ENSIIE. Simulation methods

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$\rightsquigarrow$ The term

Generating a random number x from a r.v. X or
Simulating a realization x of X or
Sampling a random number x from X

consists on mimicking the r.v. in order to generate one possible value (or observation) $X(\omega) = x$ from X.

$\rightsquigarrow$ *Example*. Let X be a Bernouilli random variable with success parameter p: $\mathbb{P}(X = 1) = p$, $\mathbb{P}(X = 0) = 1 - p$.

1. When we sample a random number x from X, $x = 1$ or $x = 0$.
2. When the sample is of size N: $X_1(\omega) = x_1, \ldots, X_N(\omega) := x_N$ are iid with $X_i \stackrel{d}{=} X$, it must be in line with the theoretical results as the Law of Large Numbers: $\bar{X}_N := \frac{X_1 + \ldots + X_N}{N} \stackrel{N \to +\infty}{\longrightarrow} \mathbb{E}(X) = p$, a.s.

$\rightsquigarrow$ There are many simulation techniques: the inversion method, the rejection method, the transformation method, etc ...

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Simulation of r.v.: inversion method

Proposition. Let U be a r.v., uniformly distributed on $]0,1[$ and let X be a r.v. with cumulative distribution function (cdf) F and (generalized) inverse function F^{-1}:

$$F^{-1}(u) := \inf\{x \in \mathbb{R} : \quad F(x) \geq u\}, \qquad \forall u \in]0,1[.$$

Then X and $F^{-1}(U)$ have the same distribution: $X \stackrel{d}{=} F^{-1}(U)$.

Proof. We need to prove that $\forall x \in \mathbb{R},\ \mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x)$. We have

$$\forall u \in]0,1[,\ \forall x \in \mathbb{R}, \qquad F^{-1}(u) \leq x \iff u \leq F(x).$$

Then

$$\begin{aligned} \mathbb{P}(F^{-1}(U) \leq x) &= \mathbb{P}(U \leq F(x)) \qquad (F \text{ in nondecreasing}) \\ &= F(x) \end{aligned}$$

It follows that the cdf of $F^{-1}(U)$ and X are the same.

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The inversion method: discrete r.v.

Let X be a discrete r.v. taking values in $E = \{x_0, \ldots, x_n, \ldots\}$, with cdf F. Suppose that the x_k are ordered in a nondecreasing order and denote, $\forall k \geq 0$, $p_k = \mathbb{P}(X = x_k)$ and $c_k = p_0 + \ldots + p_k$. Then, for all $u \in]0, 1[$,

$$F^{-1}(u) = x_0 \mathbb{1}_{\{u \leq c_0\}} + \sum_{k \geq 1} x_k \mathbb{1}_{\{c_{k-1} < u \leq c_k\}}.$$

$\rightsquigarrow$ *When the cardinality N of E is finite* . We stock the values x_k on a table x and those of c_k on a table c. To generate a sample $X(\omega)$ of X we use the following algorithm (`rand` generate a r.n. from $U \sim \mathcal{U}(]0, 1[)$) :

k $\leftarrow$ 0; u $\leftarrow$ rand
while $(u > c[k])$ and $(k < N)$
$k \leftarrow k + 1$
end
$X(\omega) \leftarrow x[k]$

The inversion method: example of discrete r.v.

$\rightsquigarrow$ *Bernoulli distribution*. Let X be Bernoulli r.v. with success probability $p \in [0,1]$: $\mathbb{P}(X=0) = 1-p$ and $\mathbb{P}(X=1) = p$. In this case,

$$F^{-1}(u) = 0 \times \mathbb{1}_{\{u<1-p\}} + 1 \times \mathbb{1}_{\{1-p \leq u\}} = \mathbb{1}_{\{1-p \leq u\}}.$$

We generate a random number (r.n.) $X(\omega) = x$ from X using the following algorithm:

u $\leftarrow$ rand
if $(u < 1-p)$ $x \leftarrow 0$
else $x \leftarrow 1$

$\rightsquigarrow$ *Poisson distribution*. Let X be a r.v. having a Poisson distribution with parameter $\lambda > 0$, defined as:

$$p_k = \mathbb{P}(X=k) = e^{-\lambda}\,\frac{\lambda^k}{k!}, \qquad k = 0, 1, \ldots$$

We remark that

$$p_k = \frac{\lambda}{k}\, p_{k-1}, \qquad \forall\; k \geq 1.$$

The inversion method: example of discrete r.v.

