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主题

日期

星期

第一次作业：

1. 1．根据算符∇的微分性与矢量性，推导下列公式：

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B},$$

$$\boldsymbol{A}\times(\nabla\times\boldsymbol{A})=\frac{1}{2}\nabla A^2-(\boldsymbol{A}\cdot\nabla)\boldsymbol{A}.$$

(1) 左边 $= \nabla(A_1B_1+A_2B_2+A_3B_3)=\sum_{cyc}\nabla(A_iB_i)=\sum_{cyc}A_i\nabla B_i+B_i\nabla A_i$

$$=\sum_{cyc}A_i\nabla B_i+\sum B_i\nabla A_i$$

$$\nabla\times A=\sum_{cyc}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}$$

$$B\times(\nabla\times A)=\sum_{cyc}\left[B_1\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)-B_2\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\right]\cdot\vec{k}$$

$$(B\cdot\nabla)A=\frac{B_1\partial A}{\partial x}+\frac{B_2\partial A}{\partial y}+\frac{B_3\partial A}{\partial z}=\sum_{cyc}B_i\left(\frac{\partial A_1}{\partial x_i},\frac{\partial A_2}{\partial x_i},\frac{\partial A_3}{\partial x}\right)^T$$

$$\Rightarrow B\times(\nabla\times A)+(B\cdot\nabla)A=\sum_{cyc}\left(B_1\cdot\frac{\partial A_1}{\partial z}+B_2\frac{\partial A_2}{\partial z}-B_2\frac{\partial A_3}{\partial y}\right)\vec{k}$$

同理 $A\times(\nabla\times B)+(A\cdot\nabla)B=\sum_{cyc}A_1\left(\frac{\partial B_1}{\partial z}+A_2\frac{\partial B_2}{\partial z}-A_2\frac{\partial B_3}{\partial y}\right)\vec{k}$

以上两式相加即 $\nabla(A\cdot B)=B\times(\nabla\times A)+(B\cdot\nabla)A+A\times(\nabla\times B)+(A\cdot\nabla)B$ □

(2) 由(1)推知：

$$A\times(\nabla\times A)=\sum_{cyc}\left[A_1\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)-A_2\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\right]\vec{k}$$

$$\frac{1}{2}\nabla A^2=2A\nabla A\cdot\frac{1}{2}=A\nabla A=\sum_{cyc}A_i\cdot\frac{\partial A_i}{\partial x}\cdot\vec{i}$$

$$(A\cdot\nabla)A=\sum_{cyc}A_i\left(\frac{\partial A_1}{\partial x_i},\frac{\partial A_2}{\partial x_i},\frac{\partial A_3}{\partial x}\right)^T$$

以上两式相减： $A\times(\nabla\times A)=\frac{1}{2}\nabla A^2-(A\cdot\nabla)A$

主是

日期

星期

2. 设 u 是空间坐标 x,y,z 的函数，证明：

$$\nabla f(u)=\frac{\mathrm{d}f}{\mathrm{d}u}\nabla u,$$

$$\nabla\cdot\boldsymbol{A}(u)=\nabla u\cdot\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u},$$

$$\nabla\times\boldsymbol{A}(u)=\nabla u\times\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u}.$$

2.

$$\nabla f(u)=\left(\frac{\partial f(u)}{\partial x},\ \frac{\partial f(u)}{\partial y},\ \frac{\partial f(u)}{\partial z}\right)^T$$

$$=\left(\frac{df}{du}\frac{\partial u}{\partial x},\ \frac{df}{du}\frac{\partial u}{\partial y},\ \frac{df}{\partial u}\frac{\partial u}{\partial z}\right)$$

$$=\frac{df}{du}\nabla u$$

$$\nabla\cdot A(u)=\frac{\partial A(u)}{\partial x}+\frac{\partial A(u)}{\partial y}+\frac{\partial A(u)}{\partial z}=\frac{dA}{du}\frac{\partial u}{\partial x}+\frac{dA}{du}\frac{\partial u}{\partial y}+\frac{dA}{du}\frac{\partial u}{\partial z}$$

