偏微分方程（40420664-2）

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第9讲、高维波动方程的估计

9.1 初值问题的特征锥

三维波动方程初值问题

$$
\begin{cases}
u_{tt} - a^2 \Delta u = f(t,x), \qquad x \in \mathbf{R}^3,\ t > 0, \\
u|_{t=0} = \phi(x), \quad u_t|_{t=0} = \psi(x), \qquad x \in \mathbf{R}^3,
\end{cases}
\tag{1}
$$

的Kirchhoff公式

$$
\begin{aligned}
u(t,x)= \quad & \frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t}\int_{S_{at}(x)}\phi(y)dS_y\right)+\frac{1}{4\pi a^2 t}\int_{S_{at}(x)}\psi(y)dS_y \\
& +\int_0^t\left[\frac{1}{4\pi a^2(t-\tau)}\int_{S_{a(t-\tau)}(x)}f(\tau,y)dS_y\right]d\tau,
\end{aligned}
$$

其中积分区域是空心球面

$$
S_{at}(x)=\{y\in\mathbf{R}^3:|y-x|=at\}.
$$

二维波动方程初值问题

$$
\begin{aligned}
&u_{tt} - a^2(u_{x_1x_1} + u_{x_2x_2}) = f(t,x), \qquad x = (x_1, x_2) \in \mathbf{R}^2,\ t > 0,\\
&u|_{t=0} = \phi(x_1, x_2), \quad u_t|_{t=0} = \psi(x_1, x_2), \qquad x = (x_1, x_2) \in \mathbf{R}^2,
\end{aligned}
$$

的Poisson公式

$$u(t,x) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\left(\int_{\Sigma_{at}(x)}\frac{\phi(y)}{\sqrt{a^2t^2-|y-x|^2}}dy_1dy_2\right)$$

$$+\frac{1}{2\pi a}\int_{\Sigma_{at}(x)}\frac{\psi(y)}{\sqrt{a^2t^2-|y-x|^2}}dy_1dy_2$$

$$+\frac{1}{2\pi a}\int_0^t\left[\int_{\Sigma_{a(t-\tau)}(x)}\frac{f(\tau,y)}{\sqrt{a^2(t-\tau)^2-|y-x|^2}}dy_1dy_2\right]d\tau,$$

其中积分区域是实心圆盘

$$\Sigma_{at}(x) = \{y\in\mathbf{R}^3 : |y-x|\le at\}.$$

定理9.1：设$\phi\in C^3, \psi, f\in C^2$, 则由Kirchhoff公式和Poisson公式给出的是经典解。

从Kirchhoff或Poisson公式，我们立即看到波动方程初值问题解在时空点$P_0=(t_0,x_0)$ 的值由初始数据和右端在特征锥（P_0 为顶点）

$$K_{P_0}:=\{(t,x):|x-x_0|\le a(t_0-t),\quad t\in[0,t_0]\}$$

上（三维两维有区别!）的值唯一确定，与其它点的值无关。当右端$f=0$时，只依赖于初始数据在空间区域$B_{at_0}(x_0)$ 上的值。

物理上，膜上任意一点x_0在时刻t_0的位移只依赖于区域$B_{at_0}(x_0)$上的初始位移和初始速度以及特征锥K_{P_0}内外力f的值，而与它们在这些区域以外的值无关。

空间区域$B_{at_0}(x_0)$称为点$P_0=(t_0,x_0)$对初值的依赖区域（特征锥K_{P_0}与平面$t=0$的交集）。

空间区域D的决定区域（$f=0$）：依赖区域完全落在D中的时空点$P=(t,x)$ 组成的集合:

$$F_D := \{P=(t,x) : t>0,\ B_{at}(x)\subset D\}.$$

一个空间点x_0的影响区域（$f=0$）：以这个点为顶点的开口向上的圆锥

$$J_{x_0} := \{P=(t,x) : |x-x_0|\le at,\ t>0\}.$$

一个空间区域D的影响区域（$f=0$）是以这个区域内每个点为顶点的开口向上的圆锥的并集

$$J_D := \cup_{x\in D} J_x = \{P=(t,x) : B_{at}(x)\cap D \neq \emptyset\}.$$

即D内的扰动在$t>0$时刻影响到的集合。

有限传播速度a：离点x_0的距离超过at_0处的扰动在$t=t_0$时刻内传播（影响）不到该点。

两维波动与三维波动的定性区别（$f=0$）

对于三维问题，解在一时空点(t,x) （如$t=1, x=O$）的值只依赖于依赖区域的边界（球面）

$$S_{at}=\{y\in\mathbf{R}^3:|y-x|=at\}$$

上的初值，而与其内部的值无关。

对于二维问题，解在一时空点(t,x)的值依赖于整个依赖区域$B_{at}(x)$（实心圆盘）上的初值。

物理上，设想在初始时刻，初值及其速度只在空间闭区域D非零，P为D外一空间点（图）。记

$$d_{min} = \min_{Q \in D} |Q - P|,$$

$$d_{max} = \max_{Q \in D} |Q - P|.$$

设$t > 0$, 考虑$u(t,P)$:

