L1 - INTRODUCTION TO DECISION ANALYSIS

Today we will talk about consistency. **Consistency** means the following four requirements.

PROSPECTIVE ORIENTATION

A choice between alternatives should depend only on their consequences.

> "We owe them something. We will finish the task that they gave their lives for."
> — George W. Bush

e.g.

1. Ordering too much food and eating it even if you don't like it.
2. Keep digging for gold even though there probably isn't any.
3. Sitting through a bad movie.
4. Sunk cost.

TRANSITIVITY

(About preferences, $\succ$ means a complete and transitive preference $\succ$ over X)

If $A \succ B$ and $B \succ C$ then $A \succ C$.

e.g. A violation example, the money pump. $A \succ B, B \succ C$ and $C \succ A$.

Owns	Trades for	Pays	Total Cost
A	$A \to C$	€1	€1
C	$C \to B$	€1	€2
B	$B \to A$	€1	€3
A	$A \to C$	€1	€4
C	$C \to B$	€1	€5
...	...	...	...

INVARIANCE

Identical options should lead to identical choices, regardless of how the options are framed.

e.g. 75% lean $\succ$ 25% fat

e.g. Asian Disease Problem: If nothing is done to prevent an infectuous disease 600 people might die. There are two alternative programmes.

1. Option A: 200 lives will be saved.
2. Option B: $\frac{1}{3}$ prob. of saving all 600 and $\frac{2}{3}$ prob. that no one will be saved.

Out of 152 people 72% preferred option A.

INDEPENDENT FROM IRRELEVANT ALTERNATIVES

Whether A is preferred to B or vice versa should not be affected by the inclusion or exclusion of a third alternative C.

e.g. Take a Walk $\succ$ Watch TV $\succ$ do Laundry. But after waching machine broke: Watch TV $\succ$ Take a Walk.

L2 - OBJECTIVES AND ATTRIBUTES

Andreas Paulsson, September 11, 2024

ASSIGNMENT

Assignment: Conduct a decision analysis to support the selection of a new office location for a data science consultancy firm in Stockholm.

- **Number of Alternatives**: 3 (Link provided in the file)
- **Number of Objectives**: 3-5 (A broad objective tree or 3-5 fundamental objectives)
- **Group Work**: Work in pairs
- **Deliverables**:
    - Submit a report and present in a seminar.
    - Submit a reflection report (maximum 4000 words).
- **Style Requirements**: The report should be concise but engaging, meaningful, and realistic (some degree of fakeness is permissible).

OBJECTIVES

Definitions

（目标的定义：我们期望达到的结果 🎯）

- A desired outcome.
- Generally indicates the "direction" we should strive to improve.
- A specific goal we aim to achieve.
- A target towards which efforts are directed: an aim, goal, or end of action.
- A desired future state with particular necessary characteristics.

Usages

（目标的用途：不仅为了完成某个任务，还可以在决策过程中提示隐藏目标📦、发现决策机会 💰、引导信息的收集 💻、生成和评估替代方案）

According to *Value-Focused Thinking* (Keeney, 1992):

- Uncover hidden objectives.
- Identify decision-making opportunities.
- Guide information collection.
- Generate and evaluate alternatives.
- ...

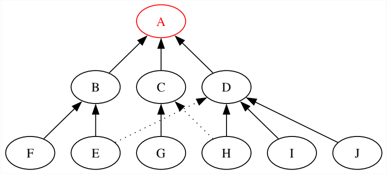
Types

（手段目标：实现其他目标的途径；根本目标：本身就是目的）

Fundamental objectives represent the core reasons why we care about a decision, while means objectives are important because they help achieve the fundamental objectives.

- **Means Objectives**: Tools for achieving other objectives.
- **Fundamental Objectives**:
  - Not a means to achieve something else.
  - Related to the specific decision context.

Hierarchies



The above image was generated using the following code:

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from graphviz import Digraph

dot = Digraph()

dot.graph_attr["rankdir"] = "BT"

dot.node("A", color = "red", fontcolor = "red")
dot.node("B")
dot.node("C")
dot.node("D")
dot.node("E")
dot.node("F")
dot.node("G")
dot.node("H")
```

