偏微分方程笔记：

1. 一阶PDE

函数　(source: 袁荣《常微分方程》)

一般形式: $F(x_1,x_2,\cdots,x_n,u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\cdots,\frac{\partial u}{\partial x_n})=0$

未知数　因变量　偏导.

一阶拟线性PDE: 最高阶导数为一阶且线性　指对 $\frac{\partial u}{\partial x_i}$ 线性 $(1\le i\le n)$

一般形式: $\sum_{i=1}^{n} a_i(x_1,x_2,\cdots,x_n,u)\frac{\partial u}{\partial x_i} = c(x_1,x_2,\cdots,u)$

一阶半线性PDE: ~且半线性

一般形式: $\sum_{i=1}^{n} a_i(x_1,x_2,\cdots,x_n)\frac{\partial u}{\partial x_i} = c(x_1,x_2,\cdots,x_n,u)$

一阶线性PDE: ~且线性　指对 $\frac{\partial u}{\partial x_i}$ 和 u 线性.

一般形式: $\sum_{i=1}^{n} a_i(x_1,x_2,\cdots,x_n)\frac{\partial u}{\partial x_i} = c(x_1,x_2,\cdots,x_n)u$

这项为零: 齐次

一阶拟线性PDE解法(以n=2为例)示例:

$$a(x,y,u)\frac{\partial u}{\partial x}+b(x,y,u)\frac{\partial u}{\partial y}=c(x,y,u)$$

考虑转化 $\begin{cases}\frac{dx}{dt}=a(x,y,u)\\ \frac{dy}{dt}=b(x,y,u)\\ \frac{du}{dt}=c(x,y,u)\end{cases}$　变为ODE

目的是将方程转为 $\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}=\frac{du}{dt}$

$$\Rightarrow \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=du$$

首次积分：考虑ODE组：

$$\begin{cases} \frac{dy_1}{dx} = f_1(x, y_1, y_2, \cdots, y_n) \\ \frac{dy_2}{dx} = f_2(x, y_1, y_2, \cdots, y_n) \\ \cdots \\ \frac{dy_n}{dx} = f_n(x, y_1, y_2, \cdots, y_n) \end{cases}$$

其中 $y_1, y_2, \cdots, y_n$ 为因变量，$f_1, f_2, \cdots, f_n$ 连续

$y_1, y_2, \cdots, y_n$ 对 x 连续可微

则如果 $\Phi(x, y_1, y_2, \cdots, y_n) = C$ 是原方程的解

就称 $\Phi(x, y_1, y_2, \cdots, y_n) = C$ 为"首次积分"

定理：上面的方程组有 n 个解，n 个首次积分，且 n 个首次积分相互独立

$$\boxed{\frac{D(\Phi_1, \Phi_2, \cdots, \Phi_n)}{D(y_1, y_2, \cdots, y_n)} = \begin{vmatrix} \frac{\partial \Phi_1}{\partial y_1} & \frac{\partial \Phi_1}{\partial y_2} & \cdots & \frac{\partial \Phi_1}{\partial y_n} \\ & \vdots & & \\ \frac{\partial \Phi_n}{\partial y_1} & \cdots & & \frac{\partial \Phi_n}{\partial y_n} \end{vmatrix} \neq 0}$$

一阶拟线性方程解法：

将 $\sum_{i=1}^{n} a_i(x_1, x_2, \cdots, x_n, u) \frac{\partial u}{\partial x_i} = c(x_1, x_2, \cdots, u)$ 转化为：

$$\begin{cases} \frac{dx_1}{dt} = a_1(x_1, x_2, \cdots, x_n, u) \\ \frac{dx_2}{dt} = a_2(x_1, x_2, \cdots, x_n, u) \\ \cdots \\ \frac{dx_n}{dt} = a_n(x_1, x_2, \cdots, x_n, u) \\ \frac{du}{dt} = c(x_1, x_2, \cdots, x_n, u) \end{cases}$$

共(n+1)个ODE，设其首次积分：

$$\begin{cases}\Phi_1(x_1,x_2,\cdots,x_n,u,t)=C_1\\ \Phi_2(x_1,x_2,\cdots,x_n,u,t)=C_2\\ \cdots\\ \Phi_n(x_1,x_2,\cdots,x_n,u,t)=C_n\\ \Phi_{n+1}(x_1,x_2,\cdots,x_n,u,t)=C_{n+1}\end{cases}$$

