奖励作业1

Prove Carathéodory's Theorem (extension theorem)

Assume that $\mathcal{E}$ is a semiring on Ω: $u(\cdot): \mathcal{E} \to [0,\infty)$ is an additive function, in other words:

(1) $u(\emptyset)=0$; (2) $\forall A_n \in \mathcal{E}$, $\sum_{n=1}^{\infty} A_n \in \mathcal{E}$, $u(\sum_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} u(A_n)$

Then: (1) u has an extension measure $\bar{u}$ on $(\Omega, \sigma(\Omega))$ s.t. $\bar{u}|_{\mathcal{E}} = u$

(2) If u is a σ-finite function. $\Omega = \sum_{n=1}^{\infty} A_n$, $A_n \in \mathcal{E}$, $u(A_n) < +\infty$ $\forall n \geq 1$. Then $\bar{u}$ is an unique extension

Proof:

Lemma 1: If $u^*(A) \triangleq \inf\{\sum_{n=1}^{\infty} u(A_n) : A_n \in \mathcal{E}, A \subseteq \bigcup_n A_n\}$ $\forall A \in \Omega$

Then (i) $u^*(\emptyset) = 0$

(ii) $A_1 \subseteq A_2 \Rightarrow u^*(A_1) \leq u^*(A_2)$

(iii) $u^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} u^*(A_n)$, $\forall A_n \subseteq \Omega$

(iv) $u^*|_{\mathcal{E}} = u$

(i) (ii) trival, omitted

(iii) Let $\delta > 0$, for each j, there exists a covering $\{B_{jk}\}$ of A_j

s.t. $\sum_k u(B_{jk}) \leq u^*(A_j) + \frac{\delta}{2}$

The double sequence $\{B_{jk}\}$ is a covering of $\bigcup_j A_j$

s.t. $\sum_j \sum_k u(B_{jk}) \leq \sum_j u^*(A_j) + \delta$

Hence for any $\delta > 0$

$$u^*(\bigcup_j A_j) \leq \sum_j u^*(A_j) + \delta$$

that establishes (iii) for u^*. since δ is arbitary small

(iv) By (iii) we know u^* is an outer measure, so a class of sets $\mathcal{F}^*$

is associated as follows:

A set $A \subseteq \Omega$ belongs to $\mathcal{F}^*$ iff for every $Z \subseteq \Omega$ we have

$$u^*(Z) = u^*(A \cap Z) + u^*(A^c \cap Z)$$

If in (iii) we change "=" into "$\leq$" the resulting inequality holds

by Lemma 1 Hence (iii) is equivalent to the reverse inequality

when "=" is changed to "$\geq$"

Lemma 2: Define $\mathcal{F}^* \triangleq \{A \subseteq \Omega : u^*(B) = u^*(A \cap B) + u^*(A^c \cap B) \ \forall B \subseteq \Omega\}$

Then: (i) $\mathcal{F}^*$ is a σ-algebra on Ω

(ii) $\mathcal{E} \subseteq \mathcal{F}^*$ ($\Rightarrow \sigma(\mathcal{E}) \subseteq \mathcal{F}^*$)

(iii) u^* is a measure on $\mathcal{F}^*$

Let $A \in \mathcal{F}_0$. For any $Z \subseteq \Omega$ and any $\varepsilon > 0$. there exists a covering

$\{B_j\}$ of Z such that

$$\sum_j u(B_j) \le u^*(Z)+\varepsilon$$

Since $A\cap B_j \in \mathcal{F}_0$. $\{A\cap B_j\}$ is a covering of AZ. $\{A^cB_j\}$ is a covering of A^cZ

Hence

$$u^*(AZ)\le \sum_j u(AB_j) \quad u^*(A^cZ)\le \sum_j u^*(A^cB_j)$$

Since u is a measure on $\mathcal{F}_0$, we have for each j:

$$u(AB_j)+u(A^cB_j)=u(B_j)$$

It follows from

$$u^*(AZ)+u^*(A^cZ)\le u^*(Z)+\varepsilon$$

Letting $\varepsilon\downarrow 0$ establishes the criterion (iii) in it's "$\geq$" form

Thus $A\in\mathcal{F}^*$ and we have proved that $\mathcal{F}_0\subseteq\mathcal{F}^*$

To prove that $\mathcal{F}^*$ is a Borel Field, it is trival that it is closed under complementation because the criterion is unaltered when A is changed into A^c.