```
dot.node("I")
dot.node("J")

dot.edge("B", "A")
dot.edge("C", "A")
dot.edge("D", "A")
dot.edge("E", "B")
dot.edge("F", "B")
dot.edge("G", "C")
dot.edge("H", "D")
dot.edge("I", "D")
dot.edge("J", "D")

dot.edge("H", "C", style = "dotted")
dot.edge("E", "D", style = "dotted")

dot.render("graph", format = "svg", view = False,
cleanup = True)
```

（层次结构：自上而下 / 自下而上 / 两者结合 ↕️）

Hierarchies help explain why certain objectives are important and provide details on others.

- Why does this objective matter to me?
- What exactly makes it significant?

This is a **description-oriented** approach.

Methods to construct a hierarchy:

- **Top-down**.
- **Bottom-up**.
- **Combined approach** (top-down and bottom-up).

Typically, as we move up the hierarchy, agreement among stakeholders increases.

Means-Ends Networks

（图形化的方式表示目标之间的关系，经常是一棵树 🌳）

A means-ends network is a graphical representation of relationships between objectives, often structured like a tree.

- One objective can serve as a means to multiple objectives.
- These networks help generate alternative solutions.

Action-Oriented.

Generating Objectives

（产生方式）

- Identifying shortcomings.
- Comparing available alternatives.
- Setting strategic goals.
- Adhering to external guidelines.
- Involving relevant stakeholders.

This process involves deliberation supported by decision analysis.

Recursive Modeling

（递归建模为目标提供了框架，使决策者能够通过反复分析、调整模型来接近目标，从而做出最佳决策）

Recursive modeling provides a framework for continuously refining objectives and improving decision-making through iterative analysis and model adjustments.

- Modeling:
  - Objectives.
  - Alternatives.
  - Preferences.
- Evaluation.

This is a deliberation process supported by decision analysis.

ATTRIBUTES

（Attributes（属性）是评估目标达成程度的具体标准。每个属性类型都与 objective 有不同的衡量关系）

Attributes are specific criteria used to measure how well objectives are being achieved. Each attribute type corresponds to a different relationship with the objective.

- **Natural Attributes**: Measure the achievement of the objective.
- **Artificial Attributes**: Constructed metrics, such as indexes.
- **Proxy Attributes**: Indicators correlated with objectives, often tied to means objectives.

Desirable Properties of Attributes:

- **Direct**.
- **Operational**.
- **Understandable**.
- **Comprehensive**.
- **Unambiguous**.

L3 - STRUCTURING DECISION PROBLEMS

Instructor: Andreas Paulsson

Date: September 18th, 2024

COMPONENTS OF DECISION MODELS

1. **Alternatives:** Possible paths forward or choices.
2. **Consequences:** Outcomes that result from choosing an alternative.
3. **Uncertainty:** Unpredictability regarding the consequences.
4. **Preferences:** How the decision-maker values different consequences.
5. **Objectives:** Goals or criteria that guide decision-making.

TYPES OF DECISION PROBLEMS

1. **Optimization Problem:** Finding the best alternative according to specific criteria.
2. **Choice Problem:** Selecting from a set of alternatives (e.g., The Group Assignment).
3. **Satisfaction Problem:** Identifying alternatives that satisfy certain constraints or criteria.

ALTERNATIVES

- **Definition:** Possible courses of action or solutions.
- **How to find alternatives:**
  - **Search for favorite alternatives** (e.g., The Group Assignment).
  - **Generate alternatives** through a goal-oriented process.
