第九周第二次作业：

一. P181-184 3. $X \sim U(-1,1)$, X, Y iid

首先计算 $E(X|X\vee Y)$：设 $X\vee Y = a$.

$$E(X|X\vee Y = a) = \int_{-1}^{a} x\,p(x|y=a)\,dx + a\cdot\int_{-1}^{a} p(a|y)\,dy$$

$$= \frac{3}{2}a - \frac{1}{2}$$

再算 $E(X|X+Y)$：设 $X+Y=a$

If $a>0$. $E(X|X+Y) = \int_{a-1}^{1} x\,p(x|x+y=a)\,dx = \frac{a}{2}$

If $a\le 0$: $E(X|X+Y) = \int_{-1}^{a+1} x\,p(x|x+y=a)\,dx = \frac{a}{2}$

$\Rightarrow E(X|X+Y) = \frac{a}{2}$

6. X, Y iid $\sim N(0,1)$, (R,θ) 表示 (X,Y) 的极坐标

(1) $p_1(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ $\quad p_1(y) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{y^2}{2})$

$p_2(x+y) = p_1(x+y)\,|J_1(x,y)| = \frac{1}{2}p_1(x+y)$

$p_2(x-y) = p_1(x-y)\,|J_1(x,y)| = \frac{1}{2}p_1(x-y)$

$\Rightarrow p_2[(x+y)(x-y)] = \frac{1}{8xy}p_1(x^2-y^2)$

$\Rightarrow p_2(x+y)\,p_2(x-y) = p_2[(x+y)(x-y)]$

$\Rightarrow X+Y$ 与 $X-Y$ independent.

$$R^2 = X^2+Y^2 = \frac{(X+Y)^2+(X-Y)^2}{2}$$

当 $X-Y=0$： $R^2=\frac{(X+Y)^2}{2}$

由 $X+Y\sim N(0,\frac{\sqrt{2}}{2})$（前面已证），故 $R^2\sim\chi^2(1)$

(2) $X=R\sin\theta,\quad Y=R\cos\theta$

if $\theta=(k+\frac{1}{2})\pi$, $k\in\mathbb{Z}$. $X=0$, $Y\sim N(0,1)$

此时 $R^2=Y^2\sim\chi^2(2)$

if $\theta\neq(k+\frac{1}{2})\pi$, $k\in\mathbb{Z}$. $\frac{X}{Y}=\tan\theta$

此时 $X=Y\cdot\tan\theta$, $R^2=X^2+Y^2=(1+\tan^2\theta)Y^2\sim\chi^2(2)$

(3) $X-Y=0\Rightarrow R^2\sim\chi^2(1)$

$\theta=\frac{\pi}{4}$ 或 $\theta=\frac{5}{4}\pi\Rightarrow R^2\sim\chi^2(2)$

事件 $A_1=\{X-Y=0\}=\{\theta=\frac{\pi}{4}\}\cup\{R=0\}$

$A_2=\{\theta=\frac{\pi}{4}$ 或 $\theta=\frac{5}{4}\pi\}=\{\theta=\frac{\pi}{4}\}\cup\{\theta=\frac{5}{4}\pi\}$

并不为同一事件.

9. $\{N_t: t\geq0\}$ 为 Possion 过程，参数为 λ，$0<s<t$

Prove: $P(N_s=k\mid N_t)=\binom{N_t}{k}(\frac{s}{t})^k(1-\frac{s}{t})^{N_t-k}\,I\{N_t\geq k\}$ a.s.

Proof: If $N_t<k\Rightarrow N_s<N_t<k\Rightarrow N_s\neq k\Rightarrow P(N_s=k\mid N_t)=0$

Otherwise we just need to prove that:

$$P(N_s=k\mid N_t)=\binom{N_t}{k}(\frac{s}{t})^k(1-\frac{s}{t})^{N_t-k}\quad a.s.$$

$$LHS = \frac{P(N_s=k, N_t=a)}{P(N_t=a)} = \frac{\frac{e^{-\lambda s}(\lambda s)^k}{k!}\,\frac{e^{-\lambda(t-s)}[\lambda(t-s)]^{a-k}}{(a-k)!}}{\frac{e^{-\lambda t}(\lambda t)^a}{a!}}$$

$$= \frac{a!}{k!(a-k)!}\cdot\frac{(\lambda s)^k[\lambda(t-s)]^{a-k}}{(\lambda t)^a}$$

$$= \binom{a}{k}\frac{s^k(t-s)^{a-k}}{t^a}$$

$$= \binom{a}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{a-k}$$

10. $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n$ iid r.v.s

$Y_{1:n}, Y_{2:n}, \cdots, Y_{n:n}$ 的次序统计量

Prove: $\sum_{i=1}^n X_iY_i \overset{d}{=} \sum_{i=1}^n X_i Y_{i:n}$ （$\overset{d}{=}$ 表示同分布）

Proof: 令 $A=\sum_{i=1}^n X_iY_i$ ， $B=\sum_{i=1}^n X_iY_{i:n}$

$\Rightarrow \frac{\partial A}{\partial Y_i} = X_i \Rightarrow \sum_{i=1}^n \frac{\partial A}{\partial Y_i} = \sum_{i=1}^n X_i$

