第十一周第二次作业.

一. P152 4. $F(x)$ 对称 $:=$ $F(x) = 1 - F(-x+0)$

Prove: 对称 $\Leftrightarrow$ ch.f. 为实偶函数

Proof: "$\Rightarrow$" $f(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x) = \int_0^{+\infty} (e^{itx} + e^{-itx})\, dF(x)$

$= \int_0^{+\infty} (e^{i|t|x} + e^{-i|t|x})\, dF(x)$ 为实偶函数

"$\Leftarrow$" 对于实偶函数 $f(t): \mathbb{R} \to \mathbb{R}$. $f(t) = f(-t)$

$dF(x) = dx \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} f(t)\, dt$

$= dx \cdot \frac{1}{2\pi} \int_0^{+\infty} (e^{itx} + e^{-itx}) f(t)\, dt$

$\Rightarrow dF(x) = -\, dF(-x)$

$\Rightarrow F(x)$ 对称

11. $\xi_1, \xi_2, \cdots, \xi_n$ iid. $\sim C(1,0)$, 求 $\bar{\xi} = \frac{1}{n}\sum_{k=1}^{n} \xi_k$ 分布

设 $\xi_1, \xi_2, \cdots, \xi_n$ 的 ch.f. 为 $f_1, f_2, \cdots, f_n$, $\bar{\xi}$ 的 ch.f. 为 $\bar{f}$

则 $\bar{f} = \frac{1}{n}\sum_{k=1}^{n} f_k = f_1 = f_2 = \cdots = f_n$

$\Rightarrow \bar{\xi} \sim C(1,0)$ 与 $\xi_1, \xi_2, \cdots, \xi_n$ 相同.

15. Prove: r.v. $\xi_1, \xi_2, \cdots, \xi_n$ 独立 $\Leftrightarrow$ 联合 ch.f. 为边缘 ch.f. 之积.

"$\Rightarrow$" $dF(x_1, x_2, \cdots, x_n) = dF(x_1)\, dF(x_2) \cdots dF(x_n)$

$f(t) = \int_{\mathbb{R}^n} e^{it_1x_1 + it_2x_2 + \cdots + it_nx_n}\, dF(x_1, x_2, \cdots, x_n)$

$$f_1(t_1)=\int_{-\infty}^{+\infty} e^{it_1x_1}\,dF(x_1)$$

$$f_2(t_2)=\int_{-\infty}^{+\infty} e^{it_2x_2}\,dF(x_2)$$

$$\cdots$$

$$f_n(t_n)=\int_{-\infty}^{+\infty} e^{it_nx_n}\,dF(x_n)$$

乘积：$\prod_{k=1}^{n} f_k(t_k)=\int_{\mathbb{R}^n} e^{it_1x_1+it_2x_2+\cdots+it_nx_n}\,dF_1(x_1)\,dF_2(x_2)\cdots dF_n(x_n)$

$$=\int_{\mathbb{R}^n} e^{it_1x_1+it_2x_2+\cdots+it_nx_n}\,dF(x_1,x_2,\cdots,x_n)$$

"$\Leftarrow$" $f(t)=f_1(t_1)f_2(t_2)\cdots f_n(t_n)$ （已知）

则 $dF(x)=dx\cdot\left(\frac{1}{2\pi}\right)^n\cdot\int_{\mathbb{R}^n} e^{-it_1x_1-it_2x_2-\cdots-it_nx_n}\,df(t_1,t_2,\cdots,t_n)$

$$=dx\cdot\left(\frac{1}{2\pi}\right)^n\cdot\int_{\mathbb{R}^n} e^{-it_1x_1-it_2x_2-\cdots-it_nx_n}\,df_1(t_1)\,df_2(t_2)\cdots df_n(t_n)$$

$$=dx\prod_{k=1}^{n}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-it_kx_k}\,df_k(t_k)$$

$$=\prod_{k=1}^{n} dF_k(x_k)$$

Pb1 5. ξ,η iid $\sim N(0,1)$

$$\zeta=\begin{cases}|\eta| & \xi>0\\ -|\eta| & \xi<0\end{cases}$$

则 ζ 服从 $N(0,1)$ 分布，但协方差阵：

$$B=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\Rightarrow |r|>1.$$ 不是二元正态分布

6. (ξ,η) 服从二元正态分布

$\xi+\eta$ 与 $\xi-\eta$ 独立 $\Leftrightarrow f(\xi^2-\eta^2)=f(\xi+\eta)f(\xi-\eta)$

$$\Leftrightarrow \iint_{\mathbb{R}^2} e^{it(x^2-y^2)} p(x,y)\,dxdy = \iint_{\mathbb{R}^2} e^{it(x+y)} p(x,y)\,dxdy \cdot \iint_{\mathbb{R}^2} e^{it(x-y)} p(x,y)\,dxdy$$

$$\left(LHS = \iint_{\mathbb{R}^2} e^{it(x^2-y^2)} p(x,y)\,dxdy \cdot \iint_{\mathbb{R}^2} p(x,y)\,dxdy \right)$$

$$\Leftrightarrow \iint_{\mathbb{R}^2} \left[e^{it(x^2-y^2)} - e^{it(x+y)\, it(x-y)} \right] p(x,y)\,dxdy = 0$$

$$\Leftrightarrow \iint_{\mathbb{R}^2} \left[e^{it(x^2-y^2)} - e^{-t^2(x^2-y^2)} \right] p(x,y)\,dxdy = 0$$

