# Applied Mathematics II

School of Humanities And

Natural Science

Adama University

Prepared by eTeaching Team based

on the script provided by

Written by:Ato Legesse Lemecha

Edited by Ato Tamirat Temesgen

# 3. POWER SERIES

## 3.0 Introduction

In unit two we have discussed about convergence and divergence of series and how to determine convergence and divergence. In this unit we will introduce a, power series representation of a function, differentiation and integration of power series. In the discussion of power series convergence is still a major question that we will be dealing with. Further more two basic questions will be raised here:

i) Given the series, to find properties of the sum function.

ii) Given a function $f$ , to find whether or not it can be represented by a power series.

Even though the answer to the first question is affirmative the answer to second question is not always possible which will be answered by Taylor series.

## Objectives:

At the end of this unit the students will be able to:

- define power series
- define Taylor series
- define Interval of convergence
- define Radius of convergence
- differentiate and integrate power series
- determine the convergence or divergence of a power series
- give a power series representation for a given function
- approximate a function by Taylor series

.

**Definition 3.1** An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_o + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$ , is called a power series

centered at x = $c$ where $a_n$ and $c$ are constants. The term $a_n(x-c)^n$ is the n$^{\text{th}}$ term and the number $\boldsymbol{c}$ is the center.

An expression of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_o + a_1 x + a_2 x^2 + \ldots$$

is a power series centered at x=0.

Is this definition meant that a power series is an infinite series?

If $x$ is a fixed number, then it reduces to series discussed in unit two. The generalized tests which have been studied in unit two can be extended to the case of power series.

Given a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, when does it converge and when does it diverge?

In example 2:36 of section 2.7 we have seen that the infinite series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ converges for all $x \in (-e, e)$ and diverges on $(-\infty, -e] \cup [e, \infty)$ . Then the interval $(-e, e)$ is called the interval of convergence for $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ and the distance r = e is the radius of convergence for this series. Now we have the following definition.

**Definition 3.2** (**Interval of convergences**)

The interval on which the power series converge is called the interval of convergence or the domain of a power series.

The interval of convergence consists of all real numbers from $c-\text{R}$ to $c+\text{R}$ for some number R where $0 \le \text{R} \le \infty$ .

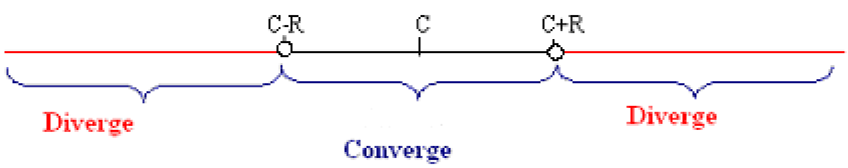


Figure 3.1

The power series diverges for all x outside the interval of convergence. The number C is called the center and R is the radius of convergence.

Remarks: i) When $\mathrm{R} = \infty$, the interval of converges is the set of all real number.

ii) When R = 0, the interval of convergence is the set containing a single point $\{C\}$

**iii)** The interval of convergence takes one and only one of the following forms [-R**,** R], [-R, R), (-R, R] or (-R, R).

**Example 3.1** Determine whether the following power series are convergent or divergent. If it converges find the interval of convergence.

$a)\sum_{n=0}^{\infty}\frac{x^n}{n!}$ b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n x^n}{3^n n}$ $c)\sum_{k=1}^{\infty}\frac{(x-2)^k}{3^k k^2}$

d) $\sum_{n=0}^{\infty} n!(2x+1)^n$ $e)\sum_{n=1}^{\infty}\frac{2^n}{n}(4x-8)^n$

**Solution: a)** if $x=0$, then $\sum\frac{x^n}{n!}=0$ the series converges trivially.

For $x\neq 0$ take $a_n=\frac{x_n}{n!}$ and $a_{n+1}=\frac{x^{n+1}}{(n+1)!}$, then use the Generalized Ratio Test.

$$\frac{a_{n+1}}{a_n}=\frac{x^{n+1}}{(n+1)!}.\frac{n!}{x^n}=\frac{x}{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x}{n+1}\right|=\frac{|x|}{n+1} \quad \text{Taking absolute value}$$

$$\mathrm{L} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{|x|}{n+1}=0<1$$

Hence by the Generalized Ratio Test $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges absolutely for all $x\in\mathbb{R}$

The interval of convergence is equal to $(-\infty,\infty)$ and radius of converges $\mathrm{R}=\infty.$

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n x^n}{3^n n}$

If x = 0, the given series is automatically convergent. For $x\neq 0$, take

$$a_n = \frac{(-1)^n 2^n x^n}{3^n n} \text{ and } a_{n+1} = \frac{(-1)^{n+1} 2^{n+1} x^{n+1}}{3^{n+1}(n+1)}$$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} 2^{n+1} x^{n+1}}{3^{n+1}(n+1)} \cdot \frac{3^n n}{(-1)^n 2^n x^n} = \frac{(-1)(2)x(n)}{3(n+1)} = \frac{-2nx}{3(n+1)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{-2nx}{3(n+1)}\right| = \frac{2n|x|}{3(n+1)}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n|x|}{n+1} = \frac{2}{3}|x|$$

So, the Generalized Ratio Test tell as that if L < 1, the series will converges absolutely, if L > 1 the series will diverge and if L = 1, we don't know what will happen. So we have,

$$\frac{2}{3}|x| < 1 \quad \Rightarrow \quad |x| < \frac{3}{2} \quad \text{series converges.}$$

$$\frac{2}{3}|x| > 1 \quad \Rightarrow \quad |x| > \frac{3}{2} \quad \text{series diverges}$$

The radius of convergence for this power series is $R = \frac{3}{2}$.

To determine the interval of convergence we need to analyses what will happen when $x = \frac{-3}{2}$ or $x = \frac{3}{2}$.

If $x = \frac{3}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n \left(\frac{3}{2}\right)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

This is alternating harmonic series and it converges by alternating series test.

If $x = \frac{-3}{2}$, the series here is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n \left(-\frac{3}{2}\right)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

This is the harmonic series and we know that it diverges. The interval of convergence for this power series is then

$$-\frac{3}{2} < x \le \frac{3}{2}$$

Therefore interval of convergence = $\left(\frac{-3}{2}, \frac{3}{2}\right]$

c) $\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^k k^2}$

The center is 2 and the power series converges if x = 2. For $x \neq 2$,

$$a_k = \frac{(x-2)^k}{3^k k^2}, \qquad a_{k+1} = \frac{(x-2)^{k+1}}{3^{k+1}(k+1)^2}$$

$$\frac{a_{k+1}}{a_k} = \frac{(x-2)^{k+1}}{3^{k+1}(k+1)^2} \cdot \frac{3^k k^2}{(x-2)^k} = \frac{(x-2)k^2}{3(k+1)^2}$$

$$\mathrm{L} = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x-2)k^2}{3(k+1)^2}\right| = \frac{|x-2|}{3}\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2 = \frac{|x-2|}{3}$$

The Generalized Ratio Test tells us that the power series converges if $\mathrm{L} < 1$ and so

$$\frac{|x-2|}{3} < 1 \quad \Rightarrow \quad |x-2| < 3 \qquad \text{Series converges.}$$

So, the radius of convergence for this power series is $\mathrm{R} = 3$. We solve this inequality to determine interval of convergence $-1 < x < 5$

If $x = -1$, the series is

$$\sum_{k=1}^{\infty} \frac{(-1-2)^k}{3^k k^2} = \sum_{k=1}^{\infty} \frac{(-3)^k}{3^k k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

and it converges by alternating series test.

If $x = 5$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(5-2)^k}{3^k k^2} = \sum_{k=1}^{\infty} \frac{(3)^k}{3^k k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Again it is convergent by p-series with p=2 > 1. Thus interval of convergence = [-1, 5].

d) $\sum_{n=0}^{\infty} n!(2x+1)^n$

If $x=-\frac{1}{2}$, the power series $\sum_{n=0}^{\infty} n!(2x+1)^{n}$ trivially converges.

For $x \neq -\frac{1}{2}$, apply Generalized Ratio Test as follows.

$$a_n = n!(2x+1)^n \qquad \text{and} \qquad a_{n+1}=(n+1)!(2x+1)^{n+1}$$

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)n!(2x+1)^n(2x+1)}{n!(2x+1)^n}=(2x+1)(n+1)$$

$$\mathrm{L}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|(2x+1)(n+1)\right|=|2x+1|\left|\lim_{n\to\infty}(n+1)\right|=\infty>1$$

Thus this power series will converge only if $x=-\frac{1}{2}$ and diverges for $x\neq-\frac{1}{2}$ . We find that the interval of convergence = $\left\{-\frac{1}{2}\right\}$ or $\left[\frac{-1}{2},\frac{-1}{2}\right]$ and Radius of convergence $\mathrm{R}=0$.

$e)\sum_{n=1}^{\infty}\frac{2^n}{n}(4x-8)^n$

If $x=2$, the given power series converges. If $x\neq 2$ apply Generalized Ratio Test.

Here $\quad a_n=\frac{2^n}{n}(4x-8)^n, \qquad a_{n+1}=\frac{2^{n+1}}{n+1}(4x-8)^{n+1}$ and

$$\frac{a_{n+1}}{a_n}=\frac{2^{n+1}}{n+1}(4x-8)^{n+1}\cdot\frac{n}{2^n(4x-8)^n}=(4x-8)\cdot\frac{2n}{n+1}$$

$$\mathrm{L}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|(4x-8)\cdot\frac{2n}{n+1}\right|=|4x-8|\lim_{n\to\infty}\left|\frac{2n}{n+1}\right|$$

$$=2|4x-8|$$

Now we can get the information about convergence or divergence from this expression.

$\mathrm{L}=2|4x-8|<1$, the given power series converges.

$\mathrm{L}=2|4x-8|>1$, the given power series diverges.

Recall that the interval of convergence requires $|x-c|<\mathrm{R}$ . In other words we must factor out 4 from absolute value in order to get radius of convergence. Consequently we have

$8|x-2|<1$ implies $|x-2|<\dfrac{1}{8}$ the power series converges

$8|x-2|>1$ implies $|x-2|>\dfrac{1}{8}$ the power series diverges.

