第十三周第一次作业:

一. P171 9. 正态分布函数列 $\{F_n(x)\}$ $F_n(x) \xrightarrow{w} F(x)$

Prove: $F(x)$ 为正态分布函数，且E.D.也收敛

Proof: $F_n(x) \xrightarrow{w} F(x) \Longleftrightarrow f_n(x) \to f(x)$ pointly

$f_n(x) = \exp\{iu_n t - \frac{\sigma_n^2 t^2}{2}\} \to f(x)$

$\Rightarrow f(x) = \exp\{iut - \frac{\sigma^2 t^2}{2}\}$, where $u = \lim_{k\to+\infty} u_k$, $\sigma = \lim_{k\to\infty} \sigma_k$

$\Rightarrow f(x)$ is the ch.f. of ξ, whose $F_\xi(\ ) = F(\cdot) = \lim_{n\to\infty} F_n(\cdot)$

10. $\{\xi_k\}$ iid ~ $\{-1,1\}$上的两点等可能分布

Prove: $\eta_n = \sum_{k=1}^{n} \frac{\xi_k}{2^k}$ 依分布收敛到区间 $(-1,1)$ 上的均匀分布

Proof: 设 F_n 为 η_n 的分布函数.

则 $F_n(x) = \mathbb{P}\{\sum_{k=1}^{n} \frac{x_k}{2^k} < x : P\{x_k=1\} = P\{x_k=-1\} = \frac{1}{2}\}$

$\lim_{n\to\infty} F_n(x) = \mathbb{P}\{\sum_{k=1}^{\infty} \frac{x_k}{2^k} < x : P\{x_k=1\} = P\{x_k=-1\} = \frac{1}{2}\}$

注意到 $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \Rightarrow \sum_{k=1}^{\infty} \frac{x_k}{2^k} = 2 \sum_{\substack{k=1 \\ x_k=1}}^{\infty} \frac{x_k}{2^k} - 1$

$\Rightarrow \mathbb{P}\{\sum_{k=1}^{\infty} \frac{x_k}{2^k} < x\} = \mathbb{P}\{\sum_{\substack{k=1 \\ x_k=1}}^{\infty} \frac{x_k}{2^k} < \frac{x+1}{2}\} = \frac{x+1}{2}$ (二进制)

$\Rightarrow F(x) = \frac{x+1}{2} \Rightarrow$ 对应 $U(-1,1)$

二. P169-170 Theorem 6.3.2: $\{\mu_n : 1 \le n < \infty\}$ be p.m.'s on $\mathbb{R}^1$ with ch.f.'s

$\{f_n : 1 \le n < \infty\}$ and:

(1) f_n converges everywhere in $\mathbb{R}^1$ and defines the limit function f_∞

(2) f_∞ is continuous at $t=0$

Prove: (1) $\mu_n \xrightarrow{v} \mu_\infty$ where μ_∞ is a p.m.

(2) f_∞ is the ch.f. of μ_∞

Lemma. $\forall A>0$, we have.

$$\mu([-2A, 2A]) \geq A\left|\int_{-\frac{1}{A}}^{\frac{1}{A}} f(t)dt\right| - 1$$

Now for $\forall \delta \in (0, \delta_0)$ we have

$$\left|\frac{1}{2\delta}\int_{-\delta}^{\delta} f_n(t)dt\right| \geq \left|\frac{1}{2\delta}\int_{-\delta}^{\delta} f(t)dt\right| - \frac{1}{2\delta}\int_{-\delta}^{\delta} |f_n(t)-f(t)|dt$$

LHS $\to 1$ $(\delta \to 0)$

For fixed δ, RHS $\to 0$ $(n \to \infty)$

Since $|f_n - f| \leq 2 \Rightarrow \forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) < \delta_0$, $n_0 = n_0(\varepsilon)$

s.t. $\forall n \geq n_0$, LHS $\geq 1-\varepsilon$

Hence by (2): $\mu_n([-2\delta^{-1}, 2\delta^{-1}]) \geq 2(1-\varepsilon) - 1 \geq 1-2\varepsilon$

$\Rightarrow \mu(\mathbb{R}^1) \geq \mu([-2\delta^{-1}, 2\delta^{-1}]) = \lim\limits_{n\to\infty} \mu_n([-2\delta^{-1}, 2\delta^{-1}]) \geq 1-2\varepsilon$

Since ε is arbitary, μ is a p.m.

Let f be the ch.f. of μ, $f_{n_k} \to f$ everywhere.

$\Rightarrow f = f_\infty \Rightarrow$ every vague limit μ considered above has the same ch.f $\Rightarrow$

$\mu_n \xrightarrow{v} \mu_\infty$

三. P174 6. $\{f_n\}$ 在 $(-\delta,\delta)$ 内一致收敛

Prove: $\{f_n\}$ 一致连续，且 $\exists$ $\{f_n\}$ 子列收敛至 ch.f.

Proof: By Ascoli-Arzelà theorem, since $|f_n(t)| \le 1$ $(\forall t)$,

$\{f_n\}$ have a subsequence $\{f_{n_k}\}$ uniformly convergent.

Next we'll show that the limit of $\{f_{n_k}\}$ is a ch.f.

$$f(0) = \lim_{k\to+\infty} f_{n_k}(0) = 1$$

$$\forall t,\ |f(t)| = \lim_{k\to\infty} |f_{n_k}(t)| \le 1$$

$\Rightarrow$ f is a ch.f.

四. P180 Theorem 6.4.6 X、Y independent, iid.

$E(X)=0$, $D(X)=1$

If $X+Y$ and $X-Y$ independent then the common

distribution of X and Y is ϕ

Proof. f, ch.f. then by (1) $f'(0)=0$, $f''(0)=-1$

ch.f. of $X+Y$ is $f(t)^2$ and $X-Y$ is $f(t)f(-t)$

Independent: ch.f. of $2X$ = ch.f of $(X+Y)$ + ch.f of $(X-Y)$

$\Rightarrow$ $f(2t) = f(t)^3 f(-t)$

suppose $\rho(t) = \dfrac{f(t)}{f(-t)}$ $\Rightarrow$ $\rho(2t) = \rho(t)^2$

$$\Rightarrow \rho(t) = \rho\left(\frac{t}{2^n}\right)^{2^n} \to 1 \quad (n \to \infty)$$

$$\Rightarrow f(t) = f(-t) \Rightarrow f(2t) = f(t)^4$$

$$\Rightarrow f(t) = f\left(\frac{t}{2^n}\right)^{4^n} = \left\{ 1 - \frac{1}{2}\left(\frac{t}{2^n}\right)^2 + o\left[\left(\frac{t}{2^n}\right)^2\right] \right\}^{4^n} \to e^{-\frac{t^2}{2}}$$