第九周第一次作业：

一. P121 例3.2.6

设 $\theta \sim U(0,2\pi)$，$\xi = \cos\theta$，$\eta = \cos(\theta+a)$ 其中 a 为常数

不难算出 $E(\xi) = E(\eta) = 0$　$D(\xi) = D(\eta) = \frac{1}{2}$

$$E(\xi\eta) = \int_0^{2\pi} \frac{1}{2\pi} \cos x \cos(x+a)\,dx = \frac{1}{2}\cos a$$

故相关系数 $r = \dfrac{E(\xi\eta)}{\sqrt{D(\xi)}\sqrt{D(\eta)}} = \cos a$

于是有：

(1) $a=0$ 时 $r=1$，此时 $\eta = \cos\theta = \xi$

(2) $a=\pi$ 时 $r=-1$，此时 $\eta = \cos(\theta+\pi) = -\cos\theta = -\xi$

(3) $a=-\frac{\pi}{2}$ 时 $r=0$. 此时 $\xi = \cos\theta$，$\eta = \sin\theta$，从而 ξ 与 η 不相关

但 ξ 与 η 不独立（取 $B_1 = B_2 = (0,\frac{1}{2})$，$P\{B_1\} = P\{B_2\} = \frac{1}{6}$. $P\{B_1B_2\} = 0$）

P133 7　$E(\alpha) = nE(\xi_1)$

$$E(\alpha^2) = E((\sum_{i=1}^{n}\xi_i)^2) = E(\sum_{i=1}^{n}\xi_i^2) + E(\sum_{1\le i<j\le n}\xi_i\xi_j) = nE(\xi_1^2) + (n^2-n)E^2(\xi_1)$$

$$\Rightarrow D(\alpha) = E(\alpha^2) - [E(\alpha)]^2 = n[E(\xi_1^2) - E^2(\xi_1)] = nD(\xi_1)$$

同理 $D(\beta) = nD(\xi_1)$

$$E(\alpha\beta) = E(\sum_{i=1}^{n}\xi_i \cdot \sum_{j=m+1}^{n+m}\xi_j) = E(\sum_{i=1}^{m}\xi_1 \cdot \sum_{j=m+1}^{m+n}\xi_j) + E(\sum_{i=m+1}^{n}\xi_i \cdot \sum_{j=n+1}^{m+n}\xi_j)$$

$$+ E(\sum_{i=m+1}^{n}\xi_i \cdot \sum_{j=m+1}^{n}\xi_j)$$

$$= mnE(\xi_1\xi_2) + (n-m)mE(\xi_1\xi_2) + [(n-m)^2 - (n-m)]E(\xi_1\xi_2)$$

$$+(n-m)E(\xi_1^2)$$

$$=(n^2-n+m)E^2(\xi_1)+(n-m)E(\xi_1^2)$$

相关系数

$$r=E\left[\frac{\alpha-E(\alpha)}{\sqrt{D(\alpha)}}\frac{\beta-E(\beta)}{\sqrt{D(\beta)}}\right]=\frac{1}{D(\xi_1)}E[E(\alpha)E(\beta)-\alpha E(\beta)-\beta E(\alpha)+\alpha\beta]$$

$$=\frac{1}{D(\xi_1)}E[n^2E^2(\xi_1)-\alpha\cdot nE(\xi_1)-\beta nE(\xi_1)+\alpha\beta]$$

$$=\frac{1}{D(\xi_1)}[n^2E^2(\xi_1)-n^2E^2(\xi_1)-n^2E^2(\xi_1)+E(\alpha\beta)]$$

$$=\frac{E(\alpha\beta)-n^2E^2(\xi_1)}{D(\xi_1)}$$

$$=1-\frac{m}{n}$$

8. 设 $P\{\xi=\xi_1\}=p$. $P\{\xi=\xi_2\}=1-p$, $P\{\eta=\eta_1\}=q$, $P\{\eta=\eta_2\}=1-q$

则 $E(\xi)=\xi_1\cdot p+\xi_2\cdot(1-p)$, $E(\eta)=\eta_1\cdot q+\eta_2(1-q)$

$E(\xi)E(\eta)=\xi_1\eta_1\cdot pq+\xi_1\eta_2\cdot p(1-q)+\xi_2\eta_1(1-p)q+\xi_2\eta_2\cdot(1-p)(1-q)$