This means that the radius of convergence is $R=\dfrac{1}{8}$. Lastly we compute the interval of convergence from the inequality that gives convergence.

$$|x-2|<\frac{1}{8}$$

$$\frac{-1}{8}<x-2<\frac{1}{8}$$

$$\frac{15}{8}<x<\frac{17}{8}$$

We know the series converges in the interval $\left(\dfrac{15}{8},\dfrac{17}{8}\right)$, but we must now test for convergence at the end points of this interval. When $x=\dfrac{15}{8}$, the series becomes

$$\sum_{k=0}^{\infty}\frac{2^n}{n}\left(\frac{15}{2}-8\right)^n=\sum_{k=0}^{\infty}\frac{2^n}{n}\left(\frac{-1}{2}\right)^n=\sum_{k=0}^{\infty}\frac{(-1)^n}{n}$$

This is the alternating harmonic series and it is convergent by alternating series test.

When $x=\dfrac{17}{8}$, the series reduced to

$$\sum_{k=0}^{\infty}\frac{2^n}{n}\left(\frac{17}{2}-8\right)^n=\sum_{k=0}^{\infty}\frac{2^n}{n}\left(\frac{1}{2}\right)^n=\sum_{k=0}^{\infty}\frac{1}{n}$$

This is harmonic series and so divergent. Therefore the power series converges for $x=\dfrac{15}{8}$ and diverges when $x=\dfrac{17}{8}$. So the interval of convergence for the given power series is $\dfrac{15}{8}\leq x<\dfrac{17}{8}$ or interval of convergence $=\left[\dfrac{15}{8},\dfrac{17}{8}\right)$.

**N.B** check always the end points to determine interval of convergence

**Example 3.2** Find the radius of convergence for the following series

**a)** $\sum_{n=1}^{\infty} \frac{1^2.3^2.5^2...(2n-1)^2 x^{2n}}{2^2.4^2.6^2...(2n)^2}$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1.3.5...(2n-3)}{2^n n!} x^n$

**Solution: a)** The power series converges if $x=0$. For $x \neq 0$ we use the Generalized Ratio Test with $a_n = \frac{1^2.3^2.5^2...(2n-1)^2 x^{2n}}{2^2.4^2.6^2...(2n)^2}$ , $a_{n+1} = \frac{1^2.3^2.5^2...(2n+1)^2 x^{2n+2}}{2^2.4^2.6^2...(2n+2)^2}$ and

$$\frac{a_{n+1}}{a_n} = \left(\frac{2n+1}{2n+2}\right)^2 x^2$$

$$\mathrm{L} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{2n+1}{2n+2}\right)^2 \left|x^2\right| = \left|x^2\right|$$

Here the power series converges if $\left|x^2\right| < 1$ and it diverges if $\left|x^2\right| > 1$. So we have

$$|x|^2 < 1$$
$$|x| < 1$$

The radius of convergence is $\mathrm{R} = 1$. Next we analyze for end points.

If $x=1$ or $x=-1$ , the power series becomes $\sum_{n=1}^{\infty} \frac{1^2.3^2.5^2...(2n-1)^2}{2^2.4^2.6^2...(2n)^2}$. Show this series converges and the interval of convergence is $[-1,1]$.

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1.3.5...(2n-3)}{2^n n!} x^n$

Using Generalized Ratio Test we see that

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2} 1.3.5...(2n-1) x^n \cdot x}{2^{n+1}(n+1)!} \frac{2^n n!}{(-1)^{n+1}\left(1.3.5...(2n-3)\right) x^n}$$

$$= \frac{(2n-1)}{2(n+1)} x$$

$$\mathrm{L} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2n-1}{2n+2}|x|$$

$$\mathrm{L} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| < 1$$

The power series converges absolutely if $\mathrm{L}<1$ and diverges if $\mathrm{L}>1$. Hence the radius of convergence is $\mathrm{R}=1$.

can you determine the interval of convergence for this series?

**Example 3.3** Find the interval and radius of convergence for the power series

$$\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}(1-ex)^n$$

**Solution:** For this power series, we have

$$\frac{a_{n+1}}{a_n} = \frac{2^n\cdot 2(n!)(1-ex)(1-ex)^n}{(n+1)^n}\frac{n^n}{2^n n!(1-ex)^n} = \frac{2(1-ex)n^n}{(n+1)^n}$$

$$\mathrm{L} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(1-ex)n^n}{(n+1)^n}\right| = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n |ex-1|$$

$$\mathrm{L} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2\left|x-\frac{1}{e}\right|$$

The Generalized Ratio Test implies that the power series $\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}(1-ex)^n$ converges if

$\mathrm{L} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2\left|x-\frac{1}{e}\right| < 1$. This implies that $\left|x-\frac{1}{e}\right| < \frac{1}{2}$. In this case the radius of convergence is $\mathrm{R}=\frac{1}{2}$. Now let us calculate the interval of convergence. We will solve the inequality that gives the convergence above.

$$\left|x-\frac{1}{e}\right| < \frac{1}{2}$$

$$\frac{2-e}{2e} < x < \frac{e+2}{2e}$$

Verify that it diverges at $x=\frac{2-e}{2e}$ and $x=\frac{2+e}{2e}$. Consequently the interval of convergence is $\left(\frac{2-e}{2e},\frac{e+2}{2e}\right)$.

**Example 3.4** Find the interval and radius of convergence for the power series

$$1+\frac{x}{3}+\frac{x^2}{4^2}+\frac{x^3}{3^3}+\frac{x^4}{4^4}+\frac{x^5}{3^5}+\frac{x^6}{4^6}+\cdots$$

**Solution:** Since

$$a_n=\begin{cases}\dfrac{1}{4^{2k}}x^{2k} & n=2k\\[2mm] \dfrac{1}{3^{2k-1}}x^{2k-1} & n=2k-1\end{cases}$$

We apply the Generalized Root Test to determine the radius of convergence and interval of convergence. In this case

$$\sqrt[n]{|a_n|}=\begin{cases}\dfrac{1}{4}|x| & n=2k\\[2mm] \dfrac{1}{3}|x| & n=2k-1\end{cases}$$

and

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\begin{cases}\dfrac{1}{4}|x| & n=2k\\[2mm] \dfrac{1}{3}|x| & n=2k-1\end{cases}$$

The series converges absolutely if $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}<1$ and the radius of convergence is the least upper bound of $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}<1$. It follows that

$$\frac{1}{3}|x|<1$$

$$|x|<3$$

$$-3<x<3$$

Verify that the series diverges when $x=-3$ and $x=3$.

Thus the interval of convergence is $(-3,3)$ and radius of converges is $\mathrm{R}=3$.

**Lemma 3.1** Suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power aeries.

i) If the series converges at $x = c$, then the series converges absolutely at $x$ whenever $|x| < |c|$.

ii) If the series diverges at $x = b$, then the series diverges at $x$ whenever $|x| > |b|$

**Proof:** i) suppose $\sum_{n=0}^{\infty} a_n c^n$ converges. By the n$^{\text{th}}$ term test the sequence $\lim_{n\to\infty} a_n c^n = 0$. This implies that the sequence of n$^{\text{th}}$ term converges. But every convergent sequence is bounded, so there exists a number M such that $|a_n c^n| < \mathrm{M}$ for all integers $n \geq 0$. Let $x$ be arbitrarily chosen with $|x| < |c|$. Then $\left|\frac{x}{c}\right| < 1$ and we need to show that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

$$|a_n x^n| = \left|a_n c^n \frac{x^n}{c^n}\right| = |a_n c^n| \left|\frac{x}{c}\right|^n < \mathrm{M}\left|\frac{x}{c}\right|^n$$

and

$$\sum_{n=0}^{\infty} |a_n x^n| < \sum_{n=0}^{\infty} \mathrm{M}\left|\frac{x}{c}\right|^n$$

However, the series $\sum_{n=0}^{\infty} \mathrm{M}\left|\frac{x}{c}\right|^n$ is a convergent geometric series with first term M and common ratio $\left|\frac{x}{c}\right| < 1$. So the Comparison Test implies that $\sum_{n=0}^{\infty} |a_n x^n|$ converges and that the series is absolutely convergent when $|x| < |c|$.

ii) Suppose the series diverges at $b$ but converges for some $x$ with $|x| > |b|$. This implies, by part (i), that $\sum_{n=0}^{\infty} a_n b^n$ converges absolutely, which contradicts the assumption of divergence at $b$.

Thus, if the series diverges at $b$, and $|x| > |b|$ the series diverges at all x. ♦

This Lemma implies that if a power series converges for some real numbers but diverges for some others, then a number R exists with the property that the series converges absolutely if $|x| < \mathrm{R}$ and diverges if $|x| > \mathrm{R}$ .

**Theorem 3.1** If $\sum_{n=0}^{\infty} a_n x^n$ is a power series, then exactly one of the following statements holds:

i) $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$

ii) $\sum_{n=0}^{\infty} a_n x^n$ converges for all real numbers $x$

iii) Let $\mathrm{R} > 0$ be the radius of convergence for the power series such that

a) $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < \mathrm{R}$;

b) $\sum_{n=0}^{\infty} a_n x^n$ diverges if $|x| > \mathrm{R}$

**Proof:** see reference book**.**

**Example 3.5** Find the domain of the function defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{(x+7)^{2n} \ln n}{(n+1)\arctan n}$$

**Solution:** Since

$$\mathrm{L} = \lim_{n\to\infty} \left| \frac{(x+7)^{2n}(x+7)^2 \ln(n+1)}{(n+2)\arctan(n+1)} \frac{(n+1)\arctan n}{(x+7)^{2n} \ln n} \right|$$

$$= \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \frac{(n+1)\arctan n}{(n+2)\arctan(n+1)} \left|(x-(-7))^2\right|$$

$= \left|(x-(-7))\right|^2$ applying L'Hopital's Rule and simplifying

The series converges if $\left|(x-(-7))\right|^2 < 1$, that is, if $\left|(x-(-7))\right| < 1$

The inequality $\left|(x-(-7))\right| < 1$ is equivalent to $-8 < x < -6$.

The series converges if $-8 < x < -6$ and diverges if $x < -8$ or $x > -6$.

At $x=-8$, the series

$$\sum_{n=1}^{\infty}\frac{\ln n}{(n+1)\arctan n}$$

diverges since the terms of the series do not approach zero for large value of n. Similarly at $x=-6$ the series diverges. Accordingly, the domain of $f$ is $(-8,-6)$.

