PDE 第十三次作业：

6. $\begin{cases} -\Delta u+u^2=0 & x\in\Omega,\ u\in C^2(\Omega)\cap C^1(\bar\Omega) \\ \frac{\partial u}{\partial n}+\alpha(x)u\big|_{\partial\Omega}=\varphi \end{cases}$

$\alpha(x)\geqslant \alpha_0>0$

$\max|u(x)|\leq \frac{1}{\alpha_0}\max_{\partial\Omega}|\varphi(x)|$

令：$v=\frac{1}{\alpha_0}\max_{\partial\Omega}|\varphi(x)| \Rightarrow \begin{cases} -\Delta v+v^3\geqslant 0 \\ \frac{\partial v}{\partial n}+\alpha(x)v\big|_{\partial\Omega}\geqslant\varphi(x) \end{cases}$

$\Rightarrow v\geqslant|u|$ 即证

10. $\begin{cases} -\Delta u=f(x,y) & (x,y)\in\Omega \\ u\big|_{\partial\Omega\setminus P_0}=\varphi(x,y) \end{cases}$

若有 u_1,u_2 均符合题意，则 $\begin{cases} -\Delta(u_1-u_2)=0 \\ u_1-u_2\big|_{\partial\Omega-P_0}=0 \end{cases}$

故只用解决 $f=\varphi=0$

$\forall\varepsilon>0$，取 r_0 充分小，s.t $\varepsilon\ln\frac{d}{r_0}>M:=\sup|u_1-u_2|$

考虑：$\Omega-B_r(P_0)$ $0<r<r_0$，$\varepsilon\ln\frac{d}{r_0}\geqslant|u|$ 在 $\partial[\Omega\setminus B_r(P_0)]$ 中

且 $-\Delta(\varepsilon\ln\frac{d}{r})=0$

$\Rightarrow$ 比较原理：$\varepsilon\ln\frac{d}{r}\geqslant|u|$

由连续性. $\varepsilon\ln\frac{d}{r}\geqslant|u|$ for $\Omega-P_0\Rightarrow\varepsilon\to0$，$u\equiv0$

27 (1) $\frac{1}{\pi R^2}\int u(x,y)\,dx\,dy = \frac{1}{\pi R^2}\int_0^R dr\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,d\theta$

$$= \frac{2}{R^2}\int_0^R dr\; r\cdot u(0,0) = u(0,0)$$

$\Rightarrow |u(0,0)| \le \frac{1}{\pi R^2}\left(\int u^2\right)^{\frac{1}{2}}\left(\int 1^2\right)^{\frac{1}{2}} = \frac{1}{\pi R^2} M^{\frac{1}{2}}(\pi R^2)^{\frac{1}{2}}$

(2) 在以(x,y)为圆心，$R-r$为半径的圆上，利用(1)即得

29. Δ的极坐标：$u_{rr}+r^{-1}u_r+u_{\theta\theta}r^{-2}=0$ 令 $r_1=\frac{R^2}{r}$

$v(r,\theta)=u(\frac{R^2}{r},\theta) \Rightarrow v_r = u_{r_1}\cdot(-\frac{R^2}{r^2}) \Rightarrow r^{-1}v_r = u_{r_1}\cdot(-\frac{R^2}{r^2})$

$v_{rr} = u_{r_1 r_1}\frac{R^4}{r^4}+u_{r_1}\cdot\frac{2R^2}{r^3}$ $(v_{\theta\theta}=u_{\theta\theta})$

$\Rightarrow v_{rr}+r^{-1}v_r+r^{-2}v_{\theta\theta} = u_{r_1r_1}\frac{R^4}{r^4}+u_{r_1}\frac{R^2}{r^3}+v_{\theta\theta}\cdot\frac{1}{R^2}$

$$=(u_{r_1r_1}+r_1^{-1}u_{r_1}+r_1^{-2}u_{\theta\theta})\frac{r_1^2}{r^2}=0$$

$\Delta v=0$

$v(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-r^2)\varphi(\alpha)}{a^2+r^2-2ar\cos(\theta-\alpha)}d\alpha = u(\frac{R^2}{r},\theta)$

$u(r,\theta)=\frac{-1}{2\pi}\int_0^{2\pi}\frac{(a^2-\frac{R^2}{r^2})\varphi(\theta)}{a^2+\frac{R^4}{r^2}-2aR^2/r\cos(\theta-\alpha)}d\alpha$