Regular graphs with extremal rigidity properties - a new section

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1 Introduction

Motevallian, Yu, and Anderson [1] investigated the family of strongly minimally 3-vertex rigid graphs $G = (V, E)$. They proved that they satisfy $|E| \geq 2|V| + 2$, assuming $|V| \geq 6$, and suggested a construction method which can be used to obtain a strongly minimally 3-vertex rigid graph on $|V|$ vertices, for each $|V| \geq 6$, for which equality holds. These graphs arise from $K_6 - e$ by a sequence of 2-extensions executed on edge pairs whose endvertices satisfy certain degree conditions. The main proof in [1], establishing that these 2-extensions preserve 3-vertex rigidity, has a major gap. In order to fill in this gap one needs an extension of Servatius' theorem to those 2-vertex rigid graphs that satisfy the $|E| = 2|V|$ count.

In this section we shall complete the proof of [1] by providing such an extension, which may be of independent interest. We shall also give a different, simpler construction.

Let $G = (V, E)$ be a graph and $X \subseteq V$. We shall use $G[X]$ to denote the subgraph of G induced by X. Let $S_G(X)$ denote the set, and $s_G(X)$ the number of edges incident with X. Thus $s_G(X) = i_G(X) + d_G(X)$.

2 Preliminaries

Lemma 1. *Suppose that $G = (V, E)$ is rigid. Then for every $X \subseteq V$ with $|V - X| \geq 2$ we have $s_G(X) \geq 2|X|$.*

Proof. Let $H = (V, F)$ be a minimally rigid spanning subgraph of G. The sparsity of H implies that $i_H(V - X) \leq 2|V - X| - 3$, which gives $s_G(X) \geq s_H(X) \geq |F| - i_H(V - X) \geq 2|V| - 3 - (2|V - X| - 3) = 2|X|$. $\square$

If $G[X]$ is a tree for which $d_G(x) = 3$ for all $x \in X$ then we call $G[X]$ a *cubic subtree* of G.

Lemma 2. *Let $G = (V, E)$ be a graph and let $G[X]$ be a cubic subtree of G for which $G[V - X]$ is rigid. Then for each pair $e, f \in S_G(X)$ we have that (i) $G - e$ is rigid, and (ii) $G - \{e, f\}$ is not rigid.*

Proof. It is not hard to see that $G - e$ can be obtained from $G[V - X]$ by a sequence of 0-extensions. This proves (i). Furthermore, the fact that $G[X]$ is a cubic tree implies $s_{G-\{e,f\}}(X) \leq 3|X| - (|X| - 1) - 2 = 2|X| - 1$. By Lemma 1 this gives (ii). $\square$

3 2-vertex rigid graphs revisited

Deleting two edges e, f from a 2-vertex rigid graph $G = (V, E)$ may destroy rigidity. It happens, for example, when e, f belong to a nontrivial 4-edge cut (cf. Lemma) or when e, f are incident with a cubic subtree (cf. Lemma 2(ii)). The next theorem asserts that there is no other case if $|E| = 2|V|$ holds. Note that Servatius' theorem asserts that the same holds if $|E| = 2|V| - 1$: in this case there exist no nontrivial 4-edge cuts (Lemma) and the cubic subtrees have at most two vertices.

Theorem 3. *Let $G = (V, E)$ be a 2-vertex rigid graph with $|E| = 2|V|$ and let $e, f \in E$. Then one of the following holds:*
(i) $G - \{e, f\}$ is rigid,
(ii) there is a nontrivial 4-edge cut X in G with $\{e, f\} \subset \delta(X)$,
(iii) there is a cubic subtree $G[X]$ with $\{e, f\} \subset S_G(X)$.

Proof. Suppose that $G - \{e, f\}$ is not rigid. Let $H = G - e$. Since G is 2-vertex rigid, it is also 2-edge rigid. Hence H is a rigid spanning subgraph of G with $2|V| - 1$ edges. Thus H can be obtained from a minimally rigid spanning subgraph B of G by adding two edges, say h_1, h_2. Let C_i be the unique rigidity circuit in $B + h_i$, for $i = 1, 2$. Now $H - f$ is not rigid, which implies, by using that rigidity circuits are 2-edge rigid, that the endvertices of f cannot belong to the same rigidity circuit C_i, $i = 1, 2$. Observe that we cannot have $|(V(C_1) \cap V(C_2)| = 1$. This follows from the fact that each rigidity circuit has minimum degree three, and hence a unique common vertex v satisfies $d_G(v) \geq 6$. But then $G - v$ has too few edges to be rigid, contradicting 2-vertex rigidity. Moreover, the gluing lemma implies that if $|(V(C_1) \cap V(C_2)| \geq 2$ holds, then $C_1 \cup C_2$ is rigid. We shall consider two cases.

