HOMEWORK 2

APC 523: Numerical Algorithm for Scientific Computing

Due: Apr.20

PROBLEM 1

(a)

Define $f(x) \equiv 1$, which can be interpolated exactly, $f(x) = \sum_{j=0}^{N} f_j L_j(x)$.

Then $1 \equiv f(x) = \sum_{j=0}^{N} f_j L_j(x) = \sum_{j=0}^{N} L_j(x)$

(b)

$$L_j(x) = \frac{x - x_j}{x - x_j} \prod_{k=0,k\neq j}^{N} \frac{x - x_k}{x_j - x_k} = \frac{1}{x - x_j} \frac{\prod_{k=0}^{N}(x - x_k)}{\prod_{k=0,k\neq j}^{N}(x_j - x_k)} = \frac{w_j^{(N)}}{x - x_j} L^{(N)}(x)$$

(c) When $x \neq x_0, x_1, \cdots, x_N$,

$$P_N(x) = \sum_{j=0}^{N} f_j L_j(x) = \sum_{j=0}^{N} f_j \frac{w_j^{(N)}}{x - x_j} L^{(N)}(x) = L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x - x_j} f_j$$

When $x = x_k$, $P_N(x) = x_k$ to avoid dividing by zero.

$$P_N(x) = \begin{cases} L^{(N)}(x) \sum_{j=0}^{N} \frac{w_j^{(N)}}{x - x_j} f_j & x \neq x_0, x_1, \cdots, x_N \\ f_k & x = x_k \end{cases}$$

(d)

From part (a), it can be shown that

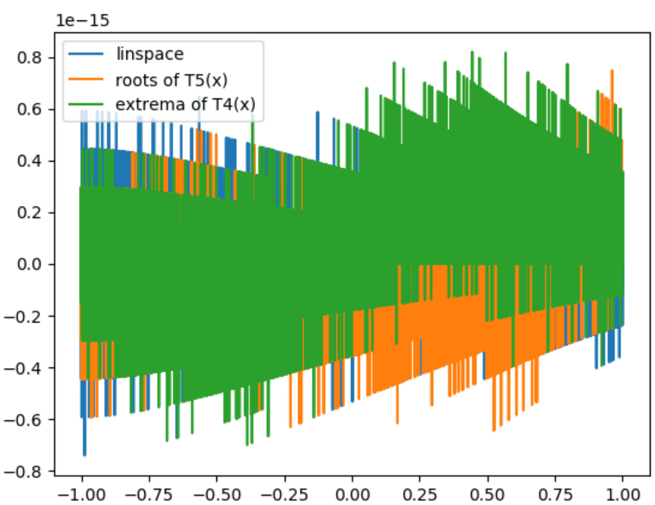
$$1 \equiv \sum_{j=0}^{N} L_j(x) = \frac{w_j^{(N)}}{x - x_j} L^{(N)}(x)$$

(e)

See attachment

(f)

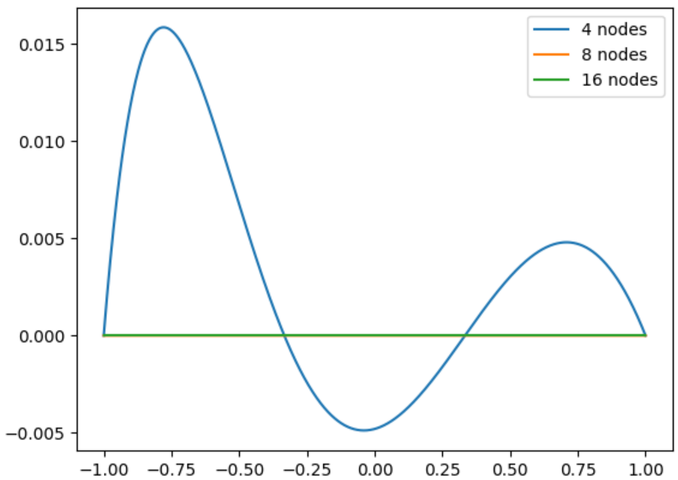
Take N = 20001, because by doubling N, the error does not change, which can be considered as max error.



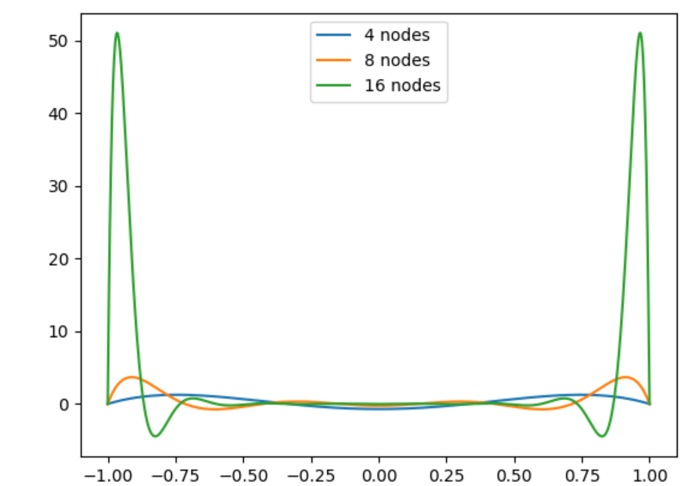
error of linspace: 7.400303277517389e-16
error of roots of T5(x): 7.477413297743957e-16
error of extrema of T4(x): 8.194533731269069e-16