$\frac{\partial B}{\partial Y_{i:n}} = X_i \Rightarrow \sum_{i=1}^n \frac{\partial B}{\partial Y_{i:n}} = \sum_{i=1}^n X_i$

$\Rightarrow F_1(a) = P\{\sum_{i=1}^n X_iY_i < a\} = \int_{\sum_{i=1}^n X_iY_i<a} \prod_{i=1}^n dP_1(X_i)\,dP_2(Y_i)$

$F_2(a) = P\{\sum_{i=1}^n X_iY_{i:n} < a\} = \int_{\sum_{i=1}^n X_iY_{i:n}<a} \prod_{i=1}^n dP_1(X_i)\,dP_2(Y_i)$

同时对 $Y=(Y_1, Y_2, \cdots, Y_n)$ 取全微分即有 $\frac{dF_1(a)}{dY} = \frac{dF_2(a)}{dY}$

$\Rightarrow \sum_{i=1}^n X_iY_i$ 与 $\sum_{i=1}^n X_iY_{i:n}$ 同分布

11 $\mathcal{C}=\{A_n: n\geq 1\}$ 为 Ω 的可数分割， $\mathcal{A}=\sigma(\mathcal{C})$ ， $\mathcal{G}$ 为 $\mathcal{A}$ 的子 σ 代数

Prove: $\mathcal{C}$ 下的正则条件概率存在

Proof: 首先证明 $\mathcal{G}=\mathcal{A}$ 的情形：

令 $P\{A_n\}=\frac{1}{2^n}$ $(n\geq 1)$ 则 $\sum_{n=1}^{+\infty} P\{A_n\}=1$

$\{A_n: n\geq 1\}$ 为 Ω 的分割 $\Rightarrow$ $A_i\cap A_j=\phi,\ i\neq j$

故 $P(A_i\cup A_j)=P(A_i)+P(A_j)=\frac{1}{2^i}+\frac{1}{2^j}$

$\Rightarrow \forall A\in \sigma(\mathcal{C}),\ A\neq\phi,\ P(A)>0$

此时 P 即为所求正则条件概率

其次，若 $\mathcal{G}\subsetneq\mathcal{A}$ 则取 $P|_{\mathcal{G}}$ 满足条件

15. (1) $X\in L_2$, Y, r.v.s $E(X|Y)=Y$ a.s. $E(Y|X)=X$ a.s.

Prove: $X=Y$ a.s.

Proof: $E(X|Y)=\int_X x\,dP(X|Y)$

$Y=\int_X Y\,dP(X|Y)$

由 $E(X|Y)=Y \Rightarrow \int_X x\,dP(x|y)=\int_X y\,dP(x|y)$

$\Rightarrow \int_X (x-y)\,dP(x|y)=0$

同理有：$\int_Y (x-y)\,dP(y|x)=0$

两式相乘：$\iint (x-y)^2\,dP(x|y)\,dP(y-x)=0$

$\Rightarrow x=y$ a.s.

(2) $\mathcal{C}_1, \mathcal{C}_2$ 为 $\mathcal{A}$ 的子 σ-代数，$X\in L_1$，$X_1=E(X|\mathcal{C}_1)$，$X_2=E(X_1|\mathcal{C}_2)$

$X=X_2$ a.s. Prove: $X_1=X_2$ a.s.

Proof: 由(1)的结论:

$$X_1 = X_2 \text{ a.s.} \Leftrightarrow X = X_1 \text{ a.s.} \Leftrightarrow E(X|X_1) = X_1 \text{ 且 } E(X_1|X) = X$$

$$E(X|X_1) = \int_X x \, dP(x|X_1) = \int_X x \cdot \left[\frac{P(\mathscr{C}_1)}{P(X_1|\mathscr{C}_1)} dP(x|\mathscr{C}_1) + \frac{1-P(\mathscr{C}_1)}{P(X_1|\mathscr{C}_1^c)} dP(x|\mathscr{C}_1^c\right.$$

$$= \frac{P(\mathscr{C}_1)}{P(X_1|\mathscr{C}_1)} E(X|\mathscr{C}_1) + \frac{1-P(\mathscr{C}_1)}{P(X_1|\mathscr{C}_1^c)} E(X|\mathscr{C}_1^c)$$

同理 $E(X_1|X_2) = \frac{P(\mathscr{C}_2)}{P(X_2|\mathscr{C}_2)} E(X_1|\mathscr{C}_2) + \frac{1-P(\mathscr{C}_2)}{P(X_2|\mathscr{C}_2^c)} E(X_1|\mathscr{C}_2^c)$

代入题中等式并化简:

$$\frac{E(X|X_1)}{E(X_1|X_2)} = \frac{X_1}{X} \Rightarrow \frac{E(X|X_1)}{E(X_1|X)} = \frac{X_1}{X} \Rightarrow X_1 = X \text{ a.s.}$$

(3) ?

