PDE 第十二周作业：

1. (1) $v(x) := C_0^{-1}\sup_{\Omega}|f(x)| \Rightarrow v|_{\partial\Omega} \geq 0 = u|_{\partial\Omega}$

$-\Delta v + c(x) v \geq \pm f(x) = -\Delta(\pm u) + c(x)(\pm u) \Rightarrow v(x) \geq \pm u(x)$

故 $C_0^{-1}\sup|f(x)| \geq \max_{\Omega}|u(x)|$

(2) 不能，$0 \in \Omega$，$d = \operatorname{diam}\Omega$

$v(x) := \sup_{\Omega}|f(x)|(d^2 - |x|^2)/2n$，$c(x) \geq 0$，$v(x) > 0$

所以 $-\Delta v + cv \geq \sup_{\Omega}|f(x)| \geq \pm f(u) = \Delta(\pm u) + c(\pm u)$

$\Rightarrow |u(x)| \leq v(x) \leq \frac{d^2}{2m}\sup|f(x)|$

(3) $u = \sin x$，$-u'' + u = 0$ $u(0) = u(\pi) = 0$，但 $u \neq 0$

5. $\forall \varepsilon > 0$，$\exists R_0 > 0$，$|x| \geq R_0$，s.t. $u(x) \leq l + \varepsilon$，任取 $x^0 \in \mathbb{R}^3 \setminus \Omega_0$

$R > R_0$. st. $x^0 \in B_R(0)$，$\Omega_0 \in B_R(0)$，在 $B_R(0) - \overline{\Omega_0}$ 用极值原理

$u(x^0) < \max(l+L, \max_{\partial\Omega_0}\varphi) \Rightarrow u(x^2) \leq \max(l, \max_{\partial\Omega_0}(\varphi))$

同理可证：$u(x^2) \geq \min(L, \min_{\partial\Omega_0}\varphi)$

4. 记左式 $= Ln$，若 u 在 Ω 内取到最小值，则 $\exists x^0 \in \Omega$ $u(x^0) = m \leq 0$

$\frac{\partial u}{\partial x}(x^0) > 0$，$u(x^0) \leq 0$，Hesse 矩阵非正定.

$\Phi[a_{ij}(x^0)]$ 正定，故 $-\sum a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \leq 0$

故 $cu(x^0) \leq 0$ 与 $Lu(x^0) = 0$ 矛盾

$\Rightarrow Lu \geq 0$ $\quad L(u+\varepsilon) = L(u) + c(x)\varepsilon > 0$（用课本 Thm.2.2）

2. $F=\sup_{\Omega}(f)$ $\Phi_1=\sup_{T_1}(\varphi_1)$, $\Phi_2=\sup_{T_2}(\varphi_2)$

$$w(x)=\phi_1/\alpha_0+\phi_2+\frac{F}{2m}\left(\frac{1+\alpha^2}{\alpha_0}+\alpha^2-|x|^2\right)\pm n$$

$$\Rightarrow Lw=-\frac{F}{2m}(-2n)+c(x)\left[\frac{\phi_1}{\alpha_0}+\phi_2+\frac{F}{2n}\left(\frac{1+\alpha^2}{\alpha_0}+\alpha^2-|x|^2\right)\right]\pm f$$

$$=F\pm f+c(x)\left[\frac{\phi_1}{\alpha_0}+\phi_2+\frac{F}{2n}\left(\frac{1+\alpha^2}{\alpha_0}+\alpha^2-|x|^2\right)\right]\geq 0$$

$$\Rightarrow w\geq 0 \Rightarrow |u|\leq\frac{\varphi}{\Omega_0}+\frac{1}{2n}\left(\frac{1+\alpha^2}{\alpha_0}+\alpha^2-|x|^2\right)F$$

3. 不妨设 $u(x^0)=0$，令 $w(x)=|x|^{-a}-r^{-a}$，$w(x^0)=0$

$$w(x_i)=-a|x|^{-a-2}x_i,\quad w_{x_ix_i}=-a(+a-2)|x|^{-a-4}x_i^2-a|x|^{-a-2}$$

$$Lw=-a(a-2)|x|^{-a-2}+an|x|^{-a-2}\sum b_ix_i+c(x)|x|^{-a}$$

$$\leq[-a(a-2)|x|+an-a\sum b_ix_i+c(x)|x|^2)|x|^{-a-2}$$

$$\leq[-a(a-2)+an+aBr+Cr^2]|x|^{-a-2}$$

$B=(\sum b_i^2)^{\frac{1}{2}}$, $C=\sup_{\Omega}c(x)$

$$L(-\varepsilon w)=0,\quad -\varepsilon w|_{\partial B}=0,\quad -\varepsilon w\,\Big|_{|x|=\frac{r}{2}}\geq u\Big|_{|x|=\frac{r}{2}}$$

$$\Rightarrow -\varepsilon w\geq u\quad \frac{r}{2}<|x|\leq r\quad \frac{\partial u}{\partial v}\geq\frac{\partial}{\partial r}(-\varepsilon w)$$

令 $\rho=|x|\Rightarrow \frac{\partial}{\partial v}(-\varepsilon w)=-\varepsilon\frac{\partial w}{\partial r}\cos(u,v)\Big|_{\rho=r}=\sum\alpha r^{-\alpha-1}\cos(u\cdot v)>0$