PDE数值解第四次作业：

1. 令 $u_j^n = v_n \cdot e^{ijwh}$，$\frac{v^{n+1}e^{ijwh} - \frac{1}{2}v_n(e^{i(j+1)wh}+e^{i(j-1)wh})}{\tau}$

$= \frac{1}{h^2} v_n [e^{i(j+1)wh} - 2e^{ijwh} + e^{i(j-1)wh}]$

故 $v_{n+1} = v_n \cdot (\cos wh + \frac{\tau}{h^2}(2\cos wh - 2))$

$|G| = |\cos wh + \frac{\tau}{h^2}(2\cos wh - 2)|$ 无条件不稳定

2. $u_j^{n-1} = u_j^{n+1} - 2\tau\partial_t u_j^{n+1} + \frac{1}{2}(-2\tau)^2\partial_{tt}u_j^{n+1} + \frac{1}{6}(-2\tau)^3\partial_{ttt}u_j^{n+1} + O(\tau^4)$

$u_j^n = u_j^{n+1} - \tau\partial_t u_j^{n+1} + \frac{1}{2}(-\tau)^2\partial_{tt}u_j^{n+1} + \frac{1}{6}(-\tau)^3\partial_{ttt}u_j^{n+1} + O(\tau^4)$

$\delta_x^2 u_j^{n+1} = u_j^{n+2} - 2u_j^{n+1} + u_j^n = h^2\partial_{xx}u_j^{n+1} + 2\cdot\frac{1}{24}h^4\partial_{xxxx}u_j^{n+1} + O(h^6)$

故截断误差为. $\theta\cdot\frac{1}{2\tau}(2\tau\partial_t - \frac{1}{2}4\tau^2\partial_{tt} + \frac{1}{6}8\tau^3\partial_{ttt})$

$+ (1-\theta)\cdot\frac{1}{\tau}(\tau\partial_t - \frac{1}{2}\cdot 3\tau^2\partial_{tt} + \frac{7}{6}\partial_{ttt})) u_j^{n+1} - \partial_{xx}u_j^{n+1} + \frac{1}{12}h^2\partial_{xxxx}u_j^{n+1} + O(h^4)$

若为二阶精度，则 $-\theta\tau - (1-\theta)\cdot\frac{3}{2}\tau = 0$，$\theta = 3$

3. $\delta_x^2 u_j^{n+1} = h^2\partial_{xx}u_j^{n+1} + 2\cdot\frac{1}{4!}h^4\partial_{xxxx}u_j^{n+1} + O(h^6)$

$\delta_x^2 u_j^n = h^2\partial_{xx}u_j^n + 2\cdot\frac{1}{4!}h^4\partial_{xxxx}u_j^n + O(h^6)$

$u_{j+1}^{n+1} - u_{j+1}^n = \tau\partial_t \cdot u_{j+1}^n + \frac{1}{2}\tau^2\partial_{tt}u_{j+1}^n + \frac{1}{6}\tau^3\partial_{ttt}u_{j+1}^n + O(\tau^4)$

$u_j^{n+1} - u_j^n = \tau\partial_t u_j^n + \frac{1}{2}\tau^2\partial_{tt}u_j^n + \frac{1}{6}\tau^3\partial_{ttt}u_j^n + O(\tau^4)$

$u_{j-1}^{n+1} - u_{j-1}^n = \tau\partial_t u_{j-1}^n + \frac{1}{2}\tau^2\partial_{tt}u_{j-1}^n + \frac{1}{6}\tau^3\partial_{ttt}u_{j-1}^n + O(\tau^4)$

$LHS = \frac{1}{12}(h^2\partial_{txx}u_j^n + \frac{1}{2}\tau h^2\partial_{ttxx}u_j^n + \frac{1}{6}\tau^2 h^2\partial_{tttxx}u_j^n + O(\tau^3 + h^4)) +$

$\partial_t u_j^n + \frac{1}{2}\tau\partial_{tt}u_j^n + \frac{1}{6}\tau^2\partial_{ttt}u_j^n + O(\tau^3)$

故截断误差 LHS-RHS $= LHS - \frac{\alpha}{2}[\partial_{xx}(u_j^{n+1}+u_j^n)+\frac{1}{12}h^2\partial_{xxxx}(u_j^{n+1}+$