17. X.Y 有界 r.v.s. $\mathscr{C}$为$\mathscr{A}$的子 σ-代数

Prove: $E[XE(Y|\mathscr{C})] = E[YE(X|\mathscr{C})]$

$$LHS = E(Y|\mathscr{C})E(X) = \int_Y y \, dF(y|\mathscr{C}) \cdot \int_X x \, dF(x) = \iint_{X,Y} xy \, dF(y|\mathscr{C}) \, dF(x)$$

$$RHS = \iint_{X,Y} xy \, dF(x|\mathscr{C}) \, dF(y)$$

又 $\frac{dF(x|\mathscr{C}_1)}{dF(y|\mathscr{C}_1)} = \frac{dF(x)}{dF(y)} \Rightarrow RHS = LHS$

故原等式成立

二. P320-321 5. If x bounded, then by Chinese Reference Book P181. 11

The problem is proved.

Otherwise Let $G_k = G \cap (-k, k)$. $\mathbb{P}\{x \in G_k\} = \sum_{i=-k+1}^{k-1} \mathbb{P}\{x = i\}$

And G_k is a Borel set as well

Consider x' is how $x = 0$ outside $(-k, k)$, then by situation 1

we have $E[E(x' | \varphi_1) Y] = E[E(Y | \varphi_1) x']$

and calculate limits for both sides We get.

$E[E(X | \varphi_1) Y] = E[E(Y | \varphi_1) X]$

6. Prove that: $\sigma^2(E_G(Y)) \le \sigma^2(Y)$

Proof: $\sigma^2(E_G(Y)) = E[(E_G(Y) - E(E_G(Y)))^2]$

$E_G(Y)$ is a r.v. on G and we have the following:

① $E[E_G(Y)] = E(Y)$

② $E[(Y - E_G(Y))^2] \ge 0$

Thus, $\sigma^2(Y) = E[var(Y | G)] + var[E(Y | G)]$

$= E[var(Y | G)] + \sigma^2(E_G(Y))$

$\ge \sigma^2(E_G(Y))$

9. $\phi(x, \eta) = \int_{-\infty}^{\eta} \varphi(x, y) dy$

$\Rightarrow \varphi(a, b) = \int_{-\infty}^{a} p(x, b) dx = \int_{-\infty}^{a} P_{\eta=b}(x) dx$

$E(Y | X = x) = \int_{-\infty}^{+\infty} y P\{Y = y | X = x\} dy$

$\Rightarrow \int_{-\infty}^{+\infty} y\varphi(x,y)dy = E(Y|X=x)$

10. G: B.F X, Y. r.v's s.t. $E(Y^2|G) = X^2$, $E(Y|G) = X$

Prove: $Y = X$ a.e

Proof: $E(Y^2|G) - E(Y|G)^2 = X^2 - X^2 = 0$

$E(Y^2|G) - E(Y|G)^2 = \sigma(Y|G)$

$\Rightarrow \sigma(Y|G) = 0 \Rightarrow$ Y is a constant on G a.e.

Also we get $Y = E(Y|G) = X$ on G a.e

11. $\forall f \in C_K$: $E[X^2|f(X)] = E[Y^2|f(X)]$ $E(X|f(X)] = E[Y|f(X)]$

Prove: $Y = X$ a.e.

Proof: By a monotone class theorem the equations hold for $f = I_B$, $B \in \mathcal{B}$

$f \in C_K \Rightarrow$ there exists countable $\{a_k\}_{k\geq 1}$ and $\{B_k\}_{k\geq 1}$

s.t. $f = \sum_{k=1}^{+\infty} a_k I_{B_k}(x)$

So $Y = X$ a.e.

13. ξ r.v. s.t. $P\{\xi > t\} = e^{-t}$ $(t>0)$

Compute $E\{\xi|\xi \wedge t\}$ and $E\{\xi|\xi \vee t\}$ for each $t>0$

Let $F(\xi)$ be the d.f. of ξ, $F(t) = P\{\xi < t\} = 1 - e^{-t}$

Let $m = \xi \wedge t$, then:

① If $m = \xi$, $E(\xi|\xi \wedge t = m) = m$.

② If $m=t$. $E(s|s\wedge t=m)=\int_m^{+\infty} s\,p(s|s\geq m)\,ds=t+1$

In conclusion. $E(s|s\wedge t=m)=\begin{cases} m & m\leq t \\ t+1 & m=t \end{cases}$

① If $m=s$, $E(s|s\vee t=m)=m$

② If $m=t$, $E(s|s\vee t=m)=\int_0^m s\,p(s|s\leq m)\,ds=\dfrac{1-(t+1)e^{-t}}{1-e^{-t}}$

三. ξ_k $k\geq 1$ iid. r.v.'s, $\xi_1\sim U(0,1)$, $Y=\max\limits_{1\leq k\leq n}\{\xi_k\}$

Compute: $E(\xi_1|Y=y)$ and $E(\xi_1|Y)$

$E(\xi_1|Y=y)=\int_{\xi_1} x_1\,p(x_1|\max\limits_{1\leq k\leq n}\{\xi_k\}=y)\,dx_1=\frac{y}{2}$ is a certain value

$E(\xi_1|Y)=\frac{Y}{2}$ is a random value