第四周第二次作业：

P65-66 2. (1) Find n r.v.s s.t. every (n-1) of them are independant but not all of them.

Consider the following n r.v.s:

$\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n$ s.t. $\sum_{i=1}^{n} \xi_i = 1$, $\xi_i \in \mathbb{R}$.

For every (n-1) of them are independant, but their sum is static,

so these n r.v.s are not independant.

(2) Find incidents Λ_1, Λ_2 and Λ_3 s.t. Λ_1 and Λ_2 are independent

Λ_1 and Λ_3 are independent, but Λ_1 and $\Lambda_2 \cup \Lambda_3$ are not independent

Consider a family with two kids, one is younger and the other one older.

Let $\Lambda_1 = \{$ The older kid is a girl $\}$

$\Lambda_2 = \{$ The younger kid is a girl $\}$

$\Lambda_3 = \{$ Two kids have the same sex $\}$

$\Rightarrow$ Λ_1 and Λ_2 independent

Λ_1 and Λ_3 independent

$P(\Lambda_1) = \frac{1}{2}$ $P(\Lambda_2 \cup \Lambda_3) = \frac{3}{4}$, $P(\Lambda_1 \cap (\Lambda_2 \cup \Lambda_3)) = \frac{1}{4} \neq P(\Lambda_1) P(\Lambda_2 \cup \Lambda_3)$

$\Rightarrow$ Λ_1 and $\Lambda_2 \cup \Lambda_3$ not independent

3. $\{E_\alpha : \alpha \in A\}$ are independent $\Leftrightarrow \{F_\alpha : \alpha \in A,\ F_\alpha = E_\alpha \text{ or } E_\alpha^c\}$ are independent

To prove the events $\bigcup_{\alpha \in A_\beta} E_\alpha$ $(\beta \in B)$ are independent.

We just need to prove that $\forall B' \subseteq B$. $B' = \{\beta_1, \beta_2, \cdots, \beta_n\}$ finite

events $\bigcup_{\alpha \in A_{\beta_i}} E_\alpha$ $(i = 1, 2, \cdots, n)$ are independent.

$$\Leftrightarrow \prod_{i=1}^{n} P\{\bigcup_{\alpha \in A_{\beta_i}} E_\alpha\} = P\{\bigcap_{i=1}^{n} \bigcup_{\alpha \in A_{\beta_i}} E_\alpha\}$$

By Exclusion Exp: $P\{\bigcup_{\alpha \in A_{\beta_i}} E_\alpha\} = \sum_{\alpha \in A_{\beta_i}} P(E_\alpha) - \sum_{\substack{\{\alpha_1, \alpha_2\} \in A_{\beta_i}^2 \\ \alpha_1 \neq \alpha_2}} P(E_{\alpha_1} \cap E_{\alpha_2}) + \cdots$

$$= \sum_{\alpha \in A_{\beta_i}} P(E_\alpha) - \sum_{\substack{\{\alpha_1, \alpha_2\} \in A_{\beta_i}^2 \\ \alpha_1 \neq \alpha_2}} P(E_{\alpha_1}) P(E_{\alpha_2}) + \cdots$$

Thus $\prod_{i=1}^{n} P\{\bigcup_{\alpha \in A_{\beta_i}} E_\alpha\} = \prod_{i=1}^{n} \left[\sum_{\alpha \in A_{\beta_i}} P(E_\alpha) - \sum_{\substack{\{\alpha_1, \alpha_2\} \in A_{\beta_i}^2 \\ \alpha_1 \neq \alpha_2}} P(E_{\alpha_1}) P(E_{\alpha_2}) + \cdots\right]$

On the other hand $P\{\bigcap_{i=1}^{n} \bigcup_{\alpha \in A_{\beta_i}} E_\alpha\} = P\{\bigcup_{(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \prod_{i=1}^{n} B_i} (E_{\alpha_1} \cap E_{\alpha_2} \cap \cdots \cap E_{\alpha_n})\}$

$$= \sum_{\vec{\alpha} \in \vec{B}} P\{E_{\alpha_1} \cap E_{\alpha_2} \cap \cdots \cap E_{\alpha_n}\} - \cdots$$

Compare RHS of the two equations, they are the same.

6. X and $f(X)$ independent $\Leftrightarrow F_1(x_0) F_2[f(x_0)] = F_3[x_0 f(x_0)]$ $\forall x_0 \in \mathbb{R}$

where $F_1 . F_2 . F_3$ are relatively d.f. of X, $f(X)$ and $(X, f(X))$

But we know that $F_1(x_0) = F_2[f(x_0)] = F_3[x_0 f(x_0)]$ a.s. for x_0

Thus $F(x) \in \{0, 1\} \Rightarrow X$ is constant with probability one and when so X and $f(X)$ are independent

7. $\bigcap_{j=1}^{\infty} E_j = \lim_{n\to\infty} \bigcap_{j=1}^{n} E_j$

So $P(\bigcap_{j=1}^{\infty} E_j) = P(\lim_{n\to\infty} \bigcap_{j=1}^{n} E_j) = \lim_{n\to\infty} P(\bigcap_{j=1}^{n} E_j)$

$\{E_j : 1 \le j < \infty\}$ independent $\Rightarrow \forall \{i_1, i_2, \cdots, i_k\} \subseteq \{1, 2, \cdots\}$.

