作业5.

P27. 12: $j(\varepsilon) := J(u+\varepsilon y) = \frac{1}{2}\int_0^1 (u'+\varepsilon y')^2 dx - 2\int_0^1 (u+\varepsilon y)\,dx - u(0) - \varepsilon y(0)$

$$\Rightarrow j'(\varepsilon) = \int_0^1 y'^2\,dx\cdot\varepsilon + \int_0^1 u'y'\,dx - 2\int_0^1 y(x)\,dx - y(0)$$

$$\Rightarrow j'(0) = \int_0^1 u'y'\,dx - 2\int_0^1 y(x)\,dx - y(0) = 0$$

$$\Rightarrow -u'(0)y(0) - \int_0^1 u''y\,dx - 2\int_0^1 y\,dx - y(0) = 0$$

$$\Rightarrow \begin{cases} u''+2=0 \\ u'(0)+1=0 \\ u(1)=0 \end{cases}$$

$$\Rightarrow u = -x^2+x+2$$

P100 2. (1) $f_1(x) = \begin{cases} 1 & (|x|<a) \\ 0 & (|x|>a) \end{cases}$

$$\bar{f}_1(\lambda) = \sqrt{\frac{2}{\pi}}\,\frac{\sin a\lambda}{\lambda}$$

$$\bar{f}_1(\lambda) = (x^2 f_1) = i^2\frac{d^2}{d\lambda^2}\bar{f}_1 = \sqrt{\frac{2}{\pi}}\ \frac{a^2\sin a\lambda}{\lambda} + \frac{2\sin a\lambda}{\lambda^3} - 2a\cos\frac{a\lambda}{\lambda^2})$$

(2) $\bar{f}(\lambda) = x g(x) = i\frac{d}{d\lambda}\left(\frac{1}{\sqrt{2\pi}}\frac{2a}{a^2+\lambda^2}\right) = -2i\sqrt{\frac{2}{\pi}}\,\frac{a\lambda}{(a^2+\lambda^2)^{\frac{1}{2}}}$

(3) $\bar{f}(\lambda) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(\lambda+ui)a}{\lambda+ui}$

(4) $\bar{f}(\lambda) = \frac{a}{\sqrt{2\pi}\,i}\left(\frac{1}{a^2+(\lambda-\lambda_0)^2} - \frac{1}{a^2+(\lambda+\lambda_0)^2}\right)$

(5). (6) 同 (3)

(7) $\bar{f}(\lambda) = \sqrt{\frac{\pi}{2}}\,\frac{1}{a}e^{-a\lambda}$

(8) $\bar{f}(\lambda) = -\sqrt{\frac{\pi}{2}}\,e^{-a\lambda}\,\mathrm{sig}(\lambda)$

(9) $\lambda>0$, $\bar{f}(\lambda)=\frac{1}{2a^3}\sqrt{\frac{\pi}{2}}e^{-a\lambda}(1+a\lambda)$

由 f 为偶函数，故 $\bar{f}(\lambda)=\frac{1}{2a^3}\sqrt{\frac{\pi}{2}}e^{-a\lambda}(1+a|\lambda|)$

4. (1) $\begin{cases}\frac{d\bar{u}}{dt}+a^2\lambda^2\bar{u}-ib\lambda u-c\bar{u}=\bar{f}(\lambda,t)\\ \bar{u}(\lambda,0)=\bar{\varphi}(\lambda)\end{cases}$

其中 $\bar{u}(\lambda,t)$ 为 $u(x,t)$ 关于 x 的 Fourier 变换

$\Rightarrow \bar{u}(\lambda,t)=\bar{\varphi}e^{(-a^2\lambda^2+ib\lambda+c t)}+\int_0^t \bar{f}(\lambda,\tau)e^{(a^2\lambda^2-ib\lambda-c)(\tau-t)}d\tau$

对上面两式反演

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{(ct-(x-\xi+bt))^2/4a^2t}d\xi$$

$$+\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\tau}}d\tau\int_{-\infty}^{+\infty}\frac{\varphi(\xi)}{(x-\xi)^2+y^2}d\xi$$

(2) $u(x,y)=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\varphi(\xi)}{(x-\xi)^2+y^2}d\xi$