## Exercise 3.1

1. Find the radius and interval of convergence for the following power series.

a) $\sum_{n=1}^{\infty}\sqrt{n}x^n$ b) $\sum_{n=1}^{\infty} n^{2n}\frac{x^n}{(n!)^2}$ c) $\sum_{n=1}^{\infty}\frac{n!(2n)!}{(3n)!}x^{2n}$ d) $\sum_{n=1}^{\infty} n^n x^{n^2}$

e) $\sum_{n=1}^{\infty} n^{2n}\frac{x^n}{(n!)^2}$ . f) $\sum_{n=1}^{\infty} n^n\frac{x^n}{n!}$ g) $\sum_{n=1}^{\infty}\frac{(\ln n)(x+1)^{3n}}{n(n+1)}$

h) $\sum_{n=0}^{\infty}\frac{(3x+4)^{2n}}{2^n}$ i) $\sum_{n=1}^{\infty}\frac{(3x)^n}{\arctan n}$ j) $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n)!}$

2. Find the domain of the function

a) $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{n\sqrt{n+1}}$ b) $g(x)=\sum_{n=1}^{\infty}\frac{(3x+1)^n}{\ln(n+1)}$

c) $f(x)=\sum_{n=1}^{\infty}\frac{3^n(1-3x)^{2n}}{\sqrt{n}}$ d) $f(x)=\sum_{n=0}^{\infty}(\ln x)^n$

3. Find the values of x for which the following series converges.

a) $\sum_{n=1}^{\infty}\left(\frac{cox}{n}\right)^n$ b) $\sum_{n=1}^{\infty}\frac{(-4)^n}{n(4-3x)^{2n}}$

c) $\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}.\frac{1}{(3x-4)^n}$ d) $\sum_{n=0}^{\infty}\frac{3^n}{(x-1)^n}$

4. Find the radius and interval of convergence of the following power series.

a) $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1.3.5...(2n-3)x^{2n}}{3^n n}$ b) $\sum_{n=2}^{\infty}\frac{2\cdot5\cdot8\cdots\cdot(3n-4)}{3\cdot6\cdot9\cdots\cdot(3n-3)}x^{n-1}$

5. Suppose the power series $\sum_{n=0}^{\infty}a_n x^n$ and $\sum_{n=0}^{\infty}b_n x^n$ have radius of convergence $R_1$ and $R_2$, respectively. What can be said about the radius of convergence R of $\sum_{n=0}^{\infty}\left(a_n x^n + b_n x^n\right)$?

6. Find the interval of convergence of the power series $\sum_{n=0}^{\infty}\frac{n^n x^n}{n!}$.

# 3.1 POWER SERIES REPRESENTATION OF FUNCTIONS

In this subsection we will be represent many functions as power series that are related to geometric series. That is we calculate a power series expression for a function by manipulating a known series.

Recall that the geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ , is convergent if } |r| < 1 \text{ and diverges if } |r| \geq 1.$$

In this series if we take $a = 1$ and $r = x$, then the series be comes

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1.$$

**Example 3.6 C**onsider a geometric series with $r = x \quad and \quad a = 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} \text{ for } |x| < 1 \qquad (1)$$

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots \qquad \text{for } |x| < 1$$

If in (1) we replace x by (-x) we get a new function with new power series representation

$$f(-x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 \ldots + (-1)^n x^n$$

Again, if we replace x by ( $x^2$ ) in (1) we get

$$f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + \ldots + x^{2n} + \ldots \qquad \text{if } |x| < 1.$$

If in (1) we replace x by ($-x^2$) we have

$$f(x) = \frac{1}{x^2+1} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \cdots \qquad \text{if } |x| < 1$$

From this example one can observe that as we vary x by different expression in (1) we get different power series representation.

**Example 3.7** Find a power series representation for the following function and determine its domain.

$$f(x) = \frac{2x^2}{5+x^3}$$

**Solution:** $f(x)=\dfrac{2x^2}{5+x^3}=2x^2\dfrac{1}{5+x^3}=2x^2\dfrac{1}{5\left(1-\left(-\dfrac{x}{\sqrt[3]{5}}\right)^3\right)}=\dfrac{2x^2}{5}\dfrac{1}{1-\left(-\dfrac{x}{\sqrt[3]{5}}\right)^3}$

Now substituting in the geometric series $\sum_{n=0}^{\infty}x^n=\dfrac{1}{1-x}$ for $|x|<1$ we find that

$$f(x)=\frac{2x^2}{5}\sum_{n=0}^{\infty}\left(-\frac{x}{\sqrt[3]{5}}\right)^{3n} \text{ provided that } \left|-\frac{x^3}{5}\right|<1$$

$$f(x)=2\sum_{n=0}^{\infty}\frac{(-1)^{3n}(x)^{3n+2}}{5^{n+1}}$$

The series converges when

$$\left|-\frac{x^3}{5}\right|<1$$

$$|x^3|<5$$

$$|x|<\sqrt[3]{5}.$$

So the interval of absolute convergence is $-\sqrt[3]{5}<x<\sqrt[3]{5}$ and hence the domain of $f$ is

$$-\sqrt[3]{5}<x<\sqrt[3]{5}.$$

**Example 3.8** Find a power series representation for the following function and determine its domain.

$$f(x)=\frac{x}{2x^3-16}$$

**Solution:** We note that $f(x)=\dfrac{x}{2x^3-16}=\dfrac{x}{-16\left(1-\dfrac{x^3}{8}\right)}=\dfrac{-x}{16}\dfrac{1}{1-\dfrac{x^3}{8}}$

Now substitute $\dfrac{x^3}{8}$ for $x$ in $\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for $|x|<1$

$$\frac{1}{1-\dfrac{x^3}{8}}=\sum_{n=0}^{\infty\infty}\frac{x^{3n}}{8^n} \qquad \text{for} \quad \left|\frac{x^3}{8}\right|<1$$

Next multiply by $-\dfrac{x}{16}$

$$f(x)=\frac{x}{2x^3-16}=\frac{-x}{16}\sum_{n=0}^{\infty}\frac{x^{3n}}{8^n}=-\sum_{n=0}^{\infty}\frac{x^{3n+1}}{2^{3n+4}} \quad \text{for} \quad \left|\frac{x^3}{8}\right|<1.$$

Thus the required power series representation of

$$\frac{x}{2x^3-16}=-\sum_{n=0}^{\infty}\frac{x^{3n+1}}{2^{3n+4}} \quad \text{for} \quad \left|\frac{x^3}{8}\right|<1$$

From $\left|\dfrac{x^3}{8}\right|<1$, we have

$$|x^3|<8$$

$$-2<x<2$$

So the interval of convergence is $(-2,2)$ and radius of convergence $\mathrm{R}=2$ .

## 3.1.1 DIFFERENTIATION OF POWER SERIES

What do we meant by derivative of a power series? Is the derivative of power series exists?

Power series are generalization of polynomials

$$p(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m=\sum_{n=0}^{m}a_nx^n$$

A power series with non zero radius of convergence is always differentiable and its derivative is obtained from $\sum_{n=0}^{m}a_nx^n$ by differentiating term by term as we differentiate polynomial functions.

**Theorem 3.2:** (**Differentiation Theorem for Power series**)

Let $\sum_{n=0}^{m}a_nx^n$ be a power series with radius of convergence $\mathrm{R}>0$. Then $\sum_{n=1}^{m}na_nx^{n-1}$ has the same radius of convergence and

$$\frac{d}{dx}\left(\sum_{n=0}^{m}a_nx^n\right)=\sum_{n=1}^{m}na_nx^{n-1}=\sum_{n=1}^{\infty}\frac{d}{dx}\left(a_nx^n\right) \quad \text{for } |x|<\mathrm{R}$$

**Proof:** omitted but if interested please consult any reference material.

Note that the initial value of this series has been changed from n = 0 to n = 1. This indicates the fact that the derivative of the first term is zero and is not in the derivative.

**Example3.8**: Show that $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2}$ *and* $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$ have the same radius of convergence.

**Solution:** Let $f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots$

$$f'(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{x^n}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+2}}{(n+2)^2}\frac{(n+1)^2}{x^{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^2 |x| = |x|$$

The power series converges absolutely if $|x| < 1$ and radius of convergence R = 1.

If $x = -1$, then the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$$

This is convergent by alternating series test. If $x = 1$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(1)^{n+1}}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

and converges by $p-$series test .

In this case the interval of convergence is $[-1,1]$. Again for the second series we find that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+2}}{(n+2)}\frac{(n+1)}{x^{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)|x| = |x|$$

Similarly the power series converges absolutely if $|x| < 1$ and radius of convergence R = 1

If $x = -1$, then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges conditionally and if $x=1$, then the series

$$\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$$ diverges.

The interval of convergence for $\sum_{n=1}^{\infty}\frac{x^n}{n+1}$ is $[-1,1)$.

Thus $f(x)=\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)^2}$ and $f'(x)=\sum_{n=1}^{\infty}\frac{x^n}{n+1}$ have the same radius of convergence but not necessarily the same interval of convergence. The differentiation theorem states that the radii of convergence of the series $\sum_{n=0}^{\infty}a_n x^n$ and $\sum_{n=1}^{\infty}na_n x^{n-1}$ are the same but does not implies that the interval of convergence are the same.

**Example 3.9** Show that $\frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots$

**Solution:** If $x\neq 0$, then

$$\mathrm{L}=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{1}{n+1}x\right|=0<1.$$

Thus the Generalized Ratio Test implies that the series converges for all $x\neq 0$ and hence the series converges for all $x$.

$$f'(x)=\sum_{k=1}^{\infty}\frac{n\quad x^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}$$

$$f'(x)=\sum_{l=0}^{\infty}\frac{x^l}{l!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}=f(x)\qquad\forall x\in\mathbb{R}$$

Recall that from exponential growth, if $f$ is continuous on $[0,\infty)$, then $f'(x)=kf(x)$ for all $x\geq 0$. In this case fix $f(0)=1$ and $k=1$, then integrating we get the only function which satisfy this property.

$$f(x)=e^x$$

Therefore

$$\boxed{e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots}\qquad(1)$$

Note that when $x=1$ we obtain the series

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=1+1+\frac{1}{2}+\frac{1}{3}+....$$

When $x=-1$ we obtain the series

$$\frac{1}{e}=e^{-1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}=1-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+...$$

When we replace $x$ by $x^2$

$$e^{x^2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}=1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots$$

If we replace $x$ by $-x$ the series be comes

$$e^{-x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots \qquad (2)$$

Subtract equation (2) from (1)

$$e^x-e^{-x}=\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots\right)-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)$$

$$=2x+\frac{2}{3!}x^3+\frac{2}{5!}x^5+\cdots+\frac{2x^{2n-1}}{(2n-1)!}+\cdots$$

$$\sinh x=\frac{e^x-e^{-x}}{2}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{x^{2n-1}}{(2n-1)!}+\cdots \qquad -\infty<x<\infty$$

Similarly show that $\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots+\frac{x^{2n}}{(2n)!}+\cdots$

**Example 3.10** Find a power series representation for the following function and determine its domain.

$$f(x)=3^x$$

**Solution:** Recall that $3^x=e^{x\ln 3}$, and $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Then the power series representation is

$$3^x=\sum_{n=0}^{\infty}\frac{(x\ln 3)^n}{n!}=1+x\ln 3+\frac{(x\ln 3)^2}{2!}+\frac{(x\ln 3)^3}{3!}+\cdots$$

Verify that it has the same radius of convergence and interval of convergence with $e^x$.

