2023 概率论期末考试:

1. $\{\eta_n, \xi_n, \xi : n\geq 1\}$ 和 $\{A_n: n\geq 1\}$ 为概率空间 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的 r.v. 列和随机事件列

(1) Prove: $\xi_n \xrightarrow{L^r} \xi \overset{①}{\Rightarrow} \xi_n \xrightarrow{P} \xi \overset{②}{\Rightarrow} \xi_n \xrightarrow{d} \xi \quad (r>0)$

① A. $\forall \varepsilon>0$. $\underbrace{\mathbb{P}\{w \mid |\xi_n-\xi|\geq\varepsilon\}}_{\mathbb{P}\text{收敛}} \leq \dfrac{\underbrace{E|\xi_n-\xi|^r}_{L^r\text{收敛}}}{\varepsilon^r}$

② $\forall x\in C(F_\xi)$, $\forall y<x<z$

$\{w \mid \xi_n(w)\leq y\} \subseteq \{w \mid \xi_n(w)\leq x\} \cup \{w \mid |\xi(w)-\xi_n(w)|\geq x-y\}$

$\mathbb{P}\{w \mid \xi_n(w)\leq y\} \leq \mathbb{P}\{w \mid \xi(w)\leq x\} + \mathbb{P}\{w \mid |\xi_n-\xi|\geq x-y\}$

由 $\xi_n \xrightarrow{\mathbb{P}} \xi$:

$F_\xi(y) \leq \lim\limits_{n\to\infty} F_{\xi_n}(x) \quad \forall y<x,\ y\uparrow x$

同理 $\limsup\limits_{n} F_{\xi_n}(x) \leq F_\xi(z)$, $\forall z>x$

$\Rightarrow F_\xi(x) \leq \liminf\limits_{n\to\infty} F_{\xi_n}(x) \leq \limsup\limits_{n\to\infty} F_{\xi_n}(x) \leq F_\xi(x)$

$\Rightarrow F_{\xi_n}(x) \longrightarrow F_\xi(x)$, $\forall x\in C(F_\xi)$

(2) $\xi_n \xrightarrow{L^r} \xi \overset{①}{\Leftarrow} \xi_n \xrightarrow{P} \xi \overset{②}{\Leftarrow} \xi_n \xrightarrow{d} \xi \quad (r>0)$ 是否成立?

均不成立: ① $\Omega=[0,1]$, $\mathcal{F}$?

$$\xi_n(w)=\begin{cases} n^{\frac{1}{w}}, & w\in[0,\frac{1}{n}] \\ 0, & w\in(\frac{1}{n},1] \end{cases}$$

Then $\xi_n \xrightarrow{P} \xi \quad E(|\xi_n-\xi|^r)=1\neq 0$

$\Rightarrow \xi_n \overset{L^r}{\not\to} \xi$

② $\{\xi_n, n\geq 1\}, \xi$ iid, $\xi \sim \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

则 $\forall n$, $F_\xi(x) = F_{\xi_n}(x)$, $\xi_n \xrightarrow{d} \xi$

但 $\forall \varepsilon < 1$, $\mathbb{P}\{w \mid |\xi_n - \xi| \geq \varepsilon\} = \mathbb{P}\{w \mid \xi_n \neq \xi\} = \frac{1}{2} \neq 0$

$\Rightarrow \xi_n \overset{P}{\not\to} \xi$

(3) 若 $\eta_n \xrightarrow{d} c$ (c 为一个常数), $\xi_n \xrightarrow{d} \xi$. 证明: $\eta_n \xi_n \xrightarrow{d} c\xi$

$\eta_n \xi_n = (\eta_n - c)\xi_n + c\xi_n$ 且 $c\xi_n \xrightarrow{d} c\xi$

WLOG, 设 $c=0$

$\forall A>0, \delta>0$, $\mathbb{P}\{w \mid |\eta_n \xi_n| \geq \delta\} \leq \mathbb{P}\{|\xi_n| > A\} \mathbb{P}\{|\eta_n| \geq \frac{\delta}{A}\}$

$\leq 1 - F_{\xi_n}(-A) + F_{\xi_n}(A) + \mathbb{P}\{|\eta_n| \geq \frac{\delta}{A}\}$