Case 1: $V(C_1) \cup V(C_2) = V$.

If $|(V(C_1) \cup V(C_2)| \geq 2$ then $C_1 \cup C_2$ is a rigid spanning subgraph of H. Since $f \notin E(C_1) \cup E(C_2)$, it follows that $H - f$ is rigid, a contradiction. Thus we may assume that $V(C_1) \cap V(C_2) = \emptyset$. Then $\delta_H(V(C_1))$ is a nontrivial edge cut in H. Let $q = |\delta_H(C_1)|$. The rigidity of G implies $q \geq 3$. Furthermore, we have $2|V| - 1 = |E(H)| \geq |E(C_1)| + |E(C_2)| + q = 2|V(C_1)| - 2 + 2|V(C_2)| - 2 + q = 2|V| - 4 + q$, which gives $q \leq 3$. On the other hand, the 2-edge rigidity of G implies $|\delta_G(V(C_1))| \geq 4$. Therefore $\delta_G(V(C_1))$ is a nontrivial 4-edge cut in G with $\{e, f\} \subset \delta_G(V(C_1))$. Hence (ii) holds.

Case 2: $V(C_1) \cup V(C_2) \neq V$.

Let $X = V(G) \setminus (V(C_1) \cup V(C_2))$. Consider a vertex $x \in X$. The 2-vertex rigidity of G implies $d_G(x) \geq 3$. To see that we must have equality suppose that $\deg_G(x) \geq 4$. Then $|E(G - x)| \leq 2|V(G)| - 2$, from which we obtain, by using the 2-vertex rigidity of G again, that $G - x$ is a rigid graph which contains at most one rigidity circuit. It contradicts the choice of x. Thus each vertex in X has degree three in G.

If $V(C_1) \cap V(C_2) = \emptyset$, then for every vertex $v \in V$ there exists a rigidity circuit in $G - v$. An argument similar to that of the previous paragraph gives that in this case we have $d_G(v) \leq 4$, for all $v \in V(G)$. Since $|E| = 2|V|$, it follows that G is 4-regular, which contradicts the existence of the degree-three vertices in X. This we may assume that $|V(C_1) \cap V(C_2)| \geq 2$. So $C_1 \cup C_2$ is rigid.

Claim 4. *$G[X]$ is a cubic subtree of G.*

Proof. We have already verified that each vertex in X has degree three in G. Suppose that for some $Y \subseteq X$ the subgraph $G[Y]$ is a cycle. Then $s_G(Y) \leq 3|Y| - |Y| = 2|Y|$, which gives $s_{G-h}(Y) \leq 2|Y| - 1$ for any edge h of the cycle. Since $G - h$ is rigid, it contradicts Lemma 1. Thus $G[X]$ is a forest. Let us assume that $G[X]$ is disconnected and consider a nontrivial partition $X = X_1 \cup X_2$ for which there is no edge in G from $G[X_1]$ to $G[X_2]$. Then the 2-edge rigidity of G and Lemma 1 yields

$$s_G(X) = s_G(X_1) + s_G(X_2) \geq 2|X_1| + 1 + 2|X_2| + 1 = 2|X| + 2.$$

Hence $|E(C_1 \cup C_2)| = |E| - s_G(X) \leq 2|V(C_1) \cup V(C_2)| - 2$ follows. But this contradicts the fact that $C_1 \cup C_2$ is rigid, and contains at least two rigidity circuits. Thus $G[X]$ is a cubic subtree, as claimed. □

Since $C_1 \cup C_2$ is rigid, $G - e$ is rigid, and $G - \{e, f\}$ is not rigid, we can now use Lemma 2 to deduce that $\{e, f\} \subset S_G(X)$. Hence (iii) holds. □

3.1 Strongly minimally 3-vertex rigid graphs

It was shown in [1] that a 3-vertex rigid graph $G = (V, E)$ with $|E| = 2|V| + 2$ and $|V| \geq 6$ has exactly four vertices of degree five, and these vertices are pairwise adjacent. We say that $W = \{v \in V : d_G(v) = 5\}$ is the *core* of G. All the other vertices are of degree four. They also show that G contains two disjoint edges $ab, cd \in E$ with $a, c \notin W$, $b, d \in W$, and $ac \notin E$. We call such an edge pair *admissible*. The key statement is that a 2-extension on an admissible edge pair preserves 3-vertex rigidity. The proof in [1] settles several simpler subcases but there is a missing case (see the Appendix for more details). With the following result we can complete the proof of their key statement.