**Example 3.11:** Find the sum of the series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{(n+3)!}$

**Solution:** Does the given series look any thing like a series that we already know? Well, it does have some ingredients in common with the series for the exponential function:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

We can make this series look more like our given series by replacing $x$ by $x+2$

$$e^{x+2}=\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!}=1+(x+2)+\frac{(x+2)^2}{2!}+\frac{(x+2)^3}{3!}+\cdots$$

But here the exponent in the numerator matches the number in the denominator whose factorial is taken. To make that happen in the given series, let's multiply and divide by $(x+2)^3$.

$$\sum_{n=1}^{\infty}\frac{(x+2)^n}{(n+3)!}=\frac{1}{(x+2)^3}\sum_{n=0}^{\infty}\frac{(x+2)^{n+3}}{(n+3)!}=\frac{1}{(x+2)^3}\left[\frac{(x+2)^3}{3!}+\frac{(x+2)^4}{4!}+\cdots\right]$$

We see that the series between brackets is just the series for $e^{x+2}$ with the first three terms missing. Consequently

$$\sum_{n=1}^{\infty}\frac{(x+2)^n}{(n+3)!}=\frac{1}{(x+2)^3}\left[e^{x+2}-1-(x+2)-\frac{(x+2)^2}{2!}\right]$$

**Example 3.12** Find a power series which defines $g(x)=\frac{x}{(1-x)^2}$

**Solution:** $f'(x)=\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)$

$$=\sum_{n=0}^{\infty}\frac{d}{dx}\left(x^n\right)=\sum_{n=0}^{\infty}nx^{n-1}\quad if\quad |x|<1$$

Now multiply $f'$ by $x$

$$xf(x)=\frac{x}{(1-x)^2}=x\sum_{n=1}^{\infty}nx^{n-1}\qquad\text{for}\quad |x|<1.$$

Therefore, $g(x)=\frac{x}{(1-x)^2}=\sum_{n=1}^{\infty}nx^n=x+2x^2+3x^3+\cdots$ for $|x|<1$

If $f\left(x\right)=\sum_{n=0}^{\infty} a_n x^n$, then $f'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}$ for all $x\in\left(-\mathrm{R},\mathrm{R}\right)$. Here $f(0)=a_0=0!a_0$ and $f'(0)=a_1=1!a_n$. Since $f'\left(x\right)$ is a power series further differentiating we get

$$f''\left(x\right)=\sum_{n=2}^{\infty} n\left(n-1\right)a_n x^{n-2}\ ,\ \ f''(0)=2!a_2$$

$$f'''\left(x\right)=\sum_{n=3}^{\infty} n\left(n-1\right)\left(n-2\right)a_n x^{n-3}\quad f'''(0)=3!a_3$$

$$\vdots$$

$$f^{(n)}(x)=\sum_{n=k}^{\infty} n!a_n x^{n-k}\ \text{ for all } k<n\ ,\ \ f^{(n)}(0)=n!a_n$$

Now we state the following theorem.

**Theorem 3.3**: Suppose a power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $\mathrm{R}>0$. Let

$$f(x)=\sum_{n=0}^{\infty} a_n x^n \quad \text{for } -\mathrm{R}<x<\mathrm{R}$$

Then $f$ has derivatives of all orders on (-R, R), and

$$f^{(n)}(0)=n!a_n \qquad \text{for all } n\geq 0$$

Consequently

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \qquad \text{for } -\mathrm{R}<x<\mathrm{R}$$

**Proof:** See from reference book.

## 3.1.2 INTEGRATION OF POWER SEIRES

Suppose $\sum a_n x^n$ has a radius of convergence R. Then $f(x)=\sum a_n x^n$ is differentiable on (-R, R) and hence f is continuous on (-R, R). Thus it is possible to integrate the function $f(x)=\sum a_n x^n$ over any closed interval in(-R, R).

**Theorem 3.4**: Let $\sum a_n x^n$ be power series with radius of convergence $\mathrm{R}>0$.

Then $\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$ has the same radius of convergence and

$$\int_0^x\left(\sum_{n=0}^{\infty}a_n t^n\right)dt=\sum_{n=0}^{\infty}\int_0^x a_n t^n dx=\sum_{n=0}^{\infty}\left[\frac{a_n x^{n+1}}{n+1}\right]\quad \text{for}\ |x|<\mathrm{R}.$$

**Proof:** Exercise.

**Example 3.13** Find the series expansion of $\ln(1+x^2)$

**Solution:** Recall that $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, if $|x|<1$. Now replace $x$ by $-x^2$ in $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, and then multiply by $2x$.

$$\frac{2x}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$$

Integrating we have

$$\int_0^x\frac{2t}{1+t^2}dt=\int_0^x\left(\sum_{n=0}^{\infty}(-1)^n 2t^{2n+1}\right)dt=\sum_{n=0}^{\infty}(-1)^n 2\left(\int_0^x t^{2n+1}dt\right)=\sum_{n=0}^{\infty}(-1)^n 2\left[\frac{t^{2n+2}}{2n+2}\right]_0^x$$

$$\left[\ln(1+t^2)\right]_0^x=\sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{2n+2}}{n+1}\right]_0^x$$

$$\ln(1+x^2)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}=x^2+\frac{x^4}{2}+\frac{x^6}{3}+\cdots\qquad \text{for}\ |x|<1.$$

Next we analyze what happen when $x=-1$ and $x=1$. At both $x=-1$ and $x=1$, the series become

$$\ln(2)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

Thus the series converges and the interval of convergence for $\ln(1+x^2)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}$ is $[-1,1]$.

**Example 3.14** Find a power series representation for the function $f(x)=\ln(1+x)$ and determine its interval of convergence.

**Solution:** Again we start with the power series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \qquad \text{for } |x|<1$$

Now replace $x$ by $-x$

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \qquad \text{for } |x|<1$$

Therefore by the above theorem we find that

$$\ln(1+x)=\int_0^x \frac{1}{1+t}\,dt=\int_0^x\left(\sum_{n=0}^{\infty}(-1)^n t^n dt\right)=\sum_{n=0}^{\infty}\int_0^x(-1)^n t^n dt$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$$

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} \qquad \text{for } |x|<1$$

At $x=-1$, the series becomes $-\sum_{n=0}^{\infty}\frac{1}{n+1}$ and it is divergent. When $x=1$, the series becomes $\ln 2=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ and it is convergent.

Hence, $\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} \quad \text{for } -1<x\le 1$

The power series expansion of $\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$, for $-1<x\le 1$ is some times known as Mercator's series.

Note that

$$\ln x=\ln(1+x-1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n} \qquad \text{for } 0<x\le 2$$

**Example 3.15** Express $\ln\left(\frac{1+x}{1-x}\right)$ as a power series for $|x|<1$.

**Solution:** Here $\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)$

But

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} \quad \text{and} \quad \ln(1-x)=\sum_{n=0}^{\infty}\frac{(-1)^{2n+1} x^{n+1}}{n+1}$$

$$\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}\left((-1)^n-(-1)^{2n+1}\right)=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\left((-1)^n-(-1)\right)$$

$$=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots+\frac{x^{2n+1}}{2n+1}+\cdots\right)$$

Therefore, $\ln\left(\frac{1+x}{1-x}\right)=2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$ for $|x|<1$.

**Example 3.16** a) Show that $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ for $|x|<1$

b) Find a series expansion of $\pi$ and approximate $\pi$ with error less than $10^{-6}$

**Solution a)** $\tan^{-1}x=\int_0^x\frac{1}{1+t^2}dt$ and $\frac{1}{1+t^2}=\frac{1}{1-(-t^2)}=\sum_{n=0}^{\infty}(-1)^n t^{2n}$ for $|t|<1$

The integration Theorem implies

$$\tan x^{-1}=\int_0^x\frac{1}{1+t^2}dt=\int_0^x\left(\sum_{n=0}^{\infty}(-1)^n t^{2n}\right)dt=\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{2n+1}\right),\ |x|<1$$

$$\boxed{\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots,|x|<1}$$

b) If $x=\frac{1}{\sqrt{3},}$, then $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}=\frac{1}{\sqrt{3}}\quad\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)}$

Thus, $\pi=\ \sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)}=2\sqrt{3}\left(1-\frac{1}{9}+\frac{1}{45}-\frac{1}{189}+\cdots\right).$

Is it possible for you to approximate ln2 to four decimal place of accuracy?

The series $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad if \quad |x|<1$ is called Gregory's **series**

Although it is not east to prove Gregory's series it holds for x=1and this yields

$$\boxed{\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$$

**Example 3.17** Show that $\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

Using this formula and Gregory's series approximate $\pi$ accurate to four decimal places

**Solution:** Let $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$ and $\beta = \tan^{-1}\left(\frac{1}{3}\right)$. Then

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha + \beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} . \frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

Taking tan inverse we obtain

$$\alpha + \beta = \tan^{-1}(-1) = \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

On the other hand from Gregory's series with $x = \frac{1}{2}$ we have

$$\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{7}\left(\frac{1}{2}\right)^7 + \left(\frac{1}{9}\right)^9 - \frac{1}{11}\left(\frac{1}{2}\right)^{11} + \frac{1}{13}\left(\frac{1}{2}\right)^{13} - \frac{1}{15}\left(\frac{1}{2}\right)^{15}$$

$\approx$ 0.5-0.04167+0.00625-0.00112+0.0022

$\approx$ 0.46365 With an error less than the 6$^{th}$ terms

Similarly, $\tan^{-1}\left(\frac{1}{3}\right) \approx 0.32174$ with an error less than the 6$^{th}$ terms.

$$\pi = 4\left(\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right)$$

$$\approx 4(0.46365 + 0.32174)$$

$$\pi \approx 3.1416$$

Thus we can approximate $\pi \approx 3.1416$ to four decimal places.

