PDE 数值解第二次作业:

1. (a) 抛物方程，特征方程为 $a\left(\frac{dy}{dx}\right)^2-2a\frac{dy}{dx}+a=0$

一条实特征线为 $y=x$，取 $\xi(x,y)=y-x$，$\eta(x,y)=x$

则 $x=\eta$，$y=\xi+\eta$，$u(x,y)=u(\xi(x,y),\eta(x,y))$

$$u_x=u_\xi\frac{\partial\xi}{\partial x}+u_\eta\frac{\partial\eta}{\partial x},\quad u_y=u_\xi\frac{\partial\xi}{\partial y}+u_\eta\frac{\partial\eta}{\partial y}$$

$$u_{xx}=u_{\xi\xi}\left(\frac{\partial\xi}{\partial x}\right)^2+u_{\xi\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+u_\xi\frac{\partial^2\xi}{\partial x^2}+u_{\eta\eta}\left(\frac{\partial\eta}{\partial x}\right)^2+u_{\xi\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+u_\eta\frac{\partial^2\eta}{\partial x^2}$$

$$u_{xy}=u_{\xi\xi}\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+u_{\xi\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+u_\xi\frac{\partial^2\xi}{\partial x\partial y}+u_{\eta\eta}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+u_{\xi\eta}\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y}+u_\eta\frac{\partial^2\eta}{\partial x\partial y}$$

故原PDE改写为:

$$a[u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}+2(-u_{\xi\xi}+u_{\xi\eta})+u_{\xi\xi}]+b(-u_\xi+u_\eta)+cu_\xi+u=0$$

故标准形式为 $au_{\eta\eta}+(c-b)u_\xi+bu_\eta+u=0$

(b) 双曲方程. 特征方程 $\left(\frac{dy}{dx}\right)^2+2\left(\frac{dy}{dx}\right)-3=0$

两条实特征线为 $y=-3x$ 和 $y=x$，取 $\xi(x,y)=3x+y$，$\eta(x,y)=x-y$

重复(a)改写.

$$(9u_{\xi\xi}+6u_{\xi\eta}+u_{\eta\eta})-2(3u_{\xi\xi}-2u_{\xi\eta}-u_{\eta\eta})-3(u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta})$$

$$+2(3u_\xi+u_\eta)+6(u_\xi-u_\eta)=0$$

即 $16u_{\xi\eta}+12u_\xi-4u_\eta=0$ (标准形式)

(c) 椭圆方程: 特征方程 $y^2\left(\frac{dy}{dx}\right)^2+x^2=0$ (无实特征线)

复特征线为 $y^2=(\pm i)x^2$，取 $\xi=y^2+ix^2$，$\eta=\bar{\xi}$

则原PDE改写为：

$$y^2(-4x^2u_{\xi\xi}+8x^2u_{\xi\eta}-4x^2u_{\eta\eta}+2iu_\xi-2iu_\eta)+x^2(4y^2u_{\xi\xi}+$$

$$8y^2u_{\xi\eta}+4y^2u_{\eta\eta}+2u_\xi+2u_\eta)=0$$

即 $8\left(\frac{\xi+\eta}{2}\right)\left(\frac{\xi-\eta}{2}\right)u_{\xi\eta}+\left[i(\xi+\eta)+\frac{\xi-\eta}{i}\right]u_\xi+\left[\frac{\xi-\eta}{i}-i(\xi+\eta)\right]u_\eta=0$

28.3. 由Taylor展开，两者均具有时空一阶精度，相容

稳定性：验证 Von-Neumann条件：

网格比 $\lambda=\frac{\tau}{h}$，则(2)格式改写为 $u_j^{n+1}-u_j^n+a\lambda u_j^{n+1}-a\tau u_{j-1}^{n+1}=0$

代入 $u_j^n=v^n e^{ijwh}$ 有：

$$v^{n+1}-v^n+a\lambda v^{n+1}-a\lambda v^{n+1}e^{-iwh}=0$$

故 $G(w,\tau)=\frac{1}{1+a\lambda(1-e^{-iwk})}$

$$\|G(w,\tau)\|^2=\left\|\frac{1}{1+a\lambda(1-e^{-iwk})}\right\|^2=\frac{1}{1+4a\frac{\tau}{h}(1+a\lambda)\sin^2\frac{wh}{2}}\le 1$$

稳定格式

对于(3)，$\|G(w,\tau)\|^2=\frac{1}{1-4a\frac{\tau}{h}(1+a\lambda)\sin^2\frac{wh}{2}}\le 1$

不满足 Von-Neumann条件 不是稳定格式

4. $\frac{3u_j^{n+1}-4u_j^n+u_j^{n-1}}{2\tau}\approx\frac{\partial u}{\partial t}+O(\tau)$ 时间一阶精度

$\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2}\approx\frac{\partial^2 u}{\partial t^2}+O(h^2)$ 空间二阶精度

稳定性： $\lambda=\frac{\tau}{h^2}$，则改写差分格式：

$$3u_j^{n+1}-4u_j^{n}+u_j^{n-1}=2a\lambda\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)$$

取 $U_j^n=\begin{pmatrix}u_j^n\\u_j^{n-1}\end{pmatrix}$

则 $$\begin{pmatrix}3u_j^{n+1}\\3u_j^{n}\end{pmatrix}=\begin{pmatrix}2a\lambda&0\\0&0\end{pmatrix}\begin{pmatrix}u_{j-1}^{n+1}\\u_{j-1}^{n}\end{pmatrix}+\begin{pmatrix}2a\lambda&0\\0&0\end{pmatrix}\begin{pmatrix}u_{j+1}^{n+1}\\u_{j+1}^{n}\end{pmatrix}$$

$$+\begin{pmatrix}-4a\lambda&0\\0&0\end{pmatrix}\begin{pmatrix}u_j^{n+1}\\u_j^{n}\end{pmatrix}+\begin{pmatrix}4&-1\\3&0\end{pmatrix}\begin{pmatrix}u_j^{n}\\u_j^{n-1}\end{pmatrix}$$

记 $U_j^n=v^n e^{ijwh}$ 得：

$$v^{n+1}=\begin{pmatrix}\frac{4}{3+8a\lambda\sin^2(\frac{wh}{2})}&\frac{-1}{3+8a\lambda\sin^2(\frac{wh}{2})}\\1&0\end{pmatrix}v^n$$

$\Rightarrow$ 特征值为 $\frac{1}{3+8a\lambda\sin^2(\frac{wh}{2})}\left(2\pm\sqrt{4-\ 3+8a\lambda\sin^2(\frac{wh}{2})}\right)$

由 V-N 条件，格式不稳定