ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

SCHOOL OF APPLIED NATURAL SCIENCES

MATHEMATICS PROGRAM

Applied Mathematics II (Math 1102)

Lecture Note

Chapter 4

Calculus of Functions of Several Variables

4.1 Definition

Definition: Let D be a set of order pairs of real numbers, D $\subseteq R^2$. Then $f: D \to R$ is said to be a function two variables x and y iff to each order pair (x, y) in D, there is a unique real number such that $f(x, y) = z$. Of course, x and y are called independent variables and z is dependent variable, since its value is determined by the values of x and y. In addition (in a simple term), this function is function of two variables.

Generally, functions of two or more variables are called functions of several variables.

4.2 Domain and Range of the functions of several Variables

Domain, D(f): The set of all values of the variables of the function f.

Range, R(f): The set of all values (images) of the function f.

Example: Find the domain and range of the function $\mathrm{f(x,y)} = \sqrt{\mathrm{x}^2 + \mathrm{y}^2 - 1}$.

Solution: Here, for the square root to be defined we must have

$\mathrm{x}^2 + \mathrm{y}^2 - 1 \geq 0 \Longrightarrow \mathrm{x}^2 + \mathrm{y}^2 \geq 1$. Therefore D(f)=$\{(\mathrm{x,y})\colon \mathrm{x}^2 + \mathrm{y}^2 \geq 1\}$

Since the square root function is nonnegative, we've

$\sqrt{\mathrm{x}^2 + \mathrm{y}^2 - 1} \geq 0$.

Moreover, $\sqrt{\mathrm{x}^2 + \mathrm{y}^2 - 1} = 0 \Longrightarrow \mathrm{x}^2 + \mathrm{y}^2 = 1$.

Therefore, R(f)= $[0, \infty)$.

Exercise: Find domain and range of the function $f(x, y) = \frac{\sqrt{4-y^2}}{\sqrt{2-lnx}}$.

Read More: Level curves and level surfaces of function f of several variables.

4.3 Limit and Continuity of Functions of Several Variables

Definition: Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

If for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

If $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$.

Note:

1. The values $f(x, y)$ of approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f.
2. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2 where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.
3. Everything that we have done in this section can be extended to functions of three or more variables.

Exercise: Show that $\lim_{(x,y)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Method of Limit evaluations

One can be sure that the computation or checking existence of limits of functions of several variables is not an easy task. Based on the type/family of the function, most common methods of the limit evaluations are:

1. **Direct substitution**
2. **Factorization**
3. **Rationalization**
4. **Polar coordinate**
5. **Squeezing theorem (refer your text books)**

Let's see one by one each of the techniques.

1. **Direct Substitution**

If f is defined at (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example: Evaluate $\lim_{(x,y)\to(0,2)} \quad e^{-xy}\cos(x^2y)$

Solution: Since the given function if defined at (0,2), so, by direct substitution, we've

$$\lim_{(x,y)\to(0,2)} e^{-xy}\cos(x^2y) = f(0,2) = e^0\cos(0) = 1$$

2. **Factorization**: sometimes the direct substitution may not work. So, we need to look for different method may be factorization method. This method used to eliminate the term that creates a problem in limit evaluation.

Example: Evaluate $\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y}$.

Solution: Here, we cannot evaluate the limit directly.

But we know that $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Thus, $\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x-y)(x^2+xy+y^2)}{x-y} = \lim_{(x,y)\to(1,1)} x^2 + xy + y^2 = 3$

3. **Rationalization**: This technique **mostly** uses when the limit under consideration involves difference of radical expressions.

Example: Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

Solution: By rationalizing the denominator,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \left(\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\right)\left(\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}\right)$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2+1-1} = \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right)$$

$$= 2$$

4. **Polar Coordinates method**: Many limit problems which cannot be evaluated by any of the above techniques can be evaluated easily using polar coordinates as $x = rcos\theta, y = rsin\theta,\ r^2 = x^2 + y^2$ and as $(x,y) \to (0,0), r \to 0^+$.

Example: Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}}$

Solution: Use $x = rcos\theta, y = rsin\theta,\ r^2 = x^2 + y^2$ and as $(x,y) \to (0,0), r \to 0^+$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{r^3cos^3\theta + r^3sin^3\theta}{\sqrt{r^2cos^2\theta + r^2sin^2\theta}} = \lim_{r\to 0^+} \frac{r^3(cos^3\theta + sin^3\theta)}{r^2\sqrt{cos^2\theta + sin^2\theta}}$$

$$= \lim_{r\to 0^+} \frac{r(cos^3\theta + sin^3\theta)}{\sqrt{cos^2\theta + sin^2\theta}} = \lim_{r\to 0^+} \frac{r(cos^3\theta + sin^3\theta)}{\sqrt{1}} = \lim_{r\to 0^+} r(cos^3\theta + sin^3\theta)$$

$$= 0(cos^3\theta + sin^3\theta) = 0$$

Reading Assignment: Limit evaluation by **Squeezing theorem** and **L'Hospital's rule.**

Definition (Continuity): A function $f(x, y)$ is continuous at a point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Otherwise, f is not continuous.

Example: Verify that the function $f(x, y) = \begin{cases} \frac{sin6xy}{3xy}, & (x, y) \neq (0,0) \\ 2, & (x, y) = (0,0) \end{cases}$ is continuous at (0,0).