$\rightsquigarrow$ To generate a r.n. from X, we first stock the values of the cdf $F(n)$, $n \in \{1, 2, \ldots, N\}$, where N is chosen such that $F[N]$ is high (for example $F[N] = 0.999$)

$\rightsquigarrow$ Then, we use the following algorithm ($pN \equiv p_N = \mathbb{P}(X = N)$):

```
u ← rand
if (u ≤ F[N])
   then
      k ← 0
      while (u > F[k]) do
         k ← k + 1
      end
   else
      k ← N, p ← pN, F ← F[N]
      while (u > F) do
         k ← k + 1, p ← λ * p/k, F ← F + p
      end
end
```

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The inversion method: example of continuous r.v.

$\rightsquigarrow$ *The exponential distribution*. If X has an exponential distribution with parameter $\lambda > 0$, with density

$$f(x) = \lambda e^{-\lambda x} \mathbb{1}_{]0,+\infty[}(x),$$

so that its cdf reads

$$F(x) = (1 - e^{-\lambda x}) \mathbb{1}_{]0,+\infty[}(x),$$

then, for any $u \in]0, 1[$, $F^{-1}(u) = -\frac{\ln(1-u)}{\lambda}$, so that if $U \sim \mathcal{U}(]0, 1[)$, then,

$$F^{-1}(U) \; = \; -\frac{\ln(1 - U)}{\lambda} \; \stackrel{d}{=} \; -\frac{\ln(U)}{\lambda} \qquad (\text{since } 1 - U \stackrel{d}{=} U).$$

$\rightsquigarrow$ *The Weibull distribution*. Let X be a Weibull distribution with parameters (λ, a), with density

$$f(x) = \lambda a x^{a-1} e^{-\lambda x^a}, \qquad \lambda, a > 0.$$

Its cdf reads

$$F(x) = (1 - e^{-\lambda x^a})\mathbb{1}_{]0,+\infty[}(x).$$

It follows that for any $u \in]0,1[$, $F^{-1}(u) = \big(-\ln(1-u)/\lambda\big)^{1/a}$, so that if $U \sim \mathcal{U}(]0,1[)$, then,

$$F^{-1}(U) \;=\; \big(-\ln(1-U)/\lambda\big)^{1/a} \;\stackrel{d}{=}\; \big(-\ln(U)/\lambda\big)^{1/a}.$$

$\rightsquigarrow$ As a consequence, if we want to generate a random number from an exponential distribution or a Weibull distribution we just have

- to generate a r.n. $u = U(\omega)$ from a uniform distribution $U \sim \mathcal{U}(]0,1[)$, and
- compute the inverse $F^{-1}(u)$ with respect to the associated distribution.

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R.n. from $U \sim \mathcal{U}(S)$

$\rightsquigarrow$ Let S be Borel set on $\mathbb{R}^d$ and let $U \sim \mathcal{U}(S)$ with density (w.r.t. the Lebesgues measure λ_d): $f(x) = (1/\lambda_d(S))\, \mathbb{1}_S(x)$. For any Borel set $A \subset S$,

$$\mathbb{P}(U \in A) = \int_A \frac{1}{\lambda_d(S)} \lambda_d(dx) = \frac{\lambda_d(A)}{\lambda_d(S)} = \frac{|A|}{|S|}.$$

If $d = 2$, we have

$$\mathbb{P}(U \in A) = \frac{\text{area}(A)}{\text{area}(S)}, \qquad A \subset S.$$

Proposition. Let $(U_n)_{n \geq 1}$ be sequence of iid r.v. with $U_1 \sim \mathcal{U}(S)$. Let $A \subset S$ and $\tau = \inf\{n \geq 1,\ U_n \in A\}$. Then $U_\tau \sim \mathcal{U}(A)$.

