

UNIT 3

Fourier Series

3.1: ORTHOGONAL FUNCTIONS

Definition: Functions $y_1(x), y_2(x), \dots$ defined on some interval are called **orthogonal** on $a \leq x \leq b$

with respect to the weight function $p(x) > 0$ if $\int_a^b p(x)y_m(x)y_n(x)dx = 0$ for $m \neq n$.

$\int_a^b p(x)y_n^2(x)dx$ is called the square of the norm of $y_n(x)$ and written as $\|y_n\|^2$. If the norm is unity, we say that the set of functions is an **orthonormal** set. If the weight function is unity, we simply say that the set is **orthogonal** on $a \leq x \leq b$.

Examples of sequences of orthogonal functions are: The set of functions $1, \cos(nx), n = 1, 2, 3, \dots$ or $1, \sin(nx), n = 1, 2, 3, \dots$ or $1, \cos(nx), \sin(nx), n = 1, 2, 3, \dots$, on the interval $a \leq x \leq a + 2\pi$ with weight functions $p(x) = 1$ for any real constant a .

The chief advantage of the knowledge of these orthogonal sets of functions is that they yield series expansions of a given function in a simple fashion. Let y_1, y_2, \dots be an orthogonal set with respect to the weight function $p(x)$ on an interval $a \leq x \leq b$. Let $f(x)$ be a given function that can be represented in terms of $y_n(x)$ by a convergent series,

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + a_2 y_2(x) + \dots$$

This is called an **orthogonal expansion** or the **generalized Fourier series**. The orthogonality of the functions helps us to find the unknown coefficients a_0, a_1, a_2, \dots in a simple fashion. These are called Fourier coefficients of $f(x)$ with respect to y_0, y_1, y_2, \dots . If we multiply both side of the above expansion by $p(x)y_n(x)$ for a fixed n , and then integrate over $a \leq x \leq b$, we obtain, assuming term by term integration is permissible,

$$\begin{aligned} \int_a^b p(x)f(x)y_n(x)dx &= \int_a^b p(x) \left(\sum_{m=0}^{\infty} a_m y_m(x) \right) y_n(x) dx \\ &= \sum_{m=0}^{\infty} a_m \int_a^b p(x)y_m(x)y_n(x)dx = a_n \int_a^b p(x)y_n^2(x)dx = a_n \|y_n(x)\|^2, \text{ all other} \end{aligned}$$

integrals being zero in the right hand side, because of the orthogonality of the set.

$$\int_a^b p(x)g(x)y_n(x)dx$$

Thus $a_n = \frac{1}{\|y_n(x)\|^2} \cdot$

Definition: A function f is said to be a periodic function with period p if p is the least positive number such that $f(x+p) = f(x)$ for all x in the domain of f . It follows that $f(x+np) = f(x)$ for all x in the domain of f and all integers n .

Example : $f(x) = \sin x$ and $g(x) = \cos x$ are the familiar periodic functions with period $p = 2\pi$.

The constant function $h(x) = c$ is a periodic function, since $h(x+p) = c = h(x)$ for all $p \in (0, \infty)$.

Proposition: If f and g are periodic functions with period p , then $H(x) = af(x) + bg(x)$, for some constants a & b , is a periodic function with period p .

Proof: $H(x+p) = a f(x+p) + b g(x+p) = af(x) + bg(x) = H(x)$ since $f(x+p) = f(x)$ & $g(x+p) = g(x)$

Therefore, $H(x)$ is a periodic function with period p .

Definition : A functional series of the form

$$\begin{aligned} & \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots + a_n \cos nx + b_n \sin nx + \dots \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ is called a } \mathbf{\text{trigonometric series}}; \text{ where } a_0, a_n, b_n \text{ (n = 1, 2, 3, ...)} \\ & \text{are real constants,) are called the } \mathbf{\text{coefficients}} \text{ of the trigonometric series.} \end{aligned}$$

If the series converges, say to a function $f(x)$, i.e., $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = f(x)$

Then f is a period function with period $p = 2\pi$, by the above proposition.

Thus $f(x+2\pi) = f(x)$ for all x in the domain of f , where $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

3.2 Fourier Series

3.2.1 Fourier Series of function with period 2π

Definition : The Fourier series for the periodic function $f(x)$ in an interval $\alpha < x < \alpha + 2\pi$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ And the coefficients}$$

a_0, a_n and b_n , $n = 1, 2, 3, \dots$ are called the **Fourier coefficients**.