选择 $A_k > 0$ s.t.: $A_k, -A_k \in C$, $A_k \to +\infty$ $(k \to +\infty)$

$\lim\limits_{n\to\infty} \mathbb{P}\{|\xi_n \eta_n| \geq \delta\} \leq 1 - \lim\limits_{n\to\infty} F_{\xi_n}(-A_k) + \lim\limits_{n\to\infty} F_{\xi_n}(A_k)$

$\underset{\uparrow}{=} 1 - F_\xi(-A_k) + F_\xi(A_k)$

$= \mathbb{P}\{|\xi| > A_k\}$

$\to 0$

$\Rightarrow \xi_n \eta_n \xrightarrow{d} 0$

(4) $\{A_n, n\geq 1\}$ 相互独立, 证: $\sum\limits_{n=1}^{\infty} P(A_n) = \infty \Leftrightarrow P(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} A_k) = 1$

"$\Rightarrow$" $\forall n < N$, $P(\bigcap\limits_{k=n}^{N} A_k^c) = \prod\limits_{k=n}^{N} P(A_k^c) = \prod\limits_{k=n}^{N} [1 - P(A_k)]$

$$\le \prod_{k=n}^{N} \exp[-P(A_k)] = \exp[-\sum_{k=n}^{N} P(A_k)]$$

令 $N \to +\infty$ $\quad P(\bigcap_{k=n}^{+\infty} A_k^c) = 0$, $\forall n$

$$\Rightarrow P(\bigcup_{n=1}^{+\infty}\bigcap_{k=n}^{+\infty} A_k^c) = \lim_{n\to+\infty} P(\bigcap_{k=n}^{+\infty} A_k^c) = 0$$

$$\Rightarrow P(\bigcap_{n=1}^{+\infty}\bigcup_{k=n}^{+\infty} A_k) = 1$$

"$\Leftarrow$" 假设 $\sum_{k=1}^{+\infty} P(A_k) < +\infty$

$$P(\bigcap_{n=1}^{+\infty}\bigcup_{k=n}^{+\infty} A_k) = \lim_{n\to\infty} P(\bigcup_{k=n}^{+\infty} P(A_k)) \le \limsup_{n\to\infty} \sum_{k=n}^{+\infty} P(A_k) = 0$$

矛盾!

2. ξ 和 η 是概率空间 $(\Omega, \mathcal{F}, P)$ 上的独立随机变量:

Prove: (1) $E\{|\xi+\eta|\} < +\infty \Leftrightarrow E\{|\xi|\} < +\infty$ 且 $E\{|\eta| < +\infty\}$

"$\Leftarrow$" $E\{|\xi+\eta|\} \le E\{|\xi|+|\eta|\} = E\{|\xi|\} + E\{|\eta|\}$

"$\Rightarrow$" Fubini:

$$+\infty > E\{|\xi+\eta|\} = \iint_{\mathbb{R}^2} |x+y| \, dF_{(\xi,\eta)}(x,y)$$

$$= \iint_{\mathbb{R}^2} |x+y| \, dF_\xi(x) \, dF_\eta(y)$$

$$= \int_{\mathbb{R}} dF_\xi(x) \int_{\mathbb{R}} |x+y| \, dF_\eta(y)$$

$\Rightarrow \exists y_0 \in \mathbb{R}$, s.t. $E|\xi+y_0| < +\infty$

$\Rightarrow E|\xi| \le |-y_0| + E(|\xi+y_0|) < +\infty$

同理 $E|\eta| < +\infty$

(2) 若 $E\{\eta\}=0$，$E|\xi|<+\infty \Rightarrow E\{|\xi+\eta|\} \geq E|\xi|$

$$E|\xi+\eta| = \int_{\mathbb{R}} E|\eta+x|\, dF_\xi(x) \geq \int_{\mathbb{R}} (E|\eta|+|x|)\, dF_\xi(x)$$

$$= \int_{\mathbb{R}} E|\eta|\, dF_\xi(x) + E|\xi| \geq E|\xi|$$

(3) $\{F_k: k\geq 1\}$ 为 $\mathbb{R}=(-\infty,+\infty)$ 上的分布函数，Prove: $\exists$ p.s. $(\Omega,\mathcal{F},P)$

及独立 $\{\xi_k: k\geq 1\}$ s.t. $F_k(x)=P\{w:\xi_k(w)<x\}$, $x\in\mathbb{R}$, $k\geq 1$

?

