第十四周第一次作业：

一、P184-185 3. $\{\xi_n\}$方差有界，$|i-j|\to\infty$时ξ_i与ξ_j相关系数$r_{ij}\to 0$

Prove: $\{\xi_n\}$成立大数定律

Proof. 只需证明 $\frac{1}{n}\sum_{i=1}^{n}\xi_i \to E[\xi_i]$ a.s.

定义样本平均 $\overline{\xi}_n = \frac{1}{n}\sum_{i=1}^{n}\xi_i$

方差 $D(\overline{\xi}_n) = D(\frac{1}{n}\sum_{i=1}^{n}\xi_i) = \frac{1}{n^2}D(\sum_{i=1}^{n}\xi_i)$

$= \frac{1}{n^2}[\sum_{i=1}^{n}D(\xi_i) + 2\sum_{1\le i<j\le n} cov(\xi_i,\xi_j)]$

因$\xi_i \le c$，故

$$\sum_{i=1}^{n} D(\xi_i) \le nC$$

对协方差项：记 $cov(\xi_i,\xi_j) = P_{ij}\sqrt{D(x_i)D(x_j)}$

$\Rightarrow |cov(\xi_i,\xi_j)| \le c$

当$|i-j|\le N$时：$\sum_{\substack{1\le i<j\le n \\ |i-j|\le N}} cov(\xi_i,\xi_j) \le nNC$

当$|i-j|>N$时. $\sum_{\substack{1\le i<j\le n \\ |i-j|>N}} |cov(\xi_i,\xi_j)| \le \varepsilon \cdot \sum_{\substack{1\le i<j\le n \\ |i-j|>N}} c = \varepsilon n^2\frac{c}{2}(1-\frac{N}{n})$

综上有：$D(\sum_{i=1}^{n}\xi_i) \le nC + 2nNC + \varepsilon C\frac{n^2}{2}(1-\frac{N}{n})$

$D(\overline{\xi}) = \frac{1}{n^2}D(\sum_{i=1}^{n}\xi_i) \le \frac{C}{n} + \frac{2NC}{n} + \varepsilon C(\frac{1}{2}-\frac{N}{2n})$

with $n\to+\infty$ 上式右侧为0

最后由 Chebechev 不等式：$\forall \delta>0$

$$\mathbb{P}\{|\bar{\xi}_n - \bar{\xi}| \geq \delta\} \leq \frac{D(\bar{\xi}_n)}{\delta^2} \to 0 \quad 当 n \to +\infty$$

故成立大数定律

4. $\{\xi_n\}$ iid r.v.s. $\forall n$, $D(\xi_n) < +\infty$, $|k-l|>2$ 时 ξ_k 与 ξ_l 独立

证：对 $\{\xi_n\}$ 大数定律成立

Proof：首先构造新的随机变量：$\{\xi_{3k+1}\}$. $\{\xi_{3k+2}\}$ $\{\xi_{3k}\}$

它们每组中均满足独立性，每个方差为 σ^2

只需证 $\sum_{n=1}^{\infty} \frac{D(\xi_n)}{n^2}$ 收敛：

因所有 ξ_n 方差相同，$D(\xi_n) = \sigma^2$，于是：

$$\sum_{n=1}^{\infty} \frac{\sigma^2}{n^2} = \sigma^2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\sigma^2$$

所以 $\sigma^2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \sigma^2 \frac{\pi^2}{6} < +\infty$

因此 $\sum_{n=1}^{\infty} \frac{D(\xi_n)}{n^2}$ 收敛

5. $\{\xi_n\}$ iid $\sim U(0,1)$，求证：$\left(\prod_{k=1}^{n} \xi_k\right)^{\frac{1}{n}} \xrightarrow{P} c$，并求 c

Proof：令 $Y_n = \log\left(\prod_{k=1}^{n} \xi_k\right)^{\frac{1}{n}} = \frac{1}{n}\sum_{k=1}^{n} \log \xi_k$

因为 $\xi_k \sim U(0,1)$，$\log \xi_k$ iid

$$E(\log \xi_k) = \int_0^1 \log x \, dx = -1$$

$$E[(\log \xi_k)^2] = \int_0^1 (\log x)^2 \, dx = 2$$

$\Rightarrow D(\log \xi_k) = 2 - 1 = 1$

$\Rightarrow \left(\prod_{k=1}^{n} \xi_k\right)^{\frac{1}{n}} = e^{Y_n} \xrightarrow{P} \frac{1}{e}$，$c = \frac{1}{e}$

8. 举例说明 Borel-Cantelli 引理中，命题

(1) 若 $\sum_{n=1}^{\infty} P(A_n) < +\infty$，则 $P(\overline{\lim_n} A_n) = 0$

的逆不成立

例子：概率空间 $(\Omega, \mathcal{F}, P)$，其中 $\Omega = [0,1]$，$\mathcal{F}$ 为 Borel 代数，

P 是区间上的 Lebesgue 测度

令 $A_n = [\frac{1}{2n}, 1]$

则 $P(\overline{\lim_{n\to\infty}} A_n) = 0$，但 $\sum_{n=1}^{\infty} P(A_n) = +\infty$

9. $\{\xi_n\}$ iid，方差有限 $D(\xi_n) = \sigma^2$，$\bar{\xi} = E\{\xi_k, k \geq 1\}$，$S_n^2 = D(\xi_1, \xi_2, \cdots, \xi_n)$

试证 $\{S_n^2\}$ 几乎必然收敛到 σ^2

Proof: $\bar{\xi}_n = \frac{1}{n}\sum_{i=1}^{n} \xi_i$，$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_i)^2$

