NSWER KEY.

PART I: Encircle the letter of your correct answer. (**each question worth 1 point**)

1. Which of the following is necessarily TRUE?
   - (A) Every unbounded sequence diverges
   - B. Every convergent sequence is monotonic
   - C. If $\sum_{n=m}^{\infty} (a_n + b_n)$ converges then $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ are convergent
   - D. A given power series and its derivative have the same radius of convergence and interval of convergence
   - E. None

2. Which of the following sequences (I–IV given below) is/are convergent?
   - I. $\{(-1)^n\}_{n=1}^{\infty}$
   - II. $\{(2^n + 3^n)^{\frac{1}{n}}\}_{n=1}^{\infty}$
   - III. $\{\frac{\cos n\pi}{n}\}_{n=1}^{\infty}$
   - IV. $a_1 = \sqrt{3}, a_{n+1} = \sqrt{3 + a_n}$ for $n \geq 1$
   - A. I and II only
   - B. I and III only
   - C. II only
   - (D) II, III, and IV
   - E. I, II, and IV

3. If $\sum_{k=1}^{n} a_k = 2 - \frac{1}{n}$, then find the nth term a_n.
   - A. $\frac{n}{n(n+1)}$
   - (B) $\frac{1}{n(n-1)}$
   - C. $\frac{1}{n(n+1)}$
   - D. $\frac{-n}{n(n-1)}$
   - E. None

4. If $\sum_{n=0}^{\infty} \left(\frac{\sin 2}{\pi}\right)^n = 2$, then what is the value of $\sin 2$?
   - A. $\frac{\pi}{4}$
   - B. $\frac{\pi}{3}$
   - (C) $\frac{\pi}{2}$
   - D. π
   - E. None

5. Which of the following is TRUE about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$?
   - A. It is convergent for $p > 0$
   - B. It is absolutely convergent for $p > 1$
   - C. It is divergent for $p \leq 0$
   - (D) All
   - E. None

PART II: Read carefully and give the answer in its most simplified form in the space provided. (**each blank space worth 0.5 point**)

1. For the sequence $\{\frac{e^{n+1}}{1+e^n}\}_{n=1}^{\infty}$, the limit $\lim_{n\to\infty} \frac{e^{n+1}}{1+e^n} = e$. Find the smallest natural number N that works for $\varepsilon = e^{-6}$ using ε-N definition of limit of a sequence.

   Answer: $N = \ln(e^7 - 1)$

2. Find the general term a_n of the sequence $\{1, 2, 3, 4, ...\}$ starting with $n = 1$.

   Answer: $\{n\}_{n=1}^{\infty}$

3. Find the limits of the following sequences(if it exists)

a) $\{(1 + 2^n)^{\frac{1}{n}}\}_{n=1}^{\infty}$ Answer: _____ 2 _____.

b) $\{\frac{2^n+3n}{3^n-2n}\}_{n=1}^{\infty}$ Answer: _____ 0 _____.

c) $a_1 = 2$, and $a_{n+1} = 2 - \frac{1}{a_n}$ for $n \geq 1$ Answer: _____ 1 _____.

4. For the sequence $\{a_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$, find a convergent subsequence of the given sequence $\{a_n\}_{n=1}^{\infty}$ by defining the associated sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$

Answer: $\{(-1)^{2k+1}\}_{k=1}^{\infty}$ or $\{(-1)^{2k}\}_{k=1}^{\infty}$.

5. Find the nth partial sum of the series $\sum_{n=1}^{\infty} \frac{n^2-n-1}{n!}$.

Answer: $S_n = 1 - \frac{n+1}{n!}$.

6. Find the value of β for which the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{\beta^n x^n}{\ln(n)}$ is 5.

Answer: $\beta = \frac{1}{5}$.

7. For the function $f(x) = \frac{1}{1-x}$,

a) Find a formula for the nth Taylor polynomial $P_n(x)$ generated by f at 0.

Answer: $P_n(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$

b) Compute $P_n(2)$.

Answer: $P_n(2) = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots$

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PART III: Show all the necessary steps clearly and neatly.

1. Determine the following series is absolutely convergent, conditionally convergent or divergent. (each question worth 2 points)

   a. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$

   b. $\sum_{n=1}^{\infty} \frac{\cos(n)+\sin(n)}{n^2+2n+1}$

a.

Soln.

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{e^{n^2}} \right| = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

Let $a_n = \frac{n}{e^{n^2}}$, $\forall n \geq 1$, then

$a_{n+1} = \frac{n+1}{e^{(n+1)^2}}$ and $a_n \geq 0$, $\forall n \geq 1$

Now

$r = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n \to \infty} \left(\frac{n+1}{e^{(n+1)^2}} * \frac{e^{n^2}}{n} \right)$

$= \lim_{n \to \infty} \left(\frac{n+1}{n} \cdot e^{n^2 - (n^2+2n+1)} \right)$

$= \lim_{n \to \infty} \left(\frac{n+1}{n} e^{-2n-1} \right)$

$= \lim_{n \to \infty} \left(\frac{n+1}{n} \cdot \frac{1}{e^{2n+1}} \right) = 0$

By ratio test, $\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{e^{n^2}} \right| = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

Converges since $0 \leq r = 0 < 1$,

Hence, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$ converges

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$ Converges absolutely

b.

