Chapter Two
Solutions of Nonlinear Equations

2.1. Introduction

A frequently occurring problem in science and engineering is **the root finding problem** for either a nonlinear equation $f(x) = 0$ involving a single independent variable x or a coupled system of two nonlinear equations $f(x, y) = 0$ in two independent variables (x, y). So in this unit we shall discuss different numerical methods used to solve equations of the form f(x) = 0.

Definition:

The expression of the form

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where a's are constants with $a_n \neq 0$ and n is a positive integer, is called a polynomial in x of degree n, and the equation $f(x) = 0$ is called an ***algebraic equation*** of degree n. If $f(x)$ contains some other functions like exponential, trigonometric, logarithmic etc., then $f(x) = 0$ is called a ***transcendental equation***.

Example: $x^3 - 3x + 5 = 0$, $x^5 - 5x^4 + 24x^3 - x^2 + x + 6 = 0$ are algebraic equations

$x^2 - 3\cos x + 1 = 0$, $xe^x - 1 = 0$, $x \log_{10}^{x} = 1.2$, etc., are transcendental equations.

Examples of transcendental equations can be found in many applications. In the theory of diffraction of light, we need the roots of the equation $x - \tan x = 0$. In the calculation of planetary orbits, we need the roots of Kepler's equation $x - a\sin x = b$ for various values of a and b.

Remark:

For a function of a single independent variable $y = f(x)$, a point $x = \alpha$ is called a root or a zero of $f(x)$ if the value of the function is zero at the point, meaning $f(\alpha) = 0$. In Figure 2.1 the points $x = x_1$, $x = x_2$, $x = x_3$ are all roots of the function $f(x)$.

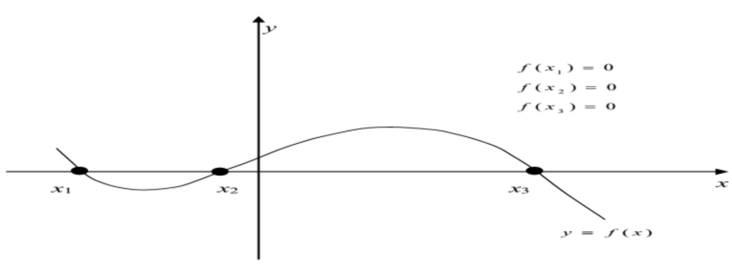

Figure 2.1: Roots of a Function $f(x)$

Locating Roots

Theorem 1.2.1 Intermediate Value Theorem (IVT)

If $f \in C[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there exist a number c in (a, b) for which $f(c) = k$

Theorem 1.2.2 Locating root

If $f\ (x\ )$ is continuous on [a , b], $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one number x_0 in $(a\ ,b)$ such that $f\ (x_0) = 0$.

An equation $f\ (x) = 0$, where $f\ (x)$ is a real continuous function, has at least one root between x_ℓ and x_u if $f\ (x_\ell)\ f\ (x_u) < 0$ (i.e. $f\ (x_\ell)$ *and* $f\ (x_u)$ differ in sign. (See Figure 2.2).

Note: If $f\ (x_\ell)\ f\ (x_u) > 0$, there may or may not be any root between x_ℓ and x_u (Figures 2.3 and 2.4). If $f\ (x_\ell)\ f\ (x_u) < 0$, then there may be more than one root between x_ℓ and x_u (Figure 2.5). So the theorem only guarantees one root between x_ℓ and x_u.

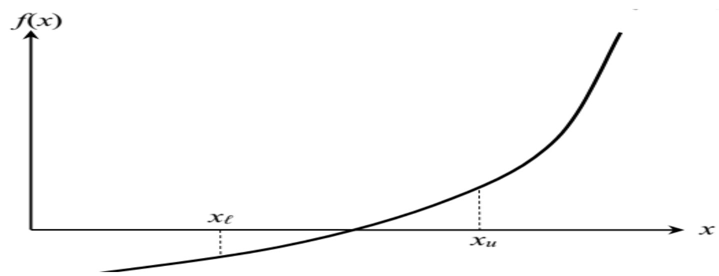

Figure 2.2: At least one root exists between the two points if the function is real, continuous, and changes sign.

