第七周第二次作业：

一. 1. ξ,η ind, $\xi \sim \Gamma(\frac{1}{2},\frac{n}{2})$, $\eta \sim \Gamma(\frac{1}{2},\frac{m}{2})$

let $\alpha=\xi+\eta$, $\beta=\frac{\xi}{\eta}$, then:

$$\begin{cases} u=x+y \\ v=\frac{x}{y} \end{cases} \Rightarrow \begin{cases} x=\frac{uv}{1+v} \\ y=\frac{u}{1+v} \end{cases} \quad (u,v>0)$$

$$|J(x,y|u,v)|=\frac{-u}{(1+v)^2}$$

So combined d.f: $q(u,v)=\frac{1}{2^{\frac{m+n}{2}}\Gamma(\frac{m+n}{2})}u^{\frac{m+n}{2}-1}e^{-\frac{u}{2}}\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\frac{v^{\frac{m}{2}-1}}{(1+v)^{\frac{m+n}{2}}}$

$\Rightarrow$ α,β iid, $\alpha \sim \chi^2(m+n)$

$$q_2(v)=\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\frac{v^{\frac{m}{2}-1}}{(1+v)^{\frac{m+n}{2}}} \quad (v>0)$$

So the d.f. of $Y=\frac{n}{m}\beta$ is

$$p(y)=\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\frac{(\frac{m}{n})^{\frac{m}{2}}y^{\frac{m}{2}-1}}{(1+\frac{m}{n}y)^{\frac{m+n}{2}}} \quad (y>0)$$

$$F_Y(y)=\int_0^y p(t)\,dt$$

2. $\{\xi_k\}$ iid r.v.s. $\xi_1 \sim U(0,1)$

$X \triangleq \min\limits_{1\le k\le n}\{\xi_k\}$. $Y \triangleq \max\limits_{1\le k\le n}\{\xi_k\}$

① $F_{(\xi_1,Y)}(z,y)=P\{\xi_1<z \text{ and } \max\limits_{1\le k\le n}\{\xi_k\}<y\}$

$=P\{\xi_1<\min\{z,y\} \text{ and } \xi_k<y \ (2\le k\le n)\}$

$=\min\{z,y\}\, y^{n-1}$

$F_{(\xi_1,X)}(z,x) = P\{\xi_1 < z \text{ and } \min_{1\le k\le n} \xi_k < x\}$

if $z \le x$. $P\{\xi_1 < z \text{ and } \min_{1\le k\le n} \xi_k < x\} = P\{\xi_1 < z\} = z$

if $z > x$, $P\{\xi_1 < z \text{ and } \min_{1\le k\le n} \xi_k < x\} = P\{\xi_1 < x\} + P\{x \le \xi_1 < z\} P\{\min_{2\le k\le n} \xi_k < x\}$

$= x + (z-x)\cdot[1-(1-x)^{n-1}]$

$= z - (z-x)(1-x)^{n-1}$

Therefore $F_{\xi_1,X}(z,x) = \begin{cases} z & \text{if } z \le x \\ z-(z-x)(1-x)^{n-1} & \text{if } z > x \end{cases}$

$F_{(X,Y)}(x,y) = P\{\min_{1\le k\le n} \xi_k < x \text{ and } \max_{1\le k\le n} \xi_k < y\}$ $(x<y)$

$= P\{\max_{1\le k\le n} \xi_k < y\} - P\{x \le \xi_k < y : 1\le k\le n\}$

$= y^n - (y-x)^n$

② $P\{\xi_1 = Y\} = P\{\xi_2, \xi_3, \cdots, \xi_n \le \xi_1\} = \int_0^1 p_{\xi_1}(x) x^{n-1} dx$

$= \int_0^1 x^{n-1} dx = \frac{1}{n}$

$P\{\xi_1 = x\} = P\{\xi_2, \xi_3, \cdots, \xi_n \ge \xi_1\} = \int_0^1 p_{\xi_1}(x)(1-x)^{n-1} dx$

$= \int_0^1 (1-x)^{n-1} dx = \frac{1}{n}$

③ $F_{\xi_1|Y=y}(x|Y=y) = P\{\xi_1 < x | Y=y\} = P\{\xi_1 < x | \xi_1 < y\} = \frac{x}{y}$

3. $\{\xi_k\}_{k\ge1}$ iid $\sim Exp(1)$, $\{\eta_k\}_{k\ge1}$ iid $\sim U(0,1)$

$\{\xi_k\}$ and $\{\eta_k\}$ indep

Let $X = \frac{\xi_1+\xi_2+\cdots+\xi_m}{\xi_1+\xi_2+\cdots+\xi_n}$ $(m<n)$. $Y = \frac{(n-m)X}{m(1-X)}$, $\zeta \triangleq (\eta_1\eta_2)^{\eta_3}$

$F_X(x) = P\{X < x\} = P\{(\xi_1+\xi_2+\cdots+\xi_m)(\frac{1}{x}-1) < (\xi_{m+1}+\cdots+\xi_n)\}$

Let $u_1 = \sum_{k=1}^{m} \xi_k$, $u_2 = \sum_{k=m+1}^{n}$. We have $u_1 \sim \Gamma(1, m)$. $u_2 \sim \Gamma(1, n-m)$

$\Rightarrow p_{u_1}(t) = t^{m-1} e^{-t}$, $p_{u_2}(t) = t^{n-m-1} e^{-t}$

$$\Rightarrow F_X(x) = \int_0^{+\infty} p_{u_1}(t) \cdot F_{u_2}(t(\tfrac{1}{x}-1))\, dt = \int_0^{+\infty} t^{m-1} e^{-t} dt \int_0^{t(\frac{1}{x}-1)} s^{n-m-1} e^{-s} ds$$