Proof. We have for any $B \subset A$,

$$\mathbb{P}(U_\tau \in B) = \sum_{k=1}^{+\infty} \mathbb{P}(U_k \in B | \tau = k) \mathbb{P}(\tau = k)$$

R.n. from $U \sim \mathcal{U}(S)$

Now, it follows from the independence of the U_k's that

$$\begin{aligned}
\mathbb{P}(U_k \in B | \tau = k) &= \mathbb{P}(U_k \in B | \{U_1 \notin A\} \cap \ldots \cap \{U_{k-1} \notin A\} \cap \{U_k \in A\}) \\
&= \frac{\mathbb{P}(\{U_k \in B\} \cap \{U_k \in A\})}{\mathbb{P}(U_k \in A)} \\
&= \frac{\mathbb{P}(U_k \in B)}{\mathbb{P}(U_k \in A)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbb{P}(\tau = k) &= \mathbb{P}(\{U_1 \notin A\} \cap \ldots \cap \{U_{k-1} \notin A\} \cap \{U_k \in A\})) \\
&= \mathbb{P}(U_1 \notin A)^{k-1} \mathbb{P}(U_k \in A).
\end{aligned}$$

Then, for any Borel set $B \subset A$,

$$\mathbb{P}(U_\tau \in B) = \sum_{k=1}^{+\infty} \Big(1 - \frac{|A|}{|S|}\Big)^{k-1} \frac{|B|}{|S|} = \frac{|B|}{|A|} \quad \Longrightarrow \quad U_\tau \sim \mathcal{U}(A).$$

R.n. from $U \sim \mathcal{U}(S)$

Example. *Random numbers uniformly distributed on the unit circle*. Let X be an uniform distribution on the unit sphere $A = \{(x,y) \in \mathbb{R}^2,$ $x^2 + y^2 \leq 1\}$ and let S be the square $]-1,+1[^2$ on $\mathbb{R}^2$.

- We have $A \subset S$.
- If $U_1, U_2 \sim \mathcal{U}(]-1,1[)$, are independent then, $(U_1, U_2) \sim \mathcal{U}(S)$.
- To sample a r.n. from $\mathcal{U}(A)$ we use the algorithm:

```
do   u1 ← 2*rand -1
     u2 ← 2*rand -1
while (u1*u1 + u2*u2 > 1)
end
U1 ← u1 and U2 ← u2
```

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General rejection method

Let f and g be explicit probability densities in $\mathbb{R}^d$, $c \geq 1$, and let

$$\begin{aligned} A_f &= \{(x,u) \in \mathbb{R}^d \times \mathbb{R}^+ :\ 0 \leq u \leq f(x)\}, \\ A_{cg} &= \{(x,u) \in \mathbb{R}^d \times \mathbb{R}^+ :\ 0 \leq u \leq cg(x)\}. \end{aligned}$$

We suppose that

- we can simulate a r.n. from the density g but not from f.
- $A_f \subset A_{cg}$ or equivalently, $f(x) \leq cg(x)$, for any $x \in \mathbb{R}^d$.

Then, the following algorithm generates a r.v. X with density f:

1. generate a r.n x from X with density g and a r.n u from $U \sim \mathcal{U}(]0,1[)$
2. if $c \times u \times g(x) \leq f(x)$, go to 3., otherwise go to 1.
3. return $X(\omega) = x$.

This previous procedure follows from the result below.

Proposition. *Let f and g be two densities and let $c \geq 1$ be so that $f \leq cg$. Let $(X_k)_{k\geq 1}$ be an iid sequence of r.v. with density g and let $(U_k)_{k\geq 1}$ be an iid sequence of with distribution $\sim \mathcal{U}(]0,1[)$, independent from X_1. Let us define a r.v. Z as*

$$Z = \begin{cases} X_1 & \text{if} \quad cU_1 g(X_1) \leq f(X_1) \\ X_\tau & \text{otherwise, where } \tau = \inf\{k \geq 1,\ cU_k g(X_k) \leq f(X_k)\}. \end{cases}$$

Then Z has density f and τ has a geometric distribution with success parameter $1/c$.

Proof. *We have for every $x \in \mathbb{R}$,*

$$\mathbb{P}(Z \leq x) = \mathbb{P}(X_\tau \leq x) = \sum_{k=1}^{+\infty} \mathbb{P}(X_k \leq x | \tau = k)\mathbb{P}(\tau = k)$$

General rejection method

⇝ Now, we have (letting $h(x) = f(x)/(cg(x))$)

$$\mathbb{P}(\tau = k) = \big(\mathbb{P}(U_1 > h(X_1))\big)^{k-1}\mathbb{P}(U_k \le h(X_k))$$

and

$$\mathbb{P}(U_1 > h(X_1)) = \int_{-\infty}^{+\infty} g(t)dt \int_{h(t)}^{1} du = 1 - 1/c.$$