  - **Use objectives as guides** to find ideal alternatives.
  - **Context enlargement:** Expanding the scope to find better alternatives.
  - **Documentation of tasks:** Divide tasks into subcomponents and solve them individually before combining results.

When to Stop Deliberating:

- Time constraints.
- Resource limitations.
- Another decision renders further deliberation unnecessary.

- Preselection of alternatives (with options for re-selection if necessary).

Set of Alternatives

- Denoted as: $A = (a, b, c, \cdots)$ or $A = (a_1, a_2, \cdots, a_n)$.
- **Mutually exclusive:** If $a \in A$ and $b \in A$, a and b are mutually exclusive.
- The set A must contain at least two alternatives.
- **Coarsening of continuous variables:** Simplifying continuous variables into discrete alternatives.

One-Level & Multi-Level Alternatives

- **One-Level Alternatives:** A single decision (e.g., selecting an office location).
- **Multi-Level Alternatives:** Future decisions or predetermined actions requiring forward planning.

CONSEQUENCES

- **Complexity of Consequences:** Consequences can vary in complexity, such as the quantity of an outcome or its utility.

PREFERENCES

Preference Relations:

- **Strict preference:** $a \succ b$ (a is preferred to b).
- **Indifference:** $a \sim b$ (indifference between a and b).
- **Weak preference:** $a \succeq b$ (a is preferred to b or there is indifference).
- **Dependence on outcomes:** Preferences over alternatives depend on the desirability of the outcomes.

Strength of Preferences:

- **Indifference:** $a \sim b$ (no preference between a and b).
- **Strict preference:** $a \succ b$.
- **Much stronger preference:** $a \gg b$.
- **Comparative strength:** a is more preferred to b than b to c.

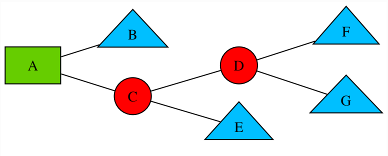
Modeling Preferences:

- **Value Functions:** Assign a value to each alternative.
- **Utility Functions:** Quantify the satisfaction or utility derived from each alternative.
- **Intervals and Differences:** Representing preferences across different levels or intervals.

DECISION TREES

Decision trees visually represent the structure of decision problems. The components include:

- **Green Square:** Objectives.
- **Red Circle:** Choices.
- **Blue Triangle:** Attributes (consequences, criteria).



DECISION MATRIX

Example:

Objectives	o_1	o_2	o_3	$\cdots$
Weights	$w(o_1)$	$w(o_2)$	$w(o_3)$	$\cdots$
A	$v_{o_1}(A)$	$v_{o_2}(A)$	$v_{o_3}(A)$	$\cdots$
B	$v_{o_1}(B)$	$v_{o_2}(B)$	$v_{o_3}(B)$	$\cdots$

THE ADDITIVE MODEL

The value of an alternative is computed as the sum of weighted attributes:

$$v(a) = \sum_{r=1}^{m} w_r v_r(a_r) \tag{1}$$

Where:

- $v(a)$ is the value of alternative a.
- w_r is the weight of the r-th attribute.
- The weights must sum to 1:

$$\sum_{r=1}^{m} w_r = 1 \tag{2}$$

COMPUTER SIMULATION

Steps to simulate decision problems:

1. Construct the value function.
2. Generate random numbers over specified intervals while ensuring constraints (e.g., weights sum to 1).
3. Apply the value function to the generated numbers.
4. Compare the value of alternatives.
5. Save and iterate until the results stabilize.

Class Discussion:

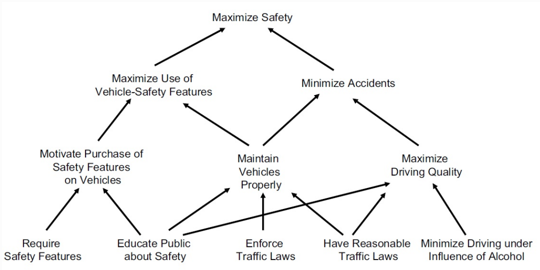
1. **Rent considerations:** Rent values are not reliable for precise decision-making and can be considered in ranges.
2. **Group Assignment:** Groups have distinct roles:
   - One group provides preferences (e.g., Group 07).

- Another group performs decision analysis and writes the report (e.g., Group 22).

L4 - PRESELECTION, VALUE FUNCTIONS

MEASURES AND DECISION COMPOSITION

Combinations of Measures



- **Measures**: Means objectives that:
  - Don't have any means to achieve them - closest to implementation. (叶子节点，直接执行)
  - Not mutually exclusive. (并不互相排斥)
- In the graph above, the five measures are RSF, EPS, ETL, HRL, MDI, Combinations: $2^5 = 32$ combinations

	~~RSF HRL~~	~~RSF EPS ETL~~	~~EPS HRL MDI~~
RSF	RSF MDI	~~RSF EPS HRL~~	ETL HRL MDI
EPS	~~EPS ETL~~	RSF EPS MDI	RSF EPS ETL HRL
~~ETL~~	~~EPS HRL~~	RSF ETL HRL	~~RSF EPS ETL MDI~~
~~HRL~~	EPS MDI	~~RSF ETL MDI~~	~~RSF EPS HRL MDI~~
MDI	ETL HRL	~~RSF HRL MDI~~	RSF ETL HRL MDI
RSF EPS	~~ETL MDI~~	EPS ETL HRL	EPS ETL HRL MDI
~~RSF ETL~~	~~HRL MDI~~	~~EPS ETL MDI~~	RSF EPS ETL HRL MDI

- ETL, HRL dependent. One useless without the other.

- RSF, EPS dependent. One useless without the other.

<table>
<tr><td>EPS</td><td rowspan="2">EPS MDI
ETL HRL</td><td>RSF EPS MDI</td><td>ETL HRL MDI
RSF EPS ETL HRL</td></tr>
<tr><td>MDI
RSF EPS</td><td>EPS ETL HRL</td><td>EPS ETL HRL MDI
RSF EPS ETL HRL MDI</td></tr>
</table>

- Preselecting - Culling the Flock: Above are all avaliable combinations among the 32.

Decomposition of Measures

- Divide the decision problem into smaller, more specfic decision problems - **modules**.
- Modules can be woken on independently.
- For each module a limited number of alternatives are selected.
- Finally the modules are combined.

For Example: Hire new Employee, we can decomposite this measure into three modules

1. Which applicant to choose?
2. Salary?
3. Job description?

DECISION CRITERIA

Knock-Out Criteria (一票否决)

Restrictions:

1. Criteria that must be met.
2. No balcony - no moving in.
3. Danger of missing alternatives (an apartment could be almost perfect in everything except the missing balcony)

Dominance (支配性)

- No risk of removing good alternatives.
- With uncertainty. Say interest rate between -1% and 10%.
- If A is never better than B for any interest rate, B dominates A.
- Remove dominated alternatives.
- More than one objective: A is dominated by B if A isn't better than B with respect to any attribute but worse with respect to at least one.

PREFERENCES AND VALUE FUNCTIONS

Preferences

- a is preferred over b: $a \succ b$
- The decision maker is indifferent between a and b: $a \sim b$

Value Function

gives the value of an alternative - $v(a)$

$v(a) > v(b)$ if and only if $a \succ b$

$v(a) = v(b)$ if and only if $a \sim b$

Ordinal Value Functions

- Strengths of Preferences are not reflected.
- Non-Measurable
- $v(a) = 200, v(b) = 0$ means the same as $v'(a) = 10, v'(b) = -5$
- Averages of ordinal value functions are meaningless.

Requirements

- If preferences are complete and transitive there is a value function
- Complete: for any two alternatives a and b, $a \succ b$ or $a \sim b$ or $a \prec b$
- Transitive: if $a \succ b$ and $b \succ c$ then $a \succ c$

Measurable Value Functions (Cardinal Value Functions)

- Value function that do reflect the strength of preferences.
- $a \succ b$ very much and $c \succ d$ not so much. With a measurable value function $v(a) - v(b) > v(c) - v(d)$

- To exchange a for b, transition from a to b: $a \to b$
