CHAPTER ONE

VECTORS AND VECTOR SPACES

Introduction

Vectors play several important roles in software engineering, especially in areas involving mathematics, graphics, physics, and data representation.

1. Data Representation

Vectors are used to represent **data structures** such as lists, arrays, and sequences in programming.

- In languages like **C++**, `std::vector` is a **dynamic array** that can grow and shrink in size.
- Example:
- `vector<int> numbers = {1, 2, 3, 4, 5};`

  This allows efficient storage and manipulation of data elements.

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2. Computer Graphics and Game Development

Vectors are essential for representing **positions, directions, and movements** in 2D or 3D space.

Used to describe:

- The position of an object (e.g., player location)
- The direction of movement
- Lighting and shading (in rendering)

3. Machine Learning and Data Science

Vectors are used to represent **features of data**.

- Each vector element corresponds to a numerical value describing a property.

  These vectors are used in algorithms such as **neural networks** and **support vector machines**.

4. Information Retrieval and NLP (Natural Language Processing)

Vectors represent **words or documents** using **word embeddings** or **vector spaces**.

5. Physics Simulations and Game Engines

In simulation software, vectors handle:

- Force, velocity, and acceleration
- Collision detection
- Motion trajectories

6. Algorithms and Optimization

Many algorithms use vectors for:

- Gradient computations in optimization problems
- Representing multidimensional variables
- Linear algebra operations (important for AI and graphics)

7. Mathematical Modeling

In simulation or computational software, vectors represent:

- States in a system
- Variables in a function
- Input-output mappings

We know that a number x can be used to represent a point on a line. A pair of numbers (x, y) can be used to represent a point in the plane. A triple a numbers (x, y, z) can be used to represent a point in space. We can say a single number represents a point in 1-space, a couple represents a point in 2-space and a triple represents a point in 3-space and so on. **Definition**: For each positive integer n, $\mathbb{R}^n$ will denote the set of all ordered n-tuples $(u_1, u_2, \dots, u_n)$ of real numbers u_i.The set $\mathbb{R}^n$ is referred to as an **n-dimensional realcoordinate space.** The elements of $\mathbb{R}^n$ are called **n-dimensional real coordinate vectors**, or simply **vectors**. The numbers u_i in a vector $(u_1, u_2, \dots, u_n)$ is called **components** of the vector. The **Definitions:**

- A **scalar** is a physical quantity that is described by its magnitude only.
- A **vector** is a physical quantity that is described using both magnitude and direction.

For instance, velocity, displacement, and force are some examples of vectors, whereas mass and temperature are examples of scalars. Elements of $\mathbb{R}$ will be referred to as **scalars**.

1.1. Geometric and coordinate representation of vectors

Vectors in $\mathbb{R}^2$ *and* $\mathbb{R}^3$

Geometrically, a **vector** is represented as a **directed line segment** — that is, a **straight line with an arrow** on one end showing its **direction**.

A vector has:

- **Magnitude (length)** → how long the line is
- **Direction** → which way the arrow points

So, a vector shows **movement** or **displacement** from one point to another in space.

Important Terms

Term	Meaning
Initial Point (Tail)	Starting point of the vector
Terminal Point (Head)	End point of the vector (arrow tip)
Magnitude	Distance between the tail and the head
Direction	Orientation of the vector in space

A **vector** is a quantity that has **both magnitude and direction**.

To describe a vector in a coordinate system, we use its **components** along the coordinate axes (usually x, y, and z).This is called the **coordinate representation** of the vector.

Every pair of distinct points A and B in $\mathbb{R}^2$ *and* $\mathbb{R}^3$determines a directed line segment with initial point at A and terminal point at B. We call such a directed line segment a **vector** and denote it by $\overrightarrow{AB}$. The length of the line segment is the magnitude of the vector and the arrow indicates its direction.

A vector in standard position is a vector with initial point at the origin. A vector in standard position in the plane can be described as $\vec{u} = (u_1, u_2)$, where $u_1, u_2 \in \mathbb{R}$,or$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, u_1 , $u_2 \in \mathbb{R}$.

