第十一周第一次作业：

一. P152 J. ① $4[1-f(t)]-[1-f(2t)] = f(2t)-4f(t)+3$

$$= \int_{-\infty}^{+\infty} (e^{itx}-1)(e^{itx}-3)\, dF(x)$$

其中 $\text{Re}[(e^{itx}-1)(e^{itx}-3)] = [\cos(tx)-1][\cos(tx)-3]+\sin^2(tx) \geq 0$

因 $f(t)$为实值函数. 故 $\int_{-\infty}^{+\infty}(e^{itx}-1)(e^{itx}-3)\,dF(x) =$

$\text{Re}[\int_{-\infty}^{+\infty}(e^{itx}-1)(e^{itx}-3)\,dF(x)] \geq 0$

② 首先证明 $f(2t) \geq f(t)^2$

$$f(2t) = \int_{-\infty}^{+\infty} e^{2itx}\, dF(x)$$

$$f(t)^2 = [\int_{-\infty}^{+\infty} e^{itx}\, dF(x)]^2 \leq \int_{-\infty}^{+\infty} e^{2itx}\, dF(x) = f(2t)$$

因此有： $1+f(2t)-2[f(t)]^2 \geq 1+f(t)^2-2[f(t)]^2 = 1-f(t)^2 \geq 0 \quad (|f(t)| \leq 1)$

$\Rightarrow \quad 1+f(2t) \geq 2[f(t)]^2$

14. $\xi.\eta$独立，$\xi \sim$ Possion分布，$\eta \sim N(0,1)$

Prove： $\xi+\eta$为连续型

Proof：设 $\xi+\eta$ 的 d.f 为 $F(x)$

$$\Rightarrow F(x) = \sum_{k=0}^{+\infty} P_1(k) F_2(x-k), \quad P_1(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad F_2 \text{ is } \eta\text{'s d.f.}$$

$$F(x+h)-F(x) = \sum_{k=0}^{+\infty} P_1(k)[F_2(x+h-k)-F_2(x-k)]$$

$$\leq \sum_{k=0}^{+\infty} [F_2(x+h-k)-F_2(x-k)]$$

$$= \sum_{k=0}^{+\infty} \mathbb{P}_2\{\eta \in (x-k, x+h-k)\}$$

If $h<1$ we have: $(x-k, x+h-k)$ 两两不交

从而 $\lim_{h\to 0} \sum_{k=0}^{+\infty} \mathbb{P}_2\{\eta \in (x-k, x+h-k)\} = 0$

$\Rightarrow F(x)$ 连续

16. (ξ,η) density: $p(x,y) = \frac{1}{4}[1+cxy(x^2-y^2)]$ $|x|<1$. $|y|<1$ $c\in(-1,1)$

(1) $f(s,t) = \int_{-1}^{1}\int_{-1}^{1} e^{i(sx+ty)} p(x,y)\,dx\,dy =$

$= \frac{\sin s}{s}\frac{\sin t}{t} + \frac{c}{s^4t^2}[(3s^2-6)\sin s-(s^3-6s)\cos s](\sin t - t\cos t)$

$- \frac{c}{s^2t^4}[(3t^2-6)\sin t - (t^3-6t)\cos t](\sin s - s\cos s)$

(2) $p_1(x) = \int_{-1}^{1} \frac{1}{4}[1+cxy(x^2-y^2)]\,dy = \frac{1}{2}$

$f_1(s) = \int_{-1}^{1} e^{isx}\frac{1}{2}\,dx = \frac{\sin s}{s}$

同理 $f_2(t) = \frac{\sin t}{t}$

(3) 设 p 为 $x+y$ 的密度函数:

若 $z\le 0$: $p(z) = \int_{-1}^{z+1} p(x, z-x)\,dx = \frac{1}{4} + \frac{z^2c}{4}x^2\Big|_{-1}^{z+1} = \frac{c}{4}z^4 + \frac{c}{2}z^3 + \frac{1}{4}z^2 + \frac{1}{2}$

若 $z>0$. $p(z) = p(-z) = \frac{c}{4}z^4 - \frac{c}{2}z^3 + \frac{1}{4}z^2 + \frac{1}{2}$ (对称性)

