第十六周作业：

P238-239. 1. $\Delta = (xy)^2 + y^2(l+x)$

双曲型：$\begin{cases} y^2 > 0 \\ x^2+x+l>0 \end{cases}$

$1-4l<0 \Rightarrow D=\{(x,y): y\neq 0\}$

$1-4l=0 \Rightarrow D=\{(x,y): y\neq 0, x\neq -\frac{1}{2}\}$

$1-4l>0 \Rightarrow D=\{(x,y): y\neq 0, x<-\frac{1+\sqrt{1+4l}}{2}$ 或 $x>\frac{-1+\sqrt{1-4l}}{2}\}$

椭圆、抛物线型可类似讨论

2. 证问题简化为常函数的情形：

$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+b_1u_x+b_2u_y+cu+d=0$

$\begin{cases} \xi = a_1x+b_1y \\ \eta = a_2x+b_2y \end{cases}$ $J=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

$\Rightarrow \Delta' = J^2\Delta$. $J^2>0 \Rightarrow$ 命题得证

3. (1) 无界；(2) 常数；(3) 无界；(4) 常数

31. $f\in C^1(R)$, $f'(x)\in C^\infty(\mathbb{R})$, $u\in H^1(\Omega)$

$u_n\in C^1(\Omega)$, $\|u_n-u\|_{H^1(\Omega)}\to 0$

$|f\circ u - f\circ u_n| = |f'(\xi)|\cdot\|u_n(x)-u(x)\|$

故 $\|f_0(u)-f_0(u_n)\|_{L^2}\to 0$

$|f'(u_n(x))_{x_1}-f'(u(x))_{x_1}| = |f'(u_n(x_1))-f'(u(x_1))u_{x_1}(x)+$

$f'(u_n(x))(u_{n x_1}(x)-u_{x_1}(x_1))|$

$\Rightarrow |f'(u_n(x))_{x_1}-f'(u(x))_{x_1}|_{L^2}\to 0$

$\Rightarrow f_0 u\in H$

34. $u\in H^1[a,b]$, $J(W)=\min J(v)$

$J(v)=\frac{1}{2}\int_a^b [k(x)(\frac{dv}{dx})^2+p(x)v^2]+\frac{\alpha}{2}v^2(b)+\frac{\beta}{2}v^2(a)$

$-\int_a^b f(x)v(x)dx-g_1v(b)-g_2v(a)$

$k\geq k_0>0,\quad p\geq p_0>0$

$J(v)\geq\frac{1}{2}\int_a^b k_0v'^2+pv^2-\int_a^b f^2\int_a^b v^2-c>-\infty$

故下界存在

记 $J(v_k)\leq\inf J(v)+\frac{1}{k}$, $v_k(a)=v_l(a)$, $v_k(b)=v_l(b)$

$\Rightarrow J(v_k)-J(v_l)\geq\frac{1}{2}\cdot M\cdot\|v_k-v_l\|_{H_1}$

$\Rightarrow \{v_k\}$ 为 Cauchy 列，解存在

唯一性: 若 $J(u_1)=J(u_2)=\pm\infty\Rightarrow\|\nabla(u_1-u_2)\|_{L^2}=J(u_1)+J(u_2)$

$-2J(\frac{u_1+u_2}{2})\leq 0$，故 $u_1=u_2$ a.e.

32. $u-\varphi\in H_0^2$

$J(v)=\frac{1}{2}\int_\Omega|\nabla v|^2dx-\int_\Omega fv\,dx\geqslant\frac{1}{2}\int_\Omega|\nabla v|^2dx-\|f\|_{L^2}\cdot\|v\|_{L^2}>-\infty$

(Poincare. $\|\nabla u\|_{L^2}^2\geqslant c\|u\|_{L^2}^2$)

类似34证明：完备性，解存在，故基本解存在

唯一性用 $\|\nabla(u_1-u_2)\|^2=J(u_1)+J(u_2)-2J(\frac{u_1+u_2}{2})\leqslant 0$

$-\Delta u=f \Leftrightarrow J(u+\varepsilon v)=J(v)\quad J'(\varepsilon)=0$

$\Rightarrow\langle\nabla u,\nabla\Phi\rangle=-\langle\Delta u,\phi\rangle=\langle f,\phi\rangle=-\Delta u=f$

