数值分析作业3

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3. Doolittle分解:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{6} & \frac{1}{5} & 1 & 0 \\ -\frac{1}{6} & \frac{1}{10} & -\frac{9}{37} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 6 & 2 & 1 & -1 \\ 0 & \frac{10}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{37}{10} & -\frac{9}{10} \\ 0 & 0 & 0 & \frac{191}{74} \end{bmatrix}$$

$\det A = \det L \cdot \det U = 1 \times 191 = 191$

设 $y = Ux$ 则解 $Ly = b \Rightarrow y = (6, -1, \frac{21}{5}, -\frac{213}{74})^T$

解 $Ux = y \Rightarrow x = (\frac{151}{191}, -\frac{69}{191}, \frac{165}{191}, -\frac{213}{191})^T$

8 系数矩阵 $A = \begin{pmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{pmatrix}$

Cholesky分解: $A = LL^T$ 其中 $L = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -3 & 3 \end{pmatrix}$

求解 $Ly = b \Rightarrow y = (-1, 2, 6)^T$

求解 $L^Tx = y \Rightarrow x = (-\frac{9}{4}, 4, 2)^T$

11. $L = \begin{pmatrix} l_1 & & & \\ m_2 & l_2 & & \\ & \ddots & \ddots & \\ & & m_n & l_n \end{pmatrix}$ $\quad L^T = \begin{pmatrix} l_1 & m_2 & & \\ & l_2 & \ddots & \\ & & \ddots & m_n \\ & & & l_n \end{pmatrix}$

$$LL^T = \begin{bmatrix} l_1^2 & l_1 m_2 & 0 & \cdots & 0 \\ l_1 m_2 & m_2^2 + l_2^2 & l_2 m_3 & \cdots & 0 \\ 0 & l_2 m_3 & m_3^2 + l_3^2 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & & \cdots & m_n^2 + l_n^2 \end{bmatrix}$$

得到方程组:
$$\begin{cases} l_k m_{k+1} = a_{k+1} \quad (k=1,2,\cdots,n-1) \\ l_1^2 = b_1 \\ l_k^2 + m_k^2 = b_k \quad (k=2,3,\cdots,n) \end{cases}$$

解得:
$$l_1 = \sqrt{b_1}$$
$$\begin{cases} m_{k+1} = \dfrac{a_{k+1}}{l_k} = \dfrac{a_{k+1}}{\sqrt{b_k - m_k^2}} \\ l_k = \sqrt{b_k - m_k^2} \end{cases}$$

12. 证. $\|A\|_2 = \sqrt{\rho(A^TA)} = \rho(A)$

$\|L\|_2^2 = (\sqrt{\rho(L^TL)})^2 = \rho(L^TL) = \rho(A)$

故 $\|A\|_2 = \|L\|_2^2$

15. $A = \begin{pmatrix} 1 & 1 \\ -5 & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix}$, $\mathrm{cond}(A)_\infty = \|A\|_\infty \|A^{-1}\|_\infty = 6 \times 1 = 6$

$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ 特征值 $\lambda_1 = 2+\sqrt{2}$, $\lambda_2 = 2$, $\lambda_3 = 2-\sqrt{2}$

$\Rightarrow \rho(B) = \lambda_1 = 2+\sqrt{2}$

$\rho(B^{-1}) = \lambda_3^{-1} = 1 + \frac{\sqrt{2}}{2}$

$\mathrm{cond}(B)_2 = \|B\|_2 \|B^{-1}\|_2 = \rho(B)\rho(B^{-1}) = 3+2\sqrt{2}$

18. 证：$\|A^{-1}\delta A\| \le \|A^{-1}\|\cdot\|\delta A\| < 1 \Rightarrow I + A^{-1}\delta A$ 非异

$\Rightarrow A(I+A^{-1}\delta A) = A+\delta A$ 非异 $\Rightarrow (A+\delta A)^{-1}$ 存在

$$\begin{aligned} A^{-1} - (A+\delta A)^{-1} &= A^{-1}[(A+\delta A)-A](A+\delta A)^{-1} \\ &= A^{-1}\cdot\delta A\,(A+\delta A)^{-1} \\ &= A^{-1}\cdot\delta A\cdot A^{-1}(I+A^{-1}\delta A)^{-1} \end{aligned}$$

$$\Rightarrow \|A^{-1}-(A+\delta A)^{-1}\| \le \|A^{-1}\|^2\,\|\delta A\|\cdot\|I+A^{-1}\delta A\|^{-1} \le \frac{\|A^{-1}\|^2\|\delta A\|}{1-\|A^{-1}\|\cdot\|\delta A\|}$$

$$\Rightarrow \frac{\|A^{-1}-(A+\delta A)^{-1}\|}{\|A^{-1}\|} \le \frac{\mathrm{cond}(A)\frac{\|\delta A\|}{\|A\|}}{1-\mathrm{cond}(A)\frac{\|\delta A\|}{\|A\|}}$$

19. $(1+\alpha)A(x+\delta x) = (1+\beta)b$

$$\Rightarrow \delta x = \frac{1+\beta}{1+\alpha}A^{-1}b - x = \frac{\beta-\alpha}{1+\alpha}x$$

$$\Rightarrow \frac{\|\delta x\|_2}{\|x\|_2} = \left|\frac{\beta-\alpha}{1+\alpha}\right| \le \frac{|\alpha|+|\beta|}{1-|\alpha|}$$

21 B奇异 A非异 $\Rightarrow A^{-1}B$ 奇异 $\Rightarrow \|I-A^{-1}B\| \ge 1$

$$\Rightarrow \|A^{-1}\|\cdot\|A-B\| \ge \|I-A^{-1}B\| \ge 1$$

$$\Rightarrow \frac{1}{\mathrm{cond}(A)} \le \frac{\|A-B\|}{\|A\|}$$