3. (1) 设 $\xi_1,\xi_2,\cdots,\xi_n$，$(\Omega,\mathcal{F},P)$ 上 iid，$\xi_1\sim N(a,\sigma^2)$

Prove: $\bar{\xi} \triangleq \frac{1}{n}\sum\limits_{k=1}^{n}\xi_k$ 与 $S_n^2 \triangleq \frac{1}{n-1}\sum\limits_{k=1}^{n}(\xi_k-\bar{\xi})^2$ 独立 r.v.

且 $\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$

记 $\vec{a}=(a_1,a_2,\cdots,a_n)^T$，$B\triangleq \sigma^2 I_{n\times n}$

$\vec{\xi}=(\xi_1,\xi_2,\cdots,\xi_n)^T \sim N(\vec{a},B)$

$\exists\, C$ s.t. $CC^T=C^TC=I_{n\times n}$

$$C=\begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ & & * & & \end{pmatrix} \triangleq (C_{ij})_{n\times n} = (\vec{c}_1,\vec{c}_2,\cdots,\vec{c}_n)^T$$

$\vec{\eta}=C\vec{\xi} \sim N(C\vec{a},\sigma^2 I_{n\times n}) \Rightarrow \eta_1,\eta_2,\cdots,\eta_n$ 独立

$\eta_1 \sim N(\frac{1}{\sqrt{n}}a,\sigma^2)$，$\bar{\xi}=\frac{1}{n}\sum\limits_{k=1}^{n}\xi_k=\frac{1}{\sqrt{n}}\eta_1$

$|\vec{c}_1|=1$，$\langle c_k,c_1\rangle=0$，$k=2,3,\cdots,n$

$\Rightarrow \sum\limits_{j=1}^{n} c_{k,j}=0$

$\Rightarrow \eta_k = \sum_{j=1}^{n} c_{kj} \xi_j \sim N(0, \sigma^2)$

$\Rightarrow \sum_{k=2}^{n} \eta_k^2 = \sum_{k=1}^{n} \eta_k^2 - \eta_1^2 = \sum_{k=1}^{n} \xi_k^2 - n\bar{\xi}^2 = \sum_{k=1}^{n} (\xi_k - \bar{\xi})^2 = (n-1)S_n^2$

$\Rightarrow \frac{(n-1)S_n^2}{\sigma} = \sum_{k=2}^{n} \left(\frac{\eta_k}{\sigma}\right)^2 \sim \chi_{n-1}$

(2) ξ 为 $(\Omega, \mathcal{F}, P)$ 上非负随机变量，证明 $\forall s>0$，有：

$$\int_0^{\infty} x^s \, dF_\xi(x) = \int_0^{\infty} s x^{s-1} P\{w: \xi(w) \geq x\} \, dx$$

$\int_0^{\infty} s x^{s-1} P\{\xi > x\} \, dx = \int_0^{\infty} s x^{s-1} \left[\int_x^{\infty} dF_\xi(y)\right] dx = \int_0^{\infty} \left(\int_0^{y} s x^{s-1} dx\right) dF_\xi(y) = LHS$

(3) 设 $\{f_n(t): n \geq 1\}$ 是一列特征函数，$f_n(t) \to f(t)$，$t \in \mathbb{R}$

若 $f(t)$ 在 0 点连续且 $f(0)=1$，证：$\exists$ d.f. $F(\cdot)$ s.t. $f(t) = \int_{\mathbb{R}} \exp\{itx\} dF(x)$

$\forall t \in \mathbb{R}$

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4. $\xi_1, \xi_2, \cdots, \xi_n$ 为 p.s. $(\Omega, \mathcal{F}, P)$ 上 iid 随机变量且 $\xi_1 \sim U(0,1)$

定义：$X \triangleq \min\{\xi_k, 1 \leq k \leq n\}$，$Y \triangleq \max\{\xi_k, 1 \leq k \leq n\}$. 求.