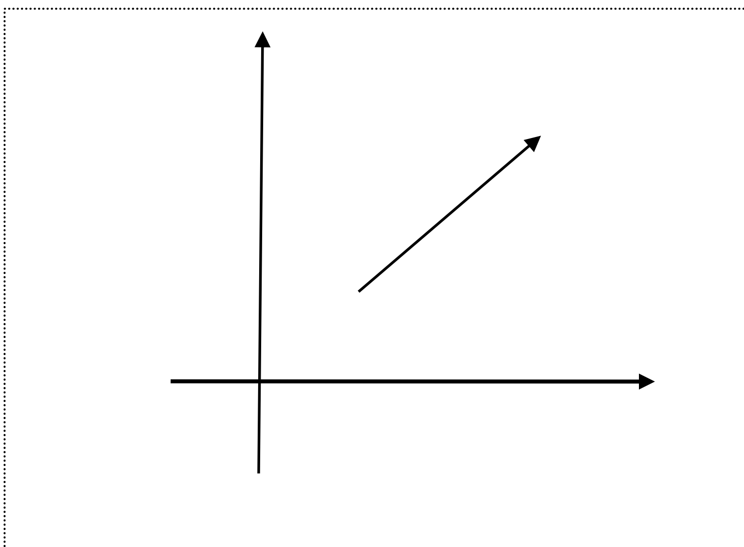
Similarly, a vector in the space $\mathbb{R}^3$ can be described as a triple of numbers $\vec{v} = (v_1, v_2, v_3)$ where $v_1, v_2, v_3 \in \mathbb{R}$.

Definition: Two vectors $\vec{u}$ *and* $\vec{v}$ in $\mathbb{R}^2/\mathbb{R}^3$ are said to be equal (or equivalent) if they have the same magnitude and direction, and is denoted by $\vec{u} = \vec{v}$.

That is, if$\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$ *in* $\mathbb{R}^2$, $\vec{u} = \vec{v}$ *iff* $u_1 = v_1$ *and* $u_2 = v_2$.

Definitions:

1. A located vector is a vector $\overrightarrow{AB}$ defined as an arrow whose initial point is at point A and whose terminal point is at B.
2. Position vector is a vector whose initial point is at the origin.



$$b_1 = a_1 + (b_1 - a_1)$$

$$b_2 = a_2 + (b_2 - a_2)$$

$$\mathrm{B} = A + (\mathrm{B} - \mathrm{A})$$

Definition(Parallel Vectors)

Two non – zero vectors $\vec{u}$ *and* $\vec{v}$ of the same dimension are said to be **parallel** if they are scalar multiples of one another.

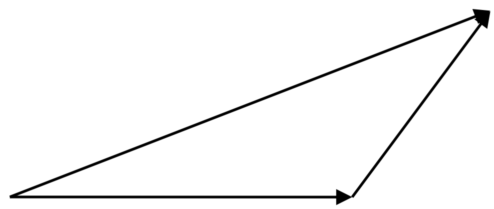
In other words, the two vectors $\vec{u}$ *and* $\vec{v}$ are said to be parallel, denoted by $\vec{u}//\vec{v}$ if there exists a scalar c such that $\vec{u} = c\,\vec{v}$.

1.2. Vector addition and Scalar multiplication

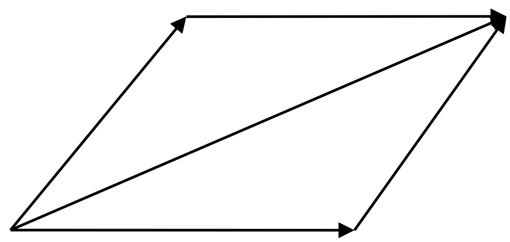
Definition: For any vector $\vec{u} = (u_1, u_2)$ *and any scalar* c, we define $c\,\vec{u}$ as $c\,\vec{u} = (cu_1, cu_2)$.

Definition: For any two vectors $\vec{u} = (u_1, u_2)$ *and* $\vec{v} = (v_1, v_2)$ in $\mathbb{R}^2$, we define their sum to be $\vec{u} + \vec{v} = (u_1 + v_1\,, u_2 + v_2)$.

Geometrically, if we represent the two vectors $\vec{u}$ *and* $\vec{v}$ by $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$ respectively, then $\vec{u} + \vec{v}$ is represented by $\overrightarrow{AC}$, as shown in the diagram below:



a) The Triangular Law



b) The Parallelogram Law

Properties of Vector addition & Scalar Multiplication

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ and c & m are scalars. Then:

a) $\vec{u} + \vec{v} \in \mathbb{R}^2$
b) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
c) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}, where\ \overline{0} = (0,0) \in \mathbb{R}^2$.
d) There exists $w \in \mathbb{R}^2$ such that $\vec{u} + \vec{w} = 0$ for every $u \in \mathbb{R}^2$.
e) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
f) $c\,(m\,\vec{u}) = (cm)\vec{u}$
g) $(c + m)\vec{u} = c\vec{u} + m\vec{u}$

h) $1.\vec{u} = \vec{u}$

Remark: The properties described above also hold true for vectors in $\mathbb{R}^3$, where $\overline{0} = (0,0,0) \in \mathbb{R}^3$, replaces the zero vector $\overline{0}$ in $\mathbb{R}^2$.