- $v(b) - v(a) > v(d) - v(c)$ if and only if $(a \to b) \succ (c \to d)$

Example - The Gangrene Patient

The Gangrene Patient, possible outcomes:

- dead now (dn)
- dead later (dl)
- amputated above the knee (aa)
- amputated below the knee (ab)
- fully recovered (fr)

If

$$fr \succ ab \succ aa \succ dl \succ dn \tag{1}$$

Then the value functions

$$\begin{array}{ccccc} v(dn) = 0 & v(dl) = 1 & v(aa) = 2 & v(ab) = 3 & v(fr) = 4 \\ v'(dn) = -50 & v'(dl) = -10 & v'(aa) = 2 & v'(ab) = 3 & v'(fr) = 100 \end{array} \tag{2}$$

both reflect the preferences.

$v(aa) - v(dn) = 2 - 0 = 2 > v(fr) - v(ab) = 4 - 3 = 1$, so if v is measurable, $(dn \to aa) \succ (ab \to fr)$. One-legged much better than being dead. 😅

$v'(fr) - v'(ab) = 100 - 3 = 97 > v'(aa) - v'(dn) = 2 - (-50) = 52$, so if v' is measurable, $(ab \to fr) \succ (dn \to aa)$. Going from dead to alive beats any other transition.

METHODS FOR DETERMINING VALUE FUNCTIONS

1. Direct rating, just ask
2. Beginning with the worst and best consequence x^- and x^+
3. Order by preference
4. Assign points in order
5. Normalise so that $0 \leq v \leq 1$

Normalization

Example: we have

$$v'(dn) = -50, \quad v'(dl) = -10, \quad v'(aa) = 2, \quad v'(ab) = 3, \quad v'(fr) = 100 \tag{3}$$

Transform these value go from 0 to 1:

$$-50 + 50 = 0, \quad -10 + 50 = 40, \quad 2 + 50 = 52, \quad 3 + 50 = 53, \quad 100 + 50 = 150 \tag{4}$$

Then divide all the above values with 150:

$$\frac{0}{150} = 0, \quad \frac{40}{150} = 0.267, \quad \frac{52}{150} = 0.347, \quad \frac{53}{150} = 0.352, \quad \frac{150}{150} = 1 \tag{5}$$

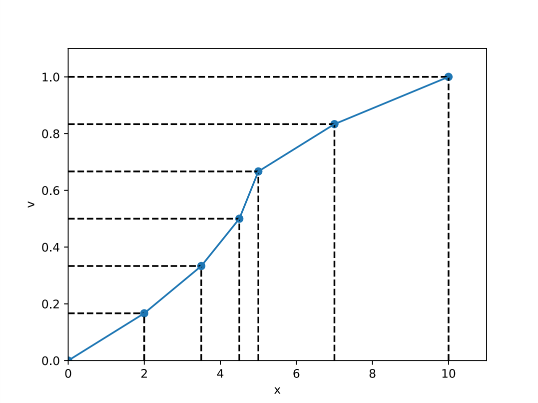
i.e.

$$v''(dn) = 0, \quad v''(dl) = 0.267, \quad v''(aa) = 0.347, \quad v''(ab) = 0.352. \quad v''(fr) = 1 \tag{6}$$

Difference Standard and Bisection

Difference Standard Method

1. Assume continuous attribute.
2. Need €5 for food. Possible range of income is between €0 and €10
3. Make a fifth of the range.
   1. The first increase: €2, let $x_! = 2$, $v(€2) = 1$, standard
   2. Find x_2 such that $(€0 \to €2) \sim (€2 \to €x_2)$
   3. Say $x_2 = €3.5$, $(€0 \to €2) \sim (€0 \to €3.5)$, $v(€3.5) = 2$
   4. Find x_3 such that $(€2 \to €3.5) \sim (€3.5 \to x_3)$, get $x_3 = 4.5$, $v(€4.5) = 3$
   5. Find x_4 such that $(€3.5 \to €4.5) \sim (€4.5 \to €5)$, get $x_4 = 5$, $v(€5) = 4$
   6. Find x_5 such that $(€4.5 \to €5) \sim (€5 \to €7)$, get $x_5 = 7$, $v(€7) = 5$
   7. Find x_6 such that $(€5 \to €7) \sim (€7 \to €10)$, get $x_6 = 10$, $v(€10) = 6$
4. Normalize $0, 1, 2, 3, 4, 5, 6$ to $0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6} = 1$



Bisection Method

1. Find $x_{0.5}$ such that $(x^- \to x_{0.5}) \sim (x_{0.5} \to x^+)$
2. Find $x_{0.25}$ such that $(x^- \to x_{0.25}) \sim (x_{0.25} \to x_{0.5})$
3. Find $x_{0.75}$ such that $(x_{0.5} \to x_{0.75}) \sim (x_{0.75} \to x^+)$
4. Find $x_{0.125}$ such that $(x^- \to x_{0.125}) \sim (x_{0.125} \to x_{0.25})$
5. Find $x_{0.375}$ ...

Comparation

Direct Rating: easy on the analyst, hard on the decision maker

Difference standard and bisection methods: a little more advanced, easier for the decision maker

L5 - MULTIATTRIBUTE VALUE FUNCTIONS

Many Attributes, One Value Function, for $v_1(a_1), v_2(a_2), \cdots, v_n(a_n)$:

$$\begin{aligned} & v(a_1, a_2, \cdots, a_n) = w_1 v_1(a_1) + \cdots + w_n v_n(a_n) = \sum_{r=1}^{n} w_r v_r(a_r) \\ & w_1, w_2, \cdots, w_n > 0, \quad \sum_{r=1}^{n} w_r = 1 \end{aligned} \tag{1}$$