Solution: Let's evaluate the limit,

$$\lim_{(x,y)\to(0,0)} \frac{sin6xy}{3xy} = \lim_{(x,y)\to(0,0)} \frac{2sin6xy}{2(3xy)} = \lim_{(x,y)\to(0,0)} \frac{2sin6xy}{6xy} = 2 \lim_{(x,y)\to(0,0)} \frac{sin6xy}{6xy}$$

$= 2(1) = 2$ (**Hint**: $\lim_{x\to 0} \frac{sinx}{x} = 1$). Moreover, $f(0,0) = 2$.

Since, $\lim_{(x,y)\to(0,0)} f(x, y) = f(0,0) = 2$ then f is continuous at $(x, y) = (0,0)$.

Exercise: Verify whether the function $f(x, y) = \begin{cases} \frac{4xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$ is continuous or not at (0,0).

4.4 Partial Derivatives

In this section we will introduce the idea of partial derivatives as well as the standard notations and how to compute them. Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$, represents the rate of change of the function as x changes. This is an important interpretation of derivatives and we are not going to want to lose it with functions of more than one variable. The problem with functions of more than one variable is that there is more than one variable.

Definition: Let $z = f(x, y)$. Then, the partial derivatives of with respect to x and y are the functions denoted by f_x and f_y respectively and defined by

$$f_x\,(x, y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \text{and}$$

$$f_y\,(x, y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \quad \text{provided that these limits exist.}$$

Given the function $z = f(x, y)$ the following are all equivalent notations,

$$f_x\,(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial (f(x,y))}{\partial x} = z_x = \frac{\partial z}{\partial x} = D_x f \quad \text{(The partial derivative of } f \text{ with respect to } x\text{)}$$

$$f_y\,(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial (f(x,y))}{\partial y} = z_y = \frac{\partial z}{\partial y} = D_y f \quad \text{(The partial derivative of } f \text{ with respect to } y\text{)}$$

Note that these two partial derivatives are sometimes called the **first order partial derivatives**.

Moreover, the **partial derivative** of $f(x, y)$ with respect to *x* at a given point (a, b) is given by

$$f_x\,(a, b) = \frac{\partial f}{\partial x}(a, b) = \frac{\partial z}{\partial x}\Big|_{(a,b)} \qquad \text{and}$$

The **partial derivative** of $f(x, y)$ with respect to *y* at a given point (a, b) is given by

$$f_y\,(a, b) = \frac{\partial f}{\partial y}(a, b) = \frac{\partial z}{\partial y}\Big|_{(a,b)}$$

The **rule** is that when we want to find f_x we hold *y* fixed and allow *x* to vary and to find f_y we hold *x* fixed and allow *y* to vary. Besides, all derivatives rules can be used in the usual way as of in functions of single variable on applied mathematics I.

Note: We can define the partial derivatives of three variables in a similar way as we did for two variables here above.

Example: Find all of the first order partial derivatives for

a. $f(x,y) = \cos\left(\frac{4}{x}\right) e^{x^2y-5y^3}$

b. $f(x,y,z) = ln\left(\sqrt{x^2+y^2+z^2}\right)$

Solution:

a. In this case both the cosine and the exponential contain x's and so we have really got a product of two functions involving x's and so we will need to product rule of derivatives this up. Here is the derivative with respect to x.

$$f_x(x,y) = -sin\left(\frac{4}{x}\right)\left(\frac{-4}{x^2}\right)e^{x^2y-5y^3} + \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}(2xy)$$

Do not forget the chain rule for functions of one variable!

Now, let's differentiate with respect to y. *I*n this case we don't have a product rule to worry about since the only place that the y shows up is in the exponential. Therefore, since x's are considered to be constants for this derivative, the cosine in the front will also be thought of as a multiplicative constant.

So, $f_y(x,y) = (x^2 - 15y^2)\cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}$.

b. $f_x(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{\sqrt{x^2+y^2+z^2}}.\frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{x^2+y^2+z^2}$

Similarly, $f_y(x,y,z) = \frac{y}{x^2+y^2+z^2}$, $f_z(x,y,z) = \frac{z}{x^2+y^2+z^2}$

4.4.1 Partial Derivative at a point and its interpretation

As with functions of single variables partial derivatives represent the rates of change of the functions as the variables change. As we saw in the previous section, $f_x(x,y)$ represents the rate of change of the function $f(x,y)$ as we change x and hold y fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change y and hold x fixed.

Example: Determine if $f(x,y) = \frac{x^2}{y^3}$ is increasing or decreasing at (2,5).

a. if we allow x to vary and hold y fixed.

b. if we allow y to vary and hold x fixed.

Solution:

a. if we allow x to vary and hold y fixed.

In this case we will first need $f_x(x,y)$ and its value at the point.

$$f_x(x,y) = \frac{2x}{y^3} \qquad \Rightarrow \qquad f_x(2,5) = \frac{4}{125} > 0$$

So, the partial derivative with respect to x is positive and so if we hold y fixed the function is increasing at $(2,5)$ as we vary x.

b. if we allow y to vary and hold x fixed.

In this case we will first need $f_y(x,y)$ and its value at the point.

$$f_y(x,y) = \frac{-3x^2}{y^4} \qquad \Rightarrow \qquad f_y(2,5) = \frac{-12}{625} < 0$$

So, the partial derivative with respect to y is negative and so the function is increasing at $(2,5)$ as we vary y and hold x fixed.

4.4.2 Higher Order Partial Derivatives

Just as we had higher order derivatives with functions of one variable we will also have higher order derivatives of functions of more than one variable.

