第七周第一次作业：

一. (A) 中文教材 P86-87

11. (ξ,η,ζ) 联合密度函数 $p(x,y,z)=xze^{-(x+xy+z)}\quad(x,y,z>0)$

$$p_\xi(x)=\int_0^{+\infty}\int_0^{+\infty}xze^{-(x+xy+z)}\,dy\,dz=e^{-x}\quad(x>0)$$

$$p_\eta(y)=\int_0^{+\infty}\int_0^{+\infty}xze^{-(x+xy+z)}\,dx\,dz=\frac{1}{(1+y)^2}\quad(y>0)$$

$$p_{\xi,\zeta}(x,z)=\int_0^{+\infty}xze^{-(x+xy+z)}\,dy=ze^{-x-z},\quad(x,z>0)$$

P92 6. $p_1(x,y)=4xy,\ 0<x,y<1$

$\Rightarrow p_{11}(x)=\int_0^1 4xy\,dy=2x,\ p_{12}(y)=\int_0^1 4xy\,dx=2y$

$\Rightarrow p_1(x,y)=p_{11}(x)p_{12}(y)\Rightarrow \xi,\eta$ 独立

$p_2(x,y)=8xy,\ 0<x<y<1$

$\Rightarrow p_{21}(x)=\int_x^1 8xy\,dy=4x(1-x^2)\quad p_{22}(y)=\int_0^y 8xy\,dx=4y^3$

$\Rightarrow p_2(x,y)\neq p_{21}(x)p_{22}(y)$ a.s.

$\Rightarrow \xi,\eta$ 不独立

(B) 证明定理 2.6.2 (P90):

ξ,η iid $\Leftrightarrow P\{\xi\in B_1,\eta\in B_2\}=P\{\xi\in B_1\}P\{\eta\in B_2\}\quad\forall B_1,B_2\in\mathcal{B}$

由 $F(x_1,x_2,\cdots,x_n)=F_1(x_1)\cdots F_n(x_n)$. 充分性得证. 下证必要性： (*)

固定 x_2. 记 $\mathcal{E}=\{B_1\subseteq\mathbb{R}: P\{\xi\in B_1,\eta<x_2\}=P\{\xi\in B_1\}P\{\eta<x_2\}\}$

可证明$\mathcal{E}$为λ类，事实上，由$\{\xi\in\mathbb{R}\}=\Omega$，有$\mathbb{R}\in\mathcal{E}$

再由概率的可减性：$\mathcal{E}$关于真差封闭.

又由概率下连续性 $\Rightarrow$ $\mathcal{E}$关于上升极限封闭.

由(*)式：　$\mathcal{E}\supset\mathcal{P}=\{(-\infty,x_1):x_1\in\mathbb{R}\}$

注意到$\mathcal{P}$为生成$\mathcal{B}$的π类，故由单调类定理$\mathcal{E}\supset\sigma(\mathcal{P})=\mathcal{B}$

$\Rightarrow P\{\xi\in B_1,\eta<x_2\}=P\{\xi\in B_1\}P\{\eta<x_2\}$，$B_1\in\mathcal{B}$. $x_2\in\mathbb{R}$

固定$B_1\in\mathcal{B}$ 记

$$\mathcal{G}=\{B_2\subseteq\mathbb{R}:P\{\xi\in B_1,\eta\in B_2\}=P\{\xi\in B_1\}P\{\eta\in B_2\}\}$$

类似可以证明$\mathcal{G}$为λ类，又$\mathcal{P}\subseteq\mathcal{G}\Rightarrow\mathcal{B}\subseteq\mathcal{G}$

从而必要性得证.

二. 1. $(\xi,\eta)\sim U_D$，$D=\{(x,y):x^2+y^2\le1\}$

$$\Rightarrow\ p_\xi(x\mid\eta=y)=\frac{p(x,y)}{p_\eta(y)}=\frac{\pi}{\frac{dF_\eta(y)}{dy}}=\frac{1}{2\sqrt{1-y^2}}$$

$$\Rightarrow\ \xi|_{\eta=y}\sim U(-\sqrt{1-y^2},\sqrt{1-y^2})$$

2. $\xi\sim\eta\sim N\left(\begin{pmatrix}a_1\\a_2\end{pmatrix}\begin{pmatrix}\sigma_1^2 & r\sigma_1\sigma_2\\ r\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right)$

$$\Rightarrow p_\xi(x\mid\eta=y)=\frac{p(x,y)}{p_\eta(y)}$$

$$p(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\exp\left\{-\frac{1}{2(1-r^2)}\left[\left(\frac{x-a_1}{\sigma_1}\right)^2-2r\cdot\frac{x-a_1}{\sigma_1}\cdot\frac{x-a_2}{\sigma_2}+\left(\frac{y-a_2}{\sigma_2}\right)^2\right]\right\}$$

$$p_\eta(y)=\int_{-\infty}^{+\infty}p(x,y)\,dx=\frac{1}{\sqrt{2\pi}\,\sigma_2}\exp\left[-\frac{(y-a_2)^2}{2\sigma_2^2}\right]$$

$$\Rightarrow p_\xi(x|\eta=y) = \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-r^2}} \exp\left\{-\frac{\{x-[a_1+r\cdot\frac{\sigma_1}{\sigma_2}(y-a_2)]\}^2}{2\sigma_1^2(1-r^2)}\right\}$$

3. $\xi \sim \Gamma(\lambda, r_1)$. $\eta \sim \Gamma(\lambda, r_2)$, $\xi.\eta$ iid.

Then $p_{\xi+\eta}(x) = \int_{-\infty}^{+\infty} p_{\xi,\eta}(t, x-t)\,dt = \int_{-\infty}^{+\infty} p_\xi(t)\,p_\eta(x-t)\,dt$

$$= \int_{-\infty}^{+\infty} \frac{\lambda^{r_1}}{\Gamma(r_1)} t^{r_1-1} e^{-\lambda t} \cdot \frac{\lambda^{r_2}}{\Gamma(r_2)} (x-t)^{r_2-1} e^{-\lambda(x-t)}\,dt$$

$$= \int_{-\infty}^{+\infty} \frac{\lambda^{r_1+r_2}}{\Gamma(r_1+r_2)} t^{r_1-1} (x-t)^{r_2-1} e^{-\lambda x}\,dt$$

$$= \frac{\lambda^{r_1+r_2}}{\Gamma(r_1+r_2)} e^{-\lambda x} \int_{-\infty}^{+\infty} t^{r_1-1} (x-t)^{r_2-1}\,dt$$

$$= \frac{\lambda^{r_1+r_2}}{\Gamma(r_1+r_2)} x^{r_1+r_2-1} e^{-\lambda x}$$

$\Rightarrow \xi+\eta \sim \Gamma(\lambda, r_1+r_2)$

4. $\xi \sim C(\lambda_1, u_1)$. $\eta \sim C(\lambda_2, u_2)$, $\xi.\eta$ iid

$\Rightarrow p_{\xi+\eta}(x) = \int_{-\infty}^{+\infty} p_\xi(t)\,p_\eta(x-t)\,dt$

$$= \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{\lambda_1}{\lambda_1^2+(t-u_1)^2}\,\frac{1}{\pi}\frac{\lambda_2}{\lambda_2^2+(x-t-u_2)^2}\,dt$$

$$= \frac{1}{\pi}\frac{\lambda_1+\lambda_2}{(\lambda_1+\lambda_2)^2+(x-u_1-u_2)^2}$$

即 $\xi+\eta \sim C(\lambda_1+\lambda_2, u_1+u_2)$