Soln. consider

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)+\sin(n)}{n^2+2n+1} \right| -$$

We know that

$$0 < \left| \frac{\cos(n)+\sin(n)}{n^2+2n+1} \right| = \frac{|\cos(n)+\sin(n)|}{n^2+2n+1}$$

$$\leq \frac{|\cos(n)|+|\sin(n)|}{n^2+2n+1}$$

$$\leq \frac{2}{n^2+2n+1} \leq \frac{2}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is a convergent p-series with $p=2>1$.

Thus, $\sum_{n=1}^{\infty} \left| \frac{\cos(n)+\sin(n)}{n^2+2n+1} \right|$ converges

by comparison test,

and hence, $\sum_{n=1}^{\infty} \frac{\cos(n)+\sin(n)}{n^2+2n+1}$ converges

$\therefore \sum_{n=1}^{\infty} \frac{\cos(n)+\sin(n)}{n^2+2n+1}$ converges absolutely

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c. $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n\ln(n)}$

Sol^n $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n\ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$

Considering

$\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{n\ln(n)} \right| = \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n\ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Let $a_n = \frac{1}{n\ln(n)}$, $\forall n \geq 2$

Let $f(x) = \frac{1}{x\ln(x)}$, $\forall x \geq 2$, then $f(n) = \frac{1}{n\ln n} = a_n$, $\forall n \geq 2$ **(0.5 pts)**

Here, $a_n \geq 0$, $\forall n \geq 2$

• $f(x)$ is defined on $[2, \infty)$

• $f(x)$ is decreasing since $f'(x) = -\frac{(\ln x + 1)}{x^2 (\ln x)^2} \leq 0$, $\forall x \geq 2$ **(0.5 pts)**

We can apply Integral test,

$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x\ln x} dx = \lim_{r \to \infty} \left(\int_2^r \frac{1}{x\ln x} dx \right)$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$

and $\int \frac{1}{x\ln x} dx = \int \frac{1}{u} du = \ln|u| + c = \ln(\ln x) + c$ **(0.5 pts)**

Now,

$\int_2^{\infty} f(x)dx = \lim_{r \to \infty} \int_2^r \frac{1}{x\ln x} dx$

$= \lim_{r \to \infty} \left(\ln(\ln x) \big/_2^r \right) = \lim_{r \to \infty} \left(\ln(\ln r) - \ln(\ln 2) \right) = +\infty$

Hence $\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{n\ln(n)} \right| = \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n\ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges by Integral test //

Again, $\sum_{n=2}^{\infty} \frac{\cos n\pi}{n\ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ is an AS, use AST to check the conv.

Let $b_n = \frac{1}{n\ln n}$, $\forall n \geq 2$ • $b_n > 0$, $\forall n \geq 2$

• $\{b_n\}_{n=1}^{\infty}$ is Decreasing (shown in above)

• $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n\ln n} = 0$

Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ converges by AST, **(0.5 pts)**

∴ $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n\ln(n)}$ Converges Conditionally //

2. Find a power series representation(centered at 0) for the function $f(x) = \frac{x^2}{(1-2x)^2}$ and determine the radius and interval of convergence. **(4 points)**

Sol^n We know that $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, $|t| < 1$

Taking $t = 2x$, $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$, $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$ (0.5 pts)

Differentiating both sides, we get

$$\frac{d}{dx}\left(\frac{1}{1-2x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} 2^n x^n\right)$$

$$\Rightarrow \frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1}$$

Hence $\frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1}$ (0.5 pts)

Now,

$$f(x) = \frac{x^2}{(1-2x)^2} = x^2 \cdot \frac{1}{(1-2x)^2} = x^2 \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1}$$

∴ The PS representation of $f(x) = \frac{x^2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1}$ (1 pt)

To find interval of convergence,

For $x = 0$, the PS converges trivially

For $x \neq 0$, let $a_n = 2^{n-1} n x^{n+1}$, $\forall n \geq 1$, then $a_{n+1} = 2^n (n+1) x^{n+2}$

Now

$$r = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left|\frac{2^n (n+1) x^n \cdot x^2}{2^n \cdot 2^{-1} \cdot n \cdot x^n \cdot x}\right| = 2|x| \lim_{n \to \infty} \frac{n+1}{n} = 2|x|$$ (1 pt)

By Generalized ratio test, the series converges if $r = 2|x| < 1$

i.e $2|x| < 1 \Rightarrow |x| < \frac{1}{2}$

When $x = \frac{1}{2}$, $\sum_{n=1}^{\infty} 2^{n-1} n \cdot \frac{1}{2^{n+1}} = \sum 2^n \cdot 2^{-1} \cdot n \cdot \frac{1}{2^n \cdot 2} = \sum_{n=1}^{\infty} \frac{1}{4} n$ diverges by Divergence test

When $x = -\frac{1}{2}$, $\sum_{n=1}^{\infty} 2^{n-1} n \frac{(-1)^n}{2^n \cdot 2} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{4} n$ diverges by Divergence test (1 pt)

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∴ The Interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$ and radius of convergence is $R = \frac{1}{2}$ //