1.3. Dot (Scalar) product, Magnitude of a vector, Angle between two Vectors, Orthogonal Projection, Direction angles and direction cosines.

1.3.1. Dot (Scalar) Product

Definition: Let $\vec{u} = (u_1, u_2, u_3)$ be a vector in $\mathbb{R}^3$. Then the **magnitude (norm)** of $\vec{u}$, denoted by $\|\vec{u}\|$ is defined by:

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Similarly, for a vector $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$, its norm is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

Examples: a) If $\vec{v} = (-1,4,3)$, then find $\|\vec{v}\|$.

b) If $\|\vec{u}\| = 6, find\ x$ such that $\vec{u} = (-1, x, 5)$.

Remarks: (i) $\|\vec{v}\| \neq 0\ if\ \vec{v} \neq \overline{0}$

(ii) $\|\vec{v}\| = \|-\vec{v}\|$.

Theorem: If $c \in \mathbb{R}$, then $\|c\ \vec{v}\| = |c|\,\|\vec{v}\|$.

Definition (Unit Vector)

Any vector $\vec{u}$ satisfying $\|\vec{u}\| = 1$ is called a **unit vector**.

Examples: The vectors $(0,1)$, $(-1,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $(1,0,0)$ are examples of unit vectors.

<u>**N.B**</u>**:** 1. All unit vectors in $\mathbb{R}^2$ are of the form $(\cos\theta\ , \sin\theta), where\ \theta \in \mathbb{R}$.

2. For any non-zero vector $\vec{v}$, the unit vector $\hat{u}$ corresponding to $\vec{v}$ in the direction of $\vec{v}$ can be obtained as: $\hat{u} = \frac{\vec{V}}{\|\vec{V}\|}$

3. For two points $P(u_1, u_2)\ and\ Q\ (v_1, v_2)$ on the plane $\mathbb{R}^2$, we calculate the distance $d\ (\mathrm{P}, Q)$ between the two points, as:

$d\ (\mathrm{P}, Q) = \|\overrightarrow{\mathrm{P}Q}\| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2},\ where\ \overrightarrow{\mathrm{P}Q}$is the vector with initial point P and terminal point Q, ; $\overrightarrow{\mathrm{P}Q} = (v_1 - u_1\ , v_2 - u_2)$.

1.3.2. **The Dot Product (or Scalar Product)**

Definition: Suppose that $\vec{u} = (u_1, u_2)\ and\ \ \vec{v} = (v_1, v_2)$ are vectors in $\mathbb{R}^2$, and that $\theta \in [0, \pi]$ represents the angle between them. We define the dot product of $\vec{u}\ and\ \vec{V}$ denoted by $\vec{u} \cdot \vec{v}$ by:

$$\vec{u} \cdot \vec{v} = \begin{cases} \|\vec{u}\|\|\vec{v}\| \cos\theta, & if\ \vec{u} \neq \overline{0} \& \vec{v} \neq \overline{0} \\ 0 & if\ \vec{u} = \overline{0}\ or\ \vec{v} = \overline{0} \end{cases}$$

Or alternatively, we write

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2}$$

Thus $\|\vec{u}\|\|\vec{v}\| \cos\theta = u_1 v_1 + u_2 v_2$,for anon – zero vector $\vec{u}\ and\ \vec{v}\ in\ \mathbb{R}^2$.

Similarly, if $\vec{u} = (u_1, u_2, u_3)\ and\ \vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$,we define $\vec{u} \cdot \vec{v}$ as bellow:

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

Remark: The dot product of two vectors is a scalar quantity, and its value is maximum when $\theta = 0^{\circ}$ and minimum if $\theta = 180^{\circ}\ or\ \pi$ radians.

Example: If $\vec{u} = i - 2j + 3k\quad and\ \ \vec{v} = \langle 0,1,-5\rangle$, then find:

a) $\vec{u} \cdot \vec{v}$ b) $\vec{u} \cdot \vec{u}$ c)$(\vec{u} + \vec{v}) \cdot \vec{v}$

Properties of the dot product

If $\vec{u}, \vec{v}, and\ \vec{w}$ are vectors in the same dimension, and $c \in \Re$, then:

1. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$
2. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
4. $\overline{0}\ \cdot \vec{u} = 0$
5. $(c\ \vec{u}) \cdot \vec{v} = c\ (\vec{u} \cdot \vec{v}) = \vec{v} \cdot (c\ \vec{u})$
6. $\vec{u} \cdot \vec{u} \geq 0\ and\ \vec{u} \cdot \vec{u} = 0\ iff\ \vec{u} = 0$.

1.3.3. Angle between two vectors

If θ is the angle between two non – zero vectors $\vec{u}\ and\ \vec{v}$, then, the angle between the two vectors can be obtained by:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$\Rightarrow \theta = cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right), and\ \theta \in [0, \pi]$.