EXAMPLE - THE ROAD

Attributes Levels

	West	**Central**
Cost	$v_c(21) = 0.95$	$v_c(27) = 0.61$
Time	$v_t(55) = 0.01$	$v_t(30) = 0.14$
Emissions	$v_e(18) = 0.11$	$v_e(20) = 0.06$

Weighting them Together

Arbitrary weights:

$$w_c = 0.5,\ w_t = 0.2,\ w_e = 0.3 \tag{2}$$

Multiattribute Value for the western option:

$$v(21, 55, 18) = 0.5 \times 0.95 + 0.2 \times 0.01 + 9,3 \times 0.11 = 0.51 \tag{3}$$

Multiattribute Value for the central option:

$$v(27, 30, 20) = 0.5 \times 0.61 + 0.2 \times 0.14 + 0.3 \times 0.06 = 0.35 \tag{4}$$

So the western position of the road has the highest value.

INDEPENDENCE

Preferential Independence (偏好独立)

任意一个属性，偏好独立于其它属性

- Definition: $a = (a_1, a_2, a_3)$, $b = (a_1, b_2, a_3)$ and $c = (c_1. a_2, c_3)$, $d = (c_1, b_2, c_3)$
  If $a \succ b \iff c \succ d$ then attribute 2 is preferentially independent from the other attributes.
- No need to know how the environment is effected to evaluate the cost.

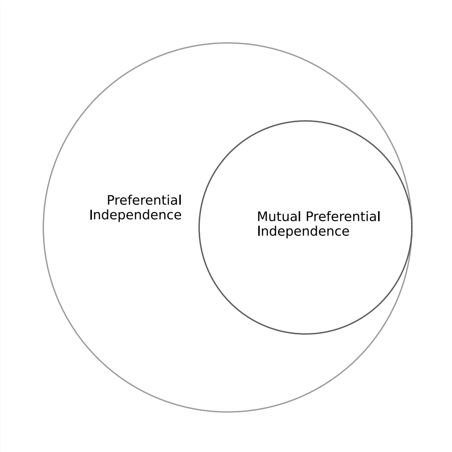
Mutual Preferential Independence (相互偏好独立)

任意若干属性，偏好独立于其它属性

- Definition: Every subset of the attributes is oreferentially independent from the other attributes.
- **Not enough for measurable value function**

Relationships

Mutual Preferential Independence ⇒ Preferencial Independence ⇒ Additive Value Function



PARAMETER METHOD

- Difference Independence (差异独立):
  $((a_1, a_2) \to (b_1, a_2)) \sim ((a_1, c_2) \to (b_1, c_2))$
- Measurable (可测量的价值函数，即有一个函数值)
- Prop. Difference Independence ⇒ measurable value function

The Trade-off Method

- Need individual value functions
- Find artficial alternatives that differ in only two attributes like $(a, b, c) \sim (d, e, c)$
- Say that $(a, b, c, d, e) \sim (f, b, c, d, e)$
- Then $v(a, b, c, d, e) = v(f, b, c, g, e)$ and if $v(\cdot)$ is additive:

$$\begin{aligned} &w_1v_1(a) + w_2v_2(b) + w_3v_3(c) + w_4v_4(d) + w_5v_5(e) = \\ &w_1v_1(f) + w_2v_2(b) + w_3v_3(c) + w_4v_4(g) + w_5v_5(e) \end{aligned} \tag{5}$$

⇒

$$w_1v_1(a) + w_4v_4(d) = w_1v_1(f) + w_4v_4(g) \tag{6}$$

$v_1(a)$, $v_1(f)$, $v_4(d)$, $v_4(g)$ are known so we can solve for w_1 or w_4:

$$w_1 = \frac{v_4(g) - v_4(d)}{v_1(a) - v_1(f)} w_4 \tag{7}$$

- With three more indifferent statements we have enough relations between weights so that together with $w_1 + w_2 + w_3 + w_4 + w_5 = 1$. (In general: one less comparsion than number of attributes)

The Swing Method

- No need for individual value functions.
- Say four attributes, range x_i^- and x_i^+ best for the i-th attribute
- Five artificial alternatives

$$\begin{aligned} &a = (x_1^-, x_2^-, x_3^-, x_4^-) \\ &b_1 = (x_1^+, x_2^-, x_3^-, x_4^-), \quad b_2 = (x_1^-, x_2^+, x_3^-, x_4^-), \\ &b_3 = (x_1^-, x_2^-, x_3^+, x_4^-), \quad b_4 = (x_1^-, x_2^-, x_3^-, x_4^+) \end{aligned} \tag{8}$$

- Zero points to $(x_1^-, x_2^-, x_3^-, x_4^-)$ rank b_1, b_2, b_3 and b_4 and give 100 points to whichever is best
- Give points between 0 and 100 to the other three alternatives according to the ranking
- Normalize the points

$$W_i = \frac{\text{points for } b_i}{\text{points for } a + \text{points for } b_1 + \text{points for } b_2 + \text{points for } b_3 + \text{points for } b_4} \tag{9}$$

L6 - MORE ABOUT PROBABILITIES

CONDITIONAL PROBABILITY

$$\begin{aligned} p(A,B) &= p(A|B)p(B) \\ p(A,B) &= p(B|A)p(A) \end{aligned} \tag{1}$$

