PDE第十四周作业：

17. (1) $(\xi,\eta)$ 关于 $y=0$ 的对称点 $(\xi,-y)$

$\Rightarrow$ Green Function: $G(x,y;\xi,\eta)=\Gamma(x,y;\xi,\eta)-\Gamma(x,y;\xi,-\eta)$

$$=\frac{1}{2\pi}\ln\sqrt{\frac{(x-\xi)^2+(y+\eta)^2}{(x-\xi)^2+(y-\eta)^2}}$$

(2) 对称性：

$$G(x,y;\xi,\eta)=\Gamma(x,y;\xi,\eta)+\Gamma(x,y;-\xi,\eta)-\Gamma(x,y;-\xi,-\eta)-\Gamma(x,y;\xi,-\eta)$$

$$=\frac{1}{2\pi}\ln\sqrt{\frac{(x-\xi)^2+(y+\eta)^2(x+\xi)^2+(y+\eta)^2}{(x+\xi)^2+(y-\eta)^2(x-\xi)^2+(y-\eta)^2}}$$

(3) $$G(x,y;\xi,\eta)=\sum_{n=1}^{\infty}\left[\Gamma(x,y;\xi,\eta+na)-\Gamma(x,y;-\xi,-\eta-na)\right]$$

$$=\sum_{n=1}^{\infty}\frac{1}{2\pi}\ln\frac{(x-\xi)^2+(y+\eta+na)^2}{(x-\xi)^2+(y-\eta-na)^2}$$

18. $$\begin{cases}-\Delta u=f(x)\\ u|_{\partial B(R)}=\varphi\end{cases}$$

由 slide: $u(\xi)=\int_{x_d>0}G(x;\xi)f(x)\,dx+\frac{2\xi_d}{w_d}\int_{x_d=0}\frac{\varphi(x)}{(x-\xi)^d}\,ds$

19. $$u(\xi,\eta)=\int_{B^+(R)}SfG\,dxdy+\int_{\partial B^+(R)}G\frac{\partial u}{\partial n}-u\frac{\partial G}{\partial u}\,ds$$

$$=\int_{D^+(R)}f\cdot G\,dxdy+\int_{\partial B^+(R)\cap\{y>0\}}\left(G\cdot\frac{\partial u}{\partial n}-\varphi\frac{\partial G}{\partial n}\right)ds$$

$$+\int_{\{y=0\}\cap\{-R<x<R\}}\left(G\psi-u\frac{\partial G}{\partial n}\right)ds$$

要求 G: $\begin{cases} G|_{\partial B^+(R)\cap\{y>0\}}=0 \\ \frac{\partial G}{\partial n}=-\frac{\partial G}{\partial y}\Big|_{\{y=0\}\cap\{-R<x<R\}}=0 \end{cases}$

则 $G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}-\frac{1}{2\pi}\ln\frac{R}{\sqrt{(x-\xi^*)^2+(y-\eta^*)^2}}$

$\frac{1}{2a}\ln\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}-\frac{1}{2a}\ln\frac{R}{\sqrt{(x-\xi^*)^2+(y-\eta^*)^2}}$

$(\xi^*,\eta^*)$ 是关于 $\partial B^+(R)\cup\partial B^-(R)$ 与 $(\xi,\eta)$ 的反演点、

21. (1) $\varphi(x,y)=\varphi(\alpha)=\varphi(R\cos\alpha,R\sin\alpha)$

$\varphi(2\pi-\alpha)=-\varphi(\alpha) \qquad 0<\alpha<\pi$

$$u(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\alpha)}\varphi(\alpha)\,d\alpha$$

(2) $\varphi(\alpha)=\varphi(-\alpha)$

$$u(\rho,\theta)=\frac{1}{2\pi}\int_0^{\pi}\left[\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\alpha)}+\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta+\alpha)}\right]\varphi(\theta)\,d\alpha$$

$$u(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\alpha)}\varphi(\alpha)\,d\alpha$$

$$u(\rho,-\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\alpha)}\varphi(\alpha)\,d\alpha$$

$$\overset{\alpha=-\alpha}{=}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(R^2-\rho^2)\varphi(-\alpha)}{R^2+\rho^2-2R\rho\cos(\theta-\alpha)}\,d\alpha$$

$$=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(R^2-\rho^2)\varphi(-\alpha)}{R^2+\rho^2-2R\rho\cos(\theta-\alpha)}\,d\alpha$$

$\partial B^+(R)\cap\{y>0\}$ 的边值用定理 2.6 的结果，对 $y=0$ 边值利用 $n$ 在 $y=0$ 连续性

23. $$\frac{\partial u}{\partial r}=-\frac{a}{2\pi}\int_0^{2\pi}\varphi(\alpha)\frac{2r-2a\cos(\alpha-\theta)}{a^2+r^2-2ar\cos(\alpha-\theta)}\,d\alpha$$

由 $\int_0^{2\pi}\varphi(\alpha)\,d\alpha=0$，故 $r\frac{\partial u}{\partial r}=-\frac{a}{2\pi}\int_0^{2\pi}\varphi\cdot\frac{2r^2-2ar\cos(\alpha-\theta)}{a^2+r^2-2ar\cos(\alpha-\theta)}\,d\alpha$

$$= \frac{a}{2\pi} \int_0^{\pi} \varphi \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\alpha - \theta)} d\alpha \Rightarrow \lim_{\substack{r \to a \\ \theta \to 0}} \frac{\partial u}{\partial r} = \varphi(\theta_0)$$