PDE 数值解作业3：

1. 对 $\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{u_{j+1}^{n+1}-u_{j+1}^n}{\tau}+\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2h}+\frac{u_{j+1}^{n}-u_{j-1}^{n}}{2h}$

$$u_j^{n+1}=u_j^n+\tau\frac{\partial}{\partial t}u_j^n+\frac{1}{2}\tau^2\frac{\partial^2}{\partial t^2}u_j^n+O(\tau^3)$$

$$u_{j+1}^{n+1}=u_{j+1}^n+\tau\cdot\frac{\partial}{\partial t}u_{j+1}^n+\frac{1}{2}\tau^2\frac{\partial^2}{\partial t^2}u_{j+1}^n+O(\tau^3)$$

$$u_{j+1}^{n+1}=u_j^{n+1}+h\cdot\frac{\partial}{\partial x}u_j^{n+1}+\frac{1}{2}h^2\frac{\partial^2}{\partial x^2}u_j^{n+1}+\frac{1}{6}h^3\frac{\partial^3}{\partial x^3}u_j^{n+1}+O(h^4)$$

$$u_{j+1}^{n}=u_j^n+h\frac{\partial}{\partial x}u_j^n+\frac{1}{2}h^2\frac{\partial^2}{\partial x^2}u_j^n+\frac{1}{6}h^3\frac{\partial^3}{\partial x^3}u_j^n+O(\tau^2)$$

$$\frac{u_{j+1}^{n+1}-u_{j+1}^n}{\tau}=\frac{\partial}{\partial t}u_{j+1}^n+\frac{1}{2}\tau\frac{\partial^2}{\partial t^2}u_{j+1}^n+O(\tau^2)$$

$$\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2h}=\frac{\partial}{\partial x}u_j^{n+1}+\frac{1}{6}h^2\frac{\partial^3}{\partial x^3}u_j^{n+1}+O(h^3)$$

$$\frac{u_{j+1}^{n}-u_{j-1}^{n}}{2h}=\frac{\partial}{\partial x}u_j^{n}+\frac{1}{6}h^2\frac{\partial^3}{\partial x^3}u_j^{n}+O(h^3)$$

相加为：$\frac{1}{2}\tau\frac{\partial^2}{\partial t^2}u_j^n+O(\tau^2)+\frac{\partial}{\partial t}u_{j+1}^n+\frac{1}{2}\tau\frac{\partial^2}{\partial t^2}u_{j+1}^n+\frac{\partial}{\partial x}u_j^{n+1}+\frac{1}{6}h^2\frac{\partial^3}{\partial x^3}u_j^{n+1}+$

$\frac{1}{6}h^2\frac{\partial^3}{\partial x^3}u_j^n+O(h^3)$

但 $\frac{\partial}{\partial t}u_{j+1}^n+\frac{\partial}{\partial x}u_j^{n+1}=\frac{\partial}{\partial t}(u_j^n+h\frac{\partial}{\partial x}u_j^n+\frac{1}{2}h^2\frac{\partial^2}{\partial x^2}u_j^n+O(h^3))+\frac{\partial}{\partial x}(u_j^n+$

$\tau\frac{\partial}{\partial t}u_j^n+\frac{1}{2}\tau^2\frac{\partial^2}{\partial t^2}u_j^n+O(\tau^3))$

故为 $\frac{1}{2}\tau\partial_{tt}u_j^n+O(\tau^2)+\frac{1}{2}\tau\partial_{tt}u_{j+1}^n+(\tau+h)\partial_{xx}u_j^n+\frac{1}{2}h^2\partial_{txx}u_j^n+\frac{1}{2}\tau^2\partial_{ttx}u_j^n$

$+\frac{1}{6}h^2\partial_{xxx}u_j^{n+1}+\frac{1}{6}h^2\partial_{xxx}u_j^n+O(h^3)=h\partial_{tx}u_j^n+\frac{1}{2}\tau h\partial_{ttx}u_j^n+O(\tau h^2)$

