第八周第二次作业：

English Textbook (Kai Lai Chung):

P46-47 Ex 2, 4, 5, 7

P51-52 Ex 10, 11, 12, 14, 19, 20

P65-66 Ex 9, 10, 11

P46-47 2. $E(|X|)<+\infty$, $\lim_{n\to\infty} P(\Lambda_n)=0$

Then $\lim_{n\to+\infty} \int_{\Lambda_n} X\,dP=0$

$$\Rightarrow E(|X|)=\int_{-\infty}^{+\infty} X\,dP \geq \int_{-\infty}^{+\infty} \sum_{k=1}^{+\infty} 1_{\Lambda_k} X\,dP=0$$

$$=\sum_{k=1}^{+\infty} \int_{\Lambda_k} X\,dP$$

$$\Rightarrow \lim_{n\to+\infty} \sum_{k=n}^{+\infty} \int_{\Lambda_k} X\,dP=0$$

$$\Rightarrow \lim_{n\to+\infty} \int_{\Lambda_n} X\,dP=0$$

In particular, let $\Lambda_k=\{|X|>k\}$

Then $\lim_{n\to+\infty} \int_{\{|X|>n\}} X\,dP = \lim_{n\to+\infty} \int_{\Lambda_n} X\,dP=0$

4. c is a fixed constant, $c>0$

Prove that $E(|X|)<+\infty \Leftrightarrow \sum_{n=1}^{+\infty} P(|X|\geq cn)<+\infty$

Proof: $\sum_{n=1}^{+\infty} P(|X|\geq cn)=\sum_{n=1}^{+\infty} \int_{|X|\geq cn} dP = \int_{-\infty}^{+\infty} dP \sum_{n=1}^{[\frac{|X|}{c}]} 1$

$$=\int_{-\infty}^{+\infty} \left[\frac{|X|}{c}\right] dP \leq \int_{-\infty}^{+\infty} \frac{|X|}{c}\,dP = \frac{1}{c}\int_{-\infty}^{+\infty} |X|\,dP = \frac{1}{c}E(|X|)$$

So $\sum_{n=1}^{+\infty} P(|X| \geq cn) < +\infty \iff \frac{1}{c} E(|X|) < +\infty \iff E(|X|) < +\infty$

5. Prove: $\forall r > 0$, $E(|X|^r) < +\infty \iff \sum_{n=1}^{+\infty} n^{r-1} P(|X| \geq n) < +\infty$

Proof. $\sum_{n=1}^{+\infty} n^{r-1} P(|X| \geq n) = \sum_{n=1}^{+\infty} n^{r-1} \int_{|X| \geq n} dP$

$$= \int_{-\infty}^{+\infty} dP \sum_{n=1}^{[|X|]} n^{r-1} \leq \int_{-\infty}^{+\infty} dP \int_0^{|X|} s^{r-1} ds = \int_{-\infty}^{+\infty} dP \frac{|X|^r}{r}$$

$$= \frac{1}{r} \int_{-\infty}^{+\infty} |X|^r dP = \frac{1}{r} E(|X|^r)$$

So $\sum_{n=1}^{+\infty} n^{r-1} P(|X| \geq n) < +\infty \iff \frac{1}{r} E(|X|^r) < +\infty \iff E(|X|^r) < +\infty$

7. X r.v. $E(X)$ finite, $\varepsilon > 0$

Prove: $\exists$ simple r.v. X_ε s.t. $E(|X - X_\varepsilon|) < \varepsilon$

Proof: $E(X) = \int_{-\infty}^{+\infty} x \, dP(x) = \sum_{n=1}^{+\infty} \int_{n-1 \leq |x| < n} x \, dP(x)$

$$E(|X - X_\varepsilon|) = \sum_{n=1}^{+\infty} \int_{n-1 \leq |x| < n} |X - X_\varepsilon| \, dP(x)$$

For $\forall n$, let $A_n = \{x \mid n-1 \leq |x| < n\}$, $M_n = \sup_{x \in A_n} \frac{dP(x)}{dx}$, $\varepsilon_n = \frac{\varepsilon}{2^n}$