若 $\xi.\eta$ 不相关 则 $E(\xi)E(\eta)=E(\xi\eta)\Rightarrow\xi_1\eta_1,\ \xi_2\eta_2,\ \xi_1\eta_2,\ \xi_2\eta_1$ 两两不等

故对 $\forall$ $\mathbb{R}$上的可积Borel函数 $f.g$

$E[f(\xi)]=f(\xi_1)\cdot p+f(\xi_2)\cdot(1-p)$

$E[g(\eta)]=g(\eta_1)\cdot q+g(\eta_2)\cdot(1-q)$

$$E[f(\xi)g(\eta)]=f(\xi_1)g(\eta_1)\cdot pq+f(\xi_1)g(\eta_2)p(1-q)+$$
$$f(\xi_2)g(\eta_1)\cdot(1-p)q+f(\xi_2)g(\eta_2)(1-p)(1-q)$$

$$=E[f(\xi)]E[g(\eta)]\Rightarrow f\ g\ 独立$$

9 $E(\xi)E(\xi^2) = E(\xi)[E^2(\xi)+D(\xi)] = E^3(\xi)+E(\xi)D(\xi) = u^3+u6$

$E(\xi^3) = \int_{-\infty}^{+\infty} x^3 p(x)\,dx = \int_{-\infty}^{+\infty} x^3 \cdot e^{-\frac{(x-u)^2}{26^2}}\,dx = u^3+u6$

$\Rightarrow E(\xi)E(\xi^2) = E(\xi^3) \Rightarrow$ ξ和ξ^2不相关.

但若取 $f(x)=2x$，$g(x)=x+1$.

则 $E[f(\xi)g(\xi^2)] \neq E[f(\xi)]E[g(\xi^2)]$. 故 ξ和ξ^2不独立.

10. $E(\xi) = \int_{-\infty}^{+\infty} xp(x)\,dx = \int_{-\infty}^{0} xp(-x)\,dx + \int_{0}^{+\infty} xp(x)\,dx$

$= \int_{0}^{+\infty} (-x+x)\,p(x)\,dx = 0$

$E(\xi\cdot|\xi|) = \int_{-\infty}^{+\infty} x|x|p(x)\,dx = \int_{-\infty}^{0} x|x|p(-x)\,dx + \int_{0}^{+\infty} x|x|p(x)\,dx$

$= \int_{0}^{+\infty} (-x|x|+x|x|)\,p(x)\,dx = 0$

$\Rightarrow E(\xi)E(|\xi|) = E(\xi|\xi|) \Rightarrow$ ξ与$|\xi|$不相关

取 $f(x)=|x|$，$g(x)=x$.

此时 $E[f(\xi)] = \int_{-\infty}^{+\infty} |x|p(x)\,dx = E[g(x)]$

$E[f(\xi)g(\xi)] = \int_{-\infty}^{+\infty} |x|^2 px\,dx \leq \left(\int_{-\infty}^{+\infty} |x|p(x)\,dx\right)^2$ （Cauchy 不等式）

但因 $|x|$与$p(x)$不成正比例. 故不取等.

从而 $|\xi|$和ξ不独立

14 $\xi_1 \sim U(0,1)$, $\xi_{k+1} \sim U(\xi_k, x_{k+1})$

$\Rightarrow E(\xi_{k+1}) = E[E(\xi_k) + x_{k+1}] = E(\xi_k) + \frac{1}{2}$

故由归纳法 $E(\xi_n) = \frac{n}{2}$

15 $D(\xi|\eta) = E(\xi^2|\eta) - E^2(\xi|\eta)$

$\Rightarrow E[D(\xi|\eta)] = E[E(\xi^2|\eta) - E^2(\xi|\eta)]$

$$= \int_{-\infty}^{+\infty} \left(E(\xi^2|\eta) - E^2(\xi|\eta)\right) p(\eta)\, d\eta$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \xi^2 p(\xi)\, d\xi - \left(\int_{-\infty}^{+\infty} \xi p(\xi)\, d\xi\right)^2\right] p(\eta)\, d\eta$$

$D(E(\xi|\eta)) = E[E^2(\xi|\eta)] - E^2[E(\xi|\eta)]$

$$= \int_{-\infty}^{+\infty} p(\eta)\, d\eta \left(\int_{\infty}^{+\infty} \xi p(\xi)\, d\xi\right)^2 - \left[\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \xi p(\xi)\, d\xi\right) p(\eta)\, d\eta\right]^2$$