$$=\nabla u\cdot\frac{dA}{du}$$

$$\nabla\times A(u)=\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}+\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\vec{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\vec{j}$$

$$=\nabla u\times\frac{dA}{du}$$

(2) 求 $\nabla\cdot\boldsymbol{r}$，$\nabla\times\boldsymbol{r}$，$(\boldsymbol{a}\cdot\nabla)\boldsymbol{r}$，$\nabla(\boldsymbol{a}\cdot\boldsymbol{r})$，$\nabla\cdot[\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})]$ 及 $\nabla\times[\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})]$，其中 $\boldsymbol{a}$、$\boldsymbol{k}$ 及 $\boldsymbol{E}_0$ 均为常矢量．

3.(2)

$$\nabla\cdot\vec{r}=\frac{\partial r_1}{\partial x}+\frac{\partial r_2}{\partial y}+\frac{\partial r_3}{\partial z}=3$$

$\nabla\times\vec{r}=0$ (无旋场)

$$(\vec{a}\cdot\nabla)\vec{r}=\frac{a_1\partial r}{\partial x}+\frac{a_2\partial r}{\partial y}+\frac{a_3\partial r}{\partial z}=(a_1,a_2,a_3)^T=\vec{a}$$

$$\nabla\cdot[E_0\sin(k\cdot\vec{r})]=\vec{k}\,\vec{E_0}\cdot\cos(k\vec{r})$$

7. 有一内外半径分别为 r_1 和 r_2 的空心介质球，介质的电容率为 ε．使介质内均匀带静止自由电荷 ρ_f，求

(1) 空间各点的电场；

(2) 极化体电荷和极化面电荷分布．

7.

(1) 设空间中一点与圆心构成的向量为 $\vec{r}$

若 $|\vec{r}|<r_1$. $\vec{E}=0$

若 $r_1<|\vec{r}|<r_2$, $\quad Q_r=\frac{4(r^3-r_1^3)\pi}{3}\cdot\rho f$

$$\vec{E}=\frac{\vec{r}}{r}\cdot\frac{1}{4\pi\varepsilon}\cdot Q_r\cdot\frac{1}{r^2}=\frac{(r^3-r_1^3)\rho f}{3\varepsilon r^3}$$

主題

日期

星期

若 $|\vec{r}|>r_2$， $q_r=\frac{4(r_2^3-r_1^3)\pi}{3}\rho_f$

$$\vec{E}=\frac{\vec{r}}{r}\cdot\frac{q_r}{4\pi\varepsilon r^2}=\frac{(r_2^3-r_1^3)\rho_f}{3\varepsilon r^3}$$

(2) 先求极化体电荷：

$$\rho_p=-\nabla P=-(\varepsilon-\varepsilon_0)\nabla\cdot E=-(\varepsilon-\varepsilon_0)\cdot\frac{1}{\varepsilon}\rho_f$$

$$=-(1-\frac{\varepsilon_0}{\varepsilon})\rho_f$$

再求内、外极化面电荷 σ_{p_1} σ_{p_2}

$\sigma_{p_1}=0$ (电荷排斥力)

$$\begin{cases}\sigma_{p_2}+\sigma_{f_2}=\sigma_2 & ① \\ \sigma_2\cdot S=\rho_f\cdot V & ② \\ \sigma_{p_2}:\sigma_{f_2}=(\varepsilon-\varepsilon_0):\varepsilon_0 & ③\end{cases}$$

联立以上三式解得 $\sigma_{p_2}=\frac{r_2^3-r_1^3}{3r_2^2}(1-\frac{\varepsilon_0}{\varepsilon})\rho_f$

8. 若 $r<r_1$. $B=0$

若 $r>r_2$. 记横截面为 S. 则 $B=\iint_S\frac{\mu_0 ds\ J_f\times(\vec{r}-\vec{r_0})}{2\pi|\vec{r}-\vec{r_0}|}$

$$=\frac{\mu_0 J_f}{2\pi}\times\iint_S\frac{\vec{r}-\vec{r_0}}{|\vec{r}-\vec{r_0}|}ds$$

$$=\frac{\mu_0(r_2^2-r_1^2)}{2r^2}J_f\times\vec{r}$$

若 $r_1<r<r_2$. 则将 μ_0 换为 μ. r_2 换为 r 得. $B=\frac{\mu(r^2-r_1^2)}{2r^2}J_f\times\vec{r}$