⇝ In the other hand, $\mathbb{P}(X_k \le x | \tau = k) = \frac{\mathbb{P}(X_k \le x; U_k \le h(X_k))}{\mathbb{P}(U_k \le h(X_k))}$ and

$$\mathbb{P}(X_k \le x; U_k \le h(X_k)) = \int_{-\infty}^{x} g(t)dt \int_{0}^{h(t)} du = \int_{-\infty}^{x} g(t)h(t)dt = \frac{1}{c}\int_{-\infty}^{x} f(t)dt.$$

It follows that

$$\mathbb{P}(Z \le x) = \frac{1}{c}\int_{-\infty}^{x} f(t)dt \sum_{k=1}^{+\infty}\Big(1 - \frac{1}{c}\Big)^{k-1} = \int_{-\infty}^{x} f(t)dt,$$

so that Z has density f.

$\rightsquigarrow$ *The Gamma distribution*. Let $\lambda, a > 0$ and let X be a r.v. with Gamma distribution $\Gamma(\lambda, a)$, with pdf

$$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} \qquad \text{where} \qquad \Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} dx.$$

We want to generate a r.n. from the distribution of $X \sim \Gamma(\lambda, a)$. First note that if $Z \sim \Gamma(1, a)$, then $X = Z/\lambda \sim \Gamma(\lambda, a)$, so that it is enough to say how to simulate a r.n. from Z.

- When $a = n$ is an integer number then $Z \stackrel{d}{=} E_1 + \ldots + E_n$, where the E_k's are iid exponentially distributed r.v. with param. 1: $E_k \sim \mathcal{E}(1)$.
- If $a \in]0, 1[$ (and $\lambda = 1$), we have $f(x) \leq cg(x)$, where

$$c = \frac{e + a}{ae\Gamma(a)} \quad \text{and} \qquad g(x) = \frac{ae}{e + a}\left[x^{a-1}\mathbb{1}_{]0,1[}(x) + e^{-x}\mathbb{1}_{[1,+\infty[}\right].$$

We can apply the rejection algorithm to generate a r.n. from Z.

General rejection method examples

In fact, if X has pdf g its inverse function reads for every $u \in]0,1[$,

$$G^{-1}(u) = \left(\frac{e+a}{e}u\right)^{\frac{1}{a}} \mathbb{1}_{]0,\frac{e}{e+a}[}(u) - \ln\left((1-u)\frac{e+a}{ae}\right) \mathbb{1}_{]\frac{e}{e+a},1[}(u).$$

and $h(x) = f(x)/(cg(x))$ reads

$$h(x) = e^{-x}\mathbb{1}_{]0,1[}(x) + x^{a-1}\mathbb{1}_{[1,+\infty[}.$$

Then, to generate a r.n. from $Z \sim \Gamma(1,a)$, $a \in]0,1[$,

1. we generate a r.n. V from $\mathcal{U}(]0,1[)$, we compute $X = G^{-1}(V)$ and generate another r.n. U from $\mathcal{U}(]0,1[)$, independent from V.
2. If $U \leq h(X)$, we set $Z = X$, otherwise, return to step 1.

$\rightsquigarrow$ *The Beta distribution*. The Beta distribution with parameters $a, b > 0$ has pdf

$$f(x) = B(a,b)x^{a-1}(1-x)^{b-1}\mathbb{1}_{]0,1[}(x) \quad \text{with} \quad B(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

For $a, b > 1$, we have

$$f(x) \le cg(x) \quad \text{where} \quad c = \Big(\frac{a-1}{a+b-2}\Big)^{a-1}\Big(\frac{b-1}{a+b-2}\Big)^{b-1}$$

$$\text{and} \qquad g(x) = \mathbb{1}_{]0,1[}(x).$$

Then, to generate a r.n. from $Z \sim B(a,b)$,

1. we generate a r.n. X from $\mathcal{U}(]0,1[)$ and generate another r.n. U from $\mathcal{U}(]0,1[)$, independent from X.
2. If $U \le h(X)$, we set $Z = X$, otherwise, return to step 1.

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Transformation method: the principle

$\rightsquigarrow$ Some times, the random variable to generate reads as a function of easy generable random variables. This method is specific to some random variables and we are going to give examples of the Gamma distribution and the Gaussian distribution.