- 当$t < d_{min}/a$时，D与(t,P)的依赖域$B_{at}(P)$不相交，所以$u(t,P) = 0$.
- 当$d_{min}/a < t < d_{max}/a$时，$D$与$(t,P)$的依赖域$B_{at}(P)$相交，所以$u(t,P) \neq 0$.
- 当$d_{max}/a < t$时，三维情况下D与(t,P)的依赖域$B_{at}(P)$不相交，所以$u(t,P) = 0$; 而在二维情况下D与(t,P) 的依赖域$B_{at}(P)$仍然相交，所以$u(t,P) \neq 0$.

3维波传播有清晰的波前和波后（无后效现象，可以清楚地听到对方的谈话）

2维波传播只有清晰的波前，没有波后。

9.2 二维无外力波动方程Cauchy问题解的衰减估计

定理9.2：设初值和初始速度有紧支集，分别是二次、一次连续可微的。那么存在正常数C使得当时间变量t趋于无穷大时，解有如下衰减估计

$$|u(t,x)| \le Ct^{-1/2}.$$

证明：在极坐标$y = x + \rho(\cos\theta, \sin\theta)$变换下，Poisson公式成为

$$\begin{aligned} u(t,x) = &\quad \tfrac{1}{2\pi a}\int_0^{at}\int_0^{2\pi} \tfrac{\rho\psi(x+\rho v)}{\sqrt{a^2t^2-\rho^2}} d\theta d\rho + \tfrac{1}{2\pi a}\tfrac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi} \tfrac{\rho\phi(x+\rho v)}{\sqrt{a^2t^2-\rho^2}} d\theta d\rho, \\ \equiv &\quad \tfrac{1}{2\pi a}(I_1 + I_2), \end{aligned}$$

其中$v = (\cos\theta, \sin\theta)$. 注意被积函数在$\rho = at$有奇性。

不妨假设t充分大。那么第一项I_1有估计

$$
\begin{aligned}
|I_1| \leq\ & \int_0^{at}\int_0^{2\pi}\frac{1}{\sqrt{at+\rho}}\frac{\rho|\psi(x+\rho v)|}{\sqrt{at-\rho}}d\theta d\rho \\
\leq\ & \frac{1}{\sqrt{at}}\left[\int_0^{at-1}+\int_{at-1}^{at}\right]\int_0^{2\pi}\frac{\rho|\psi(x+\rho v)|}{\sqrt{at-\rho}}d\theta d\rho \\
\equiv\ & I_{11}+I_{12}.
\end{aligned}
$$

而

$$
\begin{aligned}
|I_{11}| \leq\ & \frac{1}{\sqrt{at}}\int_0^{at-1}\int_0^{2\pi}\rho|\psi(x+\rho v)|d\theta d\rho \\
\leq\ & \frac{1}{\sqrt{at}}\int_0^{\infty}\int_0^{2\pi}\rho|\psi(x+\rho v)|d\theta d\rho = \frac{1}{\sqrt{at}}\int_{R^2}|\psi(y)|dy.
\end{aligned}
$$

对于I_{12}，注意到$\psi = \psi(x)$的支集是紧的，那么有

$$\psi(x + \rho v) = -\int_{\rho}^{\infty} \frac{d\psi(x + sv)}{ds} ds = -\int_{\rho}^{\infty} v \cdot \nabla\psi(x + sv) ds.$$