Let $\int_{n-1 \leq |x| < n} |X - X_\varepsilon| \, dP(x)$

$$\leq \int_{n-1 \leq |x| < n} \frac{\varepsilon_n}{2} dP(x)$$

$$\leq \varepsilon_n = \frac{\varepsilon}{2^n}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \int_{n-1 \leq |x| < n} x \, dP(x) \leq \sum_{n=1}^{+\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

Hence there exists a sequence of simple r.v.'s X_m such that

$$\lim_{m \to \infty} E(|X - X_m|) = 0$$

P51-52. 10. $0 \le r < r'$ and $E(|X|^{r'}) < +\infty$

Prove that (1) $E(|X|^r) < +\infty$

(2) $E(|X|^r) < +\infty \Leftrightarrow E(|X-a|^r) < +\infty \quad \forall a$

Proof: (1) $E(|X|^r) = \int_{-\infty}^{+\infty} |x|^r p(x)\,dx$

$$= \int_{|x|>1} |x|^r p(x)\,dx + \int_{0 \le |x| \le 1} |x|^r p(x)\,dx$$

$$< \int_{|x|>1} |x|^{r'} p(x)\,dx + \int_{0 \le |x| \le 1} |x|^r p(x)\,dx$$

By Cauchy inequalities we have:

$$\left(\int_{0 \le |x| \le 1} |x|^r p(x)\,dx\right)^{r'} \le \left(\int_{0 \le |x| \le 1} |x|^{r'} p(x)\,dx\right)^r \cdot \left(\int_{0 \le |x| \le 1} p(x)\,dx\right)^{r'-r}$$

$$\Rightarrow \int_{0 \le |x| \le 1} |x|^r p(x)\,dx < +\infty$$

Also $\int_{|x|>1} |x|^{r'} p(x)\,dx \le \int_{-\infty}^{+\infty} |x|^{r'} p(x)\,dx < +\infty$

So $E(|X|^r) < +\infty$

(2) "$\Leftarrow$": let $a=0$, obvious

"$\Rightarrow$": $E(|X-a|^r) = \int_{-\infty}^{+\infty} |x-a|^r\,dP(x) = \int_{-\infty}^{+\infty} |x|^r\,dP(x+a)$

$$E(|X|^r) = \int_{-\infty}^{+\infty} |x|^r\,dP(x)$$

$$E(|X-a|^r) - E(|X|^r) = \int_{-\infty}^{+\infty} |x|^r [dP(x+a) - dP(x)]$$

$$= \int_{-\infty}^{+\infty} |x|^r\,d[P(x+a) - P(x)]$$

$$= \int_{-\infty}^{+\infty} |x|^r \, d\mathbb{P}\{x < t \le x+a\}$$

$$< +\infty$$

$\Rightarrow E(|X-a|^r) < +\infty$

11. $E(X^2)=1$, $E(|X|) \geq a > 0$

Prove: $P\{|X| \geq \lambda a\} \geq (1-\lambda)^2 a^2 \quad (0 \leq \lambda \leq 1)$

Proof: $\dfrac{dP\{|X| \geq \lambda a\}}{da} \leq 0 \qquad \dfrac{d[(1-\lambda)^2 a^2]}{da} = 2(1-\lambda)^2 a \geq 0$

So we just need to prove the situation where $a = E(|X|)$

Let $Y=|X|$. we just need to prove:

$$P\{Y \geq \lambda E(Y)\} \geq (1-\lambda)^2 E(Y)^2 \quad (Y \geq 0)$$

?

12. $X \geq 0$, $Y \geq 0$, $p \geq 0$

prove: (1) $E[(X+Y)^P] \leq 2^P [E(X^P)+E(Y^P)]$

(2) if $p>1$, the factor 2^P may be replaced by 2^{P-1}

(3) if $0 \leq p \leq 1$, it may be replaced by 1.

