming algorithm.
- *Bipartite matching*: $O(nk)$ *max-flow* based algorithm.
- *Independent set*: **NP**-*complete*.
- *Competitive facility location*: **PSPACE**-*complete*.

Homework

- Read Chapter 1 of the textbook.
- Exercises 4 & 5 in Chapter 1.