数值分析作业-1

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2\. (1) $1-\cos 2^\circ \approx 6\times 10^{-4}$, $|(1-\cos 2^\circ)-6\times 10^{-4}| \approx 9.173\times 10^{-6}$

$0.5\times 10^{-5} < 9.173\times 10^{-6} < 0.5\times 10^{-4}$

故有一位有效数字

(2) $|(1-\cos 2^\circ) - 2\times 0.0175^2| \approx 3.33\times 10^{-6}$

$0.5\times 10^{-6} < 3.33\times 10^{-6} < 0.5\times 10^{-5}$

故有2位有效数字

(3) $\left|(1-\cos 2^\circ) - \frac{0.0349^2}{1+0.9994}\right| \approx 1.48\times 10^{-8}$

$0.5\times 10^{-8} < 1.48\times 10^{-8} < 0.5\times 10^{-7}$

故有4位有效数字

(4) $1-\cos x = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(2k)!}\left(\frac{\pi}{90}\right)^{2k}$ (Taylor)

$\left|(1-\cos 2^\circ) - \sum_{k=1}^{2} \frac{(-1)^{k-1}}{(2k)!}\cdot\left(\frac{\pi}{90}\right)^{2k}\right| \approx 9.04\times 10^{-13}$

$0.5\times 10^{-12} < 9.04\times 10^{-13} < 0.5\times 10^{-11}$

故有8位有效数字

3\. $\left|e^{-5} - \sum_{i=0}^{9}(-1)^i \frac{5^i}{i!}\right| = \left|\sum_{i=10}^{\infty}(-1)^i\frac{5^i}{i!}\right| = \sum_{k=5}^{\infty}\left[\frac{5^{2k}}{(2k)!} - \frac{5^{2k+1}}{(2k+1)!}\right] \geqslant \frac{5^{10}}{10!} - \frac{5^{11}}{11!}$

$\left|e^{-5} - \left(\sum_{i=0}^{9}\frac{5^i}{i!}\right)^{-1}\right| = \left|(e^5)^{-1} - \left(\sum_{i=0}^{9}\frac{5^i}{i!}\right)^{-1}\right| = e^{-5}\left(\sum_{i=0}^{9}\frac{5^i}{i!}\right)^{-1}\cdot\left|e^5 - \sum_{i=0}^{9}\frac{5^i}{i!}\right|$

上式 $= \frac{6}{11}\cdot\frac{5^{10}}{10!}$

下式中，$\left|e^5 - \sum_{i=0}^{9}\frac{5^i}{i!}\right| = \frac{1}{9!}\int_0^5 e^t(5-t)^9\,dt$ (积分余项)

$$= \frac{5^{10}}{9!}\int_0^1 e^x (1-x)^9 dx \le \frac{5^{10}}{9!}\int_0^1 (1-x)^4 dx = \frac{5^9}{9!}$$

注意到 $\frac{5^9}{9!} < \frac{5^{10}}{10!}\cdot\frac{6}{11}$，故 $\left|e^{-5} - \sum_{i=0}^{9}(-1)^i \frac{5^i}{i!}\right|$ 更大，计算更粗略

5. (1) 当 $N$ 较大时 $\arctan(N+1)$ 与 $\arctan N$ 相差较小，不宜直接相减

故 $\arctan(N+1) - \arctan N$

$$= \arctan(\tan(\arctan(N+1) - \arctan N))$$

$$= \arctan\left[\frac{(N+1)-N}{1+(N+1)\cdot N}\right]$$

$$= \arctan\left(\frac{1}{N^2+N+1}\right)$$

(2) 误差原因同上：

$$\sqrt{x+\frac{1}{x}} - \sqrt{x-\frac{1}{x}} = \frac{\frac{2}{x}}{\sqrt{x+\frac{1}{x}}+\sqrt{x-\frac{1}{x}}} = \frac{2}{x}\,\frac{1}{\sqrt{x+\frac{1}{x}}+\sqrt{x-\frac{1}{x}}}$$

(3) $\ln(x+1) - \ln x = \ln\frac{x+1}{x} = \ln\left(1+\frac{1}{x}\right)$

设 $\frac{1+a}{1-a} = \frac{x+1}{x}$，解得 $a = \frac{1}{2x+1}$

故由Taylor展开式 $\ln\left(1+\frac{1}{x}\right) = \ln\left(\frac{1+a}{1-a}\right) = \sum_{k=0}^{\infty}\frac{2}{(2k+1)!}a^{2k+1}$

(4) $x \approx \frac{\pi}{4}$ 时 $\cos x$ 与 $\sin x$ 均接近于 $\frac{\sqrt{2}}{2}$

$\cos^2 x - \sin^2 x = \cos 2x$ 即可算

7. (1) 不准确（直接验证）

(2) $I_n - \tilde{I}_n = (1 - nI_{n-1}) - (1 - n\tilde{I}_{n-1}) = n(\tilde{I}_{n-1} - I_{n-1})$

$$= (-n)\cdot(I_{n-1} - \tilde{I}_{n-1})$$

$\Rightarrow \varepsilon_n = n!\cdot(-1)^n \varepsilon_0 \Rightarrow |\varepsilon_n| = n!\,|\varepsilon_0|$