To evaluate the Fourier Coefficients, the following integrals, involving sine and cosine functions are useful.

$$\text{i) } \int_{\alpha}^{\alpha+2\pi} \cos nx dx = \int_{\alpha}^{\alpha+2\pi} \sin nx dx = 0, n = 1, 2, 3, \dots$$

$$\text{ii) } \int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx dx = \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x] dx$$

$$= \frac{1}{2} \left[\frac{1}{m+n} \sin(m+n)x + \frac{1}{m-n} \sin(m-n)x \right] \Big|_{\alpha}^{\alpha+2\pi} = 0, m \neq n.$$

$$\text{iii) } \int_{\alpha}^{\alpha+2\pi} \cos mx \sin nx dx = 0$$

$$\text{iv) } \int_{\alpha}^{\alpha+2\pi} \cos^2(nx) dx = \int_{\alpha}^{\alpha+2\pi} \sin^2(nx) dx = \pi, n \neq 0$$

$$\int_{\alpha}^{\alpha+2\pi} \cos^2(nx) dx = \int_{\alpha}^{\alpha+2\pi} \frac{1+\cos(2nx)}{2} dx = \frac{1}{2} \left[x + \frac{1}{2n} \sin 2x \right] \Big|_{\alpha}^{\alpha+2\pi} = \frac{1}{2} [a + 2\pi - a] = \pi$$

In addition to these properties of integrals involving sine and cosine functions, we often need the following trigonometric functions for particular arguments.

$$\text{i) } \sin(2n+1) \frac{\pi}{2} = \cos n\pi = (-1)^n \text{ and (ii) } \sin n\pi = \cos(2n+1) \frac{\pi}{2} = 0, n = 1, 2, \dots$$

Theorem : (Euler's Formulae): The Fourier coefficients in

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

are given by

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx, a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx \text{ and } b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx$$

Corollary : 1. If $\alpha = 0$, the interval becomes $0 < x < 2\pi$, and Euler's formulae are given by:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx.$$

2. If $\alpha = -\pi$, then the interval becomes $-\pi < x < \pi$, and the Euler's Formulae are given by:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

Examples If $f(x) = \left(\frac{\pi-x}{2}\right)^2$ in the range $(0, 2\pi)$, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

Solution: The Fourier series for f in $(0, 2\pi)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_{\pi}^{2\pi} \left(\frac{\pi-x}{2}\right)^2 dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 - 2\pi x + x^2) dx = \frac{1}{4\pi} \left[\pi^2 x - \pi x^2 + \frac{1}{3} x^3 \right] \Big|_0^{2\pi} = \frac{\pi^2}{6}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi-x}{2} \right)^2 \cos nx dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi-x)^2 \cos nx dx \\
&= \frac{1}{4\pi} \left[\frac{(\pi-x)^2}{n} \sin(nx) + 2(\pi-x) \left(\frac{-\cos(nx)}{n^2} \right) + 2 \left(\frac{-\sin(nx)}{n^3} \right) \right]_0^{2\pi} = \frac{1}{4\pi} \left(\frac{2\pi}{n^2} + \frac{2\pi}{n^2} \right) = \frac{1}{n^2} \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi-x}{2} \right)^2 \sin(nx) dx \\
&= \frac{1}{4\pi} \left[(\pi-x)^2 \left(\frac{-\cos(nx)}{n^2} \right) + 2(\pi-x) \left(\frac{-\sin(nx)}{n^2} \right) + \frac{2}{n^3} \cos(nx) \right]_0^{2\pi} = 0 \\
\text{Therefore, } f(x) &= \left(\frac{\pi-x}{2} \right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}
\end{aligned}$$

DIRICHLET'S CONDITIONS:

Suppose that:

- a) $f(x)$ is defined and single – valued except possibly at a finite number of points in $(\alpha, \alpha + 2\pi)$.
- b) $f(x)$ is periodic outside $(\alpha, \alpha + 2\pi)$ with period 2π .
- c) $f(x)$ and $f'(x)$ are sectionally continuous in $(\alpha, \alpha + 2\pi)$.

Then the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$ with coefficients

$a_0 = \frac{1}{\pi} \int_a^{a+2\pi} f(x) dx$, $a_n = \frac{1}{\pi} \int_a^{a+2\pi} f(x) \cos(nx) dx$ & $b_n = \frac{1}{\pi} \int_a^{a+2\pi} f(x) \sin(nx) dx$ converges to

i) $f(x)$ if x is a point of continuity

ii) $\lim_{x \rightarrow c^-} f(x) + \lim_{x \rightarrow c^+} f(x)$ if c is a point of discontinuity .

