数值分析作业14：（第八章 1.2.5.8.10.18.19题）

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1. (1) $f(x)=1$, $\int_0^1 f(x)\,dx=1$ $\frac{1}{4}f(0)+\frac{3}{4}f(\frac{2}{3})=1$

$f(x)=x$, $\int_0^1 f(x)\,dx=\frac{1}{2}$ $\frac{1}{4}f(0)+\frac{3}{4}f(\frac{2}{3})=\frac{1}{2}$

$f(x)=x^2$, $\int_0^1 f(x)\,dx=\frac{1}{3}$ $\frac{1}{4}f(0)+\frac{3}{4}f(\frac{2}{3})=\frac{1}{3}$

$f(x)=x^3$, $\int_0^1 f(x)\,dx=\frac{1}{4}$ $\frac{1}{4}f(0)+\frac{3}{4}f(\frac{2}{3})=\frac{2}{9}$

⇒ 代数精度为2

(2) $f(x)=1$ $\int_a^b f(x)\,dx=\frac{b-a}{2}[f(a)+f(b)]-\frac{(b-a)^2}{12}[f'(b)-f'(a)]=b-a$

$f(x)=x$ $\int_a^b f(x)\,dx=\frac{b-a}{2}[f(a)+f(b)]-\frac{(b-a)^2}{12}[f'(b)-f'(a)]=\frac{b^2-a^2}{2}$

$f(x)=x^2$ $\int_a^b f(x)\,dx=\frac{b-a}{2}[f(a)+f(b)]-\frac{(b-a)^2}{12}[f'(b)-f'(a)]=\frac{b^3-a^3}{3}$

$f(x)=x^3$ $\int_a^b f(x)\,dx=\frac{b-a}{2}[f(a)+f(b)]-\frac{(b-a)^2}{12}[f'(b)-f'(a)]=\frac{b^4-a^4}{4}$

$f(x)=x^4$ $\int_a^b f(x)\,dx=\frac{b^5-a^5}{5}$

$$\frac{b-a}{2}[f(a)+f(b)]-\frac{(b-a)^2}{12}[f'(b)-f'(a)]=\frac{b-a}{6}(a^4+2a^3b+2ab^3+b^4)$$

⇒ 代数精度为3

2. (1) $f(x)=1$ $\int_0^2 dx = 2 = C_0 + C_1 + C_2$

$f(x)=x$ $\int_0^2 x\,dx = 2 = C_1 + 2C_2$

$f(x)=x^2$ $\int_0^2 x^2 dx = \frac{8}{3} = C_1 + 4C_2$

$\Rightarrow$ $C_0 = \frac{1}{3}$ $C_1 = \frac{4}{3}$ $C_2 = \frac{1}{3}$

此时 $f(x)=x^3$ $\int_0^2 x^3 dx = 4 = \frac{1}{3}f(0) + \frac{4}{3}f(1) + \frac{1}{3}f(2)$

$f(x)=x^4$. $\int_0^2 x^3 dx = \frac{32}{5}$ $\frac{1}{3}f(0) + \frac{4}{3}f(1) + \frac{1}{3}f(2) = \frac{20}{3}$

$\Rightarrow$ 代数精度为3

(2) $f(x)=1$ $\int_0^1 dx = 1 = \frac{1}{2} + C_1$

$f(x)=x$ $\int_0^1 x\,dx = \frac{1}{2} = \frac{1}{2}x_0 + C_1 x_1$

$f(x)=x^2$ $\int_0^1 x^2 dx = \frac{1}{3} = \frac{1}{2}x_0^2 + C_1 x_1^2$

$\Rightarrow$ $C_1 = \frac{1}{2}$ $x_0 = \frac{1}{2} - \frac{\sqrt{3}}{6}$ $x_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}$

此时取 $f(x)=x^3$ $\int_0^1 x^3 dx = \frac{1}{4} = \frac{1}{2}f\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) + \frac{1}{2}f\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)$

$f(x)=x^4$ $\int_0^1 x^4 dx = \frac{1}{5}$

$\frac{1}{2}f\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) + \frac{1}{2}f\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) = \frac{7}{36}$

$\Rightarrow$ 代数精度为3

5. $f(x)=1$ $\int_0^1 dx = 1 = C_0 + C_1$

$f(x)=x$ $\int_0^1 x\,dx = \frac{1}{2} = C_1 + B_0$

$f(x)=x^2$ $\int_0^1 x^2\,dx = \frac{1}{3} = C_1$

$\Rightarrow C_0 = \frac{2}{3},\ C_1 = \frac{1}{3},\ B_0 = \frac{1}{6}$

此时取 $f(x)=x^3$ 得 $E(f) = -\frac{1}{12}$ $f^{3}(\xi) = 6$

$\Rightarrow k = -\frac{1}{72}$

8. 使用 Gauss 型求积公式：节点为关于 $P(x)=\sqrt{x}$ 的正交多项式零点 x_0 和 x_1

设 $w(x) = (x-x_0)(x-x_1) = x^2+bx+c$，因 $w(x)$ 与 1 与 x 正交：

$\int_0^1 \sqrt{x}\cdot w(x)\,dx = 0$ $\int_0^1 \sqrt{x}\,w(x)\cdot x\,dx = 0$

$\Rightarrow \frac{2}{7} + \frac{2}{5}b + \frac{2}{3}c = 0$ $\frac{2}{9} + \frac{2}{7}b + \frac{2}{5}c = 0$

$\Rightarrow b = -\frac{10}{9},\ c = \frac{5}{21} \Rightarrow w(x) = x^2 - \frac{10}{9}x + \frac{5}{21}$

$\Rightarrow x_0 = \frac{1}{63}(35 - 2\sqrt{70}) = 0.2900$ $x_1 = \frac{1}{63}(35 + 2\sqrt{70}) = 0.8212$

由此知公式对 1. x 均精确成立，故

$A_0 + A_1 = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}$ $A_0x_0 + A_1x_1 = \int_0^1 \sqrt{x}\cdot x\,dx = \frac{2}{5}$

$\Rightarrow A_0 = 0.2776$ $A_1 = 0.3891$

10. $x_0=\frac{\sqrt{2}}{2}$, $x_1=-\frac{\sqrt{2}}{2}$, $A_1=A_2=\frac{\pi}{2}$

$$\Rightarrow \int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}dx=\frac{\pi}{2}\left[\left(\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{\sqrt{2}}{2}\right)^2\right]=\frac{\pi}{2}$$

18 (1) $x_1=x_0-h$. $x_2=x_0+3h$, Lagrange 插值后得:

$$f(x)=\frac{x-x_2}{x_1-x_2}f(x_1)+\frac{x-x_1}{x_2-x_1}f(x_2)+\frac{f''(\xi)}{2}(x-x_1)(x-x_2)$$

对x求导: $f'(x)\Big|_{x=x_0}=\frac{1}{4h}[f(x_0+3h)-f(x_0-h)]-hf''(\xi)$

误差项为 $-f''(\xi)\cdot h$

(2) x_0, $x_1=x_0+h$, $x_2=x_0+2h$, Lagrange 插值后得.

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$
$$+\frac{f'''(\xi)}{6}(x-x_0)(x-x_1)(x-x_2)$$

对x求导: $f'(x)|_{x=x_0}=\frac{1}{2h}[4f(x_0+h)-3f(x_0)-f(x_0+2h)]+\frac{h^2}{3}f'''(\xi)$

误差项为 $\frac{h^2}{3}f'''(\xi)$