Bayes Rule

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} \tag{2}$$

The Law of Total Probability

Let $B_1, B_2, \cdots$ be mutually exclusive and collectively exhaustive.

$$\begin{aligned} p(A) &= \sum_{i\geq 1} p(A \cap B_i) \\ &= \sum_{i\geq 1} p(A|B_i)p(B_i) \end{aligned} \tag{3}$$

Bayes Rule with Extended Denominator

$$\begin{aligned} p(A|B) &= \frac{p(B|A)p(A)}{p(B)} \\ &= \frac{p(B|A)p(A)}{p(B|A)p(A) + p(B|\overline{A})p(\overline{A})} \end{aligned} \tag{4}$$

if

$$B \subset \bigcup_{i=1}^{n} A_i \tag{5}$$

then

$$p(A_i|B) = \frac{p(B|A_i)p(A_i)}{\sum_{i=1}^{n} p(B|A_i)p(A_i)} \tag{6}$$

L7 - UTILITY THEORY

Expected Value

Alternative a with n possible consequence $a_!, a_2, \cdots$ s.t. $p(a_1) = p(a_2) = \cdots$

$$\mathrm{EV}(a) = \sum_{i \geq 1} p(a_i) a_i \tag{1}$$

Value is only about consequences. Not consideration for risk.

THE SAINT PETERSBURG LOTTERY

Place a bet. Toss a coin.

- If tails, win €2 and game over.
- If heads, no payment, new coin toss.
  - If tails, win €4 and game over
  - If heads, no payment, new coin toss.
    - ...

The organizer keeps tossing the coin until it lands tails up. Then you are paid €$2^{\text{number of tosses}}$

The Saint Petersburg Paradox

Decision alternatives: save the money or play the game. Which has the highest expected value?

i tails in a row

$$P(i \text{ tails in a row}) = \frac{1}{2^i} \tag{2}$$

Expected value

$$\mathrm{EV}(\text{playing}) = \sum_{i \geq 1} \frac{1}{2^i} \times 2^i = +\infty \tag{3}$$

The series does not converge.

Thus, no matter how high the stake is, it's worth to play the game.

Solution by Bernoulli himself

Logarithmic value function

$$v(€4) = 2, \quad v(€8) = 3, \quad \cdots \tag{4}$$

Expected Value

$$\mathrm{EV}(\text{playing}) = \sum_{i \geq 1} \frac{1}{2^i} \times i = 2 \tag{5}$$

This means the problem remains with new game - pay not €2^i but €2^{2^i}

UTILITY: VON NEUMANN AND MORGENSTERN

Utility Theory: 效用理论

Utility takes into account your attitude to the value of the consequence as well as your attitude to risk.

Axioms of Utility Theory

If and only if preferences fulfill the axioms（公理）does a utility function exists and is it rational to maximise expected utility.

The axioms are all about preferences regarding lotteries:

1. Complete Ordering:
    1. Completeness: Any pair of lotteries is comparable, for all lotteries a and b, $a \succeq b$ or $b \succeq a$
    2. Trainsitivity: if $a \succeq b$ and $b \succeq c$ then $a \succeq c$
2. Continuity: If $a \succeq b$ and $b \succeq c$ there is a probability p such that
$$b \sim pa + (1-p)c \tag{6}$$
$pa + (1-b)c$, or written $(a, p; c, 1-p)$ is called a compoundlottery
3. Independence: if $a \succeq b$ then for all lotteries c and all probabilities p
$$pa + (1-p)(c) \succeq pb + (1-p)c \tag{7}$$

Axioms and Expected Utility

a has n consequences a_i, $i = 1, 2, \cdots, n$. Given $x_{\max}$ and $x_{\min}$, $x_{\max} \succeq a_i \succeq x_{\min}$ by continuity there is q_i such that

$$a_i \sim (x_{\max}, q_i; x_{\min}, 1-q_i) \tag{8}$$

By independence we can substitute a_i for $(x_{\max}, q_i; x_{\min}, 1-q_i)$

All lotteries b can be made into b''. If $a \succeq b$ then $a'' \succeq b''$

That of a'' or b'' that gives the highest probability for $x_{\max}$ is the optimal.