PROBLEM 2

(a) and (b) see attachments
(c)
(i)

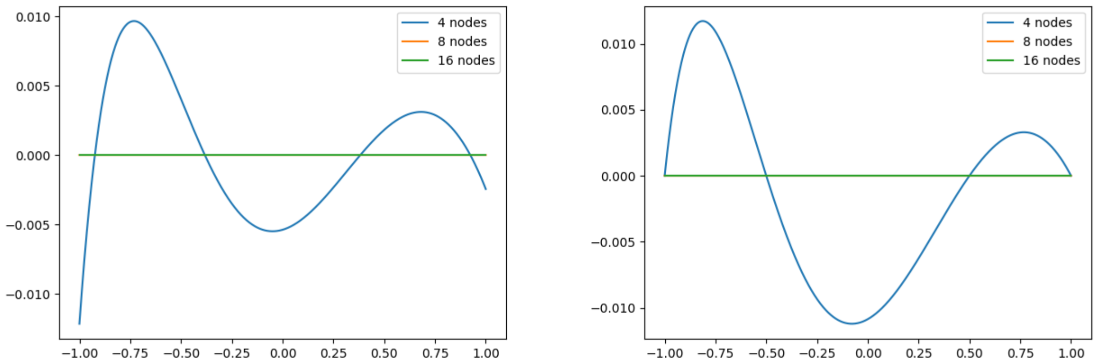


error of 4 nodes: 0.015862018637759306
error of 8 nodes: 1.6274149425682455e-06
error of 16 nodes: 7.000836595671838e-14
(ii)

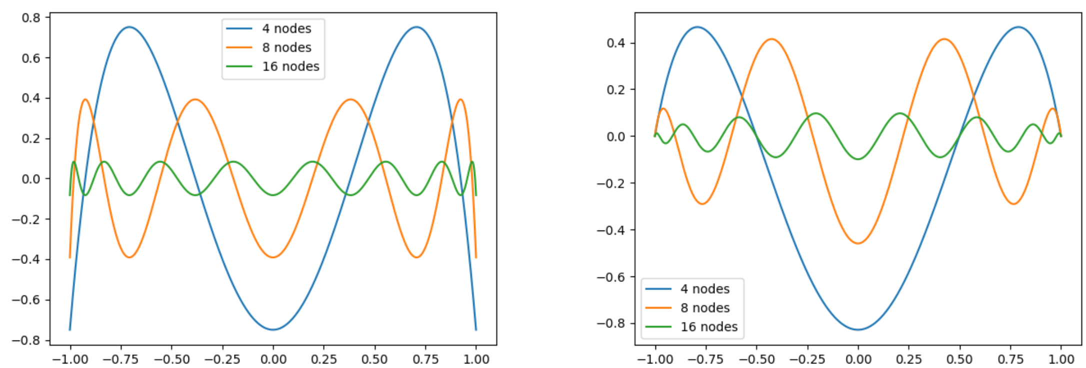


error of 4 nodes: 1.2569071519343897
error of 8 nodes: 3.6792655802995706
error of 16 nodes: 51.07694738207374

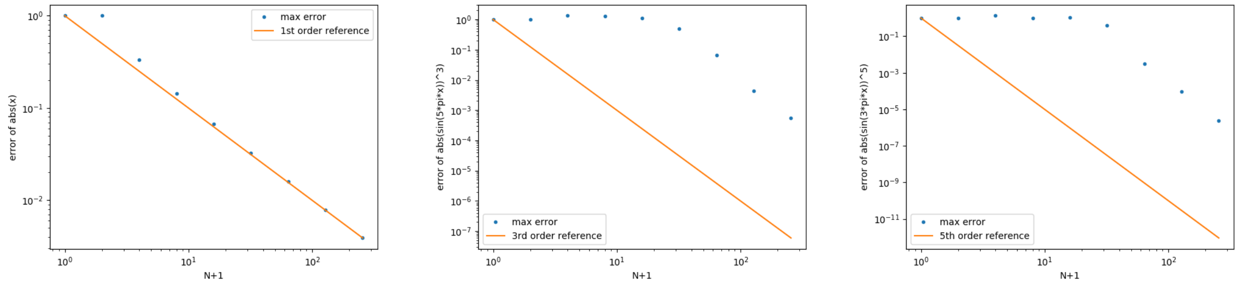
(iii)



chebyshev first kind (left):
error of 4 nodes: 0.012157034328109002
error of 8 nodes: 4.843765068041292e-07
error of 16 nodes: 9.679389495585656e-16
chebyshev second kind (right):
error of 4 nodes: 0.011716446906674694
error of 8 nodes: 5.275789099427979e-07
error of 16 nodes: 1.00776528914119e-15
(iv)