**Example 3.18** Find a power series representation for the following function using term wise integration

$$a)\quad f(x)=\int_0^x \frac{arc\tan t}{t}dt$$

$$b)\quad f(x)=\int_0^x \frac{1-e^{-t^2}}{t^2}dt$$

$$c)\quad f(x)=\tanh^{-1}x \ =\int_0^x \frac{1}{1-t^2}dt$$

**Solution**: Here $\tan^{-1}x \ =\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ , $|x|<1$. Dividing both sides by $x$ implies

$$\frac{\tan^{-1}x}{x} \ = \ 1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\ldots \quad |x|<1$$

By integration theorem we get

$$f(x)=\int_0^x \frac{\tan^{-1}t}{t}dt=\int_0^x\left[1-\frac{t^2}{3}+\frac{t^4}{5}-\frac{t^6}{7}+\ldots\right]dt$$

$$=x-\frac{x^3}{9}+\frac{x^5}{25}-\frac{x^7}{49}+\cdots$$

$$\int_0^x \frac{arc\tan t}{t}dt \ = \ \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)}$$

b) We know that $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ $\quad \forall x\in \mathbb{R}$ . Replace $x$ by $-x^2$

We find that

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$$

$$=1+x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}\cdots$$

With simply calculation

$$\frac{1-e^{-t^2}}{t^2}=\frac{1}{t^2}\left[t^2-\frac{t^4}{2!}+\frac{t^6}{3!}+\frac{t^8}{4!}+\ldots\right]=1-\frac{t^2}{2!}+\frac{t^4}{3!}-\frac{t^6}{4!}+\cdots$$

Applying integration Theorem yields

$$f(x)=\int_0^x \frac{1-e^{-t^2}}{t^2}\,dt=\int_0^x\left[1-\frac{t^2}{2!}+\frac{t^4}{3!}-\frac{t^6}{4!}+\frac{t^8}{5!}\ldots\right]dt \;=x-\frac{x^3}{2!\;3}+\frac{x^5}{3!\;5}-\frac{x^7}{4!\;7}+\frac{x^9}{5!\;9}-\cdots$$

Hence,

$$f(x)=\int_0^x \frac{1-e^{-t^2}}{t^2}\,dt=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(n+1)n!}$$

c) For $|x|<1,$ we know that $\frac{1}{1-x}=1+x+x^2+x^3+\ldots=\sum_{n=0}^{\infty}x^n$ . Now replace $x \quad by \quad t^2$

$$\frac{1}{1-t^2}=1+t^2+t^4+t^6\ldots=\sum_{n=0}^{\infty}t^{2n}$$

Integration Theorem implies

$$\int_0^x \frac{1}{1-t^2}\,dt=\int_0^x\left(1+t^2+t^4+t^6+\ldots\right)dt \;=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots$$

$$\boxed{\tanh^{-1}x=\int_0^x \frac{1}{1-t^2}\,dx=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}}$$

# Exercise 3. 2

1. Find a power series representation for the following functions

   a) $f(x)=\dfrac{1}{3-x}$ b) $f(x)=\dfrac{1+x^2}{1-x^2}$ c) $f(x)=\ln\left(\dfrac{1+x}{1+x^2}\right)$

   d) $g(x)=\int_0^x (e^t+\ln(t+1))dt$ e) $f(x)=\int_0^x e^{t^2}dt$ f) $g(x)=x\ln(1+x)$

2. For the following functions, use a power series representation to approximate the definite integral to within $10^{-4}$.

   a) $\int_0^{1/2}\dfrac{1}{1+x^3}dx$ b) $\int_{-0.01}^{0}\dfrac{(e^{-x}+x-1)}{x^2}dx$

3. Find a series representation for the definite integral $\int_0^1 xe^{x^2}dx$

4. Show that $\int_0^1 \dfrac{1}{1+t^2}dt=\sum_{n=0}^{\infty}(-1)^n\dfrac{1}{2n+1}$

5. Find a power series representation for the function and determine the radius of convergence

   a) $f(x)=\dfrac{x^2}{(1-2x)^2}$ b) $f(x)=\arctan\left(\dfrac{x}{3}\right)$

6. Evaluate the following indefinite integral as a power series

   a) $\int\dfrac{\arctan x}{x}dx$ b) $\int\tan^{-1}(x^2)dx$ c) $\int x^2\tan^{-1}(x^4)dx$

# 3.2 TAYLOR POLYNOMIALS

In this section we show how a function can be approximated as closely as desired by a polynomial, provided that the function possesses a sufficient number of derivatives.

**Definition 3.3** Let $f$ be a function such that the n$^{\text{th}}$ derivative $f^{(n)}$ exists at a in some interval containing a as interior point. Then the polynomial

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{[n]}(a)(x-a)^n}{n!} \quad \text{or}$$

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the n$^{\text{th}}$ degree **Taylor's polynomials** $f$ of at a.

**Example 3.19** Expand $g(x) = 1 + 2x^2 + 5x^3 - 9x^4$ in powers of $x-1$.

**Solution**: Here, n =4 and a=1.

$g(x) = 1 + 2x^2 + 5x^3 - 9x^4$ $\qquad$ $g(1) = -1$

$g'(x) = 4x + 15x^2 - 36x^3$ $\qquad$ $g'(1) = 4 + 15 - 36 = -17$

$g''(x) = 4 + 30x - 108$ $\qquad$ $g''(1) = 4 + 30 - 108 = -74$

$g'''(x) = 30 - 216x$ $\qquad$ $g'''(1) = 30 - 216 = -186$

$g^{[4]}(x) = -216$ $\qquad$ $g^{[4]}(1) = -216$

The Taylor's polynomials of degree 4 is given by

$$p_4(x) = g(1) + g'(1)(x-1) + \frac{g''(1)}{2!}(x-1)^2 + \frac{g'''(1)}{3!}(x-1)^3 + \frac{g^{[4]}(1)}{4!}(x-1)^4$$

$$= -1 - 17(x-1) - 37(x-1)^2 - 31(x-1)^3 - 9(x-1)^4$$

**Example 3.20** Find the fourth degree Taylor's polynomial of $f(x) = \sin x$ at $a = \frac{\pi}{2}$.

**Solution:** Here $f(x) = \sin x$ $\qquad$ $f(\frac{\pi}{2}) = 1$

$f'(x) = \cos x$ $\qquad$ $f'(\frac{\pi}{2}) = 0$

$f''(x) = -\sin x \qquad f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x \qquad f'''(\frac{\pi}{2}) = 0$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(\frac{\pi}{2}) = 1$

Thus the fourth degree polynomial is given by

$$p_4(x) = f(\pi) + f'(\frac{\pi}{2})\left(x - \frac{\pi}{2}\right) + \frac{f''(\frac{\pi}{2})}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{f'''(\frac{\pi}{2})}{3!}\left(x - \frac{\pi}{2}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{2}\right)}{4!}\left(x - \frac{\pi}{2}\right)^4$$

$$= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4$$

**Example 3.21** Let $f(x) = \int_0^x \tan(\sin t)dt$ .Then find the Taylor polynomial of order 4 at 0.

**Solution:** Here $f(x) = \int_0^x \tan(\sin t)dt \qquad f(0) = 0$

$f'(x) = \tan(\sin x) \qquad f'(0) = 0$

$f''(x) = \sec^2(\sin x)\cos x \qquad f''(0) = 1$

$f'''(x) = \sec^2(\sin x)[2\tan(\sin x)\cos^2 x - \sin x] \qquad f'''(0) = 0$

$$f^{(4)}(x) = 2\sec^2(\sin x)\tan(\sin x)\cos x[2\tan(\sin x)\cos^2 x - \sin x] + \sec^2(\sin x)[2\sec^2(\sin x)\cos^3 x - 4\tan(\sin x)\cos x\sin x - \cos x] \qquad f^{(4)}(0) = 1$$

This implies that

$$P_4(x) = \frac{1}{2}x^2 + \frac{1}{24}x^4$$

**Example 3.22** Let $f(x) = \ln(1+x)$ . Then

i) Find a formula for the n$^{th}$ Taylor polynomial of

ii) Calculate $p_6(1)$ .

**Solution**: We have $f(x) = \ln(1+x) \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1$

$$f'''(x)=\frac{(-1)^2 2!}{(1+x)^3} \qquad f'''(0)=2!$$

$$f^{[4]}(x)=\frac{(-1)^3 3!}{(1+x)^4} \qquad f^{[4]}(0)=3!$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \qquad f^{(n)}(0)=(-1)^{n-1}(n-1)!$$

Therefore the n[th] Taylor polynomial is

$$p_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n-1}x^n}{n}$$

ii) $p_6(1)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}=\frac{37}{60}\approx 0.616667$

**Example 3.23** Let $f(x)=\dfrac{1}{1-x}$. Then

i) Find a formula for the n[th] Taylor polynomial of $f$

ii) Compute $p_n(2)$

**Solution:** As in the above example

$$f(x)=\frac{1}{1-x} \qquad f(0)=1$$

$$f'(x)=\frac{1}{(1-x)^2} \qquad f'(0)=1$$

$$f''(x)=\frac{2!}{(1-x)^3} \qquad f''(0)=2!$$

$$f'''(x)=\frac{3!}{(1-x)^4} \qquad f'''(0)=3!$$

$$\vdots \qquad \vdots$$

$$f^{n}(x)=\frac{(n-1)!}{(1-x)^n} \qquad f^{(n)}(0)=(n-1)!$$

Hence we get , $p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)x^n}{n!}$

$$= 1 + x + x^2 + x^3 + \cdots + x^n$$

ii) $p_n(2) = 1 + 2 + 2^2 + 2^3 + \ldots + 2^n$

The Taylor polynomial of degree one, $P_1(x) = f(a) + f'(a)(x-a)$ is familiar with equation for the tangent line to $y = f(x)$ at a point a. Similarly,

$P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$ is the tangent parabola to $y = f(x)$ at a point a. Since the first $n$ derivatives of $P_n(x)$ at $a$ are equal to the corresponding first $n$ derivatives of $f(x)$ at $a$, then $P_n(x)$ is an approximation for $f(x)$ if $x$ is near $a$.

## Exercise 3.3

Find the n[th] degree Taylor's polynomials of the following functions at the indicated point

a) $f(x) = \cos x$ at x = $\frac{\pi}{4}$

b) g(x) = $\frac{1}{1+2x}$ at x = 2

c) $g(x) = \cosh x$ at $x = 0$.

d) h (x) = $\ln\left(\frac{1+x}{1+x^2}\right)$ at $x = 0$

e) $f(x) = \int_0^x \ln(1+x)dt$

**Definition 3.4:** Let $p_n(x)$ be the n[th] degree Taylor polynomial of the function $f$. Then **remainder term**, denoted $R_n(x)$ is given by

$$\boxed{R_n(x) = f(x) - p_n(x)}$$

Now we state and prove the following theorem.