(1) r.v.s. (ξ_1, Y) 及 (X, Y) 的 d.f.

(ξ_1, Y)

① $(x, y) \in [0,1]^2$，$F_{(\xi_1, Y)}(x, y) = P\{\xi_1 < x \wedge y\} \prod_{k=2}^{n} P\{w \mid \xi_k < y\}$

$$= \begin{cases} x y^{n-1} & x \leq y \\ y^n & x > y \end{cases}$$

② $x \notin [0,1]$ or $y \notin [0,1]$

(i) $x < 0$，$F_{(\xi_1, Y)}(x, y) = 0$

(ii) $x > 1$，$F_{(\xi_1, Y)}(x, y) = (F_{1y}, y)$

$$= \begin{cases} 0 & y<0 \\ y^n & y\in[0,1] \\ 1 & y>1 \end{cases}$$

(iii) $y<0$, $F(x,y)=0$

(iv) $y>1$, $F_{(\xi_1,Y)}(x,y) = F_{\xi_1}(x) = \begin{cases} 0 & x<0 \\ x & x\in[0,1] \\ 1 & x>1 \end{cases}$

(X,Y)

① $X\geq Y$: $F_{(X,Y)}(x,y) = F_Y(y)$

② $X<Y$: $F_{(X,Y)}(x,y) = P\{Y\leq y\} - P\{X\geq x, Y\leq y\}$

$$= F_Y(y) - [F_\xi(y) - F_\xi(x)]^n$$

(2) $P\{w: Y-X\leq \frac{1}{2}\}$

(3) $E\{\xi_i^2 \mid Y\}$

$$F_{\xi_1}(x \mid Y=y) = \begin{cases} 0 & x\leq 0 \\ \frac{n+1}{n}\frac{x}{y} & 0<x\leq y \\ 1 & x>y \end{cases}$$

?

5. (1) r.v. $(\xi_1,\xi_2,\xi_3)^T \sim N\left(\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\right)$ 求：$E\{\xi_2-\xi_3 \mid \xi_1+\xi_2+\xi_3\}$

?

(2) $\{X_n: n\geq 1\}$ r.v.s. iid. $X_1 \sim C(1,0)$. 证明：$\limsup\limits_{n\to\infty} \frac{1}{n}\left|\sum_{k=1}^{n} X_k\right| = \infty$ a.s.

?

6. 设对 $\forall n\geq 1$, $\{\xi_{n,m}: m=1,2,\cdots,m\}$ 为独立 r.v. 且：

(1) $E\{\xi_{n,m}\}=0$ (2) $\lim\limits_{n\to+\infty} \sum_{m=1}^{n} E\{\xi_{n,m}^2\}=1$

(3) 对 $\forall \varepsilon > 0$, $\lim\limits_{n\to+\infty} \sum\limits_{m=1}^{n} E\{\xi_{n,m}^2 I\{|\xi_{n,m}|>\varepsilon\}(w)\} = 0$

令 $S_n \triangleq \sum\limits_{m=1}^{n} \xi_{n,m}$, 证明: $S_n \xrightarrow{d} \xi \sim N(0,1)$

特征函数 $f_{n,m}(t) = E(e^{iwt\xi_{n,m}})$

$$\sigma_{n,m}^2 = D(\xi_{n,m})$$

$$\Rightarrow \sum_{m=1}^{n} \sigma_{n,m}^2 \to 1$$

$\max\limits_{1\le m\le n} \{\sigma_{n,m}^2\} \to 0 \quad (n\to+\infty)$ $\qquad f_{S_n}(t) = \prod\limits_{m=1}^{n} f_{n,m}(t) \quad \forall t \in \mathbb{R}$