Example: Find the angle between the vectors $\vec{u} = \langle 1,0,-1\rangle\ and\ \vec{v} = (1,1,0)$.

Solution: Let θ be the angle between the two vectors:

Then $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Definition: Two non – zero $\vec{u}\ and\ \vec{v}$ are said to be orthogonal (perpendicular)iff$\vec{u}\cdot\vec{v} = 0$, i.e, if $\theta = \frac{\pi}{2}$.

Example: Find the value (s) of x such that the vectors $A = (1\ ,4, 3)\ and$

$\mathrm{B} = (x, -1,\ 2)$are orthogonal.

Definition: If P and Q are points in 2 or 3 spaces, the distance between P and Q, denoted by $\|\mathrm{P} - Q\|$ is given by $\|\mathrm{P} - Q\| = \sqrt{(\mathrm{P} - Q).(\mathrm{P} - Q)}$

Pythagoras Theorem:

If A and B are orthogonal vectors, then $\|A + B\|^2 = \|A\|^2 + \|B\|^2$

Proof:$\|A + B\|^2 = (A + B)\cdot(A + B)$

$= \|A\|^2 + 2A\cdot B + \|B\|^2$

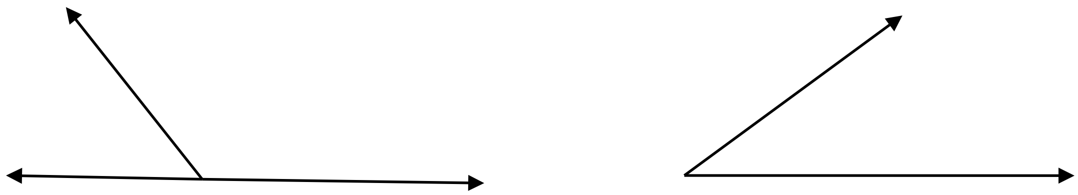
$= \|A\|^2 + \|B\|^2\ since\ A\cdot B = 0.$

Theorem: Given two vectors $\vec{u}\&\vec{v}$ in space, $\|\vec{u} + \vec{v}\| = \|\vec{u} - \vec{v}\|$ iff $\vec{u}\ and\ \vec{v}$are orthogonal vectors.

Note: If A is perpendicular to B, then it is also perpendicular to any scalar multiple of B.

1.3.4. Orthogonal Projection

Definition: Suppose S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the vector projection of B on to A, and is denoted by projectA^B.



The **scalar projection of B onto A** (also called the component of B a long A) is defined to be the length of projectA^B, which is equal to $\|\boldsymbol{B}\| \cos \boldsymbol{\theta}$ and is denoted by compA^B.

Thus: **Comp** $\boldsymbol{A^B} = \frac{\boldsymbol{A \cdot B}}{\|\boldsymbol{A}\|}$

Proj$\boldsymbol{A^B} = \left(\frac{\boldsymbol{A \cdot B}}{\|\boldsymbol{A}\|}\right) \frac{\boldsymbol{A}}{\|\boldsymbol{A}\|} = \frac{\boldsymbol{A \cdot B}}{\|\boldsymbol{A}\|^2} \boldsymbol{A}$

Example: Let $A = (-1,3,1) = -i + 3j + k$and $B = (2,4,3) = 2i + 4j + 3k$

Then find (i) ProjB^A $[ans. \frac{13}{29}(2,4,3)]$

(ii) ProjA^B $[ans. \frac{13}{11}(-1,3,1)]$

(iii) Comp A^B

1.3.5. Directional angles and cosines

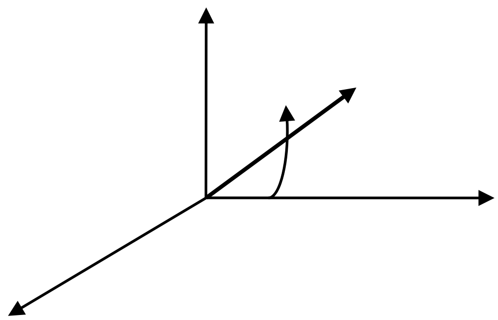
Let $A = a_1 i + a_2 j + a_3 k$ be a vector positioned at the origin in $\mathbb{R}^3$, making an angle of α, β *and* γ with the positive x, y *and* z axes respectively. Then the angles α, β *and* γ are called the **directional angles of A**, and the quantities

$\cos\alpha$, $\cos\beta$ *and* $\cos\gamma$are called the **directional cosines** of A, which can be computed as follows:

$$\cos\alpha = \frac{a_1}{\|A\|} \quad , \quad \alpha \in [0, \pi]$$

$$\cos\beta = \frac{a_2}{\|A\|} \quad , \quad \beta \in [0, \pi]$$

$$\cos\gamma = \frac{a_3}{\|A\|} \quad , \quad \gamma \in [0, \pi]$$



Remark: $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$. (Verify!)