**Theorem 3.6**: (**Taylor's Theorem with Remainder**)

If a function $f$ has $n+1$ derivatives on an open interval $I$ about $a$ and $x$ is in this interval, then a number $c$ between $x$ and $a$ exists with

$$f(x)=P_n(x)+R_n(x)$$

Where

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}\left(x-a\right)^k \quad \text{and} \quad R_n(x)=\frac{f^{(n+1)}(c)}{\left(n+1\right)!}\left(x-a\right)^{n+1}$$

**Proof:** We show that if $f^{(n+1)}(x)$ exists in [a, b], then for any number $x\in[a,b]$, there is a number $c$ in $[a,x]$ such that

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(x-a)^n}{n!}+R_n(x) \qquad (1)$$

Where

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad (2)$$

Let $x\in[a,b]$ be fixed. We define a new function $h(t)$ by

$$h(t)=f(x)-f(t)-f'(t)(x-t)-\frac{f''(t)}{2!}(x-t)^2-\cdots-\frac{f^{(n)}(t)}{n!}(x-t)^n-\frac{R_n(x)(x-t)^{n+1}}{(x-a)^{n+1}} \qquad (3)$$

Then

$$h(x)=f(x)-f(x)-f'(x)(x-x)-\frac{f''(x)}{2!}(x-x)^2-\cdots-\frac{f^{(n)}(x)}{n!}(x-x)^n-\frac{R_n(x)(x-x)^{n+1}}{(x-a)^{n+1}}=0$$

and

$$h(a)=f(x)-f(a)-f'(a)(x-a)-\frac{f''(a)}{2!}(x-a)^2-\cdots-\frac{f^{(n)}(a)}{n!}(x-a)^n-\frac{R_n(x)(x-a)^{n+1}}{(x-a)^{n+1}}$$

$$=f(x)-P_n(x)-R_n(x)=R_n(x)-R_n(x)=0.$$

Since $f^{(n+1)}(x)$ exists, $f^{(n)}$ is differentiable so that $h$, being a sum of products of differentiable functions, is also differentiable for $t$ in $(a,x)$. Remember that $x$ is fixed so $h$ is a function of $t$ only.

Recall that Rolle's Theorem which states that if $h$ is continuous on [a, b], differentiable on (a, b), and $h(a)=h(b)=0$, then there is at least one number $c$ in (a, b) such that $h'(c)=0$. We see that the condition of Rolle's Theorem hold in the interval [a, x] so that there is a number $c$ in (a, x) with $h'(c)=0$

$$\frac{d}{dt}(f'(t)(x-t)) = -f'(t) + f''(t)(x-t) \qquad \text{Why?} \tag{4}$$

Also for $1 \le k \le n$

$$\frac{d}{dt}\left[\frac{f^{(k)}(t)}{k!}(x-t)^k\right] = -\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k \tag{5}$$

Using (4) and (5) on (3) it follows that

$$h'(t) = \frac{-f^{(n+1)}(t)(x-t)^n}{n!} + \frac{(n+1)R_n(x-t)^n}{(x-a)^{n+1}}.$$

Then, setting $t=c$, we obtain

$$0 = h'(c) = \frac{-f^{(n+1)}(c)(x-c)^n}{n!} + \frac{(n+1)R_n(x)(x-c)^n}{(x-a)^{n+1}}.$$

Finally, dividing the equations above through by $(x-c)^n$ and solving for $R_n(x)$, we obtain

$$R_n(x) = \frac{f^{(n+1)}(c)(x-a)^n}{(n+1)n!} = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}.$$

This is what we wanted to prove. ♦

In the above theorem $f(x) = P_n(x) + R_n(x)$ is called Taylor's formula and $R_n$ is called Lagrange remainder formula.

**Remark:** If you go over the proof, you may observe that we didn't need to assume that $x > a$; if we replace the interval [a, x] with the interval [x, a], then all results are still valid.

**Theorem 3.7**: (**Uniqueness of the Taylor Polynomial**)

If $f(x) = T_n(x) + R_n(x)$, where $T_n$ is the n$^{\text{th}}$ -degree Taylor polynomial of $f$ at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then $f$ is equal to the sum of its Taylor series on the interval $|x-a| < R$.

.

**Proof:** Suppose that there is another nth degree polynomial $Q_n(x)$ such that

$$f(x) = P_n(x) + R_n(x) = Q_n(x) + S_n(x)$$

Where

$$\lim_{x\to a}\frac{R_n(x)}{(x-a)^n} = 0, \text{ and } \lim_{x\to a}\frac{S_n(x)}{(x-a)^n} = 0.$$

Let

$$D(x) = P_n(x) - Q_n(x) = S_n(x) - R_n(x).$$

We will show that $D(x) = 0$, which will imply that $P_n(x) = Q_n(x)$. This will show that $P_n(x)$ is unique. Since $D(x)$ is the difference of nth degree polynomials, $D(x)$ is a polynomial of degree $\le n$, and it can be written in the form

$$D(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n.$$

Now

$$\lim_{x\to a}\frac{D(x)}{(x-a)^n} = \lim_{x\to a}\frac{S_n(x)}{(x-a)^n} - \lim_{x\to a}\frac{R_n(x)}{(x-a)^n} = 0 - 0 = 0$$

Similarly, if $0 \le m < n$,

$$\lim_{x\to a}\frac{D(x)}{(x-a)^m} = \lim_{x\to a}\frac{(x-a)^{n-m}D(x)}{(x-a)^n} \qquad \text{(Multiply and divide by } (x-a)^{n-m}\text{)}$$

$$= \lim_{x\to a}(x-a)^{n-m}\lim_{x\to a}\frac{D(x)}{(x-a)^n} = 0\cdot 0 = 0$$

and we have

$$\lim_{x \to a} \frac{D(x)}{(x-a)^m} = 0 \text{ for } m = 0,1,2,\cdots,n. \qquad (6)$$

To complete the proof, we note that $b_0 = D(a) = \lim_{x \to a} D(x) = 0$ by (6) with $m = 0$.

Then

$$D(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$$

so that

$$\frac{D(x)}{x-a} = b_1 + b_2(x-a) + \cdots + b_n(x-a)^{n-1}$$

and

$$b_1 = \lim_{x \to a} \frac{D(x)}{x-a} = 0 \text{ by (6) with } m = 1$$

Suppose we have show that $b_0 = b_1 = \cdots = b_k = 0,$ where $k < n$. Then

$$D(x) = b_{k+1}(x-a)^{k+1} + b_{k+2}(x-a)^{k+2} + \cdots + b_n(x-a)^n$$

so that

$$\frac{D(x)}{(x-a)^{k+1}} = b_{k+1} + b_{k+2}(x-a) + \cdots + b_n(x-a)^{n-k-1}$$

and

$$b_{k+1} = \lim_{x \to a} \frac{D(x)}{(x-a)^{k+1}} = 0. \text{ By (6) with } m = k+1 \le n.$$

This show that $b_0 = b_1 = b_2 = \cdots = b_n = 0$, which means that $D(x) = 0$ for every x in [a, b] so that $P_n(x) = Q_n(x)$ for every x in [a, b]. Thus the Taylor polynomial is unique. ♦

This theorem helps us to show that when ever the Taylor polynomial converges it converges to itself.

**Theorem 3:8 (Taylor' Inequality)** If $f$ has $n+1$ continuous derivatives, then there exists a positive number $M_n$ such that

$$|R_n(x)| \le M_n \frac{|x-a|^{n+1}}{(n+1)!}$$

for all $x$ in $[a,b]$. Here $M_n$ is an upper bound for $(n+1)st$ derivative of $f$ in the interval$[a,b]$.

**Proof:** Since $f^{(n+1)}$ is continuous on [a, b], it is bounded above and below on that interval. That is, there is a number $M_n$ such that $|f^{(n+1)}(x)| \le M_n$ for every x in [a, b].

Since

$$R_n(x) = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}$$

With $c$ in (a, x), we see that

$$|R_n(x)| = |f^{(n+1)}(c)|\frac{(x-a)^{n+1}}{(n+1)!} \le M_n \frac{(x-a)^{n+1}}{(n+1)!}$$

And the theorem is proved. ♦

**Examples 3.24** Write a Taylor's formula for $g(x) = \tan^{-1}(x)$, $n=3$, at $a=1$

$$g(x) = \tan^{-1}(x) \qquad g(1) = \frac{\pi}{4}$$

$$g'(x) = \frac{1}{1+x^2} \qquad g'(1) = \frac{1}{2}$$

$$g''(x) = \frac{-2x}{(1+x^2)^2} \qquad g''(1) = -\frac{1}{2}$$

$$g'''(x) = \frac{-2+6x^2}{(1+x^2)^3} \qquad g'''(1) = \frac{1}{2}$$

$$g^{(4)}(x) = \frac{18x-24x^3}{(1+x^2)^4} \qquad g^{(4)}(z) = \frac{18z-24z^3}{(1+z^2)^4}$$

Therefore the required formula is given by

$$g(x) = P_4(x) + R_4(x)$$

$$= g(1) + g'(1)(x-1) + \frac{g''(1)}{2!}(1)(x-1)2 + \frac{g'''}{3!}(1)(x-1)^3 + \frac{g^{(4)}(z)}{4!}(x-1)^4$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \frac{18z-24z^4}{(1+z^2)^4}(x-1)^4$$

If in definition 6.3 $a$ is replaced by $0$, then the Taylor's Formula reduced to

$$f(x) = f(0) + f'(0) + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$$

for some $z$ between $0$ and $x$. This formula is called Maclaurin formula.

**Example 3.25:** Find the Maclaurin formula for $f(x) = e^x$.

**Solution**: Here we calculate derivative of few terms

$$\begin{array}{ll} f(x) = e^x & f(0) = 1 \\ f'(x) = e^x & f'(0) = 1 \\ \vdots & \vdots \\ f^{(n)}(x) = e^x & f^{(n)}(0) = 1 \\ f^{(n+1)}(z) = e^z & \end{array}$$

The required Maclaurin formula is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{e^z}{(n+1)!}x^{n+1}$$

## 3.4 Approximation by Taylor's polynomials

If $P_n(x)$ and $R_n(x)$ are Taylor's polynomials and the n^th remainder terms of $f$ at a respectively, then

$$f(x) = R_n(x) + P_n(x)$$

$$\left|f(x) - P_n(x)\right| = \left|R_n(x)\right|$$

If we can find a number M such that $\left|R_n(x)\right| \leq M$ then we will have

$$\left|f(x) - P_n(x)\right| \leq M$$

Or

$$P_n(x) - M \leq f(x) \leq P_n(x) + M$$

We can write

$$f(x) = P_n(x) \pm M$$

This means $f(x) \approx P_n(x)$ with an error of not more then M limits.