从而I_{12}可以被估计为

$$
\begin{aligned}
|I_{12}| = \quad & \tfrac{1}{\sqrt{at}} \int_{at-1}^{at} \int_0^{2\pi} \tfrac{\rho}{\sqrt{at-\rho}} |\psi(x+\rho v)| d\theta d\rho \\
= \quad & \tfrac{1}{\sqrt{at}} \int_{at-1}^{at} \int_0^{2\pi} \tfrac{\rho}{\sqrt{at-\rho}} \left| - \int_\rho^\infty v \cdot \nabla\psi(x+sv) ds \right| d\theta d\rho \\
\le \quad & \tfrac{1}{\sqrt{at}} \int_{at-1}^{at} \int_0^{2\pi} \tfrac{1}{\sqrt{at-\rho}} \int_\rho^\infty s |\nabla\psi(x+sv)| ds d\theta d\rho \\
\le \quad & \tfrac{1}{\sqrt{at}} \int_{at-1}^{at} \tfrac{1}{\sqrt{at-\rho}} \int_{R^2} |\nabla\psi(y)| dy d\rho \\
\le \quad & \tfrac{2}{\sqrt{at}} \int_{R^2} |\nabla\psi(y)| dy.
\end{aligned}
$$

结果我们有

$$
|I_1| \le I_{11} + I_{12} \le \frac{1}{\sqrt{at}} \left(\int_{R^2} |\psi(y)| dy + 2 \int_{R^2} |\nabla\psi(y)| dy \right) \equiv C_1 t^{-1/2}.
$$

对于I_2，借助变换$\rho = atr$我们计算

$$
\begin{aligned}
I_2 = \quad & \frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{\rho\phi(x+\rho v)}{\sqrt{a^2t^2-\rho^2}}d\theta d\rho \\
= \quad & \frac{\partial}{\partial t}\int_0^{1}\int_0^{2\pi}\frac{atr\phi(x+atrv)}{\sqrt{1-r^2}}d\theta dr \\
= \quad & a\int_0^{1}\int_0^{2\pi}\frac{r\phi(x+atrv)}{\sqrt{1-r^2}}d\theta dr + at\int_0^{1}\int_0^{2\pi}\frac{ar^2 v\cdot\nabla\phi(x+atrv)}{\sqrt{1-r^2}}d\theta dr \\
= \quad & t^{-1}\int_0^{at}\int_0^{2\pi}\frac{\rho\phi(x+\rho v)}{\sqrt{a^2t^2-\rho^2}}d\theta d\rho + t^{-1}\int_0^{at}\int_0^{2\pi}\frac{\rho^2 v\cdot\nabla\phi(x+\rho v)}{\sqrt{a^2t^2-\rho^2}}d\theta d\rho.
\end{aligned}
$$

这里的第一项和I_1具有相同的形式，

$$
\left| t^{-1}\int_0^{at}\int_0^{2\pi}\frac{\rho\phi(x+\rho v)}{\sqrt{a^2t^2-\rho^2}}d\theta d\rho \right| \le C_2 t^{-3/2}.
$$

由于

$$|\rho v \cdot \nabla \phi(x+\rho v)| \leq \rho |v| |\nabla \phi(x+\rho v)| \leq at |\nabla \phi(x+\rho v)|,$$

第二项有估计

$$\left| t^{-1} \int_0^{at} \int_0^{2\pi} \frac{\rho^2 v \cdot \nabla \phi(x+\rho v)}{\sqrt{a^2t^2-\rho^2}} d\theta d\rho \right| \leq C_3 t^{-1/2}.$$

综上，我们得到

$$\sup_{x \in R^2} |u(t,x)| \leq C t^{-1/2}.$$

证毕。

9.3 高维波动方程初值问题的能量估计

对于d维波动方程，定义其以给定点(x_0, t_0)为顶点的特征锥为(看图)

$$K = K(x_0, t_0) = \{(t, x) \in [0, t_0] \times \mathbf{R}^d : |x - x_0| \le a(t_0 - t)\}.$$

对$\tau \le t_0$, 记

$$K_\tau = K \cap \{0 \le t \le \tau\}, \qquad \Omega_\tau = K \cap \{t = \tau\}.$$

用u_t乘方程的两边并在锥台K_τ上积分得

$$\int_{K_\tau} u_t \left(u_{tt} - a^2 \Delta u\right) dxdt = \int_{K_\tau} u_t f(t,x) dxdt.$$

由于

$$u_t u_{tt} = \frac{1}{2}\partial_t (u_t)^2,$$

$$u_t u_{x_j x_j} = \partial_{x_j}\left(u_t u_{x_j}\right) - u_{t x_j} u_{x_j} = \partial_{x_j}\left(u_t u_{x_j}\right) - \frac{1}{2}\partial_t (u_{x_j})^2,$$