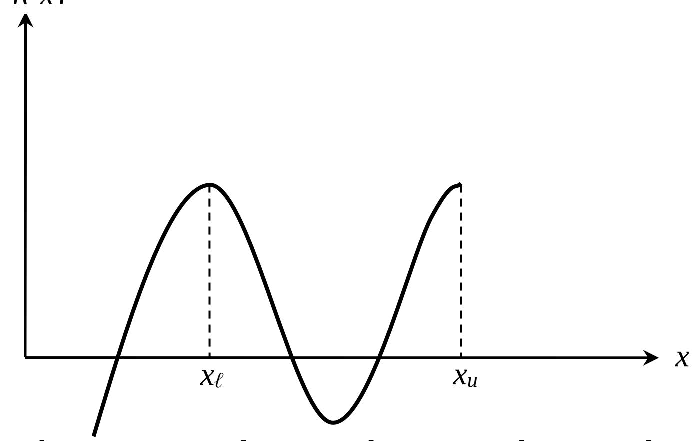

Figure 2.3 If the function $f\ (x)$ does not change sign between the two points, roots of the equation $f\ (x) = 0$ may still exist between the two points.

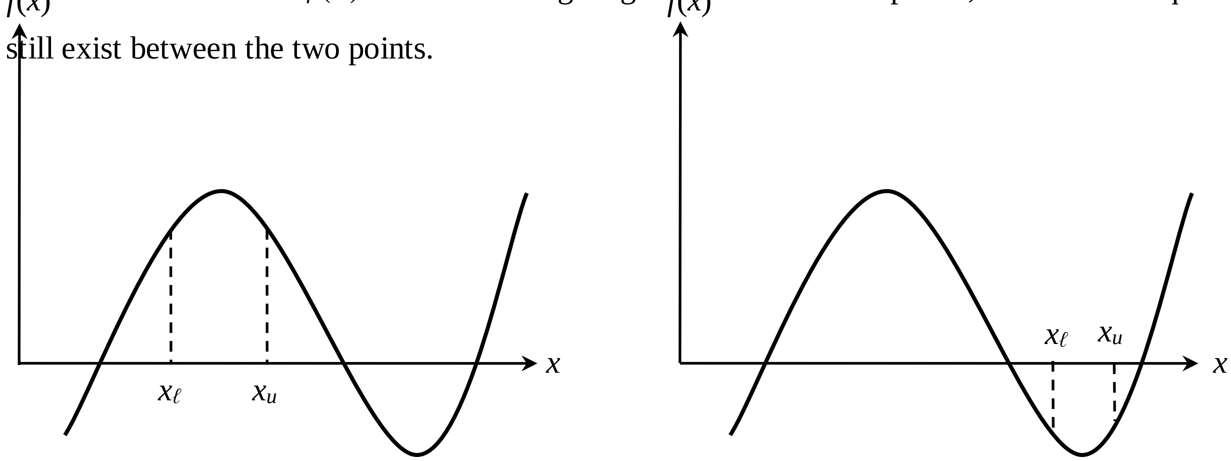

Figure 2.4 If the function $f\ (x)$ does not change sign between two points, there may not be any roots for the equation $f\ (x) = 0$ between the two points.