Consider the case of a function of two variables, $f(x,y)$, since both of the first order partial derivatives are also functions of x and y we could in turn differentiate each with respect to x or y. This means that for the case of a function of two variables there will be a total of four possible second order derivatives. Here they are and the notations to denote them:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

The **second and third** second order partial derivatives are often called mixed partial derivatives since we are taking derivatives with respect to more than one variable. Note as well that the order that we take the derivatives in is given by the notation for each these. If we are using the subscripting notation, *e.g.* f_{xy} , then we will differentiate from left to right. In other words, in this case, we will differentiate first with respect to x and then with respect to y. With the fractional notation, *e.g.* $\frac{\partial^2 f}{\partial y \partial x}$, it is the opposite. In these cases we differentiate moving along the denominator from right to left. So, again, in this case we differentiate with respect to x first and then y.

Example: Find all the second order derivatives for $f(x, y) = \cos(2x) - x^2e^{5y} + 3y^2$.

Solution: We first need the first order derivatives,

$$f_x(x, y) = -2sin(2x) - 2xe^{5y}$$

$$f_y(x, y) = -5x^2e^{5y} + 6y$$

Now, let's get the second order derivatives.

$$f_{xx} = -4\cos(2x) - 2e^{5y}$$

$$f_{xy} = -10xe^{5y}$$

$$f_{yx} = -10xe^{5y}$$

$$f_{yy} = -25x^2e^{5y} + 6$$

Now let's also notice that, in this case, $f_{xy} = f_{yx}$. This leads us to the following theorem.

Clairaut's Theorem
Suppose that f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are continuous on this disk then,

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Exercise: Verify Clairaut's Theorem for $f(x, y) = xe^{-x^2y^2}$.

Note that there are, of course, higher order derivatives as well. Here are a couple of the third order partial derivatives of function of two variables.

$$f_{xyx} = \left(f_{xy}\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$

$$f_{yxx} = \left(f_{yx}\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y}$$

Notice as well that for both of these we differentiate once with respect to y and twice with respect to x. There is also another third order partial derivative in which we can do this, f_{xxy}. There is an extension to Clairaut's Theorem that says if all three of these are continuous then they should all be equal,

$$f_{xxy} = f_{xyx} = f_{yxx}$$

In general, we can extend Clairaut's theorem to any function and mixed partial derivatives. The only requirement is that in each derivative we differentiate with respect to each variable the same number of times.

Exercise

a. Find f_{xxyzz} for $f(x, y, z) = z^3y^2\ln(x)$

b. Find $\frac{\partial^3 f}{\partial y \partial x^2}$ for $f(x, y) = e^{xy}$

4.5 Chain Rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function. For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

Case I: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\boldsymbol{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

Example: If $z = x^2y + 3xy^4$ where $x = sin2t$ and $y = cost$, find $\frac{dz}{dt}|_{t=0}$.

Solution: The Chain rule gives

$$\boldsymbol{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

$$= (\boldsymbol{2xy + 3y^4})(\boldsymbol{2cos2t}) + (\boldsymbol{x^2 + 12xy^3})\ (\boldsymbol{-sint})$$

It's not necessary to substitute the expressions for x and y in terms of t . We simply observe that when $t = 0$, we have $x = sin0 = 0$ and $y = cos0 = 1$. Therefore,

$$\frac{dz}{dt}|_{t=0} = (0 + 3)(2cos0) + (0 + 0)(-sin0) = 6$$

Case II: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of t. Then z is a differentiable function of s and t. Then

$$\boldsymbol{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}} \qquad \boldsymbol{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

This Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the dependent variable.

To remember the Chain Rule, it's helpful to draw the **tree diagram!(refer your text book).**

Note that we can consider (or extend) both cases of the chain rules for the general situation, i.e, for dependent variable z that is differentiable function of the n independent variables.

Example: If $z = e^x siny$, where $x = st^2$ and $y = s^2t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution: Applying **Case II** of the Chain Rule, we get

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x siny)(t^2) + (e^x cosy)(2st)$$

$$= t^2 e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x siny)(2st) + (e^x cosy)(s^2)$$

$$= 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t)$$

4.6 Implicit Differentiation

Before getting into implicit differentiation for multiple variable functions try to remember how implicit differentiation works for functions of one variable. Implicit differentiation works in exactly the same manner with functions of multiple variables. If we have a function in terms of three variables x, y, and z we will assume that z is in fact a function of x and y. In other words, $z = z(x, y)$. Then whenever we differentiate z's with respect to x we will use the chain rule and add on a $\frac{\partial z}{\partial x}$. Likewise, whenever we differentiate z's with respect to y we will add on a $\frac{\partial z}{\partial y}$.

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^3z^2 - 5xy^5z = x^2 + y^3$

Solution: Let's start with finding $\frac{\partial z}{\partial x}$. We first will differentiate both sides with respect to x and remember to add on a $\frac{\partial z}{\partial x}$ whenever we differentiate a z.

$$3x^2z^2 + 2x^3z\frac{\partial z}{\partial x} - 5y^5z - 5xy^5\frac{\partial z}{\partial x} = 2x$$

Remember that since we are assuming $z = z(x, y)$ then any product of x's and z's will be a product and so will need the product rule!