Theorem 5. *Let $G = (V, E)$ be a 3-vertex rigid graph with $|E| = 2|V| + 2$ and $|V| \geq 6$. Let W be the core of G, let $ab, cd \in E$ be an admissible edge pair with $b, d \in W$, and let $y \in V - W - \{a, c\}$. Then $G - y - \{ab, cd\}$ is rigid.*

Proof. Let $H = G - y$. Since G is 3-vertex rigid and $d_G(y) = 4$, H is 2-vertex rigid with $|E(H)| = 2|V(H)|$. Let $e = ab$, $f = cd$. We need to show that $G - \{e, f\}$ is rigid. By Theorem 3 it suffices to show that no nontrivial 4-edge cut or cubic subtree of H contains both edges.

First suppose that there is a nontrivial 4-edge cut $\delta_H(X)$ with $\{e,f\} \subset \delta_H(X)$. The 2-vertex rigidity of H implies that the edges in $\delta_H(X)$ are pairwise disjoint. Thus W lies entirely in one of two sides of the cut, say $W \subseteq V - X$. Hence $a, c \in X$. Since G is 3-edge rigid and each vertex of X has even degree in G, it follows that y has exactly two neighbors in X as well as in $V - X$. Thus $d_G(X) = d_G(V - X) = 6$. This shows that $|X| \geq 3$ holds, for otherwise $ac \notin E$ implies $d_G(X) = d_G(a) + d_G(c) \geq 8$, a contradiction. Let $q \in X$ with $yq \notin E$. Then $d_H(q) = 4$ holds and hence $H - q$ satisfies $|E(H - q)| = 2|V(H - q)| - 2$. Now $H - q$ is rigid, so it can be obtained from a sparse graph by adding an edge. On the other hand $W \subseteq V - X$ implies $i_{H-q}(V - X) \geq \frac{4|V-X|+4-6}{2} = 2|V - X| - 1$, a contradiction.

Next suppose that there is a cubic subtree $H[X]$ in H with $\{e,f\} \subset S_G(X)$. Then we must have $d_H(a) = d_H(c) = 3$ and $\{a,c\} \subseteq X \subseteq N_G(y)$. Moreover, since $ac \notin E$, we have $|X| \geq 3$. This implies that there exist two incident edges $jk, kl \in G[X]$. Therefore k and y are two adjacent degree-four vertices in G with two common neighbours: j and l. But then $s_{G-y-k}(\{y,k\}) = 3$, from which Lemma 1 gives that $G - y - k$ is not rigid, a contradiction. □

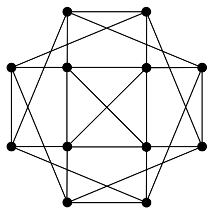

Figure 1: A 3-vertex rigid graph in which no two core vertices have a common neighbour which does not belong to the core.

We remark that not all 3-vertex rigid graphs can be obtained from $K_6 - e$ by a sequence of 2-extensions on admissible edge pairs. See Figure 1. (The example in [1] is not 3-vertex rigid.)

The main result of [1], the proof of which has been completed in this section, implies that the strongly minimally 3-vertex rigid graphs $G = (V,E)$ on at least six vertices satisfy $|E| = 2|V| + 2$. We can provide a direct construction: consider the graphs D_k, $k \geq 3$, with vertex set $\{a_i, b_i : 0 \leq i \leq k-1\}$ and edge set $\{a_i b_{i+1}, b_i a_{i+1} : 1 \leq i \leq k-1\} \cup \{a_0 b_0, a_1 b_1\}$, counting indices modulo k. See Figure 2. Note that $D_3 = K_6 - e$. It is easy to check these graphs are indeed 3-vertex rigid with $|E| = 2|V| + 2$.

References

[1] S.A. Motevallian, C. Yu, and B.D.O. Anderson, On the robustness to multiple agent losses in 2D and 3D formations, *Int. J. Robust Nonlinear Control*, 2015; 25: 1654 - 1687.

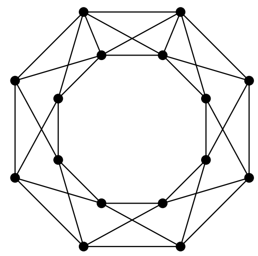

Figure 2: The graph D_8.