$\{E_{i_1}, E_{i_2}, \cdots, E_{i_k}\}$ independent $\Rightarrow P(\bigcap_{j=1}^{n} E_j) = \prod_{j=1}^{n} P(E_j)$

$\Rightarrow P(\bigcap_{j=1}^{\infty} E_j) = \lim_{n\to\infty} \prod_{j=1}^{n} P(E_j) = \prod_{j=1}^{\infty} P(E_j)$

$\bigcup_{j=1}^{\infty} E_j = \lim_{n\to\infty} \bigcup_{j=1}^{n} E_j$

So $P(\bigcup_{j=1}^{\infty} E_j) = P(\lim_{n\to\infty} \bigcup_{j=1}^{n} E_j) = \lim_{n\to\infty} P(\bigcup_{j=1}^{n} E_j)$

Since $E_1, E_2, \cdots$ independent, we have

$$P(\bigcup_{j=1}^{n} E_j) = \sum_{j=1}^{n} P(E_j) - \sum_{1\le i<j\le n} P(E_i)P(E_j) + \cdots + (-1)^{n-1} \prod_{j=1}^{n} P(E_j)$$

$$= 1 - \prod_{j=1}^{n} (1 - P(E_j))$$

$$\Rightarrow P(\bigcup_{j=1}^{\infty} E_j) = \lim_{n\to\infty} [1 - \prod_{j=1}^{n} (1 - P(E_j))] = 1 - \prod_{j=1}^{\infty} (1 - P(E_j))$$

8. Denote $M = \max\{X_1, X_2, \cdots, X_n\}$, $m = \min\{X_1, X_2, \cdots, X_n\}$

$\Rightarrow F_M(x) = P\{M \le x\} = P\{X_1 \le x, X_2 \le x, \cdots, X_n \le x\}$

Since $X_1, X_2, \cdots, X_n$ are independent, this becomes

$F_M(x) = \prod_{i=1}^{n} P(X_i \le x) = F_1(x) F_2(x) \cdots F_n(x)$

The same way we can get the minimum

$$F_m(x) = 1 - \prod_{i=1}^{n} [1 - F_i(x)]$$

17. $P(\bigcup_{j=1}^{n} E_j^{(n)}) \to 0$ as $n \to +\infty$

By $P(\bigcup_{j=1}^{n} E_j^{(n)}) \geq \sum_{j=1}^{n} P(E_j^{(n)}) - \sum_{1 \leq j < k \leq n} P(E_j^{(n)} E_k^{(n)})$

$$\Rightarrow \lim_{n \to \infty} \frac{\sum_{j=1}^{n} P(E_j^{(n)}) - \sum_{1 \leq j < k \leq n} P(E_j^{(n)}) P(E_k^{(n)})}{P(\bigcup_{j=1}^{n} E_j^{(n)})} = 1$$

On the other hand $\sum_{1 \leq j < k \leq n} P(E_j^{(n)}) P(E_k^{(n)}) = (\sum_{j=1}^{n} P(E_j^{(n)}))^2 - \sum_{j=1}^{n} P^2(E_j^{(n)}) \sim [\sum_{j=1}^{n} P(E_j^{(n)})]^2$

So $\lim_{n \to \infty} \frac{\sum_{j=1}^{n} P(E_j^{(n)})}{P(\bigcup_{j=1}^{n} E_j^{(n)})} = 1$ and $P(\bigcup_{j=1}^{n} E_j) \sim \sum_{j=1}^{n} P(E_j^{(n)})$

P60 Thm 3.3.4. finite or infinite sequence $\{\mu_j\}$ on $(\mathbb{R}^1, \mathcal{B}^1)$

and d.f.s. $\Rightarrow$ There exists a finite or infinite probability space

$(\Omega, \mathcal{F}, P)$ and a sequence of independent r.v.s $\{X_j\}$ defined on it

s.t. $\forall j$, μ_j is the p.m. of X_j.

Proof: Without loss of generality we suppose the given sequence is infinite.

$\forall n$, suppose $(\Omega_n, \mathcal{F}_n, P_n)$ be a probability space in which there exists

an r.v. X_n with μ_n as its p.m.

Define the infinite measure space $\Omega = \Omega_1 \times \Omega_2 \times \cdots$

on the collection of all points $w = \{w_1, w_2, \cdots\}$

A subset E of Ω will be called a "finite-product set" iff

it's of the form $E = \overset{\infty}{\underset{n=1}{\times}} F_n$ where each $F_n \in \mathcal{F}_n$ and all but a

finite number of these F_n's are equal to the corresponding Ω_n's.

Thus $w \in E \Leftrightarrow w_n \in F_n$, $n \geq 1$

Define a set function P on $\mathcal{F}_0$ as follows.

(1) First. $\forall E$ as finite-product set, let:

$$P(E) = \prod_{n=1}^{\infty} P_n(F_n)$$

where all but a finite number of the factors

on the right side are equal to one.

(2) Next, if $E \in \mathcal{F}_0$ and $E = \bigcup_{k=1}^{n} E^{(k)}$

where the $E^{(k)}$'s are disjoint finite-product sets, we put

$$P(E) = \sum_{k=1}^{n} P(E^{(k)})$$

If a given set E in $\mathcal{F}_0$ has two representations of the form above.

Then it is not difficult to see that the two definitions of

$P(E)$ agree. Hence the set function P is uniquely defined on $\mathcal{F}_0$

and it is clearly positive with $P(\Omega) = 1$,

and additive on $\mathcal{F}_0$ by definition.

$\Rightarrow$ $\mathbb{P}$ as defined on $\mathcal{F}_0$ is a p.m.