**Example 3.26**: Approximate sin x by a fourth degree polynomial in $x$ if $0 \leq x \leq 0.2$

**Solution:** Here $f(x) = \sin x \qquad f(0) = 0$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(z) = \cos z$$

Therefore, $P_4(x) = x - \dfrac{x^3}{6}$ and $R_4(x) = \dfrac{\cos z}{5!}x^5$. Now for $0 < z < 0.2$, take $x = 0.2$ and

$0 \leq R_4(x) \leq \dfrac{1}{120}(0.2)^5 = 2.\dot{6} \times 10^{-6} < 3 \times 10^{-6}$ since $\left|\cos z\right| \leq 1$.

Hence in the given range $\sin x$ will be approximated by

$$\sin x = x - \frac{x^3}{3!} \pm 0.000003$$

**Example 3.27:** Approximate $\sqrt[3]{e}$ to an accuracy of five decimal places.

**Solution:** Recall that the Taylor's formula for

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{e^z}{(n+1)} x^{n+1} \quad \text{and } R_n(x) = \frac{e^z}{(n+1)!} x^{n+1}, \text{ for } 0 < z < x.$$

We need to find $n$ so that $\left| R_n\left(\frac{1}{3}\right) \right| < 10^{-5}$. Clearly $e^{\frac{1}{3}} < 2$ and so

$$\left| R_4\left(\frac{1}{3}\right) \right| < \frac{2}{(n+1)!}\left(\frac{1}{3}\right)^{n+1} = \frac{2}{(n+1)!3^{n+1}}$$

If $n = 5$, then $\left| R_5\left(\frac{1}{3}\right) \right| < \frac{2}{6!3^6} = \frac{1}{360 \times 729} < 10^{-5}$.

Thus $\sqrt[3]{e} \cong 1 + \frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{3!}\left(\frac{1}{3}\right)^3 + \frac{1}{4!}\left(\frac{1}{3}\right)^4 + \frac{1}{5!}\left(\frac{1}{3}\right)^5 \cong 1.39563$

## 3.5 Taylor series

In section 6.1.1 we have seen that the natural exponential function can be represented by a power series that converges for all real numbers. In this section we discuss power series representations for other functions.

1) Which functions have power series representation?

2) Is the Taylor's series of $f$ equal to $f$ in its interval of convergence?

**Definition 3.5:** Let $f$ has derivatives of all orders at $c$. Then the Taylor's series of $f$ at $c$ is defined as

$$\sum_{n=0}^{\infty} f^{[n]}(a)\frac{(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

If a = 0, then series is called the Maclaurin series of $f$.

**Example 3.28** Find the Taylor series for $f(x) = \ln(x)$ at $x = 2$

**Solution:** Here we calculate the derivative of few terms and evaluate at $x = 2$.

$$f(x) = \ln(x) \qquad f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \qquad\qquad f''(2) = -\frac{1}{2^2}$$

$$f'''(x) = \frac{2}{x^3} \qquad\qquad f'''(2) = \frac{2}{2^3}$$

$$f^{(4)}(x) = -\frac{2(3)}{x^4} \qquad\qquad f^{(4)}(2) = -\frac{2(3)}{2^4}$$

$$f^{(5)}(x) = \frac{2(3)(4)}{x^5} \qquad\qquad f^{(5)}(2) = \frac{2(3)(4)}{2^5}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$f^{(n)}(x) = \frac{(-1)^n(n-1)!}{x^n} \qquad\qquad f^{(n)}(2) = \frac{(-1)^n(n-1)!}{2^n} \qquad\qquad \forall n \in \mathbb{N}$$

In order to find the Taylor series expansion we apply the above definition.

$$\ln(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$= f(2) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!2^n}(x-2)^n$$

$$\ln(x) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n$$

Now we state the following theorem which answer question (2).

**Theorem 3.9** If a function $f$ has derivatives of all orders on an open interval about ***a*** and if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all $x$ in that interval, then $f$ has the Taylor-series representation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

for all $x$ in the interval.

**Proof:** See reference book

**Example 3.29** Let $f(x)=\begin{cases} e^{\frac{-1}{x^2}} & , x\neq 0 \\ 0 & , x=0 \end{cases}$. Find the Taylor series of $f$ at $a=0$.

**Solution:** First show that for any $n\in\mathbb{N}$,

$$\lim_{x\to 0}\frac{1}{x^n e^{\frac{1}{x^2}}}=0 \qquad (1)$$

To see this we substitute $y=\frac{1}{x^2}$ to get

$$\lim_{x\to 0}\frac{1}{x^n e^{\frac{1}{x^2}}}=\lim_{x\to 0}\frac{1}{x^{2n} e^{\frac{1}{x^2}}}\cdot x^n=\left(\lim_{y\to\infty}\frac{y^n}{e^y}\right)\cdot\lim_{x\to 0}x^n=0$$

Next we compute $f'(0)$.

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}-0}{x}=\lim_{x\to 0}\frac{1}{xe^{\frac{1}{x^2}}}=0 \quad \text{by (1)}$$

For $x\neq 0$

$$f'(x)=\frac{d}{dx}\left(e^{-\frac{1}{x^2}}\right)=2x^{-3}e^{-\frac{1}{x^2}}$$

Thus,

$$f'(x)=\begin{cases} 2x^{-3}e^{\frac{-1}{x^2}} & , x\neq 0 \\ 0 & , x=0 \end{cases}$$

Next we compute $f''(x)$.

$$f''(x)=\lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0}=\lim_{x\to 0}\frac{2x^{-3}e^{-\frac{1}{x^2}}-0}{x}=2\lim_{x\to 0}\frac{1}{x^4 e^{\frac{1}{x^2}}}=0 \text{ by (1)}$$

Again for $x\neq 0$

$$f''(x)=\frac{d}{dx}\left(2x^{-3}e^{-\frac{1}{x^2}}\right)=\left(-6x^{-4}+4x^{-6}\right)e^{-\frac{1}{x^2}}$$

Similar calculation will lead to

$$f(0)=f'(0)=f''(0)=\cdots=0$$

Thus we have

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}=\sum_{n=0}^{\infty}\frac{0\cdot x^n}{n!}=0+0x+0x^2+\cdots=0$$

Clearly, at any $x\neq 0$, $f(x)=e^{\frac{-1}{x^2}}\neq 0$. Therefore, $f(x)\neq\sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}$

**Example 3.30** Find the Maclaurin series expansions of $f(x)=e^x$ and show it represents $e^x \quad \forall x\in\mathbb{R}$ .

**Solution:** Recall that $f^{[n]}(x)=e^x, \quad \forall n\in\mathbb{N}$ and $f^{[n]}(0)=e^1=1 \quad \forall n\in\mathbb{N}$

The Maclaurin series of $f$ is given by

$$f(x)=f(0)+f'(0)+\frac{f''}{2!}(0)x^2+\frac{f'''}{3!}(0)x^3+...+\frac{f^{(n)}}{n!}x^n+R_n(x)$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+R_n(x)$$

$$=\sum_{n=0}^{\infty}\frac{x^n}{n!}+R_n(x) \qquad \text{where } R_n(x)=\frac{e^z}{(n+1)!}x^{n+1}$$

We have shown that the series converges on $(-\infty,\infty)$. Does it converge to $e^x \ \forall x\in\mathbb{R}$ ?

$R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} = \frac{e^z x^{n+1}}{(n+1)!}$, where z lies between 0and $x$

To answer this question we need to show that $\lim_{n\to\infty}R_n(x)=0$. For this we consider the following cases.

Case 1: If $0<z<x$ , then $e^z<e^x$.

$$0\leq\lim_{n\to\infty}R_n(x)=\lim_{n\to\infty}\frac{e^z x^{n+1}}{(n+1)!}\leq\lim_{n\to\infty}\frac{e^x x^{n+1}}{(n+1)!}=0$$

Therefore, by squeezing theorem

$$\lim_{n\to\infty}R_n(x)=0$$

Case 2: If $x<z<0$ , then $e^x<e^z<1$

$$0\leq\left|R_n(x)\right|=\left|\frac{e^z x^{n+1}}{n+1}\right|=e^z\left|\frac{x^{n+1}}{n+1}\right|$$

$$0 \le \lim_{n\to\infty}\left|R_n(x)\right| = 0$$

$$\lim_{n\to\infty} R_n(x) = 0$$

Hence $\lim_{n\to\infty} R_n(x) = 0$ in both case it converges to itself and thus

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

**Example 3.31** Find the Maclaurin series of $f(x) = \sin x$ and prove that it represents f for all $x \in \mathbb{R}$

**Solution:** The Maclaurin series of expansion f(x) is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(0) = \cos x & f'(0)=1 \\ f''(0) = -\sin x & f''(0) = 0 \\ f'''(0) = -\cos x & f'''(0) = -1 \\ f^{(4)}(x) = \sin x, & f^{(4)}(0) = 0 \\ f^{(2k)}(x) = (-1)^k \sin x & f^{(2k)}(0) = 0 \\ f^{(2k+1)}(x) = (-1)^k cosx & f^{(2k+1)}(0) = (-1)^k. \end{array}$$

The required Maclaurin series is

$$\sin x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^k x^{2k+1}}{2k+1} + R_{2k+1}(x)$$

Where $R_{2k+1}(x) = \dfrac{f^{(2k+2)}(z)x^{2k+2}}{2k+2}$ for $0 < z < x$

To show that this series represents $\sin x$ for all values of $x$, we must show that

$$\lim_{n\to\infty} R_n(x) = 0$$

for all values of $x$.