Exercise: Let $A = (-1,2,2)$. Then find the directional cosines of A.

1.4. The Cross (or Vector) Product and Triple Products

Definition: Suppose that $A = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k$ and $B = (b_1, b_2, b_3) = b_1 i + b_2 j + b_3 k$ be two vectors in $\mathbb{R}^3$. Then the cross product $A \times B$ of the two vectors is defined as:

$$\boldsymbol{A \times B = (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k}$$

$$= Det \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

OR: $A \times B = \hat{n}\|A\|\|B\| \sin\theta, where\ \hat{n}$ is the unit vector in the direction of $A \times B\ and\ \theta \in [0, \pi]$ is the angle between A and B.

Example: Let $A = 4i - 3j + 2k$

$$B = 2i - 5j - k$$

Then find a) $A \times B$ b) $B \times A$

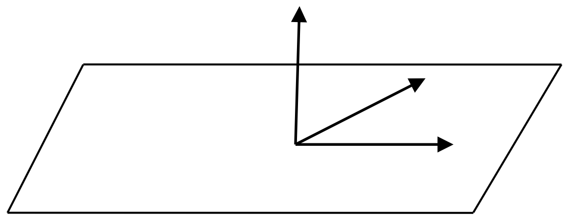
<u>Ans</u>: $A \times B = -7i + 8j + 26k$ & $B \times A = 7i - 8j - 26k = -A \times B$

Remarks: For two non – zero vectors A & B,

1. $A \times B$ is a vector which is orthogonal to both A and B.
2. $A \times B$ is not defined for $A, B \in \mathbb{R}^2$.
3. $i \times j = -j \times i = k$

$$j \times k = -(k \times j) = i$$

$k \times i = -(i \times k) = j$



Properties of Cross Product

Let A, B and C be vectors in $\mathbb{R}^3$ and α be any scalar. Then:

(1) $A \times \overline{0} = \overline{0} \times A = \overline{0}, where\ \overline{0} = \langle 0,0,0 \rangle$

(2) $A \times B = -B \times A$

(3) $A \times (B \times C) \neq (A \times B) \times C$

(4) $(\alpha A) \times B = A \times (\alpha B) = \alpha\,(A \times B)$

(5) $A \times \ (B + C) = \ A \times B + \ A \times C$

(6) $A \cdot (A \times B) = \mathrm{B} \cdot (A \times B) = 0$

(7) If A and B are parallel, then $A \times B = 0$

(8) $\|A \times \mathrm{B}\| = \|A\|\,\|\mathrm{B}\| \sin\theta \ \ , \theta \in [0, \pi]$.

(9) $\|A \times \mathrm{B}\|^2 = \|A\|^2 \|B\|^2 - (A \cdot \mathrm{B})^2$

Example: If $\|A\| = 2$, $\|\mathrm{B}\| = 4\ and\ \theta = {}^{\pi}/_{4}$ for two vectors A and B then find $\|A \times \mathrm{B}\|$.

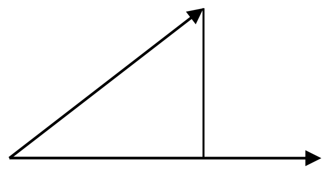
Note: The angle θ between A and B can be obtained by $\sin\theta = \frac{\|A \times \mathrm{B}\|}{\|A\|\|\mathrm{B}\|}$, for two non – zero vectors A and B.

Definition (Scalar Triple Product)

Let A, B and C is vectors in $\mathbb{R}^3$. Their scalar triple product is given by $A \cdot (B \times C)$, which is a scalar.

Applications of cross Product:

(i) **Area**: The area of a parallelogram whose adjacent sides coincides with the vectors A and B is given by $\|A \times \mathrm{B}\| = \|A\|\,\|\mathrm{B}\|\,|\sin\theta|$



So, $a(oabc) = Base\ x\ height$

$$= \|A\|\|\mathrm{B}\||\sin\theta|$$

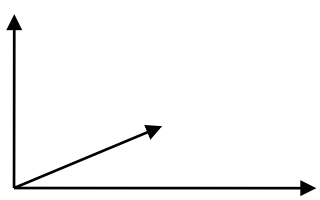
N.B: The area of the triangle formed by A and B as its adjacent sides is given by

$area = \frac{1}{2}\|A \times \mathrm{B}\|$.