Now, solve for $\frac{\partial z}{\partial x}$.

$$(2x^3z - 5xy^5)\frac{\partial z}{\partial x} = 2x - 3x^2z^2 + 5y^5z$$

$$\frac{\partial z}{\partial x} = \frac{2x - 3x^2z^2 + 5y^5z}{2x^3z - 5xy^5}$$

Now we'll do the same thing for $\frac{\partial z}{\partial y}$ except this time we'll need to remember to add on a $\frac{\partial z}{\partial y}$ whenever we differentiate a z.

$$2x^3z\frac{\partial z}{\partial y} - 25xy^4z - 5xy^5\frac{\partial z}{\partial y} = 3y^2$$

$$(2x^3z - 5xy^5)\frac{\partial z}{\partial y} = 3y^2 + 25xy^4z$$

$$\frac{\partial z}{\partial y} = \frac{3y^2 + 25xy^4z}{2x^3z - 5xy^5}$$

Theorem (Implicit Function Theorem):

Suppose z that is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Longrightarrow \frac{\partial F}{\partial x}(1) + \frac{\partial F}{\partial y}(0) + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Longrightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Longrightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \qquad \text{(provided that } \frac{\partial F}{\partial z} \neq 0\text{)}$$

The formula for $\frac{\partial z}{\partial y}$ can be obtained in a similar manner and then**,**

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Exercise: Use **implicit Function Theorem** to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

$$x^2sin(2y - 5z) = 1 + ycos(6zx)$$

4.7 Gradient Vector, Tangent Planes and Linear Approximation

We'll take a look at tangent planes to surfaces in this section as well as an application of tangent planes. In this section we will also see how the gradient vector can be used to find tangent planes and normal lines to a surface. The gradient vector is always orthogonal, or normal, to the surface at a point.

4.7.1 Gradient of a Function

The **gradient vector or gradient of $\boldsymbol{f}$** is given by

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \boldsymbol{i} + f_y \boldsymbol{j} + f_z \boldsymbol{k}, \text{ is vector quantity.}$$

Or

$$\nabla f = \langle f_x, f_y \rangle = f_x \boldsymbol{i} + f_y \boldsymbol{j}$$

The definition is only shown for functions of two or three variables, however there is a natural extension to functions of any number of variables that we would like.

4.7.2 Directional Derivatives

Definition: The rate change of $f(x, y)$ in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is called the **directional derivative** and is denoted by $D_{\vec{u}} f(x, y)$. The definition of **directional derivative** is,

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}$$

With the definition of the gradient we can now say that the **directional derivative** is denoted as $D_{\vec{u}} f$ and given by,

$$D_{\vec{u}} f(x, y) = \nabla f . \vec{u} = f_x(x, y)a + f_y(x, y)b$$ (dot product of the gradient function and unit vector)

There are similar formulas that can be derived by the same type of argument for functions with more than two variables. For instance, the directional derivative of $f(x, y, z)$ in the direction of the unit vector $\vec{u} = \langle a, b, c \rangle$ is given by,

$$D_{\vec{u}} f(x, y, z) = \nabla f . \vec{u} = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

Recall that a unit vector is a vector with length, or magnitude, of 1.

Example: Find the directional derivative, $D_{\vec{u}} f(2,0)$, where $f(x, y) = \mathrm{xe}^{\mathrm{xy}} + y$ and $\vec{u}$ is the unit vector in the direction of $\theta = \frac{2\pi}{3}$.

Solution: The gradient of f is,

$$\nabla f = \langle f_x, f_y \rangle = \langle e^{xy} + xye^{xy},\ x^2e^{xy} + 1 \rangle \qquad \text{and}$$

The unit vector giving the direction is,

$$\vec{u} = \langle cos\left(\frac{2\pi}{3}\right), sin\left(\frac{2\pi}{3}\right) \rangle = \langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \rangle$$

So, the directional derivative is,

$$D_{\vec{u}}f(x,y) = \nabla f.\vec{u} = f_x(x,y)a + f_y(x,y)b = \left(\frac{-1}{2}\right)(e^{xy} + xye^{xy}) + \left(\frac{\sqrt{3}}{2}\right)(x^2e^{xy} + 1)$$

Now, plugging in the point in question gives,

$$D_{\vec{u}}f(2{,}0) = \left(\frac{-1}{2}\right)(1) + \left(\frac{\sqrt{3}}{2}\right)(5) = \frac{5\sqrt{3} - 1}{2}$$

Exercise: Find the directional derivative, $D_{\vec{u}}f(x,y,z)$, where $f(x,y,z) = \mathrm{x}^2z + y^3z^2 - xyz$ in the direction of $\vec{v} = \langle -1{,}0{,}3 \rangle$.

Theorem: The maximum value of $D_{\vec{u}}f(\vec{x})$ (and hence then the maximum rate of change of the function $f(\vec{x})$) is given by $\|\nabla f(\vec{x})\|$ and will occur in the direction given by $\nabla f(\vec{x})$.

Example: Suppose that the height of a hill above sea level is given by $z = 1000 - 0.01x^2 - 0.02y^2$. If you are the point (60,100) in what direction is the elevation changing fast? What is the maximum rate of change of the elevation at this point?

Solution: The gradient vector of f is,

$$\nabla f(\vec{x}) = \langle -0.02x, -0.04y \rangle$$

The maximum rate of change of the elevation will then occur in the direction of

$$\nabla f(60{,}100) = \langle -1.2, -4 \rangle$$

Note: Since both of the components are negative it looks like the direction of maximum rate of change points up the hill towards the center rather than away from the hill.