36. $J(v)\geqslant C_1\|v\|_{H_1}-(\|f\|^2+\|g\|^2)\cdot C_2$ 有下界

$\|\nabla(v_1-v_2)\|_{L^2}^2\to\|v_1-v_2\|_{L^2}=J(v_1)+J(v_2)-J(\frac{v_1+v_2}{2})\cdot 2$

$\Rightarrow$ 唯一性，下证存在性：

$v\in C_0^\infty(\Omega)\Rightarrow\int_\Omega\nabla u\nabla v+uv=\int_\Omega fv$

$u\in C^2(\Omega)\cap C^1(\Omega)\Rightarrow\int_\Omega(-\Delta u+u)v=\int_\Omega fv$

$\Rightarrow -\Delta u+u=f$

再取 $v\in C^1(\Omega)$, $\int\nabla u\nabla v+uv-\int\frac{\partial u}{\partial v}v\,dl=\int_\Omega fv\,d\Omega$

$\Rightarrow\frac{\partial u}{\partial v}=g$. 即证

28. 原命题 $\Leftrightarrow\Delta v=0$, $v|_{\partial B}=0$ 的有界解为0.

$v=\varepsilon\cdot\ln\frac{R}{r}\qquad v|_{\partial B(R)}=0$

$R>r_0>0\qquad\varepsilon\cdot\ln\frac{R}{r_0}>M=\sup|u|$

考虑、$B(R)-B(r)$ $\quad v \geq \pm u \quad \Delta v=0 \quad v \geq |u|$ 在 $B(R)-B(r)$ 上成立

$\Rightarrow |u| \leq v = \varepsilon \cdot \ln \frac{R}{r}$ for $B(R) \to 0$, $\varepsilon \to 0$ 即证.

35. $u \in C^1(\bar{\Omega})$, $u(x,\xi) = \frac{1}{b}\int_0^b u(x,y)\,dy \leq \frac{1}{b}\int_0^b |u(x,y)|\,dy$

$u(x,b) = u(x,\xi) + \int_\xi^b u_y(x,y)\,dy \Rightarrow |u(x,b)| \leq |u(x,\xi)| + \int_0^b |u_y(x,y)|\,dy$

$$\leq \frac{1}{b}\int_0^b |u(x,y) + v_y(x,y)|\,dy$$

$\Rightarrow \left(\int_0^b |u(x,b)|^2\,dx\right)^{\frac{1}{2}} \leq C\|u\|_{H^1}$

$C^1(\bar{\Omega})$ 在 $H_1(\Omega)$ 中稠密

33. $v \in H_0^2$

$J(v) - J(u) = \frac{1}{2}\int_\Omega |\nabla v|^2 - |\nabla u|^2 - fv + fu$

$$= \frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 + \nabla u \cdot \nabla(v-u) - \int_\Omega f(v-u)$$

$$= \frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 \Rightarrow u \text{ 为 } \inf J(v)$$

30. $f(\xi) = \frac{1}{b-a}\int_0^b f(x)\,dx \quad (f \in C^1[a,b])$

$f(x) = f(\xi) + \int_\xi^b f'(x)\,dx$

$\Rightarrow |f(x)| \leq C(\|f\|_{L^2} + \|f'\|_{L^2})$

一般: $f \in H^1 \quad f_n \in C^1[a,b] \xrightarrow{H^1} f$

$$\Rightarrow f_n \xrightarrow{L^2} f \Rightarrow f_n \xrightarrow{a.e.} f \Rightarrow \exists g_1 \in C^1[a,b],\ g \overset{a.e.}{=} f$$

$\|g\|_{L^2} \leq M\|f\|_{L^2}$

12. $F=\sup|f|$, $G=\sup|g|$, $W=\sup|w|$, $V=\sup|v|$

$\Rightarrow W \le \frac{1}{2}(F+v)$

$\Rightarrow W\le F$, $v\le F$

$V\le \frac{1}{2}(F+W)$

14. 唯一性 $\Leftrightarrow f=0$

$\int_\Omega -u\cdot\Delta u+\sum b_i(x)u_iu+c(x)u^2=0$

$\Leftrightarrow \int_\Omega \sum u_i^2(x)+\sum b_i(x)u_iu+c(x)u^2=0$

$c\ge \frac{1}{4}\sum b_i^2$ $\quad\sum b_iuu_i\le\sum(u_i^2+\frac{1}{4}b_i^2u^2)$

$\Leftrightarrow \int(c-\sum\frac{b_i^2}{4})u^2\le 0$. 矛盾