分别考虑实虚部得：上式 $\Leftrightarrow \sigma_1^2 = \sigma_2^2 \Leftrightarrow b_{11} = b_{22}$

8. $\xi_1, \xi_2, \cdots, \xi_n$ iid. $\sim N(0,1)$, $\eta_1 = \sum_{k=1}^{n} \xi_k$, $\eta_2 = \sum_{k=1}^{m} \xi_k$ $(m<n)$

求 η_1 和 η_2 的联合分布

$D(\eta_1) = m$. $D(\eta_2) = n$

$D(\eta_1, \eta_2) = m$

故协方差阵 $B = \begin{pmatrix} m & m \\ m & n \end{pmatrix}$

联合分布 $(\eta_1, \eta_2) \sim N(\vec{0}, B)$

二. P202 1. X and Y independent r.v's, $X \sim N(u_1, \sigma^2)$, $Y \sim N(u_2, \sigma^2)$

Prove: $X+Y$ and $X-Y$ independent

Proof: $U = X+Y$, $V = X-Y \Rightarrow X = \frac{U+V}{2}$, $Y = \frac{U-V}{2}$

$\Rightarrow cov(U,V) = cov(X+Y, X-Y) = D(X) - cov(X,Y) + cov(Y,X) - D(Y)$

$= D(X) - D(Y) = \sigma^2 - \sigma^2 = 0$

Since U and V are jointly normal and $\mathrm{cov}(U, V) = 0$

Then U and V are independent

2. Suppose $X_1, X_2, \cdots, X_n$ jointly normal, $\mathrm{cov}(X_i, X_j) = 0 \quad (\forall i \neq j)$

Then $X_1, X_2, \cdots, X_n$ are independent

Proof: $X = (X_1, X_2, \cdots, X_n)^T$ random vector whose cov matrix is Σ

if $\Sigma = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)$

Then $X_1, X_2, \cdots, X_n$ uncorrelated.

Combined by X follows normal distribution,

Thus $X_1, X_2, \cdots, X_n$ independent

二. $\xi_1, \xi_2, \cdots, \xi_n$ iid $\sim N(a, \sigma^2)$

Prove: (1) $\bar{\xi}$, S_n^2 independent

(2) $\bar{\xi} \sim N(a, \frac{\sigma^2}{n})$

(3) $\frac{nS_n^2}{\sigma^2} \sim \chi^2(n-1)$

Proof: (1) $\vec{\xi} \sim N(a, \vec{B})$, $\vec{a} = (a, a, \cdots, a)$, $B = \sigma^2 I$

取 C 为 n 阶正交阵:

$$C = \begin{bmatrix} \frac{1}{\sqrt{1\cdot 2}} & \frac{-1}{\sqrt{1\cdot 2}} & 0 & \cdots & \\ \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{-2}{\sqrt{2\cdot 3}} & 0 & \cdots \end{bmatrix}$$

$$
\begin{bmatrix}
\cdots & & & \\
\frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \cdots & \frac{-(n-1)}{\sqrt{(n-1)n}} \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}}
\end{bmatrix}
$$

经 $(\xi$ 变换后各分量 $\eta_1, \eta_2, \cdots, \eta_n$ 仍独立且有相同方差 σ^2

又因 $E(\eta_1) = E(\eta_2) = \cdots = E(\eta_{n-1}) = 0$，$E(\eta_n) = \sqrt{n}a$

由正交变换保范数:

$$\sum_{k=1}^{n} \eta_k^2 = \|\vec{\eta}\|^2 = \|\vec{\xi}\|^2 = \sum_{k=1}^{n} \xi_k^2$$

又 $\eta_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \xi_k = \sqrt{n}\,\bar{\xi}$

$\Rightarrow n S_n^2 = \sum_{k=1}^{n-1} \eta_k^2$

$\eta_1, \eta_2, \cdots, \eta_n$ 独立 $\Rightarrow \bar{\xi} = \frac{\eta_n}{\sqrt{n}}$ 与 $S_n^2 = \frac{1}{n}\sum_{k=1}^{n-1} \eta_k^2$ 独立

(2) $\eta_n = \sqrt{n}\,\bar{\xi}$ 且 $\eta_n \sim N(a, \sigma^2) \Rightarrow \bar{\xi} \sim (a, \frac{\sigma^2}{n})$

(3) $\frac{n S_n^2}{\sigma^2} = \sum_{k=1}^{n-1} \left(\frac{\eta_k}{\sigma}\right)^2$，$\frac{\eta_k}{\sigma} \sim N(a, 1)$

$\Rightarrow \frac{n S_n^2}{\sigma^2} \sim \chi^2(n-1)$