

PDE 第五周作业:

$$6. \begin{cases} -\Delta u + u^2 = 0 & x \in \Omega, u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \\ \frac{\partial u}{\partial n} + \alpha(x)u|_{\partial\Omega} = \varphi \end{cases}$$

$$\alpha(x) \geq \alpha_0 > 0$$

$$\max |u(x)| \leq \frac{1}{\alpha_0} \max_{\partial\Omega} |\varphi(x)|$$

$$\text{令: } v = \frac{1}{\alpha_0} \max_{\partial\Omega} |\varphi(x)| \Rightarrow \begin{cases} -\Delta v + v^3 \geq 0 \\ \frac{\partial v}{\partial n} + \alpha(x)v|_{\partial\Omega} \geq \varphi(x) \end{cases}$$

$\Rightarrow v \geq |u|$  即证

$$10. \begin{cases} -\Delta u = f(x, y) & (x, y) \in \Omega \\ u|_{\partial\Omega \setminus P_0} = \varphi(x, y) \end{cases}$$

$$\text{若有 } u_1, u_2 \text{ 均符合题意, 则 } \begin{cases} -\Delta(u_1 - u_2) = 0 \\ u_1 - u_2|_{\partial\Omega \setminus P_0} = 0 \end{cases}$$

故只用解决  $f = \varphi = 0$

$$\forall \varepsilon > 0, \text{ 取 } r_0 \text{ 充分小, s.t. } \varepsilon \ln \frac{d}{r_0} > M := \sup |u_1 - u_2|$$

考虑:  $\Omega - B_r(P_0)$   $0 < r < r_0$ ,  $\varepsilon \ln \frac{d}{r} \geq |u|$  在  $\partial[\Omega \setminus B_r(P_0)]$  中

$$\text{且 } -\Delta(\varepsilon \ln \frac{d}{r}) = 0$$

$$\Rightarrow \text{比较原理: } \varepsilon \ln \frac{d}{r} \geq |u|$$

由连续性:  $\varepsilon \ln \frac{d}{r} \geq |u|$  for  $\Omega - P_0 \Rightarrow \varepsilon \rightarrow 0, u \equiv 0$

$$27 \quad (1) \frac{1}{\pi R^2} \int u(x, y) dx dy = \frac{1}{\pi R^2} \int_0^R dr \int_0^{2\pi} u(r \cos \theta, r \sin \theta) d\theta$$

$$= \frac{2}{R^2} \int_0^R dr \cdot r \cdot u(0, 0) = u(0, 0)$$

$$\Rightarrow |u(0, 0)| \leq \frac{1}{\pi R^2} \left( \int u^2 \right)^{\frac{1}{2}} \left( \int 1^2 \right)^{\frac{1}{2}} = \frac{1}{\pi R^2} M^{\frac{1}{2}} (\pi R^2)^{\frac{1}{2}}$$

(2) 在  $r^2(x, y)$  为圆,  $R-r$  为半径的圆上, 利用(1) 即得

$$29. \Delta \text{ 的极坐标: } u_{rr} + r^{-1} u_r + u_{\theta\theta} r^{-2} = 0 \quad \text{令 } v_1 = \frac{R^2}{r}$$

$$v(r, \theta) = u\left(\frac{R^2}{r}, \theta\right) \Rightarrow v_r = u_{v_1} \left(-\frac{R^2}{r^2}\right) \Rightarrow r^{-1} u_r = u_{v_1} \left(-\frac{R^2}{r^2}\right)$$

$$u_{rr} = u_{v_1} \cdot r_1 \cdot \frac{R^4}{r^4} + u_{v_1} \cdot \frac{2R^2}{r^3} \quad (v_{\theta\theta} = u_{\theta\theta})$$

$$\Rightarrow v_{rr} + r^{-1} v_r + v^{-2} v_{\theta\theta} = u_{v_1} r_1 \cdot \frac{R^4}{r^4} + u_{v_1} \cdot \frac{R^2}{r^3} + v_{\theta\theta} \cdot \frac{1}{r^2}$$

$$= (u_{v_1} r_1 + r_1^{-1} u_{v_1} + r_1^{-2} u_{\theta\theta}) \frac{r_1^2}{r^2} = 0$$

$$\Delta v = 0$$

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(d^2 - r^2) \varphi(\alpha)}{\alpha^2 + r^2 - 2\alpha r \cos(\theta - \alpha)} d\alpha = u\left(\frac{R^2}{r}, \theta\right)$$

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(d^2 - \frac{R^2}{r^2}) \varphi(\alpha)}{\alpha^2 + \frac{R^4}{r^2} - 2\alpha R^2 / (r \cdot \alpha) \cos(\theta - \alpha)} d\alpha$$

