数值分析作业7

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1. (1) $\lim\limits_{k\to\infty} x^{(k)} = (0,1,3)^T$

(2) $\lim\limits_{k\to\infty} x^{(k)} = (0,0,\frac{1}{2})^T$

5. (1) $B_J = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 \end{pmatrix}$ $B_G = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} \\ 0 & \frac{a_{12}a_{21}}{a_{11}a_{12}} \end{pmatrix}$

$\rho(B_J) = \sqrt{\left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|}$ $\rho(B_G) = \left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right|$

$\Rightarrow \rho(B_G) = \rho(B_J)^2$. 故 $\rho(B_J)<1 \Leftrightarrow \rho(B_G)<1$

$\Rightarrow$ J法与GS法同收敛或不收敛

(2) $\frac{R(B_J)}{R(B_G)} = \frac{-\ln\rho(B_J)}{-\ln\rho(B_G)} = \frac{1}{2}$

(3) 代入得 $\rho(B_J) = \frac{1}{|a|} \Rightarrow$ $\rho(B_J)<1 \Leftrightarrow |a|>1$

故 J法和GS法收敛 $\Leftrightarrow$ $a\in(-\infty,-1)\cup(1,+\infty)$

11. A特征值 1和4 $\Rightarrow$ 迭代矩阵 $B = I+\alpha A$ 特征值: $1+\alpha$. $1+4\alpha$

$\Rightarrow \rho(B) = \max\{|1+\alpha|, |1+4\alpha|\}$

故收敛 $\Leftrightarrow$ $\rho(B)<1$ $\Leftrightarrow$ $|1+\alpha|<1$ 且 $|1+4\alpha|<1$ $\Leftrightarrow \alpha\in(-\frac{1}{2},0)$

$$\Rightarrow \quad \rho(B) = \begin{cases} -1-4\alpha & -\frac{1}{2}<\alpha\le-\frac{2}{5} \\ 1+\alpha & -\frac{2}{5}\le\alpha<0 \end{cases}$$

当 $\alpha = -\frac{2}{3}$ 时收敛最快

12 迭代矩阵 $B = I - \omega A$ A的特征值 λ_i => B的特征值 $1-\omega\lambda_i$

$$\Rightarrow \rho(B) = \max_{1\le i\le n} |1-\omega\lambda_i|$$

迭代 $\Leftrightarrow |1-\omega\lambda_i| < 1 \ (i=1,2,\cdots,n)$

$\Leftrightarrow 0 < \omega < \frac{2}{\lambda_i} \ (i=1,2,\cdots,n)$

$\Leftrightarrow \omega \in (0, \frac{2}{\lambda_i}) \ (i=1,2\cdots n)$

$\Leftrightarrow \omega \in (0, \frac{2}{\lambda_1})$

因 $\omega > 0$. 故 $1-\omega\lambda_1 \le 1-\omega\lambda_2 \le \cdots \le 1-\omega\lambda_n$

$\Rightarrow |1-\omega\lambda_i| \le \max\{|1-\omega\lambda_1|, |1-\omega\lambda_n|\}$

$\Rightarrow \rho(B) = \max\{|1-\omega\lambda_1|, |1-\omega\lambda_n|\}$

$\Rightarrow \alpha = \frac{2}{\lambda_1+\lambda_n}$ 时 $\rho(B)$ 最小，$R(B) = -\ln \rho(B)$

17. $$\|B_J\|_\infty = \max_{1\le i\le n} \sum_{\substack{j=1\\ j\ne i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| < 1$$

$$\Rightarrow \sum_{\substack{j=1\\ j\ne i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| < 1 \quad (i=1,2,\cdots,n) \Rightarrow |a_{ii}| > \sum_{\substack{j=1\\ j\ne i}}^{n} |a_{ij}| \ (i=1,2,\cdots,n)$$

=> A是严格对角占优矩阵 => GS法收敛

18 (1) $r_0 = (0,-1)^T$，$p_0 = (0,-1)^T$

$\alpha_0 = 0.5$，$x_1 = (0,-0.5)^T$，$r_1 = (1.5,0)^T$，$\beta_0 = 2.25$，$p_1 = (1.5,-2.25)^T$

$\alpha_1 = \frac{2}{3}$，$x_2 = (1,-2)^T$，$r_2 = (0,0)^T$

$\Rightarrow x = (1,-2)^T$

(2) $r_0=(3,5,-5)^T$, $p_0=(3,5,-5)^T$

$\alpha_0=0.156914$, $x_1=(0.470745, 0.784574, -0.784574)^T$

$r_1=(-1.236702, -0.335106, -1.077127)^T$

$\beta_0=0.0474904$, $p_1=(-1.0942310, -0.0976545, -1.3145800)^T$

...

$x_3=(0,1,-1)^T$, $r_3=(0,0,0)^T$

$\Rightarrow x=(0,1,-1)^T$

19. 反证法：设有 $c_i\ (i=1,2,\cdots,k)$ s.t. $\sum_{i=1}^{k} c_i p^{(i)}=0$

左乘向量 $(p^{(j)})^T A$ 得：$\sum_{i=1}^{k} c_i p^{(j)T} A p^{(i)}=0$

因 $p^{(j)}\neq 0$，A对称正定，故 $(p^{(j)})^T A p^{(j)}>0 \Rightarrow c_j=0$

由 j 的任意性：$c^{(1)}, c^{(2)}, \cdots, c^{(k)}=0$，矛盾

故 $p^{(1)}, p^{(2)}, \cdots, p^{(k)}$ 线性无关

20. $\varphi(x^{(k+1)})=\varphi(x^{(k)}+\alpha_k p^{(k)})=\varphi(x^{(k)})-\frac{1}{2}\frac{(r^{(k)},p^{(k)})^2}{(Ap^{(k)},p^{(k)})}$

$(r^{(k)},p^{(k)})=(r^{(k)},r^{(k)})$

$\Rightarrow \varphi(x^{(k+1)})=\varphi(x^{(k)})-\frac{1}{2}\frac{(r^{(k)},r^{(k)})^2}{(Ap^{(k)},p^{(k)})}$

$\Rightarrow \varphi(x^{(k+1)})\leq\varphi(x^{(k)})$

并且若 $r^{(k)}\neq 0$ 则有 $\varphi(x^{(k+1)})<\varphi(x^{(k)})$