The maximum rate of change of the elevation at this point is,

$$\|\nabla f(60{,}100)\| = \sqrt{(-1.2)^2 + (-4)^2} = \sqrt{17.44} = 4.18$$

4.7.3 Equations of Tangent plane

The graph of a function $z = f(x,y)$ is a surface in $\mathbb{R}^3$(three dimensional space) and so we can now start thinking of the plane that is "tangent" to the surface as a point. Since the tangent plane and the surface touch at (x_0, y_0) the following point will be on both the surface and the plane.

$$(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$$

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface given by $z = f(x, y)$ at the point (x_0, y_0, z_0) is then,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Rewrite the equation of the tangent plane (since, $z_0 = f(x_0, y_0)$), is then,

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \text{ (Try to derive it!)}$$

Since we want a line that is at the point (x_0, y_0, z_0) we know that this point must also be on the line and we know that $\nabla f(x_0, y_0, z_0)$ is a vector that is normal to the surface and hence will be parallel to the line. Therefore the equation of the normal line is,

$$\overrightarrow{r(t)} = \langle x_0, y_0, z_0 \rangle + t\nabla f(x_0, y_0, z_0)$$

$$= \langle x_0, y_0, z_0 \rangle + t\langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle$$

$$= \langle x_0, y_0, z_0 \rangle + \langle tf_x(x_0, y_0, z_0), tf_y(x_0, y_0, z_0), tf_z(x_0, y_0, z_0) \rangle$$

$$= \langle x_0 + tf_x(x_0, y_0, z_0), y_0 + tf_y(x_0, y_0, z_0), z_0 + tf_z(x_0, y_0, z_0) \rangle, \text{ where } t \text{ is unknown parameter}$$

Example: Find the equation of the tangent plane and normal line to $z = \ln(2x + y)$ at $(-1,3)$.

Solution: Here $f(x, y) = ln(2x + y)$ so that $z_0 = f(-1,3) = ln(1) = 0$

$$f_x(x, y) = \frac{2}{2x + y} \qquad \Rightarrow f_x(-1,3) = 2$$

$$f_y(x, y) = \frac{1}{2x + y} \qquad \Rightarrow f_y(-1,3) = 1$$

Therefore, equation of the plane is then,

$$z = f(x_0, y_0) + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$$

$$z = 0 + 2(x + 1) + 1(y - 3)$$

$$z = 2x + y - 1$$

The normal line is,

$$\overrightarrow{r(t)} = \langle -1,3,0 \rangle + t\langle 2,1,-1 \rangle = \langle -1 + 2t, 3 + t, -t \rangle$$

4.7.4 Linear Approximation of a Function

As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point. That means, the linear approximation $L(x, y)$ becomes,

$L(x, y) = \ f(x_0, y_0) + f_x(x_0, y_0)\ (x - x_0) + f_y(x_0, y_0)\ (y - y_0)$ and as long as we are "near" (x_0, y_0) then we should have that,

$$f(x, y) \approx L(x, y) = \ f(x_0, y_0) + f_x(x_0, y_0)\ (x - x_0) + f_y(x_0, y_0)\ (y - y_0)$$

Example: Find the linear approximation to $z \ = 3 + \frac{x^2}{16} + \frac{y^2}{9}$ at $(-4,3)$.

Solution: So, we have

$$f(x, y) = 3 + \frac{x^2}{16} + \frac{y^2}{9} \Rightarrow f(-4,3) = 5$$

$$f_x(x, y) = \frac{x}{8} \Rightarrow f_x(-4,3) = \frac{-1}{2}$$

$$f_y(x, y) = \frac{2y}{9} \Rightarrow f_y(-4,3) = \frac{2}{3}$$

The tangent plane, or linear approximation, is then,

$$L(x, y) = \ f(x_0, y_0) + f_x(x_0, y_0)\ (x - x_0) + f_y(x_0, y_0)\ (y - y_0)$$

$$L(x, y) = \ 5 - \frac{1}{2}\ (x + 4) + \frac{2}{3}\ (y - 3)$$

4.8 Total Differentials

Given the function $z = f(x, y)$ the differential dz or df is given by,

$$dz = f_x dx + f_y dy \quad \text{or} \quad df = f_x dx + f_y dy$$

There is a natural extension to functions of three or more variables. For instance, given the function $w = g(x, y, z)$ the differential is given by,

$$dw = g_x dx + g_y dy + \ g_z dz$$

Example: Compute the differentials for $z = e^{x^2+y^2}\tan(2x)$

Solution: $dz = \left(2xe^{x^2+y^2}\tan(2x) + 2e^{x^2+y^2}sec^2(2x)\right)dx + 2ye^{x^2+y^2}\tan(2x)dy$

4.9 Applications of Partial Derivatives

Most of the applications will be extensions to applications to ordinary derivatives that we saw back in Applied Mathematics I. For instance, we will be looking at finding the absolute and relative extrema of a function and we will also be looking at optimization.

4.9.1 Extrema of a function

Here we will see how to identify relative (or absolute) minimums and maximums.

Definition

1. A function $f(x, y)$ has a **relative minimum** at the point (a, b) if $f(x, y) \geq f(\ a, b)$ for all points(x, y) in some region around (a, b).
2. A function $f(x, y)$ has a **relative maximum** at the point (a, b) if $f\ (x, y)\ \leq\ f(a, b)$ for all points (x, y) in some region around (a, b).

Note that this definition does not say that a relative minimum is the smallest value that the function will ever take. It only says that in some region around the point (a, b) the function will always be larger than $f(a, b)$. Outside of that region it is completely possible for the function to be smaller. Likewise, a relative maximum only says that around (a, b) the function will always be smaller than $f(a, b)$. Again, outside of the region it is completely possible that the function will be larger.

Next we need to extend the idea of **critical points** up to functions of two variables. Recall that a critical point of the function $f(x)$ was a number $x = c$ so that either $f'(c) = 0$ or $f'(c)$doesn't exist. We have a similar definition for critical points of functions of two variables.

Definition

The point (a, b) is a **critical point** (or a **stationary point**) of $f(a, b)$ provided one of the following is true,

1. $\nabla f(a, b) = \vec{0}$ (this is equivalent to saying that $f_x(a, b) = 0$ and $f_y(a, b) = 0.$)
2. $f_x(a, b)$ and/or $f_y(a, b)$ doesn't exist

Fact

If the point (a, b) is a relative extrema of the function $f(x, y)$ and the first order derivatives of $\mathrm{f}(x, y)$ exist at (a, b) then (a, b) is also a critical point of $f(x, y)$ and in fact we'll have $\nabla f(a, b) = \vec{0}$.