$\rightsquigarrow$ Let $T = (T_1, \ldots, T_d) : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a diffeomorphism whose inverse has Jacobian matrix

$$J(z) = \Big(\frac{d}{dz_j} T_i^{-1}(z)\Big)_{1 \le i,j \le d}.$$

It follows that if $Z = T(X)$, where X is an $\mathbb{R}^d$-valued random vector with pdf f_X, then, the pdf of Z reads

$$f_Z(z) = f_X(T^{-1}(z)) \times |\det(J(z))|, \qquad z \in \mathbb{R}^d.$$

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$\rightsquigarrow$ *Example of the Gamma distribution*. Let $\lambda > 0$ and $a_i > 0$, $i = 1, \ldots, n$. Let $X_i \overset{\text{iid}}{\sim} \Gamma(\lambda, a_i)$, $i = 1, \ldots, n$. Then, we know that $Z = X_1 + \ldots + X_n \sim \Gamma(\lambda, a_1 + \ldots + a_n)$. Suppose that $a_i = 1$ for any i.

- Since $\Gamma(\lambda, 1) \sim \mathrm{Exp}(\lambda)$, we can represent $Z \sim \Gamma(\lambda, n)$ as: $Z = X_1 + \ldots + X_n$, with $X_i \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda)$.
- We know from the inversion method that if $X_i \sim \mathrm{Exp}(\lambda)$ then $X_i \overset{d}{=} -\ln(U_i)/\lambda$, where $U_i \sim \mathcal{U}(]0, 1[)$.
- Then, $Z = X_1 + \ldots + X_n \sim \Gamma(\lambda, n)$ can be written as

$$Z = T(U_1, \ldots, U_n) = -\frac{1}{\lambda} \sum_{i=1}^{n} \ln(U_i) = -\frac{1}{\lambda} \ln\Big(\prod_{i=1}^{n} U_i\Big).$$

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Transformation method: example of the Gaussian vector

Let $Z = (Z_1, Z_2)$ be a two dimensional Gaussian vector. The following result, known as the *Box-Muller* method, say how to simulate Z from independent uniform random variables.

Proposition. Let $U_i \overset{\text{iid}}{\sim} \mathcal{U}(]0,1[)$, $i = 1, 2$. Then,

$$(Z_1, Z_2) = T(U_1, U_2) = \left(\sqrt{-2\ln(U_1)}\cos(2\pi U_2), \sqrt{-2\ln(U_1)}\sin(2\pi U_2)\right)$$

is a pair of indep. standard Normal distribution: $Z_i \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$, $i = 1, 2$.

Proof. Note that the transformation $T :]0,1[^2 \mapsto \mathbb{R}^2$ is bijective and if $z_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2)$ and $z_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2)$, we have $z_1^2 + z_2^2 = -2\ln(u_1)$ and $z_2/z_1 = \tan(2\pi u_2)$. Then $(z = (z_1, z_2))$

$$(u_1, u_2) = \left(T_1^{-1}(z), T_2^{-1}(z)\right) = \left(e^{-(z_1^2+z_2^2)/2}, (2\pi)^{-1}\arctan(z_2/z_1)\right).$$

The Jacobian matrix

$$J(z) = \begin{pmatrix} \frac{\partial T_1^{-1}(z)}{\partial z_1} & \frac{\partial T_1^{-1}(z)}{\partial z_2} \\ \frac{\partial T_2^{-1}(z)}{\partial z_1} & \frac{\partial T_2^{-1}(z)}{\partial z_2} \end{pmatrix} = \begin{pmatrix} -z_1 e^{-(z_1^2+z_2^2)/2} & -z_2 e^{-(z_1^2+z_2^2)/2} \\ -\frac{1}{2\pi}\frac{z_2}{z_1^2+z_2^2} & \frac{1}{2\pi}\frac{z_1}{z_1^2+z_2^2} \end{pmatrix}$$

It follows that $\det(J(z)) = \frac{1}{2\pi} e^{-(z_1^2+z_2^2)/2}$. Then

$$f_Z(z) = f_{(U_1,U_2)}\big(T^{-1}(z)\big)\,|\det(J(z))| = \frac{1}{2\pi} e^{-(z_1^2+z_2^2)/2} = \frac{1}{\sqrt{2\pi}} e^{-z_1^2/2} \frac{1}{\sqrt{2\pi}} e^{-z_2^2/2}$$

so that $(Z_1, Z_2) \sim \mathcal{N}(0_2, I_2)$: $0_2 = (0,0)$, I_2 is the 2×2 identity matrix.