其中 $\begin{cases}u=u(t)\\ x_1=x_1(t)\\ x_2=x_2(t)\\ \cdots\\ x_n=x_n(t)\end{cases}$

这样即相当于求解完毕

一阶拟线性方程的初值问题：

单独一点 $x_1=x_{10},\ x_2=x_{20},\ \cdots,\ x_n=x_{n0},\ u=u_0$ 无意义

需有特征线

特征线：$x_1=x_1(s),\ x_2=x_2(s),\ \cdots,\ u=u(s)$

$\begin{cases}u=u(t)\\ x_1=x_1(t)\\ x_2=x_2(t)\\ \cdots\\ x_n=x_n(t)\end{cases}$ 可写为 $\begin{cases}u=u(t,s)\\ x_1=x_1(t,s)\\ x_2=x_2(t,s)\\ \cdots\\ x_n=x_n(t,s)\end{cases}$

一阶拟线性PDE齐次：（通解）

$$\begin{cases} \dfrac{dx_1}{dt} = a_1(x_1, x_2, \cdots, x_n, u) \\ \dfrac{dx_2}{dt} = a_2(x_1, x_2, \cdots, x_n, u) \\ \cdots \\ \dfrac{dx_n}{dt} = a_n(x_1, x_2, \cdots, x_n, u) \\ \dfrac{du}{dt} = c(x_1, x_2, \cdots, x_n, u) = 0 \end{cases} \Rightarrow \frac{du}{dt} = 0 \Rightarrow u = C_{n+1}$$

$u = u(t, s)$

$$\begin{cases} x_1 = x_1(t, s) \\ x_2 = x_2(t, s) \\ \cdots \\ x_n = x_n(t, s) \end{cases}$$

可写为 $u = C_{n+1}$

$$\begin{cases} x_1 = x_1(0, s) \\ x_2 = x_2(0, s) \\ \cdots \\ x_n = x_n(0, s) \end{cases}$$

2. 二阶 PDE 分类

一般形式 $F(\underbrace{x_1, x_2, \cdots, x_n}_{\text{自变量}}, \underbrace{u}_{\text{未知函数}}, \underbrace{\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}}_{\text{一阶部分}}, \underbrace{\frac{\partial^2 u}{\partial x_1 \partial x_1}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \cdots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \cdots, \frac{\partial^2 u}{\partial x_n \partial x_1}, \frac{\partial^2 u}{\partial x_n \partial x_2}, \cdots, \frac{\partial^2 u}{\partial x_n \partial x_n}}_{\text{二阶部分}}) = 0$

某函数

二元线性：

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y)$$

其中 A.B. ·, G 为 x.y 的函数，与 u 无关

特殊方程.

1. $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ (一维波动方程)

2. $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$ (一维有源波动~)

3. $\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial u}{\partial t} = 0$ (一维阻尼波动~)

4. $\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial u}{\partial t} + cu = 0$ (电报~)

5. $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ (一维热传导)

6. $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$ (一维有源热传导)

7. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (二维 Laplace 方程)

8. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ (二维 Poisson 方程)

二维连续线性解法:

Step 1. $(x,y) \to (\xi,\eta)$，其中 $\xi = \xi(x,y)$，$\eta = \eta(x,y)$

将 $A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$

转化为 $a\frac{\partial^2 u}{\partial \xi^2} + 2b\frac{\partial^2 u}{\partial \xi \partial \eta} + c\frac{\partial^2 u}{\partial \eta^2} + d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu = g$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$\Downarrow$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cdots$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \cdots$$

$\Downarrow$

$\cdots$

得到的结果为:

$$a = A\left(\frac{\partial\xi}{\partial x}\right)^2 + 2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + C\left(\frac{\partial\xi}{\partial y}\right)^2$$

$$b = A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}$$

$$c = A\left(\frac{\partial\eta}{\partial x}\right)^2 + 2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} + C\left(\frac{\partial\eta}{\partial y}\right)^2$$

$$d = A\frac{\partial^2\xi}{\partial x^2} + 2B\frac{\partial^2\xi}{\partial x\partial y} + C\frac{\partial^2\xi}{\partial y^2} + D\frac{\partial\xi}{\partial x} + E\frac{\partial\xi}{\partial y}$$

$$e = A\frac{\partial^2\eta}{\partial x^2} + 2B\frac{\partial^2\eta}{\partial x\partial y} + C\frac{\partial^2\eta}{\partial y^2} + D\frac{\partial\eta}{\partial x} + E\frac{\partial\eta}{\partial y}$$

$$f = F$$

$$g = G$$