Moreover, when $f(x)$ has finite number of discontinuities in any one period, for instance if in an

interval $(\alpha, \alpha + 2\pi)$, $f(x)$ is defined by $f(x) = \begin{cases} \phi(x), & \text{for } \alpha < x < c \\ \psi(x), & \text{for } c < x < \alpha + 2\pi \end{cases}$, i.e., c is a point of discontinuity, then:

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \left(\int_a^c \phi(x) dx + \int_c^{a+2\pi} \psi(x) dx \right) \\
a_0 &= \frac{1}{\pi} \left(\int_a^c \phi(x) \cos(nx) dx + \int_c^{a+2\pi} \psi(x) \cos(nx) dx \right) \text{ and } b_n = \frac{1}{\pi} \left(\int_a^c \phi(x) \sin(nx) dx + \int_c^{a+2\pi} \psi(x) \sin(nx) dx \right)
\end{aligned}$$

Example: Find the Fourier series expansion for

$$f(x) = \begin{cases} -\pi, & \text{if } -\pi < x < 0 \\ x, & \text{if } 0 < x < \pi \end{cases} \text{ and deduce that } \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}.$$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$

$$\text{Then } a_0 = \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi dx + \int_0^\pi x dx \right] = -\int_{-\pi}^0 dx + \frac{1}{\pi} \int_0^\pi x dx = -\pi + \frac{1}{2\pi} x^2 \Big|_0^\pi = -\pi + \frac{1}{2}\pi = -\frac{1}{2}\pi$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \cos(nx) dx + \int_0^\pi x \cos(nx) dx \right] = \int_0^\pi \cos(nx) dx + \frac{1}{\pi} \int_0^\pi x \cos(nx) dx = \frac{\sin(nx)}{n} \Big|_0^\pi + \frac{1}{\pi} \left[\frac{x \sin(nx)}{n} \Big|_0^\pi - \int_0^\pi \frac{\sin(nx)}{n} dx \right] = \frac{\cos(nx)}{\pi n^2} \Big|_0^\pi = \frac{1}{\pi n^2} ((-1)^n - 1)$$

$$\text{and } b_n = \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \sin(nx) dx + \int_0^\pi x \sin(nx) dx \right] = \int_{-\pi}^0 -\sin(nx) dx + \frac{1}{\pi} \int_0^\pi x \sin(nx) dx$$

$$= \frac{\cos(nx)}{n} \Big|_{-\pi}^0 + \frac{1}{\pi} \left[\frac{-x \cos(nx)}{n} \Big|_0^\pi + \int_0^\pi \frac{\cos(nx)}{n} dx \right] = \frac{1 - (-1)^n}{n} + \frac{1}{\pi} \left[\frac{-\pi(-1)^n}{n} + \frac{\sin(nx)}{n^2} \Big|_0^\pi \right] \\ = \frac{1 - (-1)^{n+1}}{n} + \frac{(-1)^{n+1}}{n} + 0 = \frac{1 + 2(-1)^{n+1}}{n}$$

$$\text{Hence, } f(x) = \frac{-\frac{1}{2}\pi}{2} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \frac{1 + 2(-1)^{n+1}}{n} \sin(nx) \right) \\ = \frac{-\pi}{4} + \left(\frac{-2}{\pi} \cos x - \frac{2}{3^2 \pi} \cos(3x) - \frac{2}{5^2 \pi} \cos(5x) - \dots \right) + \left(3 \sin x - \frac{1}{2} \sin(2x) + \sin(3x) - \frac{1}{4} \sin(4x) + \frac{3}{5} \sin(5x) - \dots \right) \\ = \frac{-\pi}{4} - \frac{2}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \dots \right) + \left(3 \sin x - \frac{1}{2} \sin(2x) + \sin(3x) + \dots \right).$$

By Dirichlet's Condition, we have that

$$\lim_{x \rightarrow 0^-} f(x) + \lim_{x \rightarrow 0^+} f(x) = \frac{-\pi}{4} - \frac{2}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) + (0 + 0 + \dots)$$

$$\Rightarrow \frac{-\pi + 0}{2} = \frac{-\pi}{4} - \frac{2}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\Rightarrow \frac{-\pi}{4} = \frac{-2}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) \Rightarrow \left(\frac{-\pi}{4} \right) \left(\frac{-\pi}{2} \right) = \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \text{ i.e. } \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

3.2.2 Fourier Series of Functions with arbitrary period P = 2L

In many engineering problems, the period of the function required to be expanded is not 2π but some other interval say $2L$. In order to apply the foregoing discussion to functions of period $2L$, this interval must be converted to the length 2π . This involves only a proportional change in the scale.

