第八周第一次作业：

中文教材 P119

3. $\xi \sim \Gamma(\lambda, r)$, $p(x) = \dfrac{x^{r-1}\lambda^r e^{-\lambda x}}{\Gamma(r)}$

$$E(x) = \int_0^{+\infty} p(x)x\,dx = \lambda r$$

5. $\sum_{i=1}^{\infty} P\{\xi \geqslant i\} = \sum_{i=1}^{\infty} i P\{\xi = i\} = \sum_{i=0}^{\infty} i \cdot P\{\xi = i\} = E(\xi)$

6. $\int_0^{+\infty} sx^{s-1}[1-F(x)]dx = \int_0^{+\infty} sx^{s-1} \int_x^{+\infty} dF(t)\,dx$

$$= \int_0^{+\infty} dF(t) \int_0^t sx^{s-1}dx = \int_0^{+\infty} dF(t)\,t^s = \int_0^{+\infty} x^s dF(x)$$

7. $\int_0^{+\infty} [1-F(x)]dx - \int_{-\infty}^0 F(x)dx = \int_0^{+\infty} dx \int_x^{+\infty} dF(t) - \int_{-\infty}^0 dx \int_{-\infty}^t dF(t)$

$$= \int_0^{+\infty} dF(t) \int_0^t dx - \int_{-\infty}^0 dF(t) \int_t^0 dx = \int_0^{+\infty} t\,dF(x) + \int_{-\infty}^0 t\,dF(t)$$

$$= \int_{-\infty}^{+\infty} t\,dF(t) = E(\xi)$$

9. $\xi \sim C(1,0) \Rightarrow p_\xi(x) = p(x) = \dfrac{1}{\pi(1+x^2)}$

$$E(|\xi| \wedge 1) = \int_{-\infty}^{+\infty} \min\{|x|, 1\} p(x)dx$$

$$= \int_{-\infty}^{-1} p(x)dx + \int_{-1}^{1} |x| p(x)dx + \int_1^{+\infty} p(x)dx$$

$$= \frac{1}{2}\arctan x\Big|_{-\infty}^{-1} + \frac{1}{2\pi}\ln|1+x^2|\Big|_0^1 + \frac{1}{2\pi}\ln|1+x^2|\Big|_{-1}^0 + \frac{1}{2}\arctan x\Big|_1^{+\infty}$$

$$= \frac{\ln 2}{2\pi} + \frac{1}{2}$$

10. $\xi, \eta$ iid $\sim N(a, \sigma^2)$

$$E(\xi \vee \eta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \max\{x, y\} p(x)p(y)\,dxdy$$

$$= \int_{-\infty}^{+\infty} ds \left( \int_{-\infty}^{s} s \cdot p(x) p(s) dx + \int_{-\infty}^{s} s p(y) p(s) dy \right)$$

$$= \int_{-\infty}^{+\infty} 2s p(s) ds \int_{-\infty}^{s} p(x) dx$$

$$= \int_{-\infty}^{+\infty} 2s p(s) F(s) ds$$

$$= a + \frac{\sigma}{\sqrt{\pi}}$$

15. 设$\xi$为取球次数. 则 $P\{\xi = k\} = \left(\frac{b}{r+b}\right)^{k-1} \cdot \frac{r}{r+b} \quad (k \geq 1)$

$$E(\xi) = \sum_{k=1}^{+\infty} \frac{b^{k-1} r k}{(r+b)^k} = \frac{r+b+1}{r+1}$$

P127 3.2.6. $\theta \sim U(0, 2\pi)$. $\xi = \cos\theta$, $\eta = \cos(\theta + a)$

则 $E(\xi) = E(\eta) = 0$, $D(\xi) = D(\eta) = \frac{1}{2}$

$$E(\xi\eta) = \int_0^{2\pi} \frac{1}{2\pi} \cos x \cos(x+a) dx = \frac{1}{2} \cos a$$

$$\text{cov}(\xi, \eta) = \frac{E(\xi\eta)}{\sqrt{D(\xi)} \cdot \sqrt{D(\eta)}} = \cos a$$

