第六周第一次作业：

1. $\xi \sim Exp(\lambda)$, $\eta = \begin{cases} \xi & \text{if } \xi \geq 1 \\ -\xi^2 & \text{if } \xi < 1 \end{cases}$，求 $F_\eta(y)$：

$$F_\eta(y) = P\{\eta \leq y\} = \begin{cases} P\{\xi \leq y\} & \text{if } y \geq 1 \\ P\{\xi < 1\} & \text{if } 0 \leq y < 1 \\ P\{\xi < -(-y)^{\frac{1}{2}}\} + P\{(-y)^{\frac{1}{2}} < \xi < 1\} & \text{if } -1 < y < 0 \\ P\{\xi < -(-y)^{\frac{1}{2}}\} & \text{if } y \leq -1 \end{cases}$$

$\xi \sim Exp(\lambda)$: $p_\xi(x) = \lambda e^{-\lambda x}$ $\quad F_\xi(x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}$

代入上式得

$$F_\eta(y) = P\{\eta \leq y\} = \begin{cases} 1 - e^{-\lambda y} & \text{if } y \geq 1 \\ 1 - e^{-\lambda} & \text{if } 0 \leq y < 1 \\ 1 + e^{-\lambda} - e^{\lambda(-y)^{\frac{1}{2}}} - e^{-\lambda(-y)^{\frac{1}{2}}} & \text{if } -1 < y < 0 \\ 1 - e^{\lambda(-y)^{\frac{1}{2}}} & \text{if } y \leq -1 \end{cases}$$

2. $\xi, \eta$ r.v.s. independent on $(\Omega, \mathcal{F}, u)$, show:

$\xi$ continuous $\Rightarrow$ $\xi + \eta$ continuous ?

Proove by contradiction: if $\xi + \eta$ is not continuous at $a \in \mathbb{R}$

which means $F(a) - F(a^-) > 0 \Rightarrow P\{\xi + \eta = a\} > 0$

However, $P\{\xi + \eta = a\} = \int_{\mathbb{R}} P\{\eta = y\} P\{\xi = a - y\} dy = 0$. contradictory !

3. $\xi_1, \xi_2, \cdots$ iid r.v.s. $\xi_i \sim Exp(\lambda)$

$T_0 = 0, \ T_n := \sum_{k=1}^{n} \xi_k \ (n = 1, 2, \cdots)$

Define $N(t) = \max\{n \geq 0 : T_n \leq t\}, \ t \geq 0$

Prove: $\{N(t) : t \geq 0\}$ is a poisson process with parameter $\lambda$

Proof: $T_0 = 0, \ T_1 \sim Exp(\lambda), \ T_n = nT_1 \Rightarrow p_n(x) = \frac{\lambda}{n} e^{-\frac{\lambda x}{n}} \Rightarrow T_n \sim Exp(\frac{\lambda}{n})$

$P\{N(t) = k\} = P\{T_k \leq t \text{ and } T_{k+1} > t\} = P\{T_k \leq t\} - P\{T_{k+1} \leq t\}$

$= (1 - e^{-\frac{\lambda}{k} t}) - (1 - e^{-\frac{\lambda}{k+1} t}) = e^{-\frac{\lambda t}{k+1}} - e^{-\frac{\lambda t}{k}}$

$\Rightarrow N(t)$ is a Possion process with parameter $\lambda$