Critical points that are neither of the relative extrema (max. or min.) are called **saddle points.**

Theorem(Test for Relative Extrema):

Suppose that (a,b) is a critical point of $f(\mathrm{x},\mathrm{y})$ and that the second order partial derivatives are continuous in some region that contains (a,b). Next define,

$$\mathrm{D}=\mathrm{D}(\mathrm{a},\mathrm{b})=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$

We then have the following classifications of the critical point.
1. If $D > 0$ and $f_{xx}(a,b) > 0$ then there is a relative minimum at(a,b).
2. If $D > 0$ and $f_{xx}(a,b) < 0$ then there is a relative maximum at(a,b).
3. If $D < 0$ then the point (a,b) is a saddle point.
4. If $D = 0$ then the point (a,b) may be a relative minimum, relative maximum or a saddle point.

Example 2: Find and classify all the critical points for $f(x,y)=4+x^3+y^3-3xy$.

Solution: We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives.

$$f_x=3x^2-3y \qquad\qquad f_y=3y^2-3x$$

$$f_{xx}=6x \qquad f_{yy}=6y \qquad f_{xy}=-3$$

Then, critical points will be solutions to the system of equations,

$$f_x=3x^2-3y=0 \implies 3(x^2-y)=0 \implies x^2=y$$

$$f_y=3y^2-3x=0 \implies 3(y^2-x)=0 \implies y^2=x$$

Plugging the first equation into the second equation gives,

$(x^2)^2=x \implies x(x^3-1)=0 \implies x=0$ or $x=1$.

Now use the fact that $x^2=y$ to get the critical points.

$$x=0: \quad y=0^2=0 \implies (0,0)$$

$$x=1: \quad y=1^2=1 \implies (1,1)$$

Therefore, $(0,0)$ and $(1,1)$ are the only critical numbers. All we need to do now is classify them. To do this we will need D.

$$\mathrm{D}(\mathrm{x},\mathrm{y})=f_{xx}(x,y)f_{yy}(x,y)-\left[f_{xy}(x,y)\right]^2$$

$$=(6x)(6y)-(-3)^2$$

$$=36xy-9$$

To classify these critical points all that we need to do is plug in the critical points and use the fact above to classify them.

At $(0,0)$: $D = D(0,0) = -9 < 0$ $\Longrightarrow$ $(0,0)$ is a saddle point

and

At $(1,1)$: $D = D(1,1) = 36 - 9 = 27 > 0$ and $f_{xx}(1,1) = 6 > 0$

Since both D and f_{xx} are both positive at $(1,1)$, hence, we must have a relative minimum at the point.

Exercise: Find and classify all the critical points for $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

Exercise: Determine the point on the plane $4x - 2y + z = 1$ that is closest to the point $(-2, -1, 5)$.

4.9.2 Absolute Minimums and Maximums

In the previous section we were asked to find and classify all critical points as relative minimums, relative maximums and/or saddle points. In this section we want to optimize a function, that is identify the absolute minimum and/or the absolute maximum of the function, on a given region in $\mathbb{R}^2$.

Definitions

1. A region in $\mathbb{R}^2$ is called **closed** if it includes its boundary. A region is called **open** if it doesn't include any of its boundary points.
2. A region in $\mathbb{R}^2$ is called **bounded** if it can be completely contained in a disk. In other words, a region will be bounded if it is finite.

Extreme Value Theorem

If $f(x,y)$ is continuous in some closed, bounded set D in $\mathbb{R}^2$ then there are points in D, (x_1, y_1) and (x_2, y_2) so that $f(x_1, y_1)$ is the absolute maximum and (x_2, y_2) is the absolute minimum of the function in D.

Note:

1. The above theorem does NOT tell us where the absolute minimum or absolute maximum will occur. It only tells us that they will exist.
2. The absolute minimum and/or absolute maximum may occur in the interior of the region or it may occur on the boundary of the region.

Finding Absolute Extrema

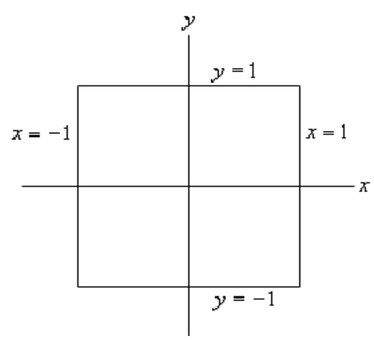
The basic process for finding absolute extrema is pretty much identical to the process that we used in Applied Mathematics I. There will however, be some procedural changes to account for the fact that we now are dealing with functions of two variables.

Thus, we need to follow the following procedures

1. Find all the critical points of the function that lie in the region *D* and determine the function value at each of these points.
2. Find all extrema of the function on the boundary.
3. The largest and smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function.

Example: Find the absolute minimum and absolute maximum of $f(x, y) = x^2 + 4y^2 - 2x^2y + 4$ on the rectangle given by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Solution: Here let's get the picture of region, D, which is a rectangle (in this case) as follow;



The boundary of this rectangle is given by the following conditions.

Right side: $x = 1, -1 \leq y \leq 1$

Left side: $x = -1, -1 \leq y \leq 1$

Upper side: $y = 1, -1 \leq x \leq 1$

Lower side: $y = -1, -1 \leq x \leq 1$

Now, let's find all the critical points that lie inside the given rectangle.