$+O(\tau^2)+O(h^3)$

又 $\tau h\leq\frac{1}{2}(h^2+\tau^2)$ 故空间精度一阶，时间精度二阶

令 $u_j^n = v^n \cdot e^{ijwh}$，则 $\frac{v^{n+1} \cdot e^{ijwh} - v^n e^{ijwh}}{\tau} + \frac{v^{n+1} e^{i(j+1)wh} - v^n e^{i(j+1)wh}}{\tau}$

$$+ \frac{v^{n+1} e^{i(j+1)wh} - v^{n+1} e^{i(j-1)wh}}{2h} + \frac{v^n e^{i(j+1)wh} - v^n e^{i(j-1)wh}}{2h} = 0$$

$v^{n+1} [(1+\cos wh + i\sin wh) \cdot \frac{1}{\tau} + \frac{1}{2h} 2i\sin wh] = v^n [(1+\cos wh + i\sin wh) \cdot \frac{1}{\tau}$

$- \frac{1}{2h} \cdot 2i\sin wh]$

$|G| \le 1 \iff |(\frac{1}{\tau} + \frac{1}{h})\sin wh| \ge |(\frac{1}{\tau} - \frac{1}{h})\sin wh|$

$\iff |\frac{1}{\tau} + \frac{1}{h}| \ge |\frac{1}{\tau} - \frac{1}{h}|$ 一直成立

2. 原方程为 $\frac{u_{j+1}^{n+1} - u_{j+1}^n}{\tau} + a \frac{u_{j+1}^{n+1} - u_j^{n+1}}{h} + \frac{u_j^{n+1} - u_j^n}{\tau} + a \frac{u_{j+1}^n - u_j^n}{h}$

$u_{j+1}^n = u_{j+1}^{n+1} - \tau \partial_t u_{j+1}^{n+1} + \frac{1}{2}\tau^2 \partial_{tt} u_{j+1}^{n+1} + O(\tau^3)$

故原式 $= -\frac{1}{2}\tau \partial_{tt} u_{j+1}^{n+1} - \frac{1}{2} ah \partial_{xx} u_{j+1}^{n+1} + \frac{1}{2}\tau \partial_{tt} u_j^n + \frac{1}{2} ah \partial_{xx} u_j^n + O(\tau^2 + h^2)$

$= -\frac{1}{2}\tau (\tau \partial_{ttt} u_j^n + h \partial_{ttx} u_j^n + O(\tau^2 + h^2)) - \frac{1}{2} ah (\tau \cdot \partial_{txx} u_j^n$

$+ h \partial_{xxx} u_j^n + O(\tau^2 + h^2)] + O(\tau^2 + h^2)$

$\Rightarrow$ 时间. 空间均为 2 阶

令 $u_j^n = v^n \cdot e^{ijwh}$ $(1+a\lambda) v^{n+1} e^{i(j+1)wh} + (1-a\lambda) v^{n+1} - (1-a\lambda) v^n e^{i(j+1)wh} -$

$(1+a\lambda) v^n e^{ijwh} = 0$

$[(1+a\lambda) e^{iwh} + (1-a\lambda)] v^{n+1} = [(1-a\lambda) e^{iwh} + (1+a\lambda)] v^n$

$v^{n+1} = \frac{(1-a\lambda)\cos wh + (1+a\lambda) + (1-a\lambda)\sin wh}{(1+a\lambda)\cos wh + (1-a\lambda) + (1+a\lambda)\sin wh}$ $\iff$ $(1+a\lambda)\sin wh]^2$

$[(1-a\lambda)\cos wh + (1+a\lambda)]^2 + [(1-a\lambda)\sin wh]^2 = [(1+a\lambda)\cos wh +$

一直成立，一直稳定

当给出初始条件：$u(0,t)=f(t)$，$u(x,0)=g(x)$ 时

我们通过 u_{j+1}^{n}，u_j^{n+1}，u_j^{n} 可得 u_{j+1}^{n+1} 的值，因此可通过 $u_0^1, u_0^2, \cdots, u_0^n$