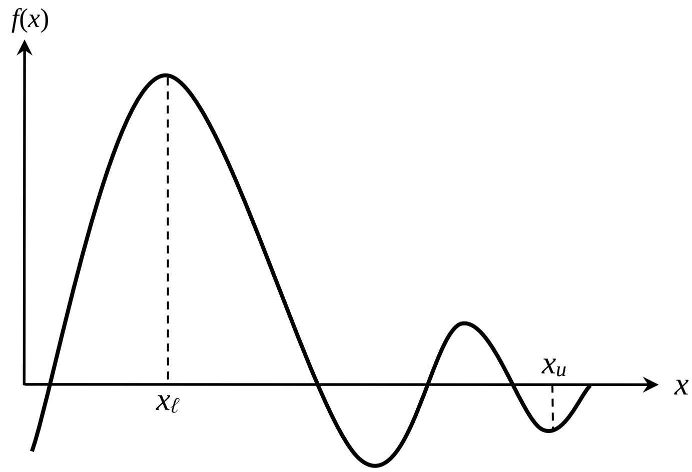

Figure 2.5 If the function $f(x)$ changes sign between the two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Method for Finding the Roots of Equations

Methods for finding roots of an equation can be classified in to two parts:

1. Direct Methods
2. Iterative methods

Direct Method

- Give exact value of the roots in a finite number of steps.
- Assumed to be free of round off errors.
- Determines all the roots at the same time.
- Are also called a closed form solution.

Example: The roots of a quadratic equation is obtained by a direct method (The General formula)

Indirect or Iterative Methods

- This methods are based on the concept of successive approximations
- The general procedure is to start with one or more initial approximation to the root and obtain a sequence of iterates, x_k, which in the limit converges to the actual or true solution or the root.
- The indirect or iterative methods are further divided into two categories:

  – **Bracketing:** require the limits between which the root lies: Bisection and False position methods

  – **Open methods**: require the initial estimation of the solution: Newton's and Fixed-point iteration.

Order (or Rate) of Convergence of Iterative Methods

Convergence of an iterative method is judged by the order at which the error between successive approximations to the root decreases.

The order of convergence of an iterative method is said to be k^{th} order convergent if k is the largest positive real number such that $\lim_{i\to\infty}\left|\frac{e_{i+1}}{e_i^k}\right| \le A$ where A is a non-zero finite number, e_i and e_{i+1} are the errors in successive approximation.

In other words, the error in any step is proportional to the k^{th} power of the error in the previous step. Physically, the k^{th} order convergence means that in each iteration, the number of significant digits in each approximation increases k times.

2.2. The Bisection Method

Suppose that $f(x)$ is continuous on an interval $[x_l, x_u]$, and $f(x_\ell)f(x_u) < 0$. Then $f(x)$ changes sign on $[x_l, x_u]$ and $f(x) = 0$ has at least one root on the interval by IVT.

Definition:

The simplest numerical procedure for finding a root is to repeatedly halve the interval $[x_l, x_u]$ keeping the half for which $f(x)$ changes sign. **This procedure is called the bisection method**, and is guaranteed to converge to a root, denoted here by α

Procedure for the Bisection Method

The steps to apply the bisection method to find the root of the equation $f(x) = 0$ are:

1. Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u.
2. Estimate the root, x_m, of the equation $f(x) = 0$ as the mid-point between x_ℓ and x_u as $x_m = \frac{x_\ell + x_u}{2}$
3. Now check the following
   a) If $f(x_\ell)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$ and $x_u = x_m$.
   b) If $f(x_\ell)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$ and $x_u = x_u$.
   c) If $f(x_\ell)f(x_m) = 0$; then the root is x_m. Stop the iteration if this is true.
4. Find the new estimate of the root and find the absolute error as:

   $$x_m = \frac{x_\ell + x_u}{2} \qquad \left|\in_a\right| = \left|x_m^{new} - x_m^{old}\right|$$, where

   x_m^{new} = estimated root from present iteration & x_m^{old} = estimated root from previous iteration
5. Compare the absolute error $\left|\in_a\right|$ with the pre-specified error tolerance (TOL). If $\left|\in_a\right| > TOL$, then go to Step 3, else stop the iteration.

   Note one should also check whether the number of iterations is more than the maximum number of iterations allowed (imax). If so, one needs to terminate the iteration and notify the user about it.