Consider the periodic function $f(x)$ defined on $(\alpha, \alpha + 2L)$. To change the problem to period 2π put

$t = \frac{\pi}{L}x$ which implies that $x = \frac{L}{\pi}t$. This gives when $x \in (\alpha, \alpha + 2L)$. Thus the function $f(x)$ of period $2L$ in $(\alpha, \alpha + 2L)$ is transformed to function $f\left(\frac{L}{\pi}t\right)$ of period 2π in $\left(\frac{\alpha\pi}{L}, \frac{\alpha\pi}{L} + 2\pi\right)$. Hence $f = \left(\frac{L}{\pi}t\right)$

can be expressed in Fourier series as:

$$g(t) = f\left(\frac{L}{\pi}t\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right), \text{ where}$$

$$a_0 = \frac{1}{\pi} \int_{\beta}^{\beta+2\pi} f\left(\frac{L}{\pi}t\right) dt, \quad a_n = \frac{1}{\pi} \int_{\beta}^{\beta+2\pi} f\left(\frac{L}{\pi}t\right) \cos(nt) dt, \text{ and } b_n = \frac{1}{\pi} \int_{\beta}^{\beta+2\pi} f\left(\frac{L}{\pi}t\right) \sin(nt) dt, \text{ where}$$

$$\beta = \frac{\alpha\pi}{L} \text{ and } x = \frac{L}{\pi}t.$$

Marking the inverse substitution $t = \frac{\pi}{L}x$ and noting that $dt = \frac{\pi}{L}dx$ in the above formula, the Fourier series Expansion of $f(x)$ in the interval $(\alpha, \alpha + 2L)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(n \frac{\pi}{L}x\right) + b_n \sin\left(n \frac{\pi}{L}x\right) \right) \text{ where}$$

$$a_0 = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) dx, \quad a_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos\left(n \frac{\pi}{L}x\right) dx \text{ and } b_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin\left(n \frac{\pi}{L}x\right) dx$$

Corollary :

- i. Putting $\alpha = 0$ in these formulae, we get the corresponding Fourier Coefficients for the interval $(0, 2L)$

$$a_0 = \frac{1}{L} \int_0^{2L} f(x) dx, \quad a_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(n \frac{\pi}{L}x\right) dx$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin\left(n \frac{\pi}{L}x\right) dx$$

ii. Putting $a = -L$ in the above formulae, we get the results:

$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$, $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(n \frac{\pi}{L} x\right) dx$ and $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx$ are the Fourier coefficients in the interval $(-L, L)$.

Example : Find the Fourier series of the periodic function $f(x)$ of period 2, where

$$f(x) = \begin{cases} -1, & \text{for } -1 < x < 0 \\ 2x, & \text{for } 0 < x < 1 \end{cases} \quad \text{and show that } 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8} \text{ and } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

Solution: $2L = 2 \Rightarrow L = 1$, and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$

$$\text{where } a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = \frac{1}{1} \int_{-1}^1 f(x) dx = \int_{-1}^0 -dx + \int_0^1 2x dx = x \Big|_0^{-1} + x^2 \Big|_0^1 = -1 + 1 = 0$$

$$\begin{aligned} a_n &= \frac{1}{1} \int_{-1}^1 f(x) \cos(n\pi x) dx = - \int_{-1}^0 \cos(n\pi x) dx + \int_0^1 2x \cos(n\pi x) dx \\ &= \frac{\sin(n\pi x)}{n\pi} \Big|_0^{-1} + \frac{2}{n\pi} (x \sin(n\pi x)) \Big|_0^1 + \frac{2}{n^2 \pi^2} \cos(n\pi x) \Big|_0^1 \\ &= \frac{2}{n^2 \pi^2} ((-1)^n - 1) = \frac{-2}{n^2 \pi^2} (1 - (-1)^n) = \frac{-2}{n^2 \pi^2} (1 + (-1)^{n+1}) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{1} \int_{-1}^1 f(x) \sin(n\pi x) dx = \int_{-1}^0 -\sin(n\pi x) dx + 2 \int_0^1 \sin(n\pi x) dx \\ &= \frac{\cos(n\pi x)}{n\pi} \Big|_{-1}^0 - \frac{2}{n\pi} \left[x \cos(n\pi x) \Big|_0^1 - \int_0^1 \cos(n\pi x) dx \right] \\ &= \frac{1}{n\pi} (1 - (-1)^n) - \frac{2}{n\pi} \left[(-1)^n - \frac{\sin(n\pi x)}{n\pi} \Big|_0^1 \right] = \frac{1 - (-1)^n - 2(-1)^n}{n\pi} = \frac{1 - 3(-1)^n}{n\pi} = \frac{1 + 3(-1)^{n+1}}{n\pi} \end{aligned}$$

