

ANSWER KEY

PART I: Encircle the letter of your correct answer. (each question worth 1 point)

- Which of the following is necessarily TRUE?
 - Every unbounded sequence diverges
 - Every convergent sequence is monotonic
 - If $\sum_{n=m}^{\infty} (a_n + b_n)$ converges then $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ are convergent
 - A given power series and its derivative have the same radius of convergence and interval of convergence
 - None
- Which of the following sequences (I–IV given below) is/are convergent?
 - $\{(-1)^n\}_{n=1}^{\infty}$
 - $\{(2^n + 3^n)^{\frac{1}{n}}\}_{n=1}^{\infty}$
 - $\{\frac{\cos n\pi}{n}\}_{n=1}^{\infty}$
 - $a_1 = \sqrt{3}, a_{n+1} = \sqrt{3 + a_n}$ for $n \geq 1$
 - I and II only
 - I and III only
 - II only
 - II, III, and IV
 - I, II, and IV
- If $\sum_{k=1}^n a_k = 2 - \frac{1}{n}$, then find the n^{th} term a_n .
 - $\frac{n}{n(n+1)}$
 - $\frac{1}{n(n-1)}$
 - $\frac{1}{n(n+1)}$
 - $\frac{-n}{n(n-1)}$
 - None
- If $\sum_{n=0}^{\infty} (\frac{\sin 2}{\pi})^n = 2$, then what is the value of $\sin 2$?
 - $\frac{\pi}{4}$
 - $\frac{\pi}{3}$
 - $\frac{\pi}{2}$
 - π
 - None
- Which of the following is TRUE about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$?
 - It is convergent for $p > 0$
 - It is absolutely convergent for $p > 1$
 - It is divergent for $p \leq 0$
 - All
 - None

PART II: Read carefully and give the answer in its most simplified form in the space provided. (each blank space worth 0.5 point)

- For the sequence $\{\frac{e^{n+1}}{1+e^n}\}_{n=1}^{\infty}$, the limit $\lim_{n \rightarrow \infty} \frac{e^{n+1}}{1+e^n} = e$. Find the smallest natural number N that works for $\varepsilon = e^{-6}$ using ε - N definition of limit of a sequence.

Answer: $N = \ln(e^7 - 1)$

- Find the general term a_n of the sequence $\{1, 2, 3, 4, \dots\}$ starting with $n = 1$.

Answer: $\{n\}_{n=1}^{\infty}$

3. Find the limits of the following sequences (if it exists)

a) $\{(1 + 2^n)^{\frac{1}{n}}\}_{n=1}^{\infty}$ Answer: 2

b) $\{\frac{2^n + 3n}{3^n - 2n}\}_{n=1}^{\infty}$ Answer: 0

c) $a_1 = 2$, and $a_{n+1} = 2 - \frac{1}{a_n}$ for $n \geq 1$ Answer: 1

4. For the sequence $\{a_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$, find a convergent subsequence of the given sequence $\{a_n\}_{n=1}^{\infty}$ by defining the associated sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$

Answer: $\{(-1)^{2k+1}\}_{k=1}^{\infty}$ or $\{(-1)^{2k}\}_{k=1}^{\infty}$

5. Find the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \frac{n^2 - n - 1}{n!}$.

Answer: $S_n = 1 - \frac{n+1}{n!}$

6. Find the value of β for which the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{\beta^n x^n}{\ln(n)}$ is 5.

Answer: $\beta = \frac{1}{5}$

7. For the function $f(x) = \frac{1}{1-x}$,

a) Find a formula for the n^{th} Taylor polynomial $P_n(x)$ generated by f at 0.

Answer: $P_n(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$

b) Compute $P_n(2)$.

Answer: $P_n(2) = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \dots$

PART III: Show all the necessary steps clearly and neatly.

1. Determine the following series is absolutely convergent, conditionally convergent or divergent. (each question worth 2 points)

a. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$

So $\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{e^{n^2}} \right| = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

Let $a_n = \frac{n}{e^{n^2}}, \forall n \geq 1$, then

$a_{n+1} = \frac{n+1}{e^{(n+1)^2}}$ and $a_n \geq 0, \forall n \geq 1$

Now

$r = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right)$

$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \cdot e^{n^2 - (n+1)^2} \right)$

$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \cdot e^{-2n-1} \right)$

$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \cdot \frac{1}{e^{2n+1}} \right) = 0$

By ratio test, $\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{e^{n^2}} \right| = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

converges since $0 \leq r = 0 < 1$.