先计算 $E(\xi_i) = u$. ξ_i iid，样本方差：

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i - \frac{1}{n}\sum_{j=1}^{n}\xi_j)^2$$

注意到 $2\bar{\xi}_n \sum_{i=1}^{n} \xi_i = 2\bar{\xi}_n \cdot n\bar{\xi}_n = 2n\bar{\xi}_n^2$

$\Rightarrow S_n^2 = \frac{1}{n-1}(\sum_{i=1}^{n}\xi_i^2 - n\bar{\xi}_n^2)$

由大数定律：$\bar{\xi}_n \xrightarrow{a.s.} u$

并且 $\bar{\xi}_n^2 \xrightarrow{a.s.} u^2$

再由大数定律 ξ_i^2：

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i^2 \sim n(\sigma^2 + u^2)$$

$\Rightarrow S_n^2 \approx \frac{1}{n-1}[n(\sigma^2+\mu^2) - n\bar{\xi}_n^2]$

由于 $\overline{\xi_n^2} \xrightarrow{a.s.} \mu^2$. 故:

$$n(\sigma^2+\mu^2) - n\bar{\xi}_n^2 \xrightarrow{a.s.} n\sigma^2$$

10 设 $\{\xi_n\}$ 为独立随机变量序列，满足条件 $E(\xi_n)\to 0$ 且 $\sum_{n=1}^{\infty}\frac{E(\xi_n^2)}{n^2} < +\infty$

证明: $\frac{1}{n}\sum_{k=1}^{n}\xi_k \xrightarrow{a.s} 0$

Proof. 令 $\eta_n = \xi_n - E(\xi_n)$，则 η_n 独立同分布，$E(\eta_n)=0$

$$E(\eta_n^2) = E[(\xi_n - E(\xi_n))^2] = E(\xi_n^2) - E^2(\xi_n)$$

因 $E(\xi_n)\to 0$，对于足够大的 n，$E(\xi_n)$ 较小

即 $E(\eta_n^2) \approx E(\xi_n^2)$

由此可得: $\sum_{n=1}^{\infty}\frac{E(\eta_n^2)}{n^2} \le 2\sum_{n=1}^{\infty}\frac{E(\xi_n^2)}{n^2} < +\infty$

由 kolmolov 强大数定律推广版本:

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k = \frac{1}{n}\sum_{k=1}^{n}[\eta_k + E(\xi_k)] = \frac{1}{n}\sum_{k=1}^{n}\eta_k + \frac{1}{n}\sum_{k=1}^{n}E(\xi_k)$$

设 $\lim_{n\to\infty} E(\xi_k) = 0$，则有:

$$\left|\frac{1}{n}\sum_{k=1}^{n}E(\xi_k)\right| \le \frac{1}{n}\sum_{k=1}^{n}|E(\xi_k)| \to 0$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n}\xi_k = \frac{1}{n}\sum_{k=1}^{n}\eta_k + \frac{1}{n}\sum_{k=1}^{n}E(\xi_k) \to 0 \quad a.s.$$

二. P111 2. Theorem 5.1.2: X_j's are uncorrelated and their second moments have a second bound, then (1) is true in L^2 and hence also pr

(1) $\frac{S_n - \mathcal{E}(S_n)}{n} \to 0$ a.e.

Under the same hypothesis we have:

$$\frac{S_n}{n^\alpha} \to 0 \text{ a.e. for any } \alpha > \frac{3}{4} \quad ?$$

P119 3. For any sequence $\{X_n\}$:

$$\frac{S_n}{n} \to 0 \text{ in pr.} \Rightarrow \frac{X_n}{n} \to 0 \text{ in pr.}$$

More generally, this is true if n is replaced by b_n,

where $\frac{b_{n+1}}{b_n} \to 1$

Proof: Assume that $\frac{S_n}{n} \to 0$ in probability, which means.

$$\forall \varepsilon > 0, \ \lim_{n\to\infty} P\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) = 0$$

We need to show that:

$$\forall \varepsilon > 0, \ \lim_{n\to\infty} P\left(\left|\frac{X_n}{n}\right| \geq \varepsilon\right) = 0$$

Note that:

$$S_n = \sum_{i=1}^{n} X_i$$

So we have $\frac{S_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i$

For any $\varepsilon > 0$.

$$P\left(\left|\frac{X_n}{n}\right| \geq \varepsilon\right) \leq P\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) + P\left(\left|\frac{S_n - X_n}{n}\right| \geq \varepsilon\right)$$

As $n \to \infty$, since $\frac{S_n}{n} \to 0$ in p. the first term on the right-hand side goes to 0.

$$\lim_{n\to\infty} P(|\frac{S_n}{n}| \geq \varepsilon) = 0$$

For the second term, observe that:

$$\frac{S_n - X_n}{n} = \frac{1}{n}\sum_{i=1}^{n-1} X_i$$

which is essentially the average of $X_1, X_2, \cdots, X_{n-1}$

which also should go to 0 in probability due to the

same reasoning applied to S_n

Thus the second term goes to 0:

$$\lim_{n\to\infty} P(|\frac{S_n}{n}| \geq \varepsilon) = 0$$

Combining these results we get:

$$\lim_{n\to\infty} P(|\frac{S_n}{n}| \geq \varepsilon) = 0$$

This shows that $\frac{X_n}{n} \to 0$ in probability

Generalization with b_n:

For a sequence $\{b_n\}$ such that $\frac{b_{n+1}}{b_n} \to 1$

Use the same arguments as before, we can show that

$\frac{X_n}{b_n} \to 0$ in probability