$$\begin{aligned} \text{Thus } f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)) \\ &= 0 + \sum_{n=1}^{\infty} \left(\frac{-2(1 + (-1)^{n+1})}{n^2 \pi^2} \cos(n\pi x) + \frac{1 + 3(-1)^{n+1}}{n\pi} \sin(n\pi x) \right) \\ &= -\frac{4}{\pi^2} \cos \pi x + \frac{4}{\pi} \sin \pi x - \frac{2}{\pi} \sin 2\pi x - \frac{4}{3^2 \pi^2} \cos 3\pi x + \frac{4}{3\pi} \sin 3\pi x \\ &\quad - \frac{2}{4\pi} \sin 4\pi x - \frac{4}{5^2 \pi^2} \cos 5\pi x + \frac{4}{5\pi} \sin 5\pi x + \dots \end{aligned}$$

$$= -\frac{4}{\pi^2} \left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \right) + \frac{2}{\pi} \left(2 \sin \pi x - \frac{\sin 2\pi x}{2} + \frac{2}{3} \sin 3\pi x - \frac{\sin 4\pi x}{4} + \dots \right)$$

When $x = 0$, the series converges to $\lim_{x \rightarrow 0^+} f(x) + \lim_{x \rightarrow 0^-} f(x) = -\frac{1}{2}$

$$\text{Therefore, } -\frac{1}{2} = \frac{-4}{\pi^2} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) + \frac{2}{\pi} (0 - 0 + 0 - \dots)$$

$$= -\frac{4}{\pi^2} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

When $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = 1$, giving $1 = \frac{-4}{\pi^2} (0 + 0 + 0 + \dots) + \frac{2}{\pi} \left(2 - \frac{0}{2} - \frac{2}{3} - \frac{0}{3} + \frac{2}{3} - \dots \right)$

$$= \frac{2}{\pi} \left(2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \dots \right) = \frac{4}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right) \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

3.2.3 Fourier Series of Odd and Even Functions

Definition: A function $f(x)$ is said to be odd iff $f(-x) = -f(x)$

A function $f(x)$ is said to be even iff $f(-x) = f(x)$

Example : The functions $\sin(nx)$ and $\tan(nx)$ are odd functions. Graph of odd function is symmetric about the origin.

Example : The functions $\cos(nx)$, x^2 , $\sec(nx)$ are even functions. Graphs of even functions are symmetric about y-axis.

Proposition : If $f(x)$ is a periodic function with period $p = 2L$, then

$$\int_{-L}^L f(x) dx = \begin{cases} 2 \int_0^L f(x) dx, & \text{if } f \text{ is even} \\ 0, & \text{if } f \text{ is odd} \end{cases}$$

Recall that a periodic function $f(x)$ defined in $(-L, L)$ can be represented by the Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

$$\text{where } a_0 = \frac{1}{L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \left(n \frac{\pi}{L} x \right) dx \quad \& \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \left(n \frac{\pi}{L} x \right) dx$$

$$\text{When } f(x) \text{ is an even function } a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = \frac{2}{L} \int_0^L f(x) dx,$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \left(n \frac{\pi}{L} x \right) dx = \frac{2}{L} \int_0^L f(x) \cos \left(n \frac{\pi}{L} x \right) dx, \quad \text{since } f(x) \cos \left(n \frac{\pi}{L} x \right)$$

is even, and $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \left(n \frac{\pi}{L} x \right) dx = 0$, since $f(x) \sin \left(n \frac{\pi}{L} x \right)$ is odd.