主題

日期

星期

$$J_M : J_f = (\mu-\mu_0):\mu_0 \Rightarrow J_M = \left(\frac{\mu}{\mu_0}-1\right)J_f \quad (r_1<r<r_2)$$

在内边界上 $\alpha_M = 0 \quad (r=r_1)$

外边界上 $\alpha_M = -\frac{\mu-\mu_0}{\mu_0}\frac{r_2^2-r_1^2}{2r_2}J_f \quad (r=r_2)$

9. 由第7题 $\rho_P = -\nabla\cdot\vec{P} = -\nabla\cdot(\varepsilon-\varepsilon_0)\vec{E} = -\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\rho_f$

11. 绝缘时: $\begin{cases} \mathscr{E} = E_1l_1+E_2l_2 & ① \\ \varepsilon_1E_1 = \varepsilon_2E_2 & ② \end{cases}$

$\Rightarrow E_1 = \mathscr{E}\cdot\frac{\varepsilon_2}{\varepsilon_2l_1+\varepsilon_1l_2} \quad E_2 = \mathscr{E}\frac{\varepsilon_1}{\varepsilon_2l_1+\varepsilon_1l_2}$

介质上电荷面密度 $\omega_f = \vec{n}\cdot(\vec{D}_2-\vec{D}_1) = \vec{n}\cdot(\varepsilon_2\vec{E}_2-\varepsilon_1\vec{E}_1)=0$

电容器两极: $\omega_{f1} = \vec{n}\cdot(\vec{D}_1-\vec{0}) = \vec{n}\cdot\varepsilon_1\vec{E}_1 = \frac{\mathscr{E}\varepsilon_1\varepsilon_2}{\varepsilon_2l_1+\varepsilon_1l_2}$

ω_{f2} 与 ω_{f1} 方向相反. $\omega_{f2} = -\frac{\mathscr{E}\varepsilon_1\varepsilon_2}{\varepsilon_2l_1+\varepsilon_1l_2}$

漏电时: 电阻 $R_1:R_2 = \sigma_2:\sigma_1$

两边电势差 $\mathscr{E}_1:\mathscr{E}_2 = \sigma_2:\sigma_1$

$\Rightarrow \begin{cases} \mathscr{E}_1 = \mathscr{E}\cdot\frac{\sigma_2}{\sigma_1+\sigma_2} \\ \mathscr{E}_2 = \mathscr{E}\cdot\frac{\sigma_1}{\sigma_1+\sigma_2} \end{cases}$

同上计算可得: 分界面上 $\omega_f = \frac{\varepsilon_2\sigma_1-\varepsilon_1\sigma_2}{\sigma_2l_1+\sigma_1l_2}\mathscr{E}$

主題

日期

星期

$$W_{f1} = \frac{\varepsilon_1 \sigma_2 \mathcal{E}}{\sigma_2 l_1 + \sigma_1 l_2}, \quad W_{f2} = -\frac{\varepsilon_2 \sigma_1 \mathcal{E}}{\sigma_2 l_1 + \sigma_1 l_2}.$$

12 (1) $\sigma_f = 0$，此时 $D_1 = D_2$

$\Rightarrow \varepsilon_1 E_1 = \varepsilon_2 E_2$

$\Rightarrow E_1 : E_2 = \varepsilon_2 : \varepsilon_1$

$$E_1 : E_2 = \frac{\tan\theta_2}{\tan\theta_1}$$

(2) 界面上有恒定电流 $\Rightarrow \vec{n} \cdot J_1 = \vec{n} \cdot J_2$

$\Rightarrow$ 电流法向分量相同.

$$\Rightarrow \frac{\tan\theta_2}{\tan\theta_1} = \frac{\sigma_2}{\sigma_1}$$