Next, to show that $\mathcal{F}^*$ is closed under union. Let $A\in\mathcal{F}^*$ and $B\in\mathcal{F}^*$. Then for any $Z\subseteq\Omega$, we have by (iii) with A replaced by B and replaced by ZA or ZA^c:

$$\mu^*(ZA) = \mu^*(ZAB) + \mu^*(ZAB^c)$$

$$\mu^*(ZA^c) = \mu^*(ZA^cB) + \mu^*(ZA^cB^c)$$

Hence by (iii) again as written:

$$\mu^*(Z) = \mu^*(ZAB) + \mu^*(ZAB^c) + \mu^*(ZA^cB) + \mu^*(ZA^cB^c)$$

Applying (iii) with Z replaced by $Z(A \cup B)$ we have:

$$\mu^*(Z(A\cup B)) = \mu^*(Z(A\cup B)A) + \mu^*(Z(A\cup B)A^c)$$

$$= \mu^*(ZA) + \mu^*(ZA^cB)$$

$$= \mu^*(ZAB) + \mu^*(ZAB^c) + \mu^*(ZA^cB)$$

Comparing the two preceding equations we see that

$$\mu^*(Z) = \mu^*(Z(A\cup B)) + \mu^*(Z(A\cup B)^c)$$

Hence $A \cup B \in \mathcal{F}^*$, and we have proved that $\mathcal{F}^*$ is a field

Now let $\{A_j\}$ be an infinate sequence of sets in $\mathcal{F}^*$, put

$$B_1 = A_1, \quad B_j = A_j \setminus \left(\bigcup_{i=1}^{j-1} A_i\right) \text{ for } j \geq 2$$

Then $\{B_j\}$ is a sequence of disjoint sets in $\mathcal{F}^*$ (because $\mathcal{F}^*$ is a field)

and has the same union as $\{A_j\}$. For any $Z \subseteq \Omega$, we have for each

$n \geq 1$:

$$\mu^*\left(Z \bigcup_{j=1}^{n} B_j\right) = \mu^*\left(Z\left(\bigcup_{j=1}^{n} B_j\right) B_n\right) + \mu^*\left(Z\left(\bigcup_{j=1}^{n} B_j\right) B_n^c\right)$$

$$= u^*(ZB_n) + u^*(Z\bigcup_{j=1}^{n-1} B_j)$$

Because $B_n \in \mathcal{F}^*$. It follows by induction on n that

$$u^*(Z\bigcup_{j=1}^{n} B_j) = \sum_{j=1}^{n} u^*(ZB_j) \qquad (a)$$

Since $\bigcup_{j=1}^{n} B_j \in \mathcal{F}^*$. we have the monotonicity of u^*:

$$u^*(Z) = u^*(Z\bigcup_{j=1}^{n} B_j) + u^*(Z(\bigcup_{j=1}^{n} B_j)^c)$$

$$\geq \sum_{j=1}^{n} u^*(ZB_j) + u^*(Z(\bigcup_{j=1}^{\infty} B_j)^c)$$

Letting $n \uparrow \infty$ and using property (iii) of u^* we obtain:

$$u^*(Z) \geq u^*(Z\bigcup_{j=1}^{\infty} B_j) + u^*(Z(\bigcup_{j=1}^{\infty} B_j)^c)$$

That establishes $\bigcup_{j=1}^{\infty} B_j \in \mathcal{F}^*$. Thus $\mathcal{F}^*$ is a Borel Field.

Finally, Let $\{B_j\}$ be a sequence of disjoint sets in $\mathcal{F}^*$. By the property (ii) of u^* and (a) with $Z = \Omega$, we have

$$u^*(\bigcup_{j=1}^{\infty} B_j) \geq \limsup_n u^*(\bigcup_{j=1}^{n} B_j) = \lim_n \sum_{j=1}^{n} u^*(B_j) = \sum_{j=1}^{\infty} u^*(B_j)$$

Combined with the property (iii) of u^*, we obtain the countable additivity of u^* on $\mathcal{F}^*$, namely the property (d) for a measure

$$u^*(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} u^*(B_j)$$

The proof of theorem 2 is complete.

Lemma 3: If u_1 and u_2 are two finite measure on $\sigma(\mathcal{E})$,

$\mathcal{E}$ is a π-system, $u_1|_{\mathcal{E}} = u_2|_{\mathcal{E}}$, then $u_1 = u_2$

Using lemma 2 (ii) $\sigma(\mathcal{E}) \subseteq \mathcal{F}^*$

Define $\bar{u} = u^*|_{\sigma(\mathcal{E})}$ Then based on lemma 2.(iii), $\bar{u}$ is a measure

on $(\Omega, \sigma(\mathcal{E}))$, using lemma 1 (iv), $\bar{u}|_{\mathcal{E}} = u$.

Thus $\bar{u}$ is an extension measure on $(\Omega, \sigma(\mathcal{E}))$ of u.

Uniqueness Assume than $u_1|_{\mathcal{E}} = u_2|_{\mathcal{E}} = u$

Define $G = \{ A \in \sigma(\mathcal{E}) : u_1(A_n \cap A) = u_2(A_n \cap A) \}$ $n \geq 1$

Based on Lemma 3. $u_1|_{A_n \cap \sigma(\mathcal{E})} = u_2|_{A_n \cap \sigma(\mathcal{E})}$

As $\Omega = \sum_{n=1}^{\infty} A_n$, we have $u_1 = u_2$