Then, $f_x = 2x - 4xy = 0 \quad \Rightarrow 2x(1 - 2y) = 0$

$$f_y = 8y - 2x^2 = 0 \Rightarrow 2(4y - x^2) = 0 \quad \Rightarrow y = \frac{x^2}{4}$$

Plugging the second equation into the first equation gives us,

$2x\left(1-2\frac{x^2}{4}\right)=2x\left(1-\frac{x^2}{2}\right)=x(2-x^2)=0 \;\Rightarrow x=0$ or $x=\pm\sqrt{2}$

But, the only value of x that will satisfy this is the first one (i.e. $x=0$) so we can ignore $x=\pm\sqrt{2}=\pm 1.414 \ldots$ for this problem because we only want critical points in the region that we are given which is $-1 \leq x \leq 1$.

Note: Don't forget to always check if the critical points are in the region (or on the boundary since that can also happen).

Plugging $x=0$ into the equation for y gives us,

$$y=\frac{x^2}{4}=\frac{0^2}{4}=0$$

The single critical point, in the region is $(0,0)$. Now, the value of the function at the critical point becomes,

$$f(0,0)=4$$

Eventually we will compare this to values of the function found in the next step and take the largest and smallest as the absolute extrema of the function in the rectangle. Next, let's find the absolute extrema of the function along the boundary of the rectangle.

i. Right side: $x=1, -1 \leq y \leq 1$

Here note we know that $x = 1$. Let's take advantage of this by defining a new function as follows,

$g(y)=f(1,y)=1^2+4y^2-2(1^2)y+4= \; 5+4y^2-2y$

We need to find the absolute extrema of $g(y)$ on the range $-1 \leq y \leq 1$. First find the critical point(s).

$g'(y)=8y-2=0 \;\; \Rightarrow \;\; y=\frac{1}{4}$.

This is in the range and so we will need the following function evaluations.

$g(-1)=f(1,-1)=11$ $\qquad$ $g(1)=f(1,1)=7$ $\qquad$ $g\left(\frac{1}{4}\right)=f\left(1,\frac{1}{4}\right)=\frac{19}{4}$

$= 4.75$

In similar fashions,

ii. Left side: $x=-1, -1 \leq y \leq 1$
$g(y)=f(-1,y)=(-1)^2+4y^2-2(-1)^2y+4= \; 5+4y^2-2y$

$$g'(y) = 8y - 2 = 0 \implies y = \tfrac{1}{4}.$$

Then the function value at the critical point and the end points are becomes,

$$g(-1) = f(-1,-1) = 11 \qquad g(1) = f(-1,1) = 7 \qquad g\left(\frac{1}{4}\right) = f\left(-1,\frac{1}{4}\right) = \frac{19}{4}$$

$$= 4.75$$

iii. Upper side: $y = 1, -1 \le x \le 1$

We define a new function except this time it will be a function of x.

Let $h(x) = f(x,1) = x^2 + 4(1)^2 - 2x^2(1) + 4 = 8 - x^2$

We need to find the absolute extrema of $h(x)$ on the range $-1 \le x \le 1$. First find the critical point(s).

$$h'(x) = 2x = 0 \implies x = 0.$$

Thus, the value of this function at the critical point and the end points is,

$$h(-1) = f(-1,1) = 7 \qquad h(1) = f(1,1) = 7 \qquad h(0) = f(0,1) = 8$$

iv. Lower side: $y = -1, -1 \le x \le 1$

Let $h(x) = f(x,-1) = x^2 + 4(-1)^2 - 2x^2(-1) + 4 = 8 + 3x^2$

Then, $h'(x) = 6x = 0 \implies x = 0.$

Thus, the value of this function at the critical point and the end points is,

$$h(-1) = f(-1,-1) = 11 \qquad h(1) = f(1,-1) = 11 \qquad h(0) = f(0,-1) = 8$$

The final step to this process is to collect up all the function values for $f(x,y)$ that we've computed in this problem. Here they are,

$$f(0,0) = 4 \qquad f(1,-1) = 11 \qquad f(1,1) = 7$$

$$f\left(1,\tfrac{1}{4}\right) = 4.75 \qquad f(-1,1) = 7 \qquad f(-1,-1) = 11$$

$$f\left(-1,\tfrac{1}{4}\right) = 4.75 \qquad f(0,1) = 8 \qquad f(0,-1) = 8$$

Consequently, the absolute minimum is at (0,0) since gives the smallest function value and the absolute maximum occurs at (1, -1) and (-1,-1) since these two points give the largest function value.

Exercise: Find the absolute minimum and absolute maximum of $f(x, y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \leq 16$.

4.10 Lagrange Multipliers

In the previous section we optimized (*i.e.* found the absolute extrema) a function on a region that contained its boundary. Finding potential optimal points in the interior of the region isn't too bad in general, all that we needed to do was find the critical points and plug them into the function. However, as we saw in the examples finding potential optimal points on the boundary was often a fairly long and messy process.

In this section we'll see how to use Lagrange Multipliers to find the absolute extrema for a function subject to a given constraint. The constraint(s) may be the equation(s) that describe the boundary of a
region although this method just requires a general constraint and doesn't really care where the constraint came from.

Let say, we want to optimize (*i.e.* find the minimum and maximum value of) a function, $f(x, y, z)$, subject to the constraint $g(x, y, z) = k$.

Method of Lagrange Multipliers

1. Solve the following system of equations.
$$\nabla f(x, y, z) = \lambda \, \nabla g(x, y, z)$$
$$\mathrm{g}(x, y, z) = k$$
2. Plug in all solutions, (x, y, z), from the first step into $f(x, y, z)$ and identify the minimum and maximum values, provided they exist.
   The constant, λ, is called the **Lagrange Multiplier**.