两式相加：

$$E[D(\xi|\eta)] + D[E(\xi|\eta)] = \iint_{\mathbb{R}^2} \xi^2 p(\xi) p(\eta)\, d\xi\, d\eta - \left[\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \xi p(\xi)\, d\xi\right) p(\eta)\, d\eta\right]^2$$

$$= E(\xi^2) - E^2(\xi)$$

$$= D(\xi)$$

二. P128 12. X.Y 独立 $\Rightarrow E(XY) = E(X)E(Y)$

(1) 若 $2|p$. $E|X+Y|^p - E|X|^p = E[(x+y)^p - x^p] = E\left[\sum_{k=0}^{p-1} \binom{p}{k} x^k y^{p-k}\right] \geq E(Y^p) = E|Y^p|$

$\Rightarrow E|X+Y|^p \geq E|X|^p + E|Y|^p \geq \max\{E|X|^p, E|Y|^p\}$

若 $2 \nmid p$ $E|X+Y|^p = E|X+Y|^{p-1} E|X+Y| \geq \max\{E|X|^{p-1}, E|Y|^{p-1}\} \max\{E|X|, E|Y|\}$

$\geq \max\{E|X|^p, E|Y|^p\}$

(2) $E|X|^p \leq E|X+Y|^p < +\infty$

$E|Y|^p \leq E|X+Y|^p < +\infty$

14. $E[Y^r] = \int_0^{+\infty} y^r \, dP\{y\}$. $E[X^r] = \int_0^{+\infty} x^r \, dP\{x\}$

$P\{Y>t\} \leq \frac{1}{t} \int_{\{Y>t\}} X \, dP = \frac{1}{t} \int_{\{Y>t\}} X \, dP\{X|Y\}$ $(\forall t>0)$

$P\{Y>t\} = \int_{\{Y>t\}} dP\{y\}$

$\Rightarrow t \int_{\{Y>t\}} dP\{y\} \leq \int_{\{Y>t\}} X \, dP\{X|Y\}$

$E[X^r] \left(\frac{r}{r-1}\right)^r = E[X^r]\left(1+\frac{1}{r-1}\right)^r \geq \int_0^{+\infty} x^r dP_x \left[1+\int_r^{+\infty} \ln(t-1) dt\right]^r$

(不会算了)

三. P121 5.3.1 $\{X_n\}$ independent r.v.s s.t.

$\forall n$ $E(X_n)=0$、$E(X_n^2) = D(X_n) < +\infty$

Then we have for every $\varepsilon>0$:

$$P\{\max_{1\leq j\leq n} |S_j| > \varepsilon\} \leq \frac{\sigma^2(S_n)}{\varepsilon^2}$$

Proof: Fix $\varepsilon>0$, $\forall w \in \Lambda := \{w: \max_{1\leq j\leq n} |S_j(w)| > \varepsilon\}$

Define $v(w) = \min\{j: 1\leq j\leq n, |S_j(w)|>\varepsilon\}$

v. is an r.v. with domain Λ. Put

$$\Lambda_k = \{w: v(w)=k\} = \{w: \max_{1\leq j\leq k-1} |S_j(w)| \leq \varepsilon, |S_k(w)|>\varepsilon\}$$

where for $k=1$, $\max_{1\leq j\leq 0} |S_j(w)| = 0$

$\Rightarrow$ ν is the "first time" that the indicated maximum exceeds ε

and Λ_k is the event that this occurs for

the first time at the k-th step

The Λ_k's are disjoint and we have $\Lambda = \bigcup_{k=1}^{n} \Lambda_k$

$$\Rightarrow \int_\Lambda S_n^2 \, dP = \sum_{k=1}^{n} \int_{\Lambda_k} [S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2] \, dP$$

Let φ_k denote the indicator for Λ_k

The $\varphi_k S_k$ and $S_n - S_k$ are independent

$$\Rightarrow \int_{\Lambda_k} S_k(S_n - S_k) \, dP = 0$$

Since $\quad E(S_n - S_k) = \sum_{j=k+1}^{n} E(X_j) = 0$

Using this we obtain:

$D(S_n) = \varepsilon^2 P(\Lambda)$