第四周第一次作业：

一. P13 Thm 1.3.2 Proof of Lebesgue Decomposition theorem.

Existence: Discreate Part $F_d(x)$ is defined as the sum of the probabilities of the atoms of F at x. This part is composed of point masses

Singular Continus Part: Construct the Cantor function $F_c(x)$ which is an example of a singular continuous distribution.

Absolutely Continuous Part: Define $F_a(x)$ as the absolutely continus part of F with respect to the Lebesgue measure.

Convex Combination: Prove that $F = \alpha F_d + \beta F_c + (1-\alpha-\beta) F_a$ for some $\alpha, \beta \in [0,1]$ where $\alpha \cdot \beta \leq 1$

Uniqueness:

Suppose there exist two decompositions of into discrete, singular continuous and absolutely continuous parts.

$F = \alpha_1 F_{d_1} + \beta_1 F_{c_1} + (1-\alpha_1-\beta_1) F_{a_1}$

$F = \alpha_2 F_{d_2} + \beta_2 F_{c_2} + (1-\alpha_2-\beta_2) F_{a_2}$

$\Rightarrow \alpha_1 = \alpha_2.\ \beta_1 = \beta_2$ by comparing the coeffients of the decompositions

P13. 6. Proof: If the distribution is concentrated on a countable set of points $\{x_1, x_2, \cdots\}$ Then the $F(x)$ would jump from 0 to p_1 at x_1, from p_1 to p_1+p_2 at x_2, $\cdots$

This satisfy the defination of a singular distribution

P15 12. First we need to prove that $G(x)$ is a distribution function

this is equivalent to:

(1) $G(x)$ is non-decreasing

(2) $\lim\limits_{x\to-\infty} G(x)=0$ and $\lim\limits_{x\to+\infty} G(x)=1$

(3) $G(x)$ is right-continuous.

Since F is already a d.f, it already satisfies the properties above.

Mutiplying $F(r_n+x)$ by 2^n doesn't change these properties.

so $G(x)$ inherits them.

Then we prove: $G(x)$ is strictly increasing for all x:

Above we have proved that $G(x)$ is non-decreasing

If there exists $a<b$. s.t. $G(a)=G(b)=A$

$$G(b)-G(a)=\sum_{k=1}^{+\infty} 2^{-k}[F(r_k+b)-F(r_k+a)]$$

$\Rightarrow F(r_k+b)=F(r_k+a) \quad \forall k\geq 1$

$\Rightarrow F$ is a constant, contradictory

$\Rightarrow G(x)$ is increasing on $\forall x \in \mathbb{R}$

Last we prove: $G(x)$ is singular.

To prove this we need to prove $G(x)$ is not continuous on at most.

Since $F(x)$ is singlar and $G(x)$ bases from linear additions of $F(x+r_n)$

$\Rightarrow G(x)$ singular.

P31 11. $\{x\}$ is a singleton $\Rightarrow \exists \delta > 0$. s.t within $(x-\delta, x+\delta)$

we have $F(x') = \begin{cases} F_1 & x' < x \\ F_2 = F_1 + \mu(\{x\}) & x' \geq x \end{cases}$

Thus $\mu(\{x\}) = x'_1 - x'_2 = F_2 - F_1 = F(x) - F(x^-)$

12. If there exists $x_0 \in \mathbb{R}$ s.t. F isn't continuous on x_0

Then $F^+(x_0) - F^-(x_0) > 0$. By right continuous we have $F^+(x_0) = F(x_0)$

$\Rightarrow F(x_0) - F^-(x_0) > 0$.

Combine ex.11 we have: $\{x_0\}$ is singleton, which is contradictory

to "without atom"

P41. 3. Yes. Let $X: \Omega \to \mathbb{R}$ to be defined as $X(\omega) = x$ if and only if

$P\{x \leq \mu\} = \mu((-\infty, x])$, $\forall x \in \mathbb{R}$

In other words, X is the inverse cumulative distribution

function of μ.

4. $\theta \sim U(0,1)$ and for each d.f. F.

Define $G(y) = \sup\{x : F(x) \le y\}$

We need to prove that $G(\theta)$ has the d.f. F

proof: $x = G(\theta) \Longleftrightarrow F(x) \le \theta$ and for $\forall \delta > 0.\ F(x+\delta) > \theta$

suppose $0 < \theta_1 < \theta_2 < 1$: Since $\{x : F(x) \le \theta_1\} \subseteq \{x . F(x) \le \theta_2\}$

$\Rightarrow G(\theta_1) \le G(\theta_2)$ which means $G(\theta)$ isn't decreasing

Thus $F_G(\theta) = \int_{-\infty}^{G^{-1}(\theta)} F(x)\,dx = F(\theta) - F(-\infty) = F(\theta)$

5. Let $A = \{F(x) : x \in \mathbb{R}\}$ Then $A \subseteq [0,1]$

When F is continuous, Then $(0,1) \in A$ and $F(x)$ follows

uniform distribution on $[0,1]$

Otherwise $F(x)$ follows uniform distribution on A

which means $\forall x \in A.\ u(x) = \frac{1}{u(A)}$

Ps2 16. $\int_{-\infty}^{+\infty} [F(x+a) - F(x)]\,dx = \int_{-\infty}^{+\infty} dx \int_{x}^{x+a} dF(x)$

$= \int_0^a dx \int_{-\infty}^{+\infty} dF(t+x) = \int_0^a dx = a$

二. Ps8 9. (1) $\frac{1+2x}{3}$ 和 1 均为单调不减的连续函数

$F(1) = \frac{1+2\times 1}{3} = 1 = F(1^+)$ 符合右连续条件.

$\lim_{x\to+\infty} F(x) = 1 \quad F(0^+) = \frac{1}{3} > 0.$ 故 $F(x)$ 是一个分布函数

(2) F 在定义域 $[0,+\infty)$ 上无突变 $\Rightarrow$ F 不是离散型

$F(0^+)=\frac{1}{3}>0$ $\Rightarrow$ F 不是连续型

令 $\xi=\frac{1}{3}\xi_1+\frac{2}{3}\xi_2$，　其中 $\xi_1=0$，$\xi_2\sim U(0,1)$

则 $F(\xi)=\frac{1}{3}F_1(\xi)+\frac{2}{3}F_2(\xi)$，$F(x)=\begin{cases}\frac{1+2x}{3} & 0<x\le 1\\ 1 & x>1\end{cases}$