$u_j^n)+o(h^4)]$

$$u_j^{n+1}=u_j^n+\tau\partial_t u_j^n+\frac{1}{2}\tau^2\partial_{tt}u_j^n+o(\tau^3)$$

$\Rightarrow$ 二阶精度

4. 如下格式: $\frac{u_j^{n+1}-u_j^n}{\tau}=\frac{1}{h^2}[\frac{1}{2}\Delta_+^x(\bar{a}_j\Delta_-^x u_j^{n+1})+\frac{1}{2}\Delta_+^x(\bar{a}_j\Delta_-^x u_j^n)]$

其中 $\Delta_+^x v_j = v_{j+1}-v_j \quad \Delta_-^x v_j = v_j - v_{j-1}$

$\bar{a}_j = \frac{h}{\int_{x_{j-1}}^{x_j}\frac{dx}{0.1+\sin^2 x}}$ ，二阶差分格式

5. $\lim_{\tau,h\to 0}\frac{u_{j+1}^n-(u_j^{n+1}+u_j^{n-1})+u_{j-1}^n}{h^2}=\lim_{\tau,h\to 0}\frac{(u_{j+1}^n-2u_j^n+u_{j-1}^n)-(u_j^{n+1}-2u_j^n+u_j^{n-1})}{h^2}$

$=\partial_{xx}u_j^n-\lim_{\tau,h\to 0}\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{h^2}=\partial_{xx}u_j^n-u_{tt}\lim_{\tau,h\to 0}\frac{\tau^2}{h^2}$

故当且仅当 $\tau \ll h$ 时相容

6. $(I-\frac{\lambda}{2}\delta_x^2)(I-\frac{\lambda}{2}\delta_y^2)u_{j+1}^{n+1}=(I+\frac{\lambda}{2}\delta_x^2)u_{j+1}^n+\lambda\delta_y^2 u_{jm}^n-\frac{\lambda}{2}\delta_y^2(I-\frac{\lambda}{2}\delta_x^2)u_{jm}^n$

$=(I+\frac{\lambda}{2}\delta_x^2)(I+\frac{\lambda}{2}\delta_y^2)u_{jm}^n$ ，故 $(I-\frac{\lambda}{2}\delta_x^2)(I-\frac{\lambda}{2}\delta_y^2)(u_{jm}^{n+1}-u_{jm}^n)$

$=\lambda(\delta_x^2+\delta_y^2)u_{jm}^n$

$=\lambda(h^2(\partial_{xx}+\partial_{yy})+\frac{1}{12}h^4(\partial_{xxxx}+\partial_{yyyy})+o(h^6))u_{jm}^n$

$(I-\frac{\lambda}{2}\delta_x^2)(I-\frac{\lambda}{2}\delta_y^2)(u_{jm}^{n+1}-u_{jm}^n)=[I-\frac{\lambda}{2}h^2\partial_{xx}-\frac{\lambda}{12}h^4\partial_{xxxx}+o(h^6)]$

$[I-\frac{\lambda}{2}h^2\partial_{yy}-\frac{\lambda}{12}h^4\partial_{yyyy})+o(h^6)][\tau\partial_t+\frac{\tau^2}{2}\partial_{tt}+o(\tau^3)]-$

$$\lambda\left(h^2(\partial_{xx}+\partial_{yy})+\frac{1}{12}h^4(\partial_{xxxx}+\partial_{yyyy})+O(h^6)\right)u_{jm}^{n}$$ 三阶精度

7. $Au_t+Bu_x=Cu_{xx}$, $A=1$, $B=-mu^{m-1}$, $C=u^m$, 故可构造:

$$\frac{u_j^{n+\frac{1}{2}}-u_j^n}{\left(\frac{\tau}{2}\right)}-m{u_j^n}^{m-1}\frac{\Delta_x^0 u_j^n}{2h}=\frac{(u_j^n)^m}{h^2}\delta_x^2 u_j^{n+\frac{1}{2}}$$

$\Rightarrow$ 计算得 $u_j^{n+\frac{1}{2}}$, 然后:

$$\frac{u_j^{n+1}-u_j^n}{2}-m(u_j^{n+\frac{1}{2}})^{m-1}\frac{\Delta_x^0(u_j^n+u_j^{n+1})}{4h}=\frac{(u_j^{n+\frac{1}{2}})^m}{2h^2}\delta_x^2(u_j^n+u_j^{n+1})$$

$\Rightarrow$ 计算得 u_j^{n+1}