ii.Volume

The volume V of a parallelepiped with the three vectors A,B and C in $\mathbb{R}^3$ as three of its adjacent edges is given by:

$$\boldsymbol{V} = |\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})| = \left|\mathbf{det}\begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \\ \boldsymbol{c_1} & \boldsymbol{c_2} & \boldsymbol{c_3} \end{pmatrix}\right|$$



$$h = \left\|proj_{A \times B}^{\ A}\right\| = \left\|\frac{A \cdot (B \times C)}{\|B \times C\|^2}(B \times C)\right\| = \frac{|A \cdot (B \times C)|}{\|B \times C\|}$$

Hence, $\boxed{\mathrm{V} = B_a h = |A \cdot (B \times C)|}$

Examples:

1. Find the area of a triangle whose vertices are $A\ (1,-1,0)$, $B\ (2,1,-1)$ $and\ C\ (-1,1,2)$.

Solution: The vectors on the sides of the triangle $\Delta\, ABC$ are $\overrightarrow{AB} = B - A = (1,2,-1)$ and $\overrightarrow{AC} = (-2,2,2) = C - A$.

So, $a\,(\Delta\, ABC) = \frac{1}{2}\left\|\overrightarrow{AB} \times AC\right\| = 3\sqrt{2}\ \ square\ units.$

2. Find the volume of the parallelepiped with edges

   $\vec{u} = i + k\ \ ,\vec{v} = 2i + j + 4k\ \ and\ \vec{w} = j + k.$

   Solution: V=$|\vec{u} \cdot (\vec{v} \times \vec{w})| = 1\ \ u^3$

<u>N.B</u>: Three vectors $A, B\ and\ C$ are coplanar iff $A \cdot (B \times C) = 0.$

1.5. Lines and Planes in $\mathbb{R}^3$

Definition: A vector $\vec{v} = \langle a, b, c\rangle$ is said to be parallel to a line $\ell\ if\ \vec{v}$ is parallel to $\overrightarrow{P_0P_1}$ for any two distinct points$P_0\ and\ P_1\ on\ \ell.$

A line $\ell\ in\ \mathbb{R}^3$ is determined by a given point $P_0(x_0, y_0, z_0)$ on ℓ and a parallel vector $\vec{v}$ (directional vector) $\vec{v} = \langle a, b, c\rangle\ to\ \ell.$

Equations of a line in space:

Let $P_0(x_0, y_0, z_0)$ be a given point on a line ℓ and $P\ (x, y, z)$ be any arbitrary point on ℓ. If $\vec{V} = \langle a, b, c\rangle$ is the parallel vector to ℓ, then

(1) The **parametric equation** of ℓ is given by

$$x = x_0 + at$$

$$y = y_0 + bt$$

$z = z_0 + ct, t \in \Re$,wheret is called the parameter.

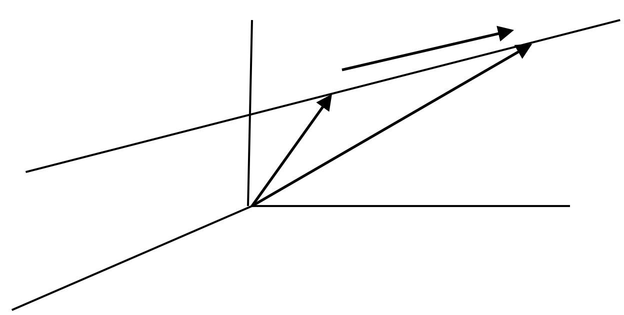
(2) The **symmetric form** of equation of ℓ is given by:

$$\boldsymbol{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, for\ \ a, b, c\ \neq 0}.$$

(3) The vector equation of ℓ is written as$\boldsymbol{\vec{r} - \vec{r}_0 = t\vec{v}}, where\ t \in \Re.$

Remarks:

1. The above equations of the line can be derived using vector algebra as follows:



From vector addition, we have $\vec{r} - \vec{r}_0 = \overrightarrow{\mathrm{P_0P}}$ & $\vec{r}_o = x_0 i + y_0 j + z_0 k$, $\vec{r} = xi + yj + zk$.

Since $\overrightarrow{\mathrm{P_0P}} \| \vec{v}$, there exists $t \in \Re$ such that $\vec{r} - \vec{r}_0 = t\vec{v} = \overrightarrow{\mathrm{P_0P}}$

$\Rightarrow \langle x - x_0, y - y_0, z - z_0 \rangle = t\,\langle a, b, c \rangle$

$\Leftrightarrow x = x_0 + at\,, y = y_0 + bt, z = z_0 + ct.$

2. If one of $a, b\ or\ c\ is\ 0$, (say for instance $b = 0$), the symmetric equation of ℓ is given as: $\frac{x - x_0}{a} = \frac{z - z_0}{c}$, $y = y_0$.