$\xi \sim N(0,1) \qquad f(t) = e^{-\frac{t^2}{2}}$

$$|f_{S_n}(t) - e^{-\frac{1}{2}t^2}| \le |f_{S_n}(t) - \prod_{m=1}^{n}[1-\tfrac{1}{2}\sigma_{n,m}^2 t^2]| \quad (L_n^{(1)})$$

$$+ |\prod_{m=1}^{n}[1-\tfrac{1}{2}\sigma_{n,m}^2 t^2] - \prod_{m=1}^{n} e^{-\frac{1}{2}\sigma_{n,m}^2 t^2}| \quad (L_n^{(2)})$$

$$+ |\prod_{m=1}^{n} e^{-\frac{1}{2}\sigma_{n,m}^2 t^2} - e^{-\frac{1}{2}t^2}| \quad (L_n^{(3)})$$

$$L_n^{(1)}(t) \le \sum_{m=1}^{n} |f_{n,m}(t) - (1-\tfrac{1}{2}\sigma_{n,m}^2 t^2)|$$

$$= \sum_{m=1}^{n} |E[e^{it\xi_{n,m}} - 1 - it\xi_{n,m} - \tfrac{1}{2}(it\xi_{n,m})^2]|$$

$$\le \sum_{m=1}^{n} E[|t\xi_{n,m}|^2 \wedge |t\xi_{n,m}|^3]$$

$$\le t^2 \sum_{m=1}^{n} E[\xi_{n,m}^2 1\{|\xi_{n,m}|\ge\varepsilon\}] + |t|^3\varepsilon \sum_{m=1}^{n}\sigma_{n,m}^2$$

令 $n\to+\infty$, $\exists \lim\limits_{n\to\infty} L_n^{(1)}(t) \le t^3\varepsilon \quad (\forall t\in\mathbb{R})$

令 $\varepsilon\to 0 \Rightarrow (?)\ L_n^{(1)}(t) = 0 \quad \forall t\in\mathbb{R}$

$$L_n^{(2)}(t) \le \sum_{m=1}^{n} \left|\frac{1}{2}\sigma_{n,m}^2 t^2\right|^2 \le t^4 \max_{1\le m\le n} \sigma_{n,m}^2 \sum_{m=1}^{n} \sigma_{n,m}^2 \to 0 \quad (\forall t\in\mathbb{R})$$

$$L_n^{(3)}(t) = \left|\exp\left(-\frac{1}{2}\sum_{m=1}^{n}\sigma_{n,m}^2 t^2\right) - \exp\left(-\frac{1}{2}t^2\right)\right| \to 0$$

$$\Rightarrow f_{S_n}(t) \to e^{-\frac{t^2}{2}} \quad (\forall t\in\mathbb{R})$$

$$\Rightarrow S_n \xrightarrow{d} \xi$$

7. $\{\xi_n : n\ge 1\}$ 是 p.s. $(\Omega, \mathcal{F}, P)$ 上的独立 r.v.s.

$E\{\xi_n\} = 0$, $(n\ge 1)$

(1) 证: 若 $\sum_{k=1}^{\infty} D(\xi_k) < \infty$，则随机级数 $\sum_{k=1}^{\infty} \xi_k$ 几乎处处收敛

?

(2) 证. 若 $\{\xi_n : n\ge 1\}$ iid. 则 $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\xi_k \xrightarrow{a.e.} 0$

?

8. (1) 设 $\xi = \{\xi_n, n\ge 1\}$ 为 p.s. $(\Omega, \mathcal{F}, P)$ 上的独立 r.v.s.

且 $E\{\xi_n^2\} < \infty$, $n\ge 1$ $\quad S_k \triangleq \sum_{m=1}^{k}\xi_m$, $k\ge 1$

证: $\forall n\ge 1, \forall \varepsilon>0$, $P\{w: \max_{1\le k\le n}\{|S_k - E\{S_k\}|\} \ge \varepsilon\} \le \frac{1}{\varepsilon^2}\sum_{k=1}^{n} D(\xi_k)$

不妨设 $E(\xi_n)=0$, $\forall n$

$B \triangleq \{w \mid \max_{1\le k\le n} |S_k| \ge \varepsilon\} = \sum_{k=1}^{n} B_k$, $B_1 = A_1$, $B_k = A_k \cap A_{k-1}^c \cap A_{k-2}^c \cap \cdots \cap A_1^c$

where $A_k = \{w \mid |\xi_k| \ge \varepsilon\}$

$$P(B) = \sum_{k=1}^{n} P(B_k) \le \sum_{k=1}^{n} \frac{1}{\varepsilon^2} E(S_k^2 I_{B_k})$$

