Lyapunov Spectrum in Backpropagation

In the context of backpropagation, we consider the following iterative dynamics:

$$x_{n+1} = f(x_n) \tag{1}$$

$$v_{n+1} = Df(x_n)v_n \tag{2}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a differentiable function and $Df(x_n)$ denotes the Jacobian matrix of f evaluated at x_n.

We define a sequence of vector spaces $V_1 \subset V_2 \subset V_3 \subset \cdots \subset T_{x_0}M$, such that for each x_0 and $v \in V_i \backslash V_{i-1}$, we have:

$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x_0)v| = \lambda_i \tag{3}$$

where λ_i represents the i-th Lyapunov exponent. The Lyapunov spectrum provides valuable information about the stability and chaotic behavior of the dynamical system described by f. Each Lyapunov exponent λ_i corresponds to the local exponential growth rate of trajectories along the corresponding direction in the tangent space.

Calculate Lyapunov Spectrums

First we choose random d vectors:

$$[e_{1,0}, e_{2,0}, \cdots, e_{d,0}] = e_0 \tag{4}$$

where $e_{i,0} \in \mathbb{R}^d$ and $e_{i,0} \neq 0$ for all i. Then we calculate e by:

$$e_{n+1} = Df(x_n)e_n \tag{5}$$

Suppose for T, $e_T = QR$, where Q is an orthogonal matrix and R is an upper triangular matrix. Then we suppose:

$$Q = LV_1 \tag{6}$$

$$R = LE \tag{7}$$

where L is a lower triangular matrix, V_1 is a diagonal matrix and E is an upper triangular matrix. We can calculate V_1 by:

Finally we get

$$Df^{NA}e = QR \tag{8}$$
$$= Q_N R_N R_{N-1} \cdots R_1 \tag{9}$$

Then we have

$$L_1 V_1 = Q_N (R_A R_{A-1} \cdots R_1) \tag{10}$$
$$L_1 E_1 \approx \frac{1}{NA} log\ diag(R_A R_{A-1} \cdots R_1) \tag{11}$$

A Good Result

Let $\epsilon_n = Df^T(x_n)\epsilon_{n+1}$, we have

$$< \epsilon_n | e_n > = < \epsilon_{n+1} | e_{n+1} > \tag{12}$$

Hence

$$\epsilon_0 \cdot e_0 = (Df^T)^N \epsilon_N \cdot e_0 = \epsilon_N \cdot e_N = \epsilon_N \cdot (Df^N) e_0 \tag{13}$$

We just consider the several front lines of ϵ and e

$$1 = \epsilon_0 \cdot e_0 = \epsilon \cdot Df^N(e_0) \tag{14}$$
$$= |\epsilon_{0,u} \cdot \frac{e_0}{|e_0|}| = |\epsilon_{N,u} \cdot \frac{Df^N(e_{0,u})}{|e_{0,u}|}| = |\epsilon_{N,u} \frac{|Df^N e_{0,u}|}{|e_{0,u}|}| \tag{15}$$