Therefore, the Fourier series expansion of a periodic even function $f(x)$ contains only the cosine terms

$$\text{whose coefficients are } a_0 = \frac{2}{L} \int_0^L f(x) dx \text{ and } a_n = \frac{2}{L} \int_0^L f(x) \cos\left(n \frac{\pi}{L} x\right) dx$$

$$\Rightarrow f(x) = \frac{1}{L} \int_0^L f(x) dx + \sum_{n=1}^{\infty} \left(\underbrace{\frac{2}{L} \int_0^L f(x) \cos\left(n \frac{\pi}{L} x\right) dx}_{\text{Coefficients}} \right) \cos\left(n \frac{\pi}{L} x\right).$$

When $f(x)$ is odd function: $a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = 0$, $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(n \frac{\pi}{L} x\right) dx = 0$, since $f(x) \cos\left(n \frac{\pi}{L} x\right)$ is odd,

$$\text{and } b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx, \text{ since } f(x) \sin\left(n \frac{\pi}{L} x\right) \text{ is even.}$$

Thus, if a period function $f(x)$ is odd, its Fourier series expansion contains only the sine terms, whose

$$\text{coefficients are } b_n = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx, \text{ so that}$$

$$f(x) = \sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx \right)}_{\text{Coefficients}} \sin\left(n \frac{\pi}{L} x\right).$$

Example: Find a series to represent $f(x) = x^2$ in the interval $(-\ell, \ell)$. Deduce the values of i)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \text{ and ii) } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Solution: Let $f(x) = x^2$ in the interval $(-\ell, \ell)$ be represented by Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{\ell} x\right) + b_n \sin\left(\frac{n\pi}{\ell} x\right) \right). \text{ Here } f(x) = (-x)^2 = x^2 = f(x), \text{i.e., } f(x) \text{ is even in } (-\ell, \ell).$$

$$\text{Therefore, } a_0 = \frac{2}{\ell} \int_0^\ell f(x) dx = \frac{2}{\ell} \int_0^\ell x^2 dx = \frac{2\ell^2}{3}$$

$$\begin{aligned} a_n &= \frac{2}{\ell} \int_0^\ell f(x) \cos\left(\frac{n\pi}{\ell} x\right) dx = \frac{2}{\ell} \int_0^\ell x^2 \cos\left(\frac{n\pi}{\ell} x\right) dx = \frac{2}{n\pi} \left[x^2 \sin\left(\frac{n\pi}{\ell} x\right) \right]_0^\ell - \int_0^\ell 2x \sin\left(\frac{n\pi}{\ell} x\right) dx \\ &= \frac{4\ell}{n^2 \pi^2} \left[x \cos\left(\frac{n\pi}{\ell} x\right) \right]_0^\ell + \int_0^\ell \cos\left(\frac{n\pi}{\ell} x\right) dx = \frac{4\ell}{n^2 \pi^2} \left[\ell(-1)^n + \frac{\ell}{n\pi} \sin\left(\frac{n\pi}{\ell} x\right) \right]_0^\ell = \frac{4\ell^2 (-1)^n}{n^2 \pi^2} \text{ and } b_n = 0 \end{aligned}$$

$$\text{Therefore, } f(x) = x^2 = \frac{\ell^2}{3} + \frac{4\ell^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi}{\ell} x\right) = \frac{\ell^2}{3} - \frac{4\ell^2}{\pi^2} \left[\frac{\cos \frac{\pi}{\ell}}{1^2} - \frac{\cos \frac{2\pi}{\ell} x}{2^2} + \frac{\cos \frac{3\pi}{\ell} x}{3^2} - \dots \right],$$

which is the required Fourier series. i) Putting $x = 0$, we get $f(0) = 0 = \frac{\ell^2}{3} - \frac{4\ell^2}{\pi^2} \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right)$

$$\text{i.e., } \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

ii) Putting $x = \ell$, we get $\frac{\lim_{x \rightarrow \ell^+} f(x) + \lim_{x \rightarrow \ell^-} f(x)}{2} = \frac{\ell^2 + \ell^2}{2} = \ell^2 = \frac{\ell^2}{3} - \frac{4\ell^2}{\pi^2} \left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \dots \right)$

$$\Rightarrow 1 - \underbrace{\frac{1}{3}}_{= \frac{2}{3}} = \frac{4}{\pi^2} \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right) \Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \left(\frac{2}{3} \right) \left(\frac{\pi^2}{4} \right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

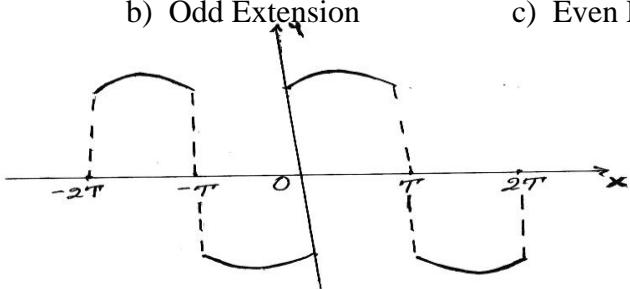
HALF – RANGE EXPANSION

In many problems of physics and engineering there is a practical need to apply a Fourier Series to a non – periodic function $F(x)$ on the interval $0 < x < T$. Because of physical or mathematical considerations, it may be possible to extend $F(x)$ over the interval $-T < x < T$, making it periodic of period $P = 2T$. The following figures illustrate the odd and even extensions of $F(x)$ which have Fourier Sine and Fourier Cosine series, respectively.

