

ANSWER KEY

PART I: Encircle the letter of your correct answer. (each question worth 1 point)

1. Which of the following is necessarily TRUE?
 A. Every unbounded sequence diverges
 B. Every convergent sequence is monotonic
 C. If $\sum_{n=m}^{\infty} (a_n + b_n)$ converges then $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ are convergent
 D. A given power series and its derivative have the same radius of convergence and interval of convergence
 E. None
2. Which of the following sequences (I–IV given below) is/are convergent?
 I. $\{(-1)^n\}_{n=1}^{\infty}$ II. $\{(2^n + 3^n)^{\frac{1}{n}}\}_{n=1}^{\infty}$ III. $\{\frac{\cos n\pi}{n}\}_{n=1}^{\infty}$
 IV. $a_1 = \sqrt{3}, a_{n+1} = \sqrt{3 + a_n}$ for $n \geq 1$
 A. I and II only B. I and III only C. II only D. II, III, and IV
 E. I, II, and IV
3. If $\sum_{k=1}^n a_k = 2 - \frac{1}{n}$, then find the n^{th} term a_n .
 A. $\frac{n}{n(n+1)}$ B. $\frac{1}{n(n-1)}$ C. $\frac{1}{n(n+1)}$ D. $\frac{-n}{n(n-1)}$ E. None
4. If $\sum_{n=0}^{\infty} \left(\frac{\sin 2}{\pi}\right)^n = 2$, then what is the value of $\sin 2$?
 A. $\frac{\pi}{4}$ B. $\frac{\pi}{3}$ C. $\frac{\pi}{2}$ D. π E. None
5. Which of the following is TRUE about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$?
 A. It is convergent for $p > 0$
 B. It is absolutely convergent for $p > 1$
 C. It is divergent for $p \leq 0$
 D. All
 E. None

PART II: Read carefully and give the answer in its most simplified form in the space provided. (each blank space worth 0.5 point)

1. For the sequence $\{\frac{e^{n+1}}{1+e^n}\}_{n=1}^{\infty}$, the limit $\lim_{n \rightarrow \infty} \frac{e^{n+1}}{1+e^n} = e$. Find the smallest natural number N that works for $\varepsilon = e^{-6}$ using ε - N definition of limit of a sequence.

Answer: $N = \ln(e^7 - 1)$

2. Find the general term a_n of the sequence $\{1, 2, 3, 4, \dots\}$ starting with $n = 1$.

Answer: 3^n $\begin{cases} \infty \\ n=1 \end{cases}$



3. Find the limits of the following sequences(if it exists)

a) $\{(1 + 2^n)^{\frac{1}{n}}\}_{n=1}^{\infty}$ Answer: 2

b) $\{\frac{2^n+3n}{3^n-2n}\}_{n=1}^{\infty}$ Answer: 0

c) $a_1 = 2$, and $a_{n+1} = 2 - \frac{1}{a_n}$ for $n \geq 1$ Answer: 1

4. For the sequence $\{a_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$, find a convergent subsequence of the given sequence $\{a_n\}_{n=1}^{\infty}$ by defining the associated sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$

Answer: $\left\{ (-1)^{2k+1} \right\}_{k=1}^{\infty}$ or $\left\{ (-1)^{2k} \right\}_{k=1}^{\infty}$

5. Find the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \frac{n^2-n-1}{n!}$.

Answer: $S_n = 1 - \frac{n+1}{n!}$

6. Find the value of β for which the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{\beta^n x^n}{\ln(n)}$ is 5.

Answer: $\beta = \frac{1}{5}$

7. For the function $f(x) = \frac{1}{1-x}$,

a) Find a formula for the n^{th} Taylor polynomial $P_n(x)$ generated by f at 0.

Answer: $P_n(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$

b) Compute $P_n(2)$.

Answer: $P_n(2) = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \dots$

PART III: Show all the necessary steps clearly and neatly.

1. Determine the following series is absolutely convergent, conditionally convergent or divergent. (each question worth 2 points)

a. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$

Sol |

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{e^{n^2}} \right| = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

 Let $a_n = \frac{n}{e^{n^2}}, \forall n \geq 1$, then
 $a_{n+1} = \frac{n+1}{e^{(n+1)^2}}$ and $a_n \geq 0, \forall n \geq 1$

Now
 $r = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$
 $= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \cdot e^{n^2 - (n^2 + 2n + 1)} \right)$
 $= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \cdot e^{-2n-1} \right)$
 $= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \cdot \frac{1}{e^{2n+1}} \right) = 0$

By ratio test, $\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{e^{n^2}} \right| = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$
 Converges since $0 \leq r = 0 < 1$.