由 $Y = \frac{(n-m)X}{m(1-X)} \Rightarrow X = \frac{mY}{n-m+mY}$

$\Rightarrow F_Y(y) = F_X(\frac{mY}{n-m+mY})$

$$F_\zeta(z) = P\{(\eta_1\eta_2)^{\eta_3} < z\} = \int_0^1\int_0^1 p(x,y)\frac{1}{\log_z xy}\, dxdy = \int_0^1\int_0^1 \frac{dxdy}{\log_z x + \log_z y}$$

4. $\xi \sim \Gamma(\frac{1}{2}, \frac{n}{2})$, $\eta \sim \Gamma(\frac{1}{2}, \frac{m}{2})$. ξ, η indepdent

$p_{\xi+\eta}(x) = \int_0^x p_\xi(t) p_\eta(x-t)\, dt$

$p_{\frac{\xi}{\xi+\eta}}(y) = \int_0^{+\infty} p_\xi(t) p_\eta(\frac{t}{y}-t)\, dt$

$$p_{\xi+\eta}(x)\, p_{\frac{\xi}{\xi+\eta}}(y) = \int_0^x\int_0^{+\infty} p_\xi(t) p_\eta(x-t) p_\xi(s) p_\eta(\tfrac{s}{y}-s)\, dt ds = p_\xi(xy)$$

$\Rightarrow \xi+\eta$ and $\frac{\xi}{\xi+\eta}$ indepdent

$\frac{\xi}{\xi+\eta} \sim \beta(\frac{m}{2}, \frac{n}{2})$, in other words $p_{\frac{\xi}{\xi+\eta}}(x) = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} x^{\frac{m}{2}-1}(1-x)^{\frac{n}{2}-1}$ $(0<x<1)$

二. $P_{106-107}.2.$

ξ_1, ξ_2 independent $\xi_1 \sim P_0(\lambda_1)$. $\xi_2 \sim P_0(\lambda_2)$

$\Rightarrow \xi_1+\xi_2 \sim P_0(\lambda_1+\lambda_2)$

$$P_{\xi_1}(k \mid \xi_1+\xi_2=n) = \frac{P\{\xi_1=k, \xi_2=n-k\}}{P\{\xi_1+\xi_2=n\}} = \binom{n}{k}\frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \quad k=0,1,2,\cdots,n$$

3 ? (表中数据有问题?)

4. $\xi \sim N(0,1)$

(1) 设 $\eta = e^{\xi}$, 则 $\xi = \ln(\eta)$

$$P_\eta(y) = P_\xi(x)\cdot \frac{d\xi}{d\eta} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{y} = \frac{1}{\sqrt{2\pi}y}\cdot y^{-\frac{\ln y}{2}} \quad y>0$$

(2) 设 $\eta = \frac{1}{\xi^2}$. 则 $\xi = \pm \eta^{-\frac{1}{2}}$

$$P_\eta(y) = P_\xi(x)\left|\frac{d\xi}{d\eta}\right| = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} y^{-\frac{3}{2}} \cdot \left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2y}} y^{-\frac{3}{2}}$$

10. $\xi+\eta \sim N(0,2)$. $\xi-\eta \sim N(0,2)$

$\Rightarrow \xi+\eta,\ \xi-\eta$ iid $\sim N(0,2)$

联合分布 $(\xi+\eta, \xi-\eta) \sim N(0,0,2,2,0)$

边缘分布 $\xi+\eta \sim N(0,2)$. $\xi-\eta \sim N(0,2)$

12. $\xi.\ \eta$ iid $\sim \exp(1)$.

$$P_{\xi+\eta}(z) = \int_0^z P_\xi(t)\cdot P_\eta(z-t)\,dt = ze^{-z}$$

$$P_{\frac{\xi}{\xi+\eta}}(w) = \int_0^{+\infty} P_{\xi+\eta}(z) P_\xi(wz)\,dz = \frac{1}{z} e^{z-zw}$$

$$\Rightarrow P_{\xi+\eta}(z) P_{\frac{\xi}{\xi+\eta}}(w) = P_\xi(zw)$$

故 $\xi+\eta$ 与 $\frac{\xi}{\xi+\eta}$ 相互独立

13. $p(x,y) = \frac{2}{\pi}(1-x^2-y^2) \quad 0<x^2+y^2<1$

由 $x = \rho\cos\theta$. $y = \rho\sin\theta$.

$$J_{x,y}(\rho,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \rho$$

$$P_{(\rho,\theta)}(\rho_0,\theta_0) = p(\rho\cos\theta, \rho\sin\theta)\,\rho = \frac{2}{\pi}(1-\rho^2)\rho = \frac{2}{\pi}(\rho-\rho^3)$$

联合密度 $p(\rho,\theta) = \frac{2}{\pi}(\rho-\rho^3)$

边缘密度 $p(\rho) = \int_0^{2\pi} p(\rho,\theta)\, d\theta = 4\rho - 4\rho^3$

$p(\theta) = \int_0^1 \frac{2}{\pi}(\rho-\rho^3)\, d\rho = \frac{1}{2\pi}$

15. ξ, η independent. $\xi \sim U(0,1)$

η: $F(y) = 1 - \frac{1}{y^2}\ (y>1)$

$\Rightarrow p(y) = \frac{2}{y^3}$

若 $z \geq 1$ $P_{\xi\eta}(z) = \int_0^1 P_\xi(x) P_\eta(\frac{z}{x})\, dx = \frac{1}{2z^3}$

若 $0<z<1$. $P_{\xi\eta}(z) = \int_0^z P_\xi(x) P_\eta(\frac{z}{x})\, dx = \frac{z}{2}$