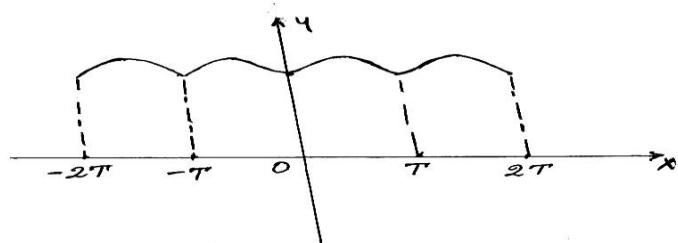
a) Original function



b) Odd Extension



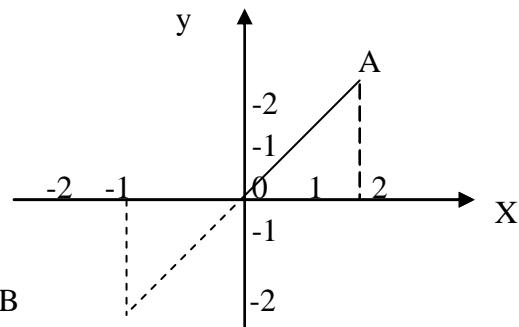
c) Even Extension



Examples : Express $f(x) = x$ as a half – range

a) Sine series in $0 < x < 2$. b) Cosine series in $0 < x < 2$.

Solutions: The graph of $f(x) = x$ in $0 < x < 2$ is the line OA. Let us extend the function $f(x)$ in the interval $-2 < x < 0$ (shown by the line BO) so that the new function is symmetric about the origin and, therefore, represents an odd function in $(-2, 2)$

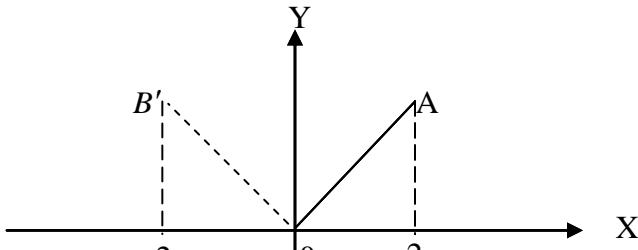


Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only sine series terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \text{ Where } b_n = \frac{2}{2} \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx = \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$

$$\begin{aligned}
&= - \left[\frac{2x \cos\left(\frac{n\pi}{2}x\right)}{n\pi} \right]_0^2 + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}x\right) \Big|_0^2 = \frac{-4(-1)^n}{n\pi} + 0 = \frac{4(-1)^{n+1}}{n\pi} \\
\Rightarrow f(x) &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right) = \frac{4}{\pi} \left[\sin\left(\frac{\pi}{2}x\right) - \frac{\sin\pi x}{2} + \frac{\sin\frac{3}{2}\pi x}{3} - \dots \right]
\end{aligned}$$

- b) The graph of $f(x) = x$ in $(0, 2)$ is the line OA. Let us extend the function $f(x)$ in the interval $(-2, 0)$ (shown by the $B'O$) so that the new function is symmetric about the y-axis and, therefore, represents an even function in $(-2, 2)$.



Hence the Fourier series $f(x) =$... will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad a_0 = \frac{2}{2} \int_0^2 f(x) dx = \int_0^2 x dx = 2 \quad \text{and} \quad a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2x \sin\left(\frac{n\pi x}{2}\right)}{n\pi} \Big|_0^2 - \frac{2}{n\pi} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx = \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) \Big|_0^2 = \frac{4}{n^2\pi^2} ((-1)^2 - 1) = \frac{4((-1)^n - 1)}{n^2\pi^2}$$

$$\text{Therefore, the desired result is: } f(x) = \frac{2}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(n\frac{\pi}{2}x\right)$$

$$= 1 + \frac{4}{\pi^2} \left(\frac{-2}{1^2} \cos\frac{\pi}{2}x - \frac{2}{3^2} \cos\frac{3\pi}{2}x - \frac{2}{5^2} \cos\frac{5\pi}{2}x + \dots \right) = 1 - \frac{8}{\pi^2} \left(\frac{\cos\frac{\pi}{2}}{1^2} + \frac{\cos\frac{3}{2}\pi}{3^2} + \frac{\cos\frac{5}{2}\pi x}{5^2} + \dots \right)$$