$f_{\xi+\eta}(t) = \int_{-2}^{2} e^{itz} p(z)\,dz = \left(\frac{\sin t}{t}\right)^2$

二. P158 3. (1) $f(t) = \frac{\alpha^2}{\alpha^2+t^2}$

$\Rightarrow p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx} f(t)\,dt = \frac{1}{\pi}\frac{\alpha}{x^2+\alpha^2}$

(2) $\frac{1}{(1-\alpha i t)^\beta} = [1+\alpha i t + (\alpha i t)^2 + \cdots]^\beta$

$$\Rightarrow p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\,[1+\alpha it+(\alpha it)^2+\cdots]^{\beta}\,dt = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}$$

(Gamma Distribution)

(3) (Pólya Distribution ?)

4. $S_n = \prod_{j=1}^{n} f_j$, $\{f_j : 1 \le j \le n\}$ ch.f.'s, then S_n is a ch.f.

$S_n \to S_\infty$ in pr.

Prove: $\prod_{j=1}^{\infty} f_j(t)$ converges in the sense of infinite product for each t and is the ch.f. of S_∞

Proof: $S_n \to S_\infty$ in pr. $\Rightarrow \lim_{n\to\infty} P\{|X_n - X| > \varepsilon\} = 0$

if for t, $\prod_{j=1}^{\infty} f_j(t)$ doesn't converge.

$f_j(t)$ is a ch.f. $\Rightarrow f_j(t)$ uniformly continuous

$\Rightarrow S_n = \prod_{j=1}^{\infty} f_j(t)$ uniformly continuous.

$\Rightarrow \int |X_n - X|\,du > 0 \Rightarrow X$

So $S_n \to S_\infty$ pointly.

$$S_n = \sum_{j=1}^{n} S_j \Rightarrow S_\infty = \lim_{n\to\infty}\sum_{j=1}^{n} S_j = \sum_{j=1}^{+\infty} S_j$$

$\Downarrow$

$$\prod_{j=1}^{n} f_j(t) = \phi_j(t) \Rightarrow \prod_{j=1}^{+\infty} f_j(t) = \phi_\infty(t) \text{ is the ch.f. of } S_\infty$$

12. $\{X_j : 1 \le j \le n\}$ independent r.v.'s, each having d.f. ϕ.

ϕ stands for Normal Distribution.

Then $(X_j)^2$ has the d.f. $\frac{1}{2\sqrt{2\pi}} e^{-\frac{y}{2}} y^{-\frac{1}{2}}$

$\Rightarrow$ ch.f. of $(X_j)^2$ is: $f(t) = \int_{-\infty}^{+\infty} e^{ity} dF(y)$

$\Rightarrow$ ch.f. of $\sum_{j=1}^{n} (X_j)^2$ is: $\prod_{j=1}^{n} f(t) = [\int_{-\infty}^{+\infty} e^{ity} dF(y)]^n$

14 $P(X=0)=0$, $P(X=1)=1$

When $X=1$, $Y=0$, when $X=0$, $P(Y=0 \mid X=0)=\frac{1}{2}$, $P(Y=2 \mid X=0)=\frac{1}{2}$

Then X and Y have the same p.m. Not independent.

$P(X+Y=0) = \frac{1}{4} = P(X+Y=2)$. $P(X+Y=1)=\frac{1}{2}$

$\Rightarrow$ $X+Y$ has the same d.f. as $X+X_1$, where X_1 and X iid.

$\Rightarrow$ $X+Y$ has the p.m. $\mu * \mu$

P104. Theorem 6.2.5.

Prove. $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt = \sum_{x\in\mathbb{R}^1} \mu(\{x\})^2$

Proof: Applying theorem 6.2.4 LHS $= (\mu * \mu')(\{0\})$

Applying (7) of Sec 6.1.

$$\text{RHS} = \int_{\mathbb{R}^1} \mu'(\{-y\})\mu(dy) = \sum_{y\in\mathbb{R}^1} \mu(\{y\})\mu(\{y\})$$

Corollary: μ is atomless (F is continuous) if and only if the limit of the LHS $= 0$.