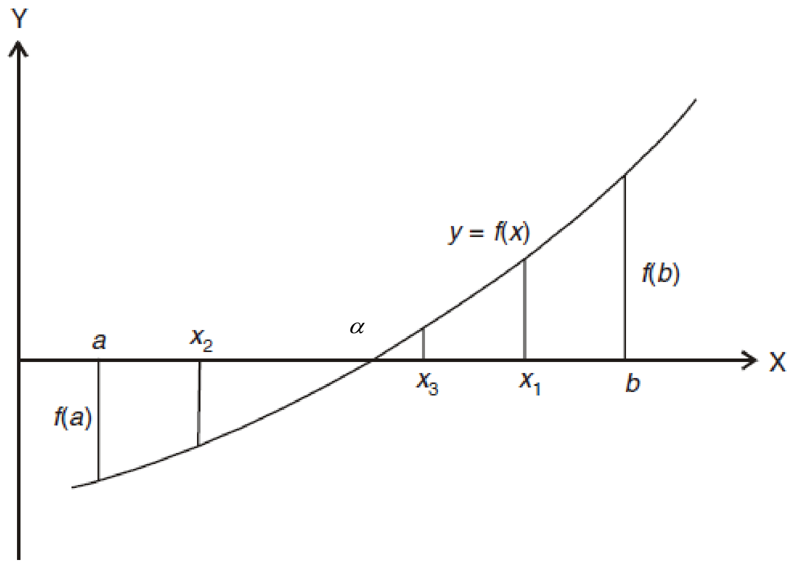

Figure 2.6 Approximation of the real root α by bisection method

Convergence of the Bisection Method

Convergence of any iteration method implies that the error in the approximation is tending to zero as the number of iteration increases.

For the bisection method, the absolute value of the error is bounded by the length of the interval in which the root lies at that particular stage.

Let α be the real root of the equation $f(x) = 0$ and the initial estimates of the root be a and b. Then

Error bound after 1st bisection $\quad |\alpha - x_1| = \varepsilon_1 \le \frac{1}{2}[b-a]$.

Error after 2nd bisection $\quad |\alpha - x_2| = \varepsilon_2 \le \frac{1}{2}\left(\frac{b-a}{2}\right) = \frac{b-a}{2^2}$

Error after 3rd bisection $\quad |\alpha - x_3| = \varepsilon_3 \le \frac{1}{2}\left(\frac{b-a}{2^2}\right) = \frac{b-a}{2^3}$

Error after n^{th} bisection $\quad |\alpha - x_n| = \varepsilon_n \le \frac{1}{2}\left(\frac{b-a}{2^{n-1}}\right) = \frac{b-a}{2^n}$

Because $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty}\left(\frac{b-a}{2^n}\right) = 0$ we conclude that the bisection method always converges to the real root α

Hence in bisection method, $e_{i+1} = 0.5 e_i$ or $\frac{e_{i+1}}{e_i} = 0.5$

Here e_i and e_{i+1} are the errors in i^{th} and $(i+1)^{th}$ iterations respectively.

Comparing the above equation with $\lim_{i\to\infty}\left|\frac{e_{i+1}}{e_i^k}\right| \le A$, we get $k = 1$ and $A = 0.5$. Thus the bisection method is ***first order convergent or linearly convergent***.

To see how many iterations will be necessary, suppose we wan

$$|\alpha - x_i| \leq \varepsilon$$

This will be satisfied if

$$\frac{b-a}{2^i} \leq \varepsilon$$

Taking logarithms both sides, we can solve this to give

$$i \geq \frac{\log\left(\frac{b-a}{\varepsilon}\right)}{\log 2}$$

Example 2.1:

Find the root of the equation $x^3 - x - 1 = 0$ b/n 1 and 2 by bisection method with two decimal places accuracy.

Solution:

The termination criteria of the iteration will be two decimal places accuracy given as tolerance (TOL).

Note: If a number is correct to *n* decimal places, then the maximum error in that approximation is given by $\frac{1}{2} x 10^{-n}$ So here $TOL = \frac{1}{2} x 10^{-2} = 0.005$. Hence we continue the iterations till $|\varepsilon_a| = |x_m^{new} - x_m^{old}| \leq$ TOL

Let $f(x) = x^3 - x - 1 = 0$.