代入上式得

$$\int_{K_\tau}\left\{\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial t}\right)^2+a^2\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2\right]-a^2\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial t}\right)\right\}dxdt$$

$$=\quad\int_{K_\tau}u_t f\,dxdt.$$

由Gauss散度定理，上式左边可以转化为锥台K_τ边界上的积分：

$$\int_{\Omega_\tau}\frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2+a^2\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2\right]dx-\int_{\Omega_0}\frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2+a^2\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2\right]dx$$

$$+\int\left\{\frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2+a^2\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2\right]n_t-a^2\sum_{j=1}^{d}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_j}\right)n_j\right\}dS_{(t,x)},$$

其中第三项的积分区域是K_τ的侧面$\{|x-x_0|=a(t_0-t),t\in[0,\tau]\}$，$(n_t,n_1,n_2,\cdots,n_d)$是该侧面的外法方向。

由于

$$n_t = \frac{a}{\sqrt{1+a^2}}, \quad n_j = \frac{x_j - x_{0j}}{a(t_0-t)\sqrt{1+a^2}},$$

上述侧面上的积分可以估计如下：

$$\begin{aligned}
& \int \left\{ \tfrac{1}{2} \left[\left(\tfrac{\partial u}{\partial t}\right)^2 + a^2 \sum_{j=1}^{d} \left(\frac{\partial u}{\partial x_j}\right)^2 \right] n_t - a^2 \sum_{j=1}^{d} \left(\frac{\partial u}{\partial t} \frac{\partial u}{\partial x_j} \right) n_j \right\} dS_{(t,x)} \\
\geq \quad & \tfrac{1}{2} \int \left\{ \left[\left(\tfrac{\partial u}{\partial t}\right)^2 + a^2 \sum_{j=1}^{d} \left(\frac{\partial u}{\partial x_j}\right)^2 \right] n_t - a^2 \sum_{j=1}^{d} \left[\left(\tfrac{\partial u}{\partial t}\right)^2 \frac{n_j^2}{n_t} + \left(\frac{\partial u}{\partial x_j}\right)^2 n_t \right] \right\} dS_{(t,x)} \\
= \quad & \tfrac{1}{2} \int \left(\tfrac{\partial u}{\partial t}\right)^2 \left(n_t - a^2 \sum_{j=1}^{d} \frac{n_j^2}{n_t} \right) dS_{(t,x)} \\
= \quad & \tfrac{1}{2} \int \left(\tfrac{\partial u}{\partial t}\right)^2 \tfrac{1}{n_t} [n_t^2 - a^2(1-n_t^2)] dS_{(t,x)} = 0.
\end{aligned}$$

这样得到

$$
\begin{aligned}
&\int_{\Omega_\tau}\left[\left(\frac{\partial u}{\partial t}\right)^2+a^2\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2\right]dx-\int_{\Omega_0}\left[\left(\frac{\partial u}{\partial t}\right)^2+a^2\textstyle\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2\right]dx\\
\le\quad & 2\int_{K_\tau}u_t f\,dxdt\le\int_{K_\tau}u_t^2dxdt+\int_{K_\tau}f^2dxdt.
\end{aligned}
$$

令

$$
G(\tau)=\int_{K_\tau}\left(u_t^2+a^2\sum_{j=1}^{d}u_{x_j}^2\right)dxdt.
$$

那么

$$\begin{aligned} \frac{dG}{d\tau} = & \int_{\Omega_\tau} \left[\left(\frac{\partial u}{\partial t}\right)^2 + a^2 \sum_{j=1}^{d} \left(\frac{\partial u}{\partial x_j}\right)^2 \right] dx \\ \leq & G(\tau) + F(\tau), \end{aligned} \tag{2}$$

其中

$$F(\tau) = \int_{\Omega_0} \left[\psi^2 + a^2 \sum_{j=1}^{d} \phi_{x_j}^2 \right] dx + \int_{K_\tau} f^2 dxdt.$$

上述不等式可以重写为

$$\frac{d(e^{-\tau} G(\tau))}{d\tau} \leq e^{-\tau} F(\tau) ds.$$

注意到$G(0) = 0$. 积分得到

$$\int_{K_\tau}\left[\left(\frac{\partial u}{\partial t}\right)^2+a^2\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2\right]dx=G(\tau)\le e^\tau\int_0^\tau e^{-s}F(s)ds\le(e^\tau-1)F(t_0).$$

代入（2）得

$$\int_{\Omega_\tau}\left[\left(\frac{\partial u}{\partial t}\right)^2+a^2\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2\right]dx\le e^\tau F(t_0).$$