Letting $u(a_i) = q_i$, $p(x_{\max}) = \sum_{i=1}^{n} p_i u(a_i)$, the expected utility of a_i is $a_i \sim (x_{\max}, q_i; x_{\min}, 1-q_i)$

Savages' Subjective Expected Utility

Probabilities not necessarily given. May be elicited by some indirect method.

Subjective expected utility theory is characterised by its independence axiom.

THE SURE THING PRINCIPLE

Set of events S, $S' \subset S$, an event is either in S' or in $S \backslash S'$

If for all $s \in S'$, $a(s) = a'(s)$ and $b(s) = b'(s)$.

And $a(s) = b(s)$ and $a'(s) = b'(s)$ for $s \in S \backslash S'$, then $a \succeq b$ if and only if $a' \succeq b'$

CERTAINTY EQUIVALENT

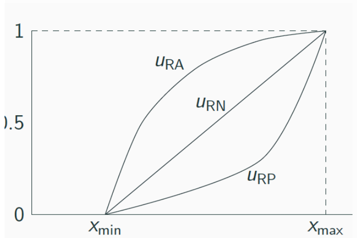
How much certain outcome is a lottery worth?

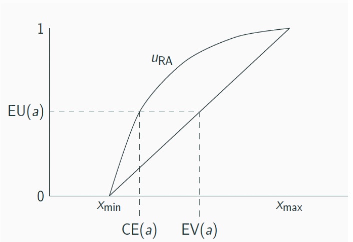
If a is a lottery, a function $\mathrm{CE}(a)$, or $u[\mathrm{CE}(a)] = \mathrm{EU}(a)$

$$\mathrm{RP}(a) = \mathrm{EV}(a) - \mathrm{CE}(a) \tag{9}$$

Say that $a = (€100, 0.5; €0, 0.5)$ and $\mathrm{CE}(a) = 40$. Even though $\mathrm{EV}(a) = €50$, a is only worth $€40$ to the decision maker.

The difference, $\mathrm{EV}(a) - \mathrm{CE}(a) = 50 - 40 = 10$ is the risk premium, that it is positive means risk aversity (风险厌恶).





Utility Function	$\mathrm{RP} = \mathrm{EV} - \mathrm{CE}$	**Risk Attitude**
concave	> 0	risk aversity
linear	$= 0$	risk neutral
convex	< 0	risk prone

BISECTION METHOD

QUANTILE, p VARIES

THE TRADE-OFF METHOD

L8 - RISK AND MULTIPLE OBJECTIVES, DESCRIPTIVE DECISION THEORY

Imprecise Probability

Give an interval instead of a precise value

$$P(I) = [p_{\min}, p_{\max}] \tag{1}$$

Interval Expected Utility (Example)

Say that $p_1 \in [0.2, 0.4]$ and $p_2 \in [0.5, 0.8]$ and $p_3 = 1 - p_1 - p_2$ and $u_1 = 0.7$, $u_2 = 0.9$, $u_3 = 0.4$

$$\begin{aligned} \mathrm{EU} &= p_1 u_1 + p_2 u_2 + p_3 u_3 \\ &= 0.7p_1 + 0.9p_2 + 0.4(1 - p_1 - p_2) \\ &= 0.3p_1 + 0.5p_2 + 0.4 \end{aligned} \tag{2}$$

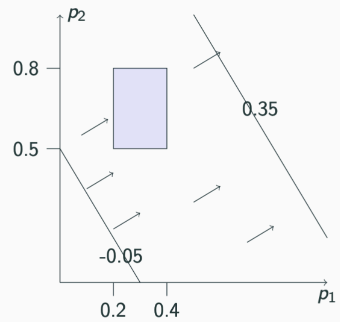
Dominance

$$a \succeq b \iff \sum_{i=1}^{n} p_i u(a_i) \geq \sum_{i=1}^{n} p_i u(b_i) \tag{3}$$

for all $p \in P(I)$

Linear Programming

$p_1 \in [0.2, 0.4]$ and $p_2 \in [0.5, 0.8]$ and
$\mathrm{EU}(a) - \mathrm{EU}(b) = 0.3p_1 + 0.5p_2 + 0.4 - (-0.2p_1 + 0.2p_2 + 0.6) = 0.5p_1 + 0.3p_2 - 0.2.$



The EU is constant along parallel lines. The EU increases in a direction perpendicular to these lines.

Stochastic Dominance

a stochastically dominates b if for every consequence x

$$p(\text{outcome} \leq x \mid a) \leq p(\text{outcome} \leq x \mid b) \tag{4}$$

and for at least one consequence y

$$p(\text{outcome} \leq y \mid a) < p(\text{outcome} \leq y \mid b) \tag{5}$$

Risk Profiles and Cumulative Risk Profiles

A risk profile shows the probabilities of the possible consequences for a given decision strategy.

A cumulative risk profile shows the corresponding cumulative probabilities.

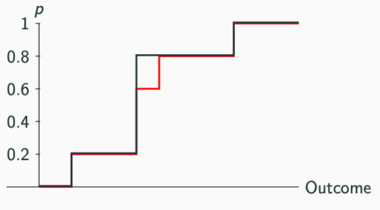
For example, if

n	$P_r(X = n)$
1	0.2
2	0.1
3	0.4
4	0.3

then

n	$P(X \leq n)$
1	0.2
2	$0.2 + 0.1 = 0.3$
3	$0.2 + 0.1 + 0.4 = 0.7$
4	$0.2 + 0.1 + 0.4 + 0.3 = 1$

There is stochastic dominance only if the risk profiles do not cross.



Sensitivity Analysis

Let a value vary, see what happens with EU.

When the Map doesn't Match the Terrain

See pp. 382-387 in Eisenführ

- Preference not according to theory
- Probabilities misreated
- Change Behavior
- Compensate
- Adapt