Hence, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$ converges

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{e^{n^2}}$ converges absolutely

(0.5 pts)

b. $\sum_{n=1}^{\infty} \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1}$

So $\sum_{n=1}^{\infty} \left| \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1} \right|$

We know that

$0 < \left| \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1} \right| = \frac{|\cos(n) + \sin(n)|}{n^2 + 2n + 1}$

$\leq \frac{|\cos(n)| + |\sin(n)|}{n^2 + 2n + 1}$

$\leq \frac{2}{n^2 + 2n + 1} \leq \frac{2}{n^2}$

and $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is a convergent p-series with $p=2 > 1$.

Thus, $\sum_{n=1}^{\infty} \left| \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1} \right|$ converges

by comparison test.

and hence, $\sum_{n=1}^{\infty} \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1}$ converges

$\therefore \sum_{n=1}^{\infty} \frac{\cos(n) + \sin(n)}{n^2 + 2n + 1}$ converges absolutely

(0.5 pts)

c. $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n \ln(n)}$

Solⁿ $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$

Considering

$$\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

Let $a_n = \frac{1}{n \ln(n)}, \forall n \geq 2$

Let $f(x) = \frac{1}{x \ln(x)}, \forall x \geq 2$, then $f(n) = \frac{1}{n \ln(n)} = a_n, \forall n \geq 2$

Here, $a_n \geq 0, \forall n \geq 2$

• $f(x)$ is defined on $[2, \infty)$

• $f(x)$ is decreasing since $f'(x) = -\frac{(\ln x + 1)}{x^2 (\ln x)^2} \leq 0, \forall x \geq 2$

We can apply Integral test,

$$\int_2^{\infty} f(x) dx = \int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{r \rightarrow \infty} \left(\int_2^r \frac{1}{x \ln x} dx \right)$$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx$

and $\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln|u| + c = \ln(\ln x) + c$

Now,

$$\int_2^{\infty} f(x) dx = \lim_{r \rightarrow \infty} \int_2^r \frac{1}{x \ln x} dx$$

$$= \lim_{r \rightarrow \infty} \left(\ln(\ln x) \Big|_2^r \right) = \lim_{r \rightarrow \infty} \left(\ln(\ln r) - \ln(\ln 2) \right) = +\infty$$

Hence $\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges by Integral test //

Again, $\sum_{n=2}^{\infty} \frac{\cos n\pi}{n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$ is an AS, use AST to check the conv.

Let $b_n = \frac{1}{n \ln(n)}, \forall n \geq 2$ • $b_n > 0, \forall n \geq 2$

• $\{b_n\}_{n=2}^{\infty}$ is Decreasing (shown in above)

• $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n \ln(n)} = 0$

Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$ converges by AST, converges conditionally //

(0.5 pts)

(0.5 pts)

(0.5 pts)

(0.5 pts)

2. Find a power series representation (centered at 0) for the function $f(x) = \frac{x^2}{(1-2x)^2}$ and determine the radius and interval of convergence. (4 points)

So we know that $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, $|t| < 1$

Taking $t=2x$, $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$, $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

Differentiating both sides, we get

$$\frac{d}{dx} \left(\frac{1}{1-2x} \right) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} 2^n x^n \right)$$

$$\Rightarrow \frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1}$$

Hence, $\frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1}$

Now,

$$f(x) = \frac{x^2}{(1-2x)^2} = x^2 \cdot \frac{1}{(1-2x)^2} = x^2 \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1}$$

∴ The ps representation of $f(x) = \frac{x^2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1}$

To find interval of convergence,

For $x=0$, the ps converges trivially

For $x \neq 0$, let $a_n = 2^{n-1} n x^{n+1}$, $\forall n \geq 1$, then $a_{n+1} = 2^n (n+1) x^{n+2}$

Now $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^n (n+1) x^{n+2}}{2^{n-1} n x^{n+1}} \right| = 2|x| \lim_{n \rightarrow \infty} \frac{n+1}{n} = 2|x|$

By Generalized ratio test, the series converges if $2|x| < 1$

i.e. $2|x| < 1 \Rightarrow |x| < \frac{1}{2}$

When $x = \frac{1}{2}$, $\sum_{n=1}^{\infty} 2^{n-1} n \cdot \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} 2^{n-1} \cdot \frac{1}{2^{n+1}} \cdot n = \sum_{n=1}^{\infty} \frac{1}{4} n$ diverges by Divergence test

When $x = -\frac{1}{2}$, $\sum_{n=1}^{\infty} 2^{n-1} n \frac{(-1)^n}{2^{n+1}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{4} n$ diverges by Divergence test

∴ The Interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$ and radius of convergence is $R = \frac{1}{2}$