Taylor's Theorem implies that for a fixed but arbitrary value of $x$ there is a number $z$ between $x$ and zero with

$$0 \le \left|R_{(2k+1)}(x)\right| = \left|\frac{f^{(2k+2)}(z)\,x^{2k+2}}{(2k+2)}\right| = f^{(2k+2)}(z)\left|\frac{x^{2k+2}}{2k+2}\right|$$

Observe that

$$\left|f^{(2k+2)}(z)\right| = \left|\cos z\right| \le 1 \ \ and \quad g(z) = \left|\sin z\right| \le 1$$

It follows that

$$0 = \lim_{n\to\infty}\left|R_n(x)\right| \le \lim_{n\to\infty}\left|\frac{x^{2n+2}}{2n+2}\right| = 0$$

By squeezing theorem $\lim_{n\to\infty} R_n(x) = 0$

Thus for all real number $x$ Maclaurin series for sin x converges to sin x.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)\,x^{2n+1}}{(2n+1)}$$

If $x = \frac{\pi}{4}$, then the values of $\sin x$ for five decimal places of accuracy is

$$\sin\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\left(\frac{\pi}{4}\right)^3}{3!} + \frac{\left(\frac{\pi}{4}\right)^5}{5!} - \frac{\left(\frac{\pi}{4}\right)^7}{7!} + \frac{\left(\frac{\pi}{4}\right)^9}{9!}$$

$$\sin\left(\frac{\pi}{4}\right) = 0.70710$$

**Example 3.32** Find the Maclaurin series expansion of $f(x) = \cos x$ and show that it represented by $f(x)$ for all x.

**Solution:** From example 6.31 we know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)\,x^{2n+1}}{(2n+1)}$$

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots\right) = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)\,x^{2n+1}}{(2n+1)}\right)$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

From Taylor’s formula for $\cos x$ with $n = 2k$ we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + R_{2k}$$

We need to show that $\lim_{n\to\infty} R_n(x) = 0$ for all x.

$$0 \le |R_{2k}(x)| = \left|\frac{f^{(2k+1)}(z)x^{2k+1}}{(2k+1)!}\right| \le \left|f^{(2k+1)}(z)\right| \frac{|x|^{2k+1}}{2k+1}$$

$$0 \le |R_{2k}(x)| = \left|\frac{(\sin z)\, x^{2k+1}}{(2k+1)!}\right| \le \frac{|x|^{2k+1}}{2k+1} \quad \text{Since } |\sin z| \le 1.$$

Computing limit we get

$$0 \le \left|\lim_{k\to\infty} R_{2k}(x)\right| \le \lim_{k\to\infty} \frac{|x|^{2k+1}}{2k+1} = 0$$

Therefore, by the squeeze theorem for limits at infinity, $\lim_{n\to\infty} R_n(x) = 0$.

Thus the Maclaurin series of $\cos x$ converges to itself.

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$$

**Example 3.33** Find the Maclaurin series for $f(x) = \sin^2 x$.

**Solution:** From trigonometric identity

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$$

From example 6.25

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

Now substituting and simplifying we obtain

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \cdots$$

$$= x^2 - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \cdots$$

$$\sin^2 x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n-1} x^{2n}}{(2n)!}$$

**Example 3.34** Find the first three terms of the Maclaurin series for $f(x) = e^x \cos x$.

**Solution:** $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

$$e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots + x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{48} - \frac{x^8}{720} + \cdots$$

$$= 1 + x + \left(-\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{24}\right)x^4 + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

**Example 3.35** Approximate $\int_0^1 e^{-x^2} dx$ with an error less than $0.0005$.

**Solution**: Here $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

By integration theorem we have

$$\int_0^1 e^{-x^2} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}$$

By alternating series test we can approximate $\int_0^1 e^{-x^2} dx$ to any desired degree of accuracy and hence $\int_0^1 e^{-x^2} dx \approx s_n$ with an error less than 0.0005. Now find n so that

$$|a_{n+1}| < 0.0005 = \frac{1}{2000}$$

$$\frac{1}{(n+1)!(2n+3)} < \frac{1}{2000}$$

Solving this we get n = 5 and

$$\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{3} + \frac{1}{2!5} - \frac{1}{3!7} + \frac{1}{4!9} - \frac{1}{5!11} = 0.7467$$

√ Check list

Mark a tick (√) against each of the following tasks that you can perform. I can

- Give definition of power series ----------------------------------------------- ☐
- Give the difference between series and power series------------------------ ☐
- Define radius of convergence and interval of convergence------------------ ☐
- Calculate radius of convergence and interval of convergence----------------- ☐
- Give power series representation of a function-------------------------------- ☐
- State differentiation theorem ---------------------------------------------------- ☐
- Differentiate power series-------------------------------------------------------- ☐
- State integration theorem -------------------------------------------------------- ☐
- Integrate power series ------------------------------------------------------------ ☐
- Give power series representation of sinx, cosx, arctanx, etc ----------------- ☐
- Define Taylor polynomials ☐
- Approximate a function by Taylor polynomial------------------------------ ☐
- Define Taylor series----------------------------------------------------------- ☐
- Define Maclaurin series-------------------------------------------------------- ☐
- Give an expression for the nth term Taylor polynomial --------------- ☐
- Give Taylor series and Maclaurin series representation of a function-- ☐
- State Taylor's Theorem-------------------------------------------------------- ☐

## Summary

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_o + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$ , is called a power series

centered at x = $c$ where $a_n$ and $c$ are constants. The term $a_n(x-c)^n$ is the n^th term and the number ***c*** is the center.

The interval on which the power series converge is called the interval of convergence or the domain of a power series.

Let $\sum_{n=0}^{m} a_n x^n$ be a power series with radius of convergence $\mathrm{R} > 0$. Then $\sum_{n=1}^{m} n a_n x^{n-1}$ has the same radius of convergence and

$$\frac{d}{dx}\left(\sum_{n=0}^{m} a_n x^n\right) = \sum_{n=1}^{m} n a_n x^{n-1} = \sum_{n=1}^{\infty} \frac{d}{dx}\left(a_n x^n\right) \quad \text{for } |x| < \mathrm{R}$$

Let $\sum a_n x^n$ be power series with radius of convergence $\mathrm{R} > 0$. Then $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ has the same radius of convergence and

$$\int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dx = \sum_{n=0}^{\infty} \left[\frac{a_n x^{n+1}}{n+1}\right] \quad \text{for} |x| < \mathrm{R} .$$

**Taylor's Theorem with Remainder**

If a function $f$ has $n+1$ derivatives on an open interval $I$ about $a$ and $x$ is in this interval, then a number $c$ between $x$ and $a$ exists with

$$f(x) = P_n(x) + R_n(x)$$

Where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad \text{and} \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

## Taylor's Series

If a function $f$ has derivatives of all orders on an open interval about $\boldsymbol{a}$ and if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all $x$ in that interval, then $f$ has the Taylor-series representation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

for all $x$ in the interval.

## Series of constants

- $1+2+3+4+\cdots+n = \frac{1}{2}n(n+1)$
- $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3+2^3+3^3+4^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+4+\cdots+n)^2$

## Series of function

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \qquad -\infty < x < \infty$
- $a^x = \sum_{n=0}^{\infty} \frac{(x\ln a)^n}{n!} = 1 + x\ln a + \frac{\left(x\ln a\right)^2}{2!} + \frac{\left(x\ln a\right)^3}{3!} + \cdots \qquad -\infty < x < \infty$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{n=0}^{\infty} \frac{\left(-1\right)x^{2n+1}}{\left(2n+1\right)}$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\left(2n\right)!}$
- $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, |x| < 1$
- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n-1}}{(2n-1)!} + \cdots$
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$
- $\tanh^{-1} x = \int_0^x \frac{1}{1-t^2}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$

**Exercise 3.4**

1. a) Find an expression for the n$^{th}$ Taylor polynomial $T_n(x)$ for $f(x)=5^x$ at x=0.

   b) Find an upper bound for $\left|f^{(n+1)}(x)\right|$ when $|x|\leq 1$.

   c) Show that $f(x)=5^x$ is equal to its Taylor series for $|x|\leq 1$.

2. Find the Maclaurin series for the following functions.

   a) $g(x)=\cosh x$ b) $f(x)=\coth x$ c) $f(x)=\int_0^x \frac{e^t-1}{t}dt$

   d) $g(x)=\frac{-3x+2}{2x^2-3x+1}$ e) $h(x)=x^2 3^x$ f) $f(x)=x\sin 3x$

3. Find the Taylor polynomial of the given degree n for the function fat the number a.

   a) $f(x)=\sqrt{x}$, $a=4,\ n=5$ b) $f(x)=\ln(\cos x),\quad a=0,\quad n=3$

4. Find the fourth Taylor polynomial a bout 3 for $f(x)=\frac{1}{x^2}$ and also determine the remainder for the polynomial.

5. a) Find the Taylor series for $f(x)=\ln(\sqrt{x})$ at $a=1$.

   b) For what values of $x$ this series have the sum equal to $\ln(\sqrt{x})$?

6. Find a power series representation for $\left(\frac{e^x-1-x}{x^2}\right)$ and use it to evaluate

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}$$

7. a) Show that $\ln\left(\frac{1}{1-x}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n}$

   b) Using (a) show that $\ln 2=\sum_{n=1}^{\infty}\frac{1}{n2^n}$

8. Approximate the value of the integral with an error less than the given error, by first using the integration theorem to express the integral as an infinite series and then approximating the infinite series by an appropriate partial sum for the following series.

   a) $f(x)=\int_0^1\frac{x^3}{2+x}dx,\quad 10^{-3}$ b) $f(x)=\int_{-1}^0 e^{x^2}dx,\quad 10^{-3}$

9. a) Give a power series representation of $\int_0^x \sin t^2 dt$

b) Using (a) approximate $\int_0^1 \sin t^2 dt$ to with in $10^{-4}$.

10. Find the values of x for which the following series converges.

a) $\sum_{n=1}^{\infty} \left( \frac{\cos x}{n} \right)^n$ b) $\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{x-1}{x} \right)^n$

11. Show that if $x > 0$, then

a) $\left| \sin x - \sum_{i=0}^{n} (-1)^i \frac{x^{2i+1}}{(2i+1)!} \right| < \frac{x^{2n+3}}{(2n+3)!}$

b) $\left| \cos x - \sum_{i=0}^{n} (-1)^i \frac{x^{2i}}{(2i)!} \right| < \frac{x^{2n+2}}{(2n+2)!}$

## References:


1. Robert Ellis and Denny Gulick: Calculus with analytic geometry, 3rd edition.
2. Thomas Fenny: Calculus
3. Edwards and Penney: Calculus and analytic geometry, 3rd edition.
4. J.Douglas Faires and Barbara T. Faires: Calculus of one variable, 2nd edition
5. Stanley I. Grossman: Multivariable calculus, Linear algebra and differential equations, 2nd edition
6. James Stewart: Multivariable Calculus, 2nd edition