(1) $a = 0$ 时. $r = 1$, 此时 $\eta = \cos\theta = \xi$

(2) $a = \pi$ 时. $r = -1$. 此时 $\eta = \cos(\theta + \pi) = -\cos\theta = -\xi$

(3) $a = -\frac{\pi}{2}$ 时 $r = 0$. 此时 $\xi = \cos\theta$, $\eta = \sin\theta$. 则$\xi$与$\eta$不相关

但我们可以证明: 取 $B_1 = B_2 = (0, \frac{1}{2})$ 则

$$P\{\eta \in B_2\} = P\{0 < \sin\theta < \tfrac{1}{2}\} = P\{\theta \in (0, \tfrac{\pi}{6}) \cup (\tfrac{5}{6}\pi, \pi)\}$$

$$= \frac{1}{6} = P\{0 < \cos\theta < \tfrac{1}{2}\} = P\{\xi \in B_1\}$$

但 $\{\xi \in B_1, \eta \in B_2\} = \emptyset$. 故 $P\{\xi \in B_1, \eta \in B_2\} = 0 \neq P\{\xi \in B_1\} P\{\eta \in B_2\}$

$\Rightarrow \xi$与$\eta$不相关

P133 2. $\xi \sim L(\lambda, u)$. $p(x)=\frac{1}{2\lambda}e^{-\frac{1}{\lambda}|x-u|}$ $(\lambda>0)$

$$\Rightarrow p(x)=\begin{cases}\frac{1}{2\lambda}e^{-\frac{x}{\lambda}+\frac{u}{\lambda}} & x\geq u\\ \frac{1}{2\lambda}e^{\frac{x}{\lambda}-\frac{u}{\lambda}} & x<u\end{cases}$$

$$E(\xi)=\int_{-\infty}^{+\infty}p(x)\cdot x\,dx=\int_{-\infty}^{u}p(x)\cdot x\cdot dx+\int_{u}^{+\infty}p(x)\cdot x\cdot dx$$

$$=\frac{1}{2}e^{(\frac{x}{\lambda}-\frac{u}{\lambda})}\Big|_{-\infty}^{u}+(-\frac{1}{2})e^{(-\frac{x}{\lambda}+\frac{u}{\lambda})}\Big|_{u}^{+\infty}$$

$$=\frac{1}{2}+\frac{1}{2}=1$$

$$E(\xi^2)=\int_{-\infty}^{+\infty}p(x^2)\,x^2dx=\int_{-\infty}^{+\infty}\frac{1}{2\lambda}e^{-\frac{1}{\lambda}|x^2-u|}x^2dx$$

$$=\int_{-\infty}^{-\sqrt{u}}\frac{1}{2\lambda}e^{-\frac{1}{\lambda}(x^2-u)}x^2dx+\int_{-\sqrt{u}}^{\sqrt{u}}\frac{1}{2\lambda}e^{-\frac{1}{\lambda}(u-x^2)}x^2dx$$

$$+\int_{\sqrt{u}}^{+\infty}\frac{1}{2\lambda}e^{-\frac{1}{\lambda}(x^2-u)}x^2dx$$

$$=2\lambda^2+1$$

$$D(\xi)=E(\xi^2)-E^2(\xi)=(2\lambda^2+1)-1=2\lambda^2$$

5. $E(\xi)=1$. $D(\xi)=\sum_{x=0}^{n}[x-E(\xi)]^2p(x)dx$

其中 $p(x)=P\{$恰有$x$个人拿对$\}=\binom{n}{x}[\frac{1}{2}-\frac{1}{3!}+\frac{1}{4!}-\cdots+(-1)^{n-x}\frac{1}{(n-x)!}]$

$$\Rightarrow D(\xi)=\sum_{x=0}^{n}(x-1)^2\frac{n!}{x!(n-x)!}[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\cdots+(-1)^{n-x}\frac{1}{(n-x)!}]$$

$$=1$$