数值分析作业4

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5. 令 $g(x)=x-\lambda f(x) \Rightarrow g'(x)=1-\lambda f'(x)$

因 $f'(x)\in[m,M]$，$\lambda>0 \Rightarrow g'(x)\in[1-\lambda M, 1-\lambda m]$

又因 $\lambda\in(0,\frac{2}{M})$，故 $g'(x)\in(1-\frac{2}{M}M, 1)=(-1,1) \Rightarrow |g'(x)|<1$

由压缩映射不动点定理：$g(x)$ 在 $\mathbb{R}$ 上有唯一不动点

又因 $g(x^*)=x^*-f(x^*)\cdot\lambda=x^*$，故 x^* 为唯一的不动点.

从而迭代 $x_{k+1}=x_k-\lambda f(x_k)$ 收敛到 x^*

7 由超线性收敛的定义：$\lim\limits_{k\to\infty}\frac{|x_{k+1}-x^*|}{|x_k-x^*|}=0$

$$\Rightarrow \lim_{k\to\infty}\frac{x_{k+1}-x_k}{x_k-x^*}-\frac{x_{k+1}-x^*}{x_k-x^*}=\lim_{k\to\infty}\frac{x_{k+1}-x_k}{x_k-x^*}$$

$$又\ \lim_{k\to\infty}\frac{x_{k+1}-x_k}{x_k-x^*}-\frac{x_{k+1}-x^*}{x_k-x^*}=\lim_{k\to\infty}\frac{-x_k+x^*}{x_k-x^*}=-1$$

$$\Rightarrow \lim_{k\to\infty}\frac{|x_{k+1}-x_k|}{|x_k-x^*|}=1$$

19. 迭代法：$\begin{cases} x_1^{n+1}=0.7\sin x_1^n+0.2\cos x_2^n \\ x_2^{n+1}=0.7\cos x_1^n-0.2\sin x_2^n \end{cases}$

证明：

取 $\varphi([x_1,x_2]^T)=(0.7\sin x_1+0.2\cos x_2, 0.7\cos x_1-0.2\sin x_2)^T$

则原方程的解为迭代法不动点

$$\varphi'(x) = \begin{pmatrix} 0.7\cos x_1 & -0.2\sin x_2 \\ -0.7\sin x_1 & -0.2\cos x_2 \end{pmatrix}$$

$\|\varphi'(x)\|_\infty \le 0.7(|\sin x_1| + |\cos x_1|) \le 0.7\sqrt{2} < 1$

故φ为压缩映射

从而全局收敛，计算得.

$x^1 = (0.5111, 0.5184)^T \quad x^2 = (0.5161, 0.5114)^T$

$x^3 = (0.5199, 0.5109)^T$

20. (1) Newton 迭代法:

$F(x) = (x_1 - 0.7\sin x_1 - 0.2\cos x_2,\ x_2 - 0.7\cos x_1 + 0.2\sin x_2)^T$

$x^0 = (0.5, 0.5)^T$

$\Phi(x) = x - (F'(x))^{-1}F(x)$ 计算得:

$$x^1 = \begin{pmatrix} 0.5268 \\ 0.5080 \end{pmatrix} \quad x^2 = \begin{pmatrix} 0.5265 \\ 0.5079 \end{pmatrix} \quad x^3 = \begin{pmatrix} 0.5265 \\ 0.5079 \end{pmatrix}$$

逆Broyden 迭代:

$$F(x^0) = \begin{pmatrix} -0.0111 \\ -0.0184 \end{pmatrix} \quad B_0 = [F'(x^0)]^{-1} = \begin{pmatrix} 2.7908 & -0.2276 \\ -0.7967 & 0.9157 \end{pmatrix}$$

$k=0$: $\quad p^0 = -B_0 F(x^0) = \begin{pmatrix} 0.0268 \\ 0.0080 \end{pmatrix}$

$$x^1 = x^0 + p^0 = \begin{pmatrix} 0.5268 \\ 0.5080 \end{pmatrix}$$

$$q^0 = F(x^1) - F(x^0) = \begin{pmatrix} 0.0112 \\ 0.0186 \end{pmatrix}$$

$$B_1 = B_0 + \frac{(p^0 - B_0 q^0)(p^0)^T B_0}{(p^0)^T B_0 q^0} = \begin{pmatrix} 2.7641 & -0.2281 \\ -0.8051 & 0.9155 \end{pmatrix}$$

$k=1$. $p^1 = 10^{-4} \times \begin{pmatrix} -3.0528 \\ -0.9519 \end{pmatrix}$

$x^2 = \begin{pmatrix} 0.5265 \\ 0.5079 \end{pmatrix}$

此时 $\|p_1\|_\infty < 10^{-3}$，算法停止

(2) Newton 迭代：

$F(x) = (x_1^2 + x_2^2 - 4,\ x_1^2 - x_2^2 - 1)^T$

$x^0 = \begin{pmatrix} 1.6 \\ 1.2 \end{pmatrix}$

$x^1 = \begin{pmatrix} 1.5813 \\ 1.2250 \end{pmatrix}$ $x^2 = \begin{pmatrix} 1.5811 \\ 1.2247 \end{pmatrix}$ $x^3 = \begin{pmatrix} 1.5811 \\ 1.2247 \end{pmatrix}$

逆 Broyden 迭代：

$B_0 = \begin{pmatrix} 0.1562 & 0.1562 \\ 0.2083 & -0.2083 \end{pmatrix}$

$k=0$: $p^0 = \begin{pmatrix} -0.0188 \\ 0.0250 \end{pmatrix}$ $x^1 = \begin{pmatrix} 1.5813 \\ 1.2250 \end{pmatrix}$

$q^0 = \begin{pmatrix} 0.0010 \\ -0.1203 \end{pmatrix}$ $B_1 = \begin{pmatrix} 0.1560 & 0.1572 \\ 0.2077 & -0.2062 \end{pmatrix}$

$k=1$: $p^1 = 10^{-4} \times \begin{pmatrix} -1.0936 \\ -2.5924 \end{pmatrix}$

$x^2 = \begin{pmatrix} 1.5811 \\ 1.2247 \end{pmatrix}$

此时 $\|p^1\|_\infty < 10^{-3}$ 算法停止