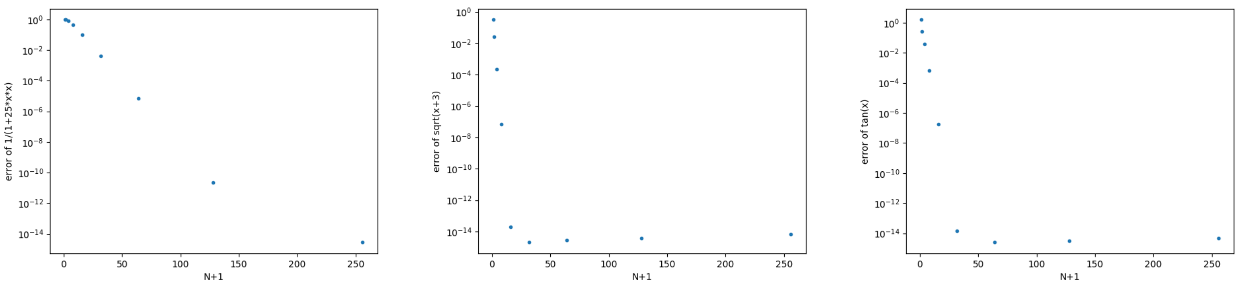


chebyshev first kind (left):
error of 4 nodes: 0.7503001200480196
error of 8 nodes: 0.39174028459023014
error of 16 nodes: 0.08310704778474848
chebyshev second kind (right):
error of 4 nodes: 0.8289124668435014
error of 8 nodes: 0.45960532477403604
error of 16 nodes: 0.09932185795194182
(d)
(i)



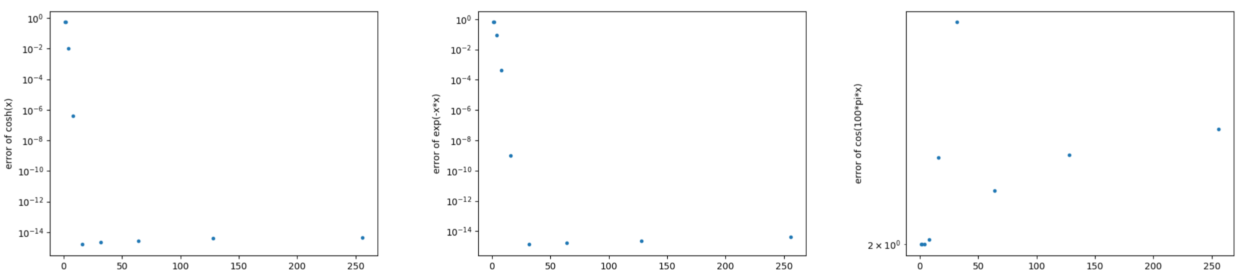
(ii)

The plots are linear until it reaches machine precision.



(iii)

The plots are linear until it reaches machine precision. The last has large error because the function is oscillating at high wavenumber.



PROBLEM 3

(a) see attachment

(b)

(i) Write down several Chebyshev polynomials.

$$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x, T_4(x) = 8x^4 - 8x^2 + 1$$

The power functions can be written as the linear combination of Chebyshev polynomials.

$$1 = T_0, x = T_1, x^2 = \frac{T_2 + T_0}{2}$$
$$x^3 = \frac{T_3 + 3T_1}{4}, x^4 = \frac{T_4 + 4T_2 + 3T_0}{8}$$

The function $f(x)$ can be written as

$$f(x) = 5T_0 + 4T_1 + 5T_2 + T_3 + T_4$$

(ii)

The result is [5. 4. 5. 1. 1.], which agrees with the theory.

(c)

It is found that $N = 2^{17} - 1$ is slower than $N = 2^{17}$, which I think is because the even N results in a complexity of $O(N \log N)$, while odd N results in complexity of $O(N^2)$. Below is an output of time complexity.

N = 1023 , t = 0.00016951560974121094

N = 8191 , t = 0.053487300872802734

N = 8192 , t = 0.0007576942443847656

N = 131071 , t = 13.24022889137268

N = 131072 , t = 0.011487722396850586

(i)

Using timeit, and run number = 10 for N = $8, 2^{13} - 1, 2^{13}, 2^{17} - 1, 2^{17}$, the runtime can be written down.

N	time
8	0.0003882179989886936
$2^{13} - 1$	0.5343649590031418
2^{13}	0.004654142001527362
$2^{17} - 1$	133.21833001599953
2^{17}	0.0784644900013518

(ii)

The runtime with respect to N in a loglog plot can be shown, the runtime should be approximately 1st order.