Proof: (1) $E[(X+Y)^P] = \iint_{\mathbb{R}_+^2} (x+y)^P p(x,y) \, dx dy$

$$E(X^P) = \int_0^{+\infty} x^P p_1(x) \, dx = \iint_{\mathbb{R}_+^2} x^P p(x,y) \, dx dy$$

$$E(Y^P) = \int_0^{+\infty} y^P p_2(y) \, dy = \iint_{\mathbb{R}_+^2} y^P p(x,y) \, dx dy$$

$$(x+y)^p \le 2^p(x^p+y^p)$$

$$左边 = \sum_{k=0}^{p} \binom{p}{k} x^k y^{p-k}$$

$$= \frac{1}{2}\sum_{k=0}^{p} \binom{p}{k} (x^k y^{p-k} + x^{p-k} y^k)$$

$$\le \sum_{k=0}^{p} \binom{p}{k} (x^p+y^p) = 右边$$

(2) If $p>1$. $\left(\frac{x+y}{2}\right)^p \le \frac{x^p+y^p}{2}$ (均值不等式)

$$\Rightarrow (x+y)^p \le 2^{p-1}(x^p+y^p)$$

(3) If $0 \le p \le 1$. By Bonoulli inequality.

$$\left(1+\frac{y}{x}\right)^p \le 1+\left(\frac{y}{x}\right)^p$$

$$\Rightarrow (x+y)^p \le x^p+y^p$$

14. $p>1$. prove that:

(1) $\left|\frac{1}{n}\sum_{j=1}^{n} x_j\right|^p \le \frac{1}{n}\sum_{j=1}^{n}|x_j|^p$, and so $E\left(\left|\frac{1}{n}\sum_{j=1}^{n} x_j\right|^p\right) \le \frac{1}{n}\sum_{j=1}^{n} E(|x_j|^p)$

(2) $E\left(\left|\frac{1}{n}\sum_{j=1}^{n} x_j\right|^p\right) \le \left\{\frac{1}{n}\sum_{j=1}^{n} E(|x_j|^p)^{\frac{1}{p}}\right\}^p$

Proof: (1) By Hölder inequalities:

$$\left|\sum_{j=1}^{n} x_j\right|^p \le \left(\sum_{j=1}^{n}|x_j|^p\right)\cdot n^{p-1} \Rightarrow \left|\frac{1}{n}\sum_{j=1}^{n} x_j\right|^p \le \frac{1}{n}\sum_{j=1}^{n}|x_j|^p$$

By linearity $E\left(\left|\frac{1}{n}\sum_{j=1}^{n} x_j\right|^p\right) \le \frac{1}{n}\sum_{j=1}^{n} E(|x_j|^p)$

(2) By Hölder inequalties.

$$\left(\sum_{j=1}^{n} E(|x_j|^p)^{\frac{1}{p}}\right)^p \ge \left(\sum_{j=1}^{n} E|x_j|^p\right)^p$$

$$\Rightarrow E(|\frac{1}{n}\sum_{j=1}^{n} x_j|^p) \le \{\frac{1}{n}\sum_{j=1}^{n} E(|x_j|^p)^{\frac{1}{p}}\}^p$$

19. $\{X_n\}$ i.d. r.v.'s with finite mean.

Prove: $\lim_{n\to+\infty} \frac{1}{n} E(\max_{1\le j\le n}|X_j|) = 0$

Proof: Let $Y_n = \max_{1\le j\le n}|X_j|$ r.v.

$$E(\max_{1\le j\le n}|X_j|) = E(Y_n) = \int_0^{+\infty} P\{Y_n > x\}\,dx = \int_0^{+\infty} [1-F_n(x)]\,dx$$

$$\lim_{n\to+\infty} \frac{1}{n} E(\max_{1\le j\le n}|X_j|) = \lim_{n\to+\infty} \frac{1}{n}\int_0^{+\infty} [1-F_n(x)]\,dx = 0$$

20. For $r>1$ prove that:

$$\int_0^{+\infty} \frac{1}{u^r} E(X\wedge u^r)\,du = \frac{r}{r-1} E(X^{\frac{1}{r}})$$

