作业4.

2. 由动量定理及已知条件

$$\int_{x_1}^{x_2} \rho\,(u_t(x,t_2)-u_t(x,t_1))\,dx = \int_{t_1}^{t_2}\int_{x_1}^{x_2} f(x,t)\,dx\,dt +$$

$$T\int_{t_1}^{t_2}(u_x(x_2,t)-u_x(x_1,t))\,dt + \int_{x_1}^{x_2}\left(\int_{t_1}^{t_2} k u_t\,dt\right)dx$$

于是有 $\rho u_{tt}-Tu_{xx}+ku_t=f(x,t)$

5. 圆锥形杆上，取 $[x,x+\Delta x]$，t 时刻，设 x 处位移 $u(x,t)$.

此时杆两端横坐标为 $x+u(x,t)$, $x+\Delta x+u(x,t)$

从而相对伸长量为

$$\frac{[x+\Delta x+u(x+\Delta x,t)]-[x+u(x,t)]}{\Delta x}=u_x(x+\theta x,t)$$

取 $\Delta x\to 0$，得相对伸长量 $u_x(x,t)$

胡克定理 $\Rightarrow$ $T(x,t)=Eu_x(x,t)$

x 处截面积

$$S(x)=\pi r^2(x)=\pi\left(\frac{h-x}{h}R\right)^2=\pi R^2\left(1-\frac{x}{h}\right)^2$$

从而由动量守恒：

$$\rho S(x)\,\Delta x\,u_{tt}(x,t)=ES(x)\,(u_x|_{x+\Delta x}-u_x|_x)$$

再令 $\Delta x\to 0$ 即

$$\rho\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial t^2}=E\frac{\partial}{\partial x}\left[\left(1-\frac{x^2}{h^2}\right)\frac{\partial u}{\partial x}\right]$$

8. 设 $u=u(x,y,z,t)$ 为 t 时刻在 (x,y,z) 处的浓度

k 为扩散系数

结合散度定理有:

$$\iiint_D u|_{t=t_2}\,dxdydz - \iiint_D u|_{t=t_1}\,dxdydz$$

$$= \int_{t_1}^{t_2} \iint_D -\vec{v}\cdot\vec{n}\,dsdt$$

$$= \iint_{\partial D} (k\nabla u)\vec{n}\,dsdt$$

$$= \int_{t_1}^{t_2} \iiint_D \mathrm{div}(k\nabla u)\,dvdt$$

$$\Rightarrow u_t - k\nabla u = 0$$

14. $(1)\Rightarrow(2)$:

$$j(\varepsilon) := J(u+\varepsilon v) = \frac{1}{2}\int_\Omega [|\nabla v+\varepsilon\nabla v|^2 + (u+\varepsilon v)^2]\,dx + \frac{1}{2}\int_\Omega f(u+\varepsilon v)\,dx$$

$$-\int_{\partial\Omega} g(u+\varepsilon v)\,ds$$

令 $j'(0)=0$. 得 $\int_\Omega (\nabla u\nabla v + uv - fv)\,dx + \int_{\partial\Omega}(a(x)uv - gv)\,dx = 0$

$(2)\Rightarrow(1)$: $\forall w\in M$. 若 u 满足 (1) 则:

$$J(w) - J(u) = \frac{1}{2}\int_\Omega (|\nabla w|^2+w^2) - (|\nabla v|^2+v^2)\,dx + \frac{1}{2}\int_{\partial\Omega} a(x)(w^2-u^2)\,ds$$

$$-\int_\Omega f(w-u)\,dx - \int_{\partial\Omega} g(w-u)\,ds$$

令 $v=w-u$. 则:

$$J(u+v) - J(u) = \text{上式} \geq 0$$

故 $J(u)$ 为最小值

16. (1) $u_x = u_\xi + u_\eta \Rightarrow u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_t = -a u_\xi + a u_\eta \Rightarrow u_{tt} = a^2 u_{\xi\xi} - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta}$

代入波动方程可得 $-4a^2 u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0$

$\Rightarrow \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0$

$\Rightarrow \frac{\partial u}{\partial \xi} = f(\xi)$

$\Rightarrow u = \int f(\xi)\, d\xi + g(\eta)$

(2) 代入 u 计算