PDE 第十三周作业：

15. $\int_\Omega (\Delta u)^2 dx = \int_\Omega f^2 dx$

由 Green 恒等式. $\int_\Omega (\Delta u)^2 dx = -\int_\Omega \nabla u \cdot \nabla(\Delta u) dx = -\sum_{i=1}^{n} \int_\Omega u_{x_i} \Delta u(x_i) dx$

$\Rightarrow$ (逐次使用 Green 恒等式) $\geq \frac{1}{n} \sum_{ij} \int_\Omega u_{ij}^2 dx$

16. $v(x,y) = X(x)Y(y)$

$X'' \cdot Y + X \cdot Y'' = 0$, $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

考虑 Sturm-Liouville 问题：

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = 0,\ Y(b) = 0 \end{cases} \Rightarrow Y_n = \sin\left(\frac{n\pi y}{b}\right),\ \lambda = \lambda_n = \left(\frac{n\pi}{b}\right)^2$$

$X'' - \lambda X = 0 \Rightarrow X_n = a_n \cdot e^{\frac{a\pi}{b} \cdot x} + b_n e^{-\frac{a\pi}{b} x}$

代入：$x=0$, $x=a$ 解得：$a_{2n} = b_{2n} = 0$

$$\begin{cases} a_{2n-1} = -\frac{4v_0}{(2n-1)\pi} \exp\left\{-\frac{2n-1}{b}\pi a\right\} \Big/ \left[\exp\left\{\frac{(2n-1)\pi a}{b}\right\} - \exp\left\{-\frac{(2n-1)\pi a}{b}\right\}\right] \\ b_{2n-1} = -\frac{4v_0}{(2n-1)\pi} \exp\left\{-\frac{2n-1}{b} a\right\} \Big/ \left[\exp\left\{\frac{(2n-1)\pi a}{b}\right\} - \exp\left\{-\frac{(2n-1)\pi a}{b}\right\}\right] \end{cases}$$

$r = |x| = \sqrt{\sum x_i^2}$, $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$

$\Delta u = 0$, $u(x) = v(r)$

$u_{x_i x_i} = v''(r) \frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r}\right)$

$\Delta u = v''(r) + \frac{n-1}{r} v'(r) = 0$

$\Rightarrow v(r)=\begin{cases} b\cdot\ln r+c & n=2 \\ \frac{b}{r^{n-2}}+c & n\geq 3 \end{cases}$

$\Rightarrow$ 基本解: $\phi(x)=\begin{cases} -\frac{1}{2\pi}\ln|x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\cdot\frac{1}{|x|^{n-2}} & n\geq 3 \end{cases}$

其中 $\alpha(n)$ 为 $\mathbb{R}^n$ 中单位球体积

6. (1) $|x|'=[xH(x)-xH(-x)]'=H(x)-H(-x)$

$|x|^{(m)}=(H(x)-H(-x))^{(m-1)}=[2\delta(x)]^{(m-2)}=2\delta^{(m-2)}(x)$

(2) $(H(x),\sin x)'=H(x)\cos x+0=H(x)$

(3) $(H(x),e^{ax})''=\delta'(x)+a\delta'(x)+a^2H(x)e^{ax}$

7. (3) $f(x)=x^2(H(x+1)-H(x-1))$

$\Rightarrow f'(x)=[H(x+1)-H(x-1)]\cdot 2x+\delta(x+1)-\delta(x-1)$

2. $\langle\partial_j(aT),\varphi\rangle=-\langle T,a\partial_i\varphi\rangle=\langle T,\partial_j a\cdot\phi\rangle-\langle T,-\partial_j(a\varphi)\rangle$

$=\langle\partial_j a\cdot T+a\partial_j T,\phi\rangle$

从而 $\partial_j(aT)=\partial_j a\cdot T+a\partial_j T$

4. $\langle\partial_j(T+S),\varphi\rangle=-\langle T+S,\partial_j\phi\rangle=\langle\partial_j T,\phi\rangle+\langle\partial_j S,\phi\rangle$

$\Rightarrow$ 线性

5. $\lim_{a\to 0}\int\frac{1}{\pi}\frac{a}{a^2+x^2}f(x)\,dx=\int\delta(x)f(x)\,dx$