Hence, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$ converges absolutely
 ~~$\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$ converges absolutely~~
 (0.5 pts) =

b. $\sum_{n=1}^{\infty} \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1}$

Sol |
 consider

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1} \right|$$

We know that

$$0 < \left| \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1} \right| = \frac{|\cos(n) + \sin(n)|}{n^2 + 2n + 1}$$

$$\leq \frac{|\cos(n)| + |\sin(n)|}{n^2 + 2n + 1}$$

$$\leq \frac{2}{n^2 + 2n + 1} \leq \frac{2}{n^2}$$

End $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is convergent
~~p-series with p=2>1.~~

Thus, $\sum_{n=1}^{\infty} \left| \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1} \right|$ converges

by comparison test.
 end hence,
 ~~$\sum_{n=1}^{\infty} \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1}$ converges~~

~~$\sum_{n=1}^{\infty} \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1}$ converges absolutely~~
 (0.5 pts)

$$\text{c. } \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n \ln(n)}$$

$$\text{So } \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$$

Considering

$$\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

$$\text{Let } \varrho_n = \frac{1}{n \ln(n)}, \forall n \geq 2$$

$$\text{Let } f(x) = \frac{1}{x \ln(x)}, \forall x \geq 2, \text{ then } f(n) = \frac{1}{n \ln(n)} = \varrho_n, \forall n \geq 2$$

$$\text{Here, } \varrho_n \geq 0, \forall n \geq 2$$

- $f(x)$ is defined on $[2, \infty)$

- $f(x)$ is decreasing since $f'(x) = -\frac{(\ln x + 1)}{x^2 (\ln x)^2} \leq 0, \forall x \geq 2$

We can apply Integral test,

$$\int_2^{\infty} f(x) dx = \int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \left(\int_2^t \frac{1}{x \ln x} dx \right)$$

$$\text{Let } u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx$$

$$\text{and } \int x \ln x dx = \int \frac{1}{u} du = \ln|u| + C = \ln(\ln x) + C$$

Now,

$$\int_2^{\infty} f(x) dx = \lim_{n \rightarrow \infty} \int_2^n \frac{1}{x \ln x} dx$$

$$= \lim_{n \rightarrow \infty} \left(\ln(\ln n) \Big|_2^n \right) = \lim_{n \rightarrow \infty} \left(\ln(\ln n) - \ln(\ln 2) \right) = +\infty$$

Hence, $\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges by Integral test //

Again, $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$ is an AS, use AST to check the conv.

$$\text{Let } b_n = \frac{1}{n \ln n}, \forall n \geq 2 \quad \bullet b_n > 0, \forall n \geq 2$$

- $\{b_n\}_{n=1}^{\infty}$ is decreasing (shown in above)

$$\bullet \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$$

Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by AST,

$\therefore \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n \ln(n)}$ converges conditionally //

(0.5 pts)

2. Find a power series representation (centered at 0) for the function $f(x) = \frac{x^2}{(1-2x)^2}$ and determine the radius and interval of convergence. (4 points)

So we know that $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, |t| < 1$

$$\text{Taking } t=2x, \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n, |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

Differentiating both sides, we get

$$\frac{d}{dx} \left(\frac{1}{1-2x} \right) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} 2^n x^n \right)$$

$$\Rightarrow \frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1}$$

$$\text{Hence, } \frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1}$$

$$\text{Now, } f(x) = \frac{x^2}{(1-2x)^2} = x^2 \cdot \frac{1}{(1-2x)^2} = x^2 \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1}$$

$$\text{The PS representation of } f(x) = \frac{x^2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1}$$

To find interval of convergence,

for $x=0$ the PS converges trivially

for $x \neq 0$, let $a_n = 2^{n-1} n x^{n+1}, \forall n \geq 1$, then $a_{n+1} = 2^n (n+1) x^{n+2}$

$$\text{Now } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^n (n+1) x^{n+2}}{2^n \cdot 2^1 \cdot n \cdot x^{n+1}} \right| = 2|x| \lim_{n \rightarrow \infty} \frac{n+1}{n} = 2|x|.$$

By generalized ratio test, the series converges if $r = 2|x| < 1$

$$\text{i.e. } 2|x| < 1 \Rightarrow |x| < \frac{1}{2}$$

$$\text{When } x = \frac{1}{2}, \sum_{n=1}^{\infty} 2^{n-1} n \cdot \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2} \cdot n \cdot \frac{1}{2^n \cdot 2} = \sum_{n=1}^{\infty} \frac{1}{4} n \text{ diverges by Divergence test}$$

$$\text{When } x = -\frac{1}{2}, \sum_{n=1}^{\infty} 2^{n-1} n \frac{(-1)^n}{2^n \cdot 2} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{4} n \text{ diverges by Divergence test}$$

$$\text{The interval of convergence is } (-\frac{1}{2}, \frac{1}{2}) \text{ and radius of convergence is } R = \frac{1}{2}$$