$\to u_1^1, u_1^2, \cdots, u_1^n \to \cdots \to u_j^1, u_j^2, \cdots, u_j^n$ 得到 u_j^n 的值

3. $\Lambda=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$，$S=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$，$S^{-1}AS=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$，$S^{-1}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}$

令 $\begin{pmatrix}u'\\v'\end{pmatrix}=S^{-1}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\frac{1}{2}u+\frac{1}{2}v\\\frac{1}{2}u-\frac{1}{2}v\end{pmatrix}$，则 $\frac{\partial u'}{\partial t}+\frac{\partial u'}{\partial x}=0$ $\quad \frac{\partial v'}{\partial t}-\frac{\partial v'}{\partial x}=0$

故 $\frac{u_j'^{n+1}-u_j'^{n}}{T}+a\frac{u_j'^{n}-u_j'^{n-1}}{h}=0$

再由 $u=u'+v'$，$v=u'-v'$ 得到 $u.v$ 的表达式

又 $u=(u,v)$，故 $u_j^{n+1}=u_j^n-\frac{h}{2}A(u_{j+1}^n-u_{j-1}^n)+\frac{\lambda}{2}(u_{j+1}^n-2u_j^n+u_{j-1}^n)$

4. $\frac{\partial}{\partial t}\int_{\mathbb{R}}u(x,t)\,dx=0$ 故 $\frac{\partial}{\partial t}\int_{-\infty}^{\xi^-(t)}u(x,t)\,dx+\frac{\partial}{\partial t}\int_{\xi(t)^+}^{\infty}u(x,t)\,dx=0$

故 $\xi'(t)\,u(\xi^-(t))-\xi'(t)\,u(\xi(t)^+,t)+\int_{-\infty}^{+\infty}f(u)_x\,dx=0$

故 $\xi'(t)u_L-\xi'(t)u_R=f(u_L)-f(u_R)$，$\xi'(t)=\frac{f(u_L)-f(u_R)}{u_L-u_R}$ 又 $\xi(0)=0$

故 $\xi(t)=\frac{f(u_L)-f(u_R)}{u_L-u_R}t$

5. $u_j^{n+1}=u_j^n-\lambda(g_{j+\frac{1}{2}}^n-g_{j-\frac{1}{2}}^n)$，其中 $g_{j+\frac{1}{2}}^n=f(R(0,u_j^n,u_{j+1}^n))$

$g_{j-\frac{1}{2}}^n=f(R(0,u_{j-1}^n,u_j^n))$

令 $v_{j+\frac{1}{2}}^n=\lambda\frac{f(u_{j+1}^n)-f(u_j^n)}{u_{j+1}^n-u_j^n}$ $\quad w_{j+\frac{1}{2}}^n=\lambda\frac{f(u_{j+1}^n)+f(u_j^n)-2f(R(0,u_j^n,u_{j+1}^n))}{u_{j+1}^n-u_j^n}$

则 $u_j^{n+1}=u_j^n-\frac{1}{2}(W_{j-\frac{1}{2}}^n+V_{j-\frac{1}{2}}^n)(u_j^n-u_{j-1}^n)+\frac{1}{2}(W_{j+\frac{1}{2}}^n-V_{j+\frac{1}{2}}^n)(u_{j+1}^n-u_j^n)$

由R的性质：$W_{j-\frac{1}{2}}^n+V_{j-\frac{1}{2}}^n\geqslant 0$，$W_{j+\frac{1}{2}}^n-V_{j+\frac{1}{2}}^n\geqslant 0$

$\frac{1}{2}(W_{j+\frac{1}{2}}^n+V_{j+\frac{1}{2}}^n+W_{j+\frac{1}{2}}^n-V_{j+\frac{1}{2}}^n)=W_{j+\frac{1}{2}}^n\leq 1$ 故为TVD

6. 令 $v_j^n=\frac{u_j^n-u_j^{n-1}}{2}$，$W_{j-\frac{1}{2}}^n=\frac{u_j^n-u_{j-1}^n}{h}$