Equation of a plane:

A plane in space is determined by a point $\mathrm{P_0}(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane. This orthogonal vector $\vec{n}$ is called a **normal vector**.

Suppose that $P\ (x, y, z)$ be any arbitrary point in the plane, and let $\vec{r}\ and\ \vec{r}_0$ be the position vectors of $P\ (x, y, z)$ and $\mathrm{P_0}(x_0, y_0, z_0)$. Then we have

$\vec{n} \cdot\ (\vec{r} - \vec{r}_0) = 0$(Since $\vec{n}$ perpendicular to any vector in the plane).

$\Leftrightarrow \vec{n} \cdot \vec{r} = \vec{n} \cdot \overrightarrow{r_o}$,which is called the vector equation of the plane.

If we let $\vec{n} = \langle a, b, c \rangle = ai + bj + ck$, we get

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$\Leftrightarrow a(x - x_0) + b(y - y_0) + c\ (z - z_0) = 0$ (point - normal form of equation of a plane)

$\boldsymbol{\Leftrightarrow ax + by + cz + d = 0,\ where\ d = -(ax_0 + by_0 + cz_0)}$

(general or standard form of the equation of a plane).

Examples:

1. Find the equations of a line that contains the point $(1,4,-1)$ and parallel to $\vec{v} = -2i + 3j$.

Solution: let $\mathrm{P_O} = \langle 1\ ,4, -1 \rangle = (x_0, y_0, z_0)$

Then the parametric form of the equation of the line is:

$$\begin{cases} x = 1 - 2t \\ y = 4 + 3t \\ z = -1 \end{cases}$$

and its symmetric form is given as:

$\frac{x-1}{-2} = \frac{y-4}{3}$, $z = -1$.

2. Find the equation of the plane through the points $\mathrm{P_0}(1,1,1)$, $\mathrm{P_1}(2,2,0)\ and\ \mathrm{P_2}(4, -6,2)$.

Solution: The vectors $\vec{A} = \overrightarrow{P_0P_1} = \langle 1,1,-1\rangle$ *and* $\vec{B} = \overrightarrow{P_1P_2} = \langle 3,-7,1\rangle$ are parallel to the plane, and hence, their cross product $\vec{n} = \vec{A} \times \vec{B} = \langle -6,-4,-10\rangle$ is normal to the plane. Thus, the equation of the plane is given by: $-6\,(x-1) - 4(y-1) - 10\,(z-1) = 0$

$\Leftrightarrow\; 3 \times +2y + 5z - 10 = 0$

OR:

The equation of the plane can be obtained by computing:

$$det\begin{pmatrix} x-1 & y-1 & z-1 \\ x-2 & y-2 & z-0 \\ x-4 & y+6 & z-2 \end{pmatrix} = 0$$

Distance in Space

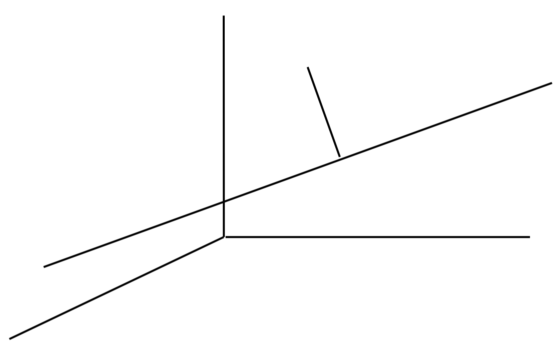
a) Distance from a point to a line

The distance D from a point P_1 (not on ℓ) to a line ℓ in space is given by:

$$\boxed{\boldsymbol{D} = \frac{\|\vec{\boldsymbol{v}} \times \overrightarrow{\mathbf{P_0P_1}}\|}{\|\vec{\boldsymbol{v}}\|}}$$

, where $\vec{v}$ is the directional vector of ℓ and P_0 is any point on ℓ.

Proof:



$$\sin\theta = \frac{D}{\|\overrightarrow{P_0P_1}\|} \Longrightarrow D = \|\overrightarrow{P_0P_1}\| \sin\theta$$

But since $\|\vec{v} \times \overrightarrow{P_0P_1}\| = \|\vec{v}\|\|\overrightarrow{P_0P_1}\||\sin\theta|, = \|\vec{v}\|D$, we have:

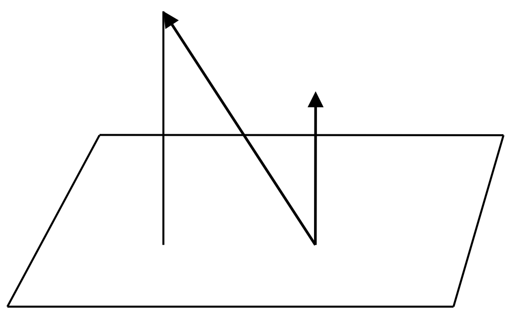
$$D = \frac{\|\vec{v}\times\overrightarrow{P_0P_1}\|}{\|\vec{v}\|}.$$

b) Distance from a point to a plane

The perpendicular distance D of a point $P_0(x_0, y_0, z_0)$ in space to the plane with the equation $ax + by + cz + d = 0$ is given by:

$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|\vec{n}\cdot\overrightarrow{o\mathrm{P}}|}{\|\vec{n}\|}$,where o is the foot of $\vec{n}$within the plane.