Notice that the system of equations actually has four equations, we just wrote the system in a simpler form. That is from the first equations, we have

$$\nabla f(x, y, z) = \lambda \, \nabla g(x, y, z)$$
$$\Longrightarrow \ \langle \mathrm{f}_x, \mathrm{f}_y, \mathrm{f}_z \rangle = \lambda \langle \mathrm{g}_x, \mathrm{g}_y, \mathrm{g}_z \rangle = \langle \lambda \mathrm{g}_x, \lambda \mathrm{g}_y, \lambda \mathrm{g}_z \rangle$$
$$\Longrightarrow \mathrm{f}_x = \lambda \mathrm{g}_x; \qquad \mathrm{f}_y = \lambda \mathrm{g}_y \quad \text{and} \quad \mathrm{f}_z = \lambda \mathrm{g}_z$$

These three equations along with the constraint, $\mathrm{g}(x, y, z) = k$, give four equations with four unknowns x, y, z, and λ.

Note:

1. If we only have functions of two variables then we won't have the third component of the gradient and so will only have three equations in three unknowns $x, y,$ and λ.
2. In some cases minimums and maximums won't exist even though the Lagrange method will seem to imply that they do.

Example: Find the dimensions of the box with largest volume if the total surface area is $64cm^2$.

Solution: First we need to identify the function that to be optimized as well as the constraint. Let's set the length of the box to be x, the width of the box to be y and the height of the box to be z. Moreover, the dimensions of a box it is safe to assume that $x, y,$ and z are all positive quantities.

We want to find the largest volume and so the function that we want to optimize is given by,

$$f(x, y, z) = xyz$$

Next we know that, the constraint is, the surface area of the box must be a constant 64. The surface area of a box is simply the sum of the areas of each of the sides so the constraint is given by

$$2xy + 2xz + 2yz = 64 \qquad \Longrightarrow \qquad xy + xz + yz = 32$$

That means, the constraint function becomes, $g(x, y, z) = xy + xz + yz$

Then, here are the four equations that we need to solve.

$$yz = \lambda(y + z) \qquad (\mathrm{f}_x = \lambda \mathrm{g}_x) \qquad (1)$$

$$xz = \lambda(x + z) \qquad (\mathrm{f}_y = \lambda \mathrm{g}_y) \qquad (2)$$

$$xy = \lambda(x + y) \qquad (\mathrm{f}_z = \lambda \mathrm{g}_z) \qquad (3)$$

$$xy + xz + yz = 32 \qquad (g(x, y, z) = 32) \qquad (4)$$

To solve this system, let's multiply equation (1) by x, equation (2) by y and equation (3) by z. This gives,

$$xyz = \lambda x(y + z) \qquad (5)$$

$$xyz = \lambda y(x + z) \qquad (6)$$

$$xyz = \lambda z(x + y) \qquad (7)$$

Now, setting equations (5) and (6) equal gives,

$$\lambda x(y + z) = \lambda y(x + z)$$

$$\Longrightarrow \quad \lambda(xz - yz) = 0 \quad \Longrightarrow \quad \lambda = 0 \text{ or } xz = yz$$

This gave two possibilities. The first, $\lambda = 0$ is not possible since if this was the case equation (1) would reduce to

$$yz = 0 \Rightarrow y = 0 \text{ or } z = 0$$

Since we are talking about the dimensions of a box neither of these are possible so we can discount $\lambda = 0$. This leaves the second possibility

$$xz = yz \Rightarrow z(x - y) = 0$$

$$\Rightarrow \quad z = 0 \text{ or } x = y$$

Since we know that $z \neq 0$ (again since we are talking about the dimensions of a box) we can cancel the z from both sides. This gives,

$$x = y \tag{8}$$

Next, let's set equations (6) and (7) equal that gives,

$$\lambda y(x + z) = \lambda z(x + y)$$

$$\Longrightarrow \quad \lambda(xy - xz) = 0 \quad \Longrightarrow \quad \lambda = 0 \text{ or } xy = xz$$

$$\Longrightarrow \quad \lambda = 0 \text{ or } x(y - z) = 0$$

$$\Longrightarrow \quad \lambda = 0 \text{ or } x = 0 \text{ or } y = z$$

As already discussed we know that $\lambda = 0$ won't work and so this leaves, $xy = xz$.
We can also say that $x \neq 0$ since we are dealing with the dimensions of a box so we must have,

$$y = z \tag{9}$$

Plugging equations (8) and (9) into equation (4) we get,

$$y^2 + y^2 + y^2 = 3y^2 = 32 \quad \Longrightarrow \quad y = \pm\sqrt{\tfrac{32}{3}} = \pm 3.266$$

However, we know that y must be positive since we are talking about the dimensions of a box. Therefore, the only solution that makes physical sense here is

$$x = y = z = 3.266$$

Since we've only got one solution we might be tempted to assume that these are the dimensions that will give the largest volume. The method of Lagrange Multipliers will give a set of points that will either maximize or minimize a given function subject to the constraint, provided there actually are minimums or maximums. The function itself, $f(x, y, z) = xyz$ will clearly have neither minimums or maximums unless we put some restrictions on the variables.

Exercise: Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Note: If we want to optimize (*i.e.* find the minimum and maximum value of) a function, $f(x, y, z)$, subject to the constraint $g(x, y, z) = c$ and $h(x, y, z) = k$.

The system that we need to solve in this case is,

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

$$g(x, y, z) = c$$
$$h(x, y, z) = k$$

Exercise: Find the maximum and minimum of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.