证毕。

定理9.3：设$u=u(t,x)\in C^2(K)$是定解问题(1)的经典解，则它满足下列两个估计：

$$\int_{\Omega_\tau}[u_t^2(\tau,x)+a^2|\nabla u|^2(\tau,x)]dx,$$

$$\begin{aligned}&\int_{K_\tau}[u_t^2+a^2|\nabla u|^2]dxdt\\ \le\ & e^\tau\left[\int_{K_\tau}f^2dxdt+\int_{\Omega_0}\left(\psi^2+a^2|\nabla\phi|^2\right)dx\right].\end{aligned}$$

唯一性，连续依赖性。

上述估计稍作修改，也适用于带有低阶项的波动方程

$$u_{tt} - a^2 \Delta u + b(t,x)u_t + \sum_{j=1}^{d} b_j(t,x)u_{x_j} + c(t,x)u = f(t,x),$$

其中诸系数$b(t,x), b_j(t,x)$和$c(t,x)$都是给定的有界函数。

事实上，由于系数有界，所以存在正常数C使得

$$|u_t b(t,x) u_t| \le C u_t^2, \quad |u_t b_j(t,x) u_{x_j}| \le C(u_t^2 + a^2 u_{x_j}^2), \quad |u_t c(t,x) u| \le C(u_t^2 + u^2).$$

这样一来，对带有低阶项的波动方程，相应的不等式（2）成为

$$\begin{aligned}
&\int_{\Omega_\tau}\left[u_t^2+a^2\sum_{j=1}^{d}u_{x_j}^2\right]dx\\
\le\ & C\int_{K_\tau}\left[u_t^2+a^2\sum_{j=1}^{d}u_{x_j}^2+u^2\right]dx\\
&+\int_{\Omega_0}\left[\psi^2+a^2\sum_{j=1}^{d}\phi_{x_j}^2\right]dx+\int_{K_\tau}f^2dxdt.
\end{aligned} \tag{3}$$

对于上式中的u^2项，我们有

$$\begin{aligned} u^2(\tau,x) - u^2(0,x) = &\int_0^\tau \frac{du^2(t,x)}{dt}dt = 2\int_0^\tau u(t,x)u_t(t,x)dt \\ \leq &\int_0^\tau [u^2(t,x) + u_t^2(t,x)]dt, \end{aligned}$$

从而

$$\begin{aligned} \int_{\Omega_\tau} u^2(\tau,x)dx &\leq \int_{\Omega_\tau} u^2(0,x)dx + \int_0^\tau \int_{\Omega_\tau} [u^2(t,x) + u_t^2(t,x)]dxdt \\ &\leq \int_{\Omega_0} u^2(0,x)dx + \int_{K_\tau} [u^2(t,x) + u_t^2(t,x)]dxdt. \end{aligned}$$

此式与不等式（3）相加得

$$
\begin{aligned}
& \int_{\Omega_\tau} \left[u_t^2 + a^2 \sum_{j=1}^d u_{x_j}^2 + u^2 \right] dx \\
\leq \quad & C \int_{K_\tau} \left[u_t^2 + a^2 \sum_{j=1}^d u_{x_j}^2 + u^2 \right] dxdt \qquad (4) \\
& + \int_{\Omega_0} \left[\psi^2 + a^2 \sum_{j=1}^d \phi_{x_j}^2 + \phi^2 \right] dx + \int_{K_\tau} f^2 dxdt.
\end{aligned}
$$

令

$$
G(\tau) = \int_{K_\tau} \left[u_t^2 + a^2 \sum_{j=1}^d u_{x_j}^2 + u^2 \right] dx.
$$

那么（4）可以写成

$$
\frac{dG}{d\tau} \leq CG(\tau) + F(\tau),
$$

其中

$$F(\tau) = \int_{\Omega_0} \left[\psi^2 + a^2 \sum_{j=1}^{d} \phi_{x_j}^2 + \phi^2 \right] dx + \int_{K_\tau} f^2 dx dt.$$

有了这个不等式，如同之前可以推得

$$\int_{K_\tau} \left[u_t^2 + a^2 \sum_{j=1}^{d} u_{x_j}^2 + u^2 \right] dx, \int_{\Omega_\tau} \left[u_t^2 + a^2 \sum_{j=1}^{d} u_{x_j}^2 + u^2 \right] dx \le C^{-1} e^{C\tau} F(t_0).$$

作业：

1、无外力，初值二次连续可微，初始速度连续可微，都有紧支集。证明三维波动方程解的衰减估计1/t .

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