Proof: Consider the diagram below:



$$D = \left\|proj_{\vec{n}}^{\overrightarrow{0\mathrm{P}}}\right\| = \frac{\left\|\overrightarrow{o\mathrm{P}}\cdot\vec{n}\right\|}{\|\vec{n}\|^2}\|\vec{n}\| = \frac{\left|\overrightarrow{o\mathrm{P}}\cdot\vec{n}\right|}{\|\vec{n}\|}$$

$\Rightarrow\ D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}},\ where\ d = -(ax_1+by_1+cz_1)$

NB: If $\mathrm{P} = (0,0,0), then\ D = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$, w/c is the distance of the plane from the origin.

Examples:

1. Find the distance of the point $\mathrm{P}_1(-1,3,0)$ from the line with symmetric equations: $\ell: x = 1,\ \frac{y-1}{3} = \frac{z+1}{2}$.
2. How far is the point $\mathrm{P}(1,2,3)$ from the plane with equation $\pi: 3x+5y-4z+37 = 0$.

Solutions:

1. Here, the directional (parallel) vector of ℓ is: $\vec{v} = \langle 0,3,2\rangle, and$

   $let\ \mathrm{P}_0(1,1,-1) \in \ell$be taken.

   Then$\overrightarrow{\mathrm{P}_0\mathrm{P}_1} = \mathrm{P}_1 - \mathrm{P}_0 = \langle -2,2,1\rangle$.

   Thus, $D = \frac{\|\mathrm{P}_0\mathrm{P}_1\times\vec{v}\|}{\|\vec{v}\|} = \frac{\|(3-4)i+(4-0)j+(0+6)k\|}{\sqrt{3^2+2^2}} = \frac{\sqrt{53}}{\sqrt{13}}$units.
2. Here $(x_0, y_0, z_0) = (1,2,3)$, $\vec{n} = \langle 3,5,4\rangle, d = 37$.

   Thus ,$D = \frac{|ax_0+by_0+cz_0+d|}{\|\vec{n}\|}$

   $= \frac{|301+5.2-4.3+37|}{\sqrt{50}} = \frac{19\sqrt{2}}{5}$units.

OR: Take $\vec{n} = \langle 3,5,4\rangle \& \ 0 = \left(0,0,\frac{37}{4}\right) \in \pi$. Then

$$D = \frac{|\vec{n}\cdot\overrightarrow{0P}|}{\|\vec{v}\|} = \frac{\left|(3,5,-4)\left(1,2,\frac{25}{4}\right)\right|}{\sqrt{50}} = \frac{19\sqrt{2}}{5}.$$

Distance between two parallel planes

Given two parallel planesπ_1 *and* π_2. Then we have normal vectors with coefficients a, b, c to be the same such that:

$\pi_1: ax + by + cz = d_1.$

$$\pi_2: ax + by + cz = d_{2.}$$

Then the distance between π_1 *and* π_2 is the same as the distance from any arbitrary point $P(x_0, y_0, z_0)$that has been taken from π_1 to the plane π_2 to the plane π_2 and is given by :

$\Rightarrow D = \frac{|ax_o + by_o + cz_o - d_2|}{\sqrt{a^2+b^2+c^2}}$. But $ax_o + by_o + cz_o = d_1$.

Thus, $\quad D = \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

Example: Find the distance between the planes $\pi_1: x + 2y - 2z = 3$ &$\pi_2: 2x + 4y - 4z = 7$

Solution: We first rewrite the equations π_1 *and* π_2 so that they have the same $\vec{n} = \langle a, b, c\rangle$.

That is: $\qquad \pi_1: x + 2y - 2z = 3$

$$\pi_2: x + 2y - 2z = {}^{7}\!/_{2} \Longrightarrow d_1 = 3 \& d_2 = {}^{7}\!/_{2}$$

Thus, $D = \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}, where \ \ a = 1, b = 2, \ c = -2$

$$= \frac{|{}^{7}\!/_{2} - 3|}{\sqrt{1+4+4}} = \frac{\frac{1}{2}}{3} = {}^{1}\!/_{6}$$