3.3 FOURIER INTEGRALS:

Consider a function $f(x)$ which satisfies the Dirichlet's conditions

in every interval $(-L, L)$ so that, we have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$, where

$a_0 = \frac{1}{L} \int_{-L}^L f(t) dt$, $a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\frac{n\pi t}{L} dt$, and $b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\frac{n\pi t}{L} dt$. Substituting the values of

a_0 a_n and b_n in the

$f(x) = \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \left(\int_{-L}^L f(t) \cos\frac{n\pi t}{L} dt \cos\frac{n\pi x}{L} + \int_{-L}^L f(t) \sin\frac{n\pi t}{L} dt \sin\frac{n\pi x}{L} \right)$ Fourier series expansion, we get the form

$$= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \int_{-L}^L f(t) \cos \frac{n\pi}{L}(t-x) dt \quad \text{----- (*)}$$

If $\int_{-\infty}^{\infty} |f(t)| dx$ converges, i.e.; $f(x)$ is absolutely integrable in $(-\infty, \infty)$, then the 1st – term on the right

side of (*) approaches 0 as $L \rightarrow \infty$, since $\left| \frac{1}{2L} \int_{-L}^L f(t) dt \right| \leq \frac{1}{2L} \int_{-\infty}^{\infty} |f(t)| dt$.

The 2nd – term on the right side of (*) tends to $\lim_{L \rightarrow \infty} \frac{1}{L} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi(t-x)}{L} dt$

$= \lim_{\delta\lambda \rightarrow 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda(t-x) dt$ on writing $\frac{\pi}{L} = \delta\lambda$. Thus as $L \rightarrow \infty$, (*) becomes

$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda$ called the Fourier Integral of $f(x)$.

Remark: 1. If function f is continuous at x , then $f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda$.

If f is not continuous at x , then $\frac{f(x+0) + f(x-0)}{2} = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda$

where $f(x+0) = \lim_{t \rightarrow x^+} f(t)$ and $f(x-0) = \lim_{t \rightarrow x^-} f(t)$.

2. Fourier sine and cosine integrals. Expanding $\cos \lambda(t-x) = \cos(\lambda t - \lambda x)$
 $= \cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x$, the Fourier integral of $f(x)$ may be written as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t dt d\lambda + \frac{1}{\pi} \int_0^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t dt d\lambda.$$

If $f(x)$ is an odd function, $f(t) \cos(\lambda t)$ is also an odd function while $f(t) \sin(\lambda t)$ is even. Then the 1st term on the right side of the above equation vanishes, and $f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin(\lambda t) dt d\lambda$, which is

known as the Fourier Sine integral.

Similarly, if $f(x)$ is even, the above integral takes the form

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos(\lambda x) \int_0^{\infty} f(t) \cos(\lambda t) dt d\lambda, \text{ known as the } \underline{\text{Fourier Cosine integral}}.$$

Example: Express $f(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq \pi \\ 0, & \text{for } x > \pi \end{cases}$ as a Fourier sine integral and hence

$$\text{evaluate } \int_0^{\infty} \frac{1 - \cos(\pi\lambda)}{\lambda} \sin(x\lambda) d\lambda.$$

Solution: The Fourier sine integral of $f(x)$ is $f(x) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \int_0^\infty f(t) \sin(\lambda t) dt$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \int_0^\pi \sin(\lambda t) dt = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \left[\frac{\cos(\lambda t)}{\lambda} \right]_0^\pi = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda \pi)}{\lambda} \sin(\lambda x) d\lambda$$

$$\text{Therefore, } f(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda \pi)}{\lambda} \sin(\lambda x) d\lambda \Rightarrow \int_0^\infty \frac{1 - \cos(\lambda x)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2}, & \text{for } 0 \leq x \leq \pi \\ 0, & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, then the value of the above integral is

$$\frac{\pi}{2} \left(\frac{\lim_{x \rightarrow \pi^+} f(x) + \lim_{x \rightarrow \pi^-} f(x)}{2} \right) = \frac{\pi}{2} \left(0 + \frac{1}{2} \right) = \frac{\pi}{4}, \text{ i.e., } \int_0^\infty \frac{1 - \cos(\lambda \pi)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{4}, \text{ at } x = \pi$$