$S_n - S_k$ 与 $\xi_k I_{B_k}$ 独立

$$E[(S_n-S_k)S_k I_{B_k}] = E(S_n-S_k)E(S_k I_{B_k})=0$$

$$P(B) \le \frac{1}{\varepsilon^2}\sum_{k=1}^{n} E(S_k^2 I_{B_k} + (S_n-S_k)^2 I_{B_k})$$

$$= \frac{1}{\varepsilon^2}\sum_{k=1}^{n} E((S_n-S_k+S_k)^2 I_{B_k})$$

$$= \frac{1}{\varepsilon^2}E(S_n^2)$$

$$= \frac{1}{\varepsilon^2}D(S_n) \le \frac{\sum_{k=1}^{n} D(\xi_k)}{\varepsilon^2}$$

(2) 证明：若 $\{\xi_n : n\ge 1\}$ 是 iid r.v.s. $E\{\xi_1^2\}=1$, $E\{\xi_1\}=0$

则：$\frac{1}{\sqrt{\nu_n}}\sum_{k=1}^{\nu_n}\xi_k \xrightarrow{d} \xi \sim N(0,1)$，其中 $\{\nu_n\}$ 为取正整数值的随机列且 $\frac{\nu_n}{n} \xrightarrow{d} 1$

$$P\left\{\left|\frac{S_{\nu_n}-S_n}{\sqrt{n}}\right| \ge \varepsilon\right\} = P\{|S_{\nu_n}-S_n| \ge \varepsilon\sqrt{n}\}$$

$$2P\left\{\left|\frac{S_{\nu_n}-S_n}{\sqrt{n}}\right| \ge \varepsilon : \nu_n \notin [a_n,b_n]\right\} \le \varepsilon + P\{\nu_n \in [a_n,b_n], |\cdot| \ge \varepsilon\sqrt{n}\}$$

$$\le \varepsilon + P\{\max_{a_n\le k\le b_n} |S_k-S_n| \ge \varepsilon\sqrt{n}\}$$

$$\le \varepsilon + \frac{1}{n\varepsilon^2}D\left(\sum_{j=n+1}^{B_0}\xi_j\right)$$

$$= \varepsilon + \frac{1}{n\varepsilon^2}D(\xi_1)(b_n-n-1)$$

$$\le \varepsilon + \frac{D(\xi_1)}{n\varepsilon^2}\varepsilon^{\alpha}\cdot n$$

$$= \varepsilon + D(\xi_1)\varepsilon^{\alpha-2}$$

$$\le \varepsilon(1+D(\xi_1))$$

令 $\varepsilon\to 0 \Rightarrow$ 成立

9.（太难了，不看了）

9. 设$\xi=\{\xi_n, n\geq 1\}$是概率空间$(\Omega,\mathcal{F},\mathbb{F},\mathbf{P})$上一个sub-martingale(下央), 其中滤子 $\mathbb{F}=\{\mathcal{F}_n\}_{n\geq 1}$ 满足:$\mathcal{F}_n\subseteq\mathcal{F}_{n+1}\subseteq\mathcal{F}$, $\mathcal{F}_n$ 是σ-代数(field). **S**和**T** 是关于$\mathbb{F}$的两个随机停时. 证明

(1) $\mathcal{F}_{\mathbf{T}}\triangleq\{A\in\mathcal{F}\mid A\cap\{\omega:T(\omega)\leq n\}\in\mathcal{F}_n, n\geq 1\}$ 是 Ω上的一个σ-代数(field).

(2) $\xi_{\mathbf{T}}$ 是 $\mathcal{F}_{\mathbf{T}}$-可测. (3) 如 **T** 有界, 则 $\xi_{\mathbf{T}}$ 可积. (4) 如 **T**有界且 $\mathbf{S}\leq\mathbf{T}$, 则 $\mathbf{E}\{\xi_{\mathbf{T}}\mid\mathcal{F}_{\mathbf{S}}\}\overset{a.s.}{\geq}\xi_{\mathbf{S}}$.