差分格式 $\Leftrightarrow$ $\frac{V_j^{n+1}-V_j^n}{\tau}=\frac{W_{j+\frac{1}{2}}^{n+1}-W_{j-\frac{1}{2}}^{n+1}}{4h}+\frac{W_{j+\frac{1}{2}}^{n}-W_{j-\frac{1}{2}}^{n}}{2h}+\frac{W_{j+\frac{1}{2}}^{n-1}-W_{j-\frac{1}{2}}^{n-1}}{4h}$

$\frac{W_{j-\frac{1}{2}}^{n+1}-W_{j-\frac{1}{2}}^{n}}{\tau}=\frac{1}{h}(V_j^{n+1}-V_{j-1}^{n+1})$

再令 $w_j^n=s_j^{n+1}$，令 $v_j^n=a_n e^{ijwh}$，$w_j^n=b_n\cdot e^{ijwh}$，$s_j^n=c_n e^{ijwh}$

则 $\frac{a_{n+1}-a_n}{\tau}=\frac{b_{n+1}}{4h}(e^{\frac{1}{2}iwh}-e^{-\frac{1}{2}iwh})+\frac{b_n}{2h}(e^{\frac{1}{2}iwh}-e^{-\frac{1}{2}iwh})$

$+\frac{c_n}{4h}(e^{\frac{1}{2}iwh}-e^{-\frac{1}{2}iwh})$

$\frac{b_{n+1}-b_n}{\tau}=\frac{1}{h}a_{n+1}(e^{\frac{1}{2}iwh}-e^{-\frac{1}{2}iwh})$，$c_{n+1}=b_n$

令 $e^{\frac{1}{2}iwh}-e^{-\frac{1}{2}iwh}=2i\sin(\frac{1}{2}wh)=\alpha$

则 $a_{n+1}=a_n+\frac{\alpha\lambda}{4}b_{n+1}+\frac{\alpha\lambda}{2}b_n+\frac{\alpha\lambda}{4}c_n$，$b_{n+1}=b_n+\alpha\lambda a_{n+1}$

故 $a_{n+1}=a_n+\frac{\alpha\lambda}{4}(b_n+\alpha\lambda a_{n+1})+\frac{\alpha\lambda}{2}b_n+\frac{\alpha\lambda}{4}c_n$

$\Rightarrow$ 特征值 ≤ 1 $\Leftrightarrow$ $\alpha\lambda$ 为纯虚数 $\Leftrightarrow$ λ 为实数

故一直稳定

7. $E_j^n = \tau \partial_t u_{jm}^n + O(\tau^2) + \frac{a\lambda}{2}(h \partial_x u_{jm}^n + \frac{1}{2}h^2 \partial_{xx} u_{jm}^n + \frac{1}{6}h^3 \partial_{xxx} u_{jm}^n$

$+ h \partial_x u_{jm}^n - \frac{1}{2}h^2$

$\partial_{xx} u_{jm}^n + \frac{1}{6}h^3 \partial_{xxx} u_{jm}^n + O(h^4)) - (\frac{a\lambda}{2})^2 (h^2 \partial_{xx} u_{jm}^n + O(h^4))$

$+ \frac{b\lambda}{2}(2h \partial_y u_{jm}^n + \frac{1}{3}h^3 \partial_{yyy} u_{jm}^n + O(h^4)) - \frac{(b\lambda)^2}{2}(h^2 \partial_{yy} u_{jm}^n + O(h^4))$

$= \tau \partial_t + a\tau \partial_x + b\tau \partial_y + \frac{1}{6} a h^2 \tau$

$\partial_{yyy} - \frac{b^2\tau^2}{2} \partial_{yy} = O(\tau^2 + h^2)$

8. 考虑格式 $\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2} - a(u_j^n) \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} = 0$

由冻结系数法，有其在 $\sup|a| \ \lambda \le 1$ 时稳定，$\lambda = \frac{\tau}{h}$