Proof: By Ex.17 $E(X\wedge u^r) = \int_0^{u^r} P(X>x)\,dx = \int_0^{u} P(X^{\frac{1}{r}} > v)\, r v^{n-1}\,dv$

$$\int_0^{+\infty} \frac{1}{u^r} E(X\wedge u^r)\,du = \int_0^{+\infty} \frac{1}{u^r}\cdot du \cdot \int_0^{u} P(X^{\frac{1}{r}}>v) r v^{n-1}\,dv$$

$$= \int_0^{+\infty} P(X^{\frac{1}{r}}>v) r v^{n-1}\,dv \int_v^{+\infty} \frac{1}{u^r}\,du$$

$$= \int_0^{+\infty} P(X^{\frac{1}{r}}>v) r v^{n-1} \cdot \frac{1}{r-1} v^{1-r}\,dv$$

$$= \int_0^{+\infty} \frac{r}{r-1} P(X^{\frac{1}{r}}>v) v^{n-r}\,dv$$

$$= \frac{r}{r-1} E(X^{\frac{1}{r}})$$

P65-66 9. X. Y independent, $E(X)$ exists

Prove: $\forall B\in\mathcal{B}$, $\int_{\{Y\in B\}} X\,dP = E(X)P\{Y\in B\}$

Proof: $E(X)P\{Y\in B\} = \int_{-\infty}^{+\infty} X\,dP \cdot P\{Y\in B\}$

$$= \int_{-\infty}^{+\infty} X\, dP \int_{\{Y\in B\}} dP(Y)$$

$$= \int_{\{Y\in B\}} dP(Y) \int_{-\infty}^{+\infty} X\, dP$$

$$= \int_{\{Y\in B\}} X\, dP$$

10. $X.Y$ independent, $p>0$

Prove: If $E(|X+Y|^p)<+\infty$, then $E(|X|^p)<+\infty$ and $E(|Y|^p)<+\infty$

Proof: Suppose $P_1.P_2$ be the probability function of X and Y.

Then we have:

$$E(|X+Y|^p) = \iint_{\mathbb{R}^2} |X+Y|^p\, dP_1 dP_2$$

$$= \int_{-\infty}^{+\infty} dP_1 \int_{-\infty}^{+\infty} |X+Y|^p\, dP_2$$

$$= \int_{-\infty}^{+\infty} dP_1 \int_{-\infty}^{+\infty} |Z|^p\, dP_2(Z-X)$$

$$= \int_{-\infty}^{+\infty} E(|Y|^p)\, dP_2(Y)$$

$\Rightarrow E(|Y|^p)<+\infty$. 同理 $E(|X|^p)<+\infty$

11. $X.Y$ independent, $E(|X|^p)<+\infty$, $p\geq 1$. $E(Y)=0$

Prove: $E(|X+Y|^p) \geq E(|X|^p)$

Proof: $E(|X+Y|^p) = \int_{-\infty}^{+\infty} dP_1 \int_{-\infty}^{+\infty} |x+y|^p\, dP_2$

$E(|X|^p) = \int_{-\infty}^{+\infty} |x|^p\, dP_1$

$$E(|X+Y|^p) - E(|X|^p) = \int_{-\infty}^{+\infty} dP_1 \int_{-\infty}^{+\infty} |x+y|^p\, dP_2 - \int_{-\infty}^{+\infty} |x|^p\, dP_1$$

$$= \int_{-\infty}^{+\infty} dP_1 \int_{-\infty}^{+\infty} (|x+y|^p - |x|^p)\, dP_2$$

$$= \int_{-\infty}^{+\infty} E(|X+Y|^p) - E(|X|^p) \, dP_1$$

$$\geq \int_{-\infty}^{+\infty} \left(E(|X|^p) - E(|X|^p)\right) dP_1 + \int_{-\infty}^{+\infty} y^p \, dP_1$$

$$\geq \left(\int_{-\infty}^{+\infty} y \, dP_1\right)^p$$

$$= 0$$