$\rightsquigarrow$ *Generating a Gaussian vector* $X \sim \mathcal{N}(0_d, I_d)$. One way of generating a r.n. from $X \sim \mathcal{N}(0_d, I_d)$ is

- to call d times the function generating a gaussian pair $(Z_1, Z_2) \sim \mathcal{N}(0_2, I_2)$ (using for example the *Box-Muller* method)
- and to set $X = (Z_1^1, \ldots, Z_1^d)$, where Z_1^i is the value of Z_1 at the i-th call of the function generating (Z_1, Z_2).

Transformation method: example of the Gaussian vector

$\rightsquigarrow$ *Drawing a Gaussian vector* $Z \sim \mathcal{N}(\mu, \Sigma)$, *where* $\mu \in \mathbb{R}^\ell$ *and* Σ *is a* $\ell \times d$ *matrix*. We can write $Z = \mu + \Sigma^{1/2} X$, where $X \sim \mathcal{N}(0_d, I_d)$. We have seen how to draw a sample from X. It remains to say how to compute $\Sigma^{1/2}$. Several methods exist but we recall here the two main methods.

1. In the non degenerated case where Σ is positive-definite we may use the *Cholesky* decomposition: find a lower triangular matrix L so that $LL^T = \Sigma$ and $\Sigma^{1/2} = L$.
2. In general (including the degenerated case) we may use $L = UA^{1/2}$ where A and U are obtained from the spectral (or eigenvalue) decomposition $\Sigma = UAU^{-1}$ of Σ. Then, $\Sigma^{1/2} = L$.

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Mixed density: the principle

$\rightsquigarrow$ Let X be an $\mathbb{R}^d$-valued random variable with pdf $f = \sum_{n\geq 0} p_n f_n$, where $\mathbb{Q} := (p_n, n \geq 0)$ is a probability on $\mathbb{N}$ and f_n is pdf for every $n \geq 0$.

$\rightsquigarrow$ Let $(X_n)_{n\geq 0}$ be an iid sequence of r.v. such that for every $n \geq 0$, X_n has pdf f_n.

$\rightsquigarrow$ Let $\nu : \Omega \mapsto \mathbb{N}$ be a r.v. with distribution $\mathbb{Q}$, independent from $(X_n)_{n\geq 0}$.

Proposition. The random variable X_ν has pdf f.

Proof. For any Borel set $A \subset \mathbb{R}^d$, we have

$$\begin{aligned}
\mathbb{P}(X_\nu \in A) &= \sum_{n\geq 0} \mathbb{P}(\nu = n)\mathbb{P}(X_\nu \in A|\nu = n) \\
&= \sum_{n\geq 0} p_n \int_A f_n(x)dx \\
&= \int_A \sum_{n\geq 0} p_n f_n(x)dx = \int_A f(x)dx.
\end{aligned}$$

Mixed density: example

Example. Let $(p_1, p_2, p_3) = (1/6, 1/3, 1/2)$ and X be a random variable with pdf

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + p_3 f_3(x)$$

where

$$f_1(x) = \mathbb{1}_{]0,1]}(x), \quad f_2(x) = \frac{1}{2}(2x-1)\mathbb{1}_{]1,2]}(x), \quad f_3(x) = \frac{2}{3}(-3x+9)\mathbb{1}_{]2,3]}(x).$$

Propose an algorithm to generate a sample from X.

Example. Let $X_1 \sim \mathcal{N}(-3, 1)$ and $X_2 \sim \mathcal{N}(3, 1)$ be two independent r.v. with resp. pdf f_1 and f_2. Let X be a r.v. with pdf

$$f(x) = p_1 f_1(x) + p_2 f_2(x), \qquad p_1, p_2 \in [0, 1],\ p_1 + p_2 = 1.$$

- Plot the graphs of f for $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$, $(p_1, p_2) = (\frac{1}{4}, \frac{3}{4})$, $(p_1, p_2) = (\frac{3}{4}, \frac{1}{4})$.
- Plot in the same graph (w.r.t (p_1, p_2)) the densities estimates from a sample of f.