First Approximation:

Since $f(1) = 1^3 - 1 - 1 = -1$, which is negative and $f(2) = 2^3 - 2 - 1 = 5$, which is positive, at least one real root will lie between 1 and 2. So taking a = 1 and b = 2, we have

$$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$$

Error of the first approximation is: $\varepsilon_a = |x_1 - a| = |1.5 - 1| = 0.5$ which is not less than TOL.

So we continue the iteration.

Second Approximation:

Since $f(x_1) = (1.5)^3 - 1.5 - 1 = 0.875 > 0$, that is positive, the root lies between a=1 and $x_1 = 1.5$. So,

$$x_2 = \frac{a+x_1}{2} = \frac{1+1.5}{2} = 1.25$$

Error of the second approximation is: $\varepsilon_a = |x_2 - x_1| = |1.25 - 1.5| = 0.25$ which is not less than 0.005.

Third Approximation:

Since $f(x_2) = (1.25)^3 - 1.25 - 1 = -0.297 < 0$, that is negative, the root lies between $x_1 = 1.5$ and $x_2 = 1.25$. So,

$$x_3 = \frac{x_1+x_2}{2} = \frac{1.25+1.5}{2} = 1.375$$

Error of the second approximation is: $\varepsilon_a = |x_3 - x_2| = |1.375 - 1.25| = 0.125$ which is not less than 0.005.

Proceeding in this manner we have the following tables of approximations with the corresponding error.

iterations	a	b	x_m	$f(a)$	$f(b)$	$f(x_m)$	**Error (ε_a)**
1	1	2	1.5	-1	5	0.875	0.5
2	1	1.5	1.25	-1	0.875	-0.297	0.25
3	1.5	1.25	1.375	0.875	-0.297	0.2246	0.125
4	1.25	1.375	1.313	-0.297	0.2246	– 0.0494	0.0625
5	1.375	1.313	1.344	0.2246	– 0.0494	0.0837	0.03125
6	1.313	1.344	1.329	– 0.0494	0.0837	0.0183	0.015625
7	1.313	1.329	1.321	– 0.0494	0.0183	– 0.0158	0.0078125
8	1.329	1.321	1.325	0.0183	– 0.0158	0.0012	0.00390625

At the 8th iteration, the error $\varepsilon_a = |x_8 - x_7| = 0.0039 < TOL = 0.005$

Hence we can stop the iteration and the root of $f(x) = x^3 - x - 1 = 0$ up to two decimal places is 1.325, which is of the desired accuracy.

Example 2.2:

Use bisection method to find the real root of equation $x \log_{10}^{x} = 1.2$ up to two decimal places accuracy.

Solution:

Let $f(x) = x \log_{10}^{x} - 1.2$ *and TOL = two decimal places accuracy = 0.5*10^{-2}= 0.005.*

To find initial guesses of the root: $f(1) = (1)\log_{10}^{1} - 1.2 = -1.2 < 0$, $f(2) = (2)\log_{10}^{2} - 1.2 = 0.602 - 1.2 = -0.598 < 0$

, and $f(3) = (3)\log_{10}^{3} - 1.2 = 0.2313 > 0$.

Thus f (2) is negative and f (3) is positive, therefore, by the intermediate value theorem, the root lies between a = 2 and b = 3.

First Approximation:

The first approximation to the root is: $x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$.

Absolute error of the first approximation $\varepsilon_a = |2.5 - 2| = 0.5$

Second Approximation:

Since $f(2.5) = (2.5)\log_{10}^{2.5} - 1.2 = 0.9948 - 1.2 = -0.2052 < 0$

Thus, the root lies between x_1= 2.5 and b = 3.

Therefore the second approximation to the root is: $x_2 = \frac{x_1+b}{2} = \frac{2.5+3}{2} = 2.75$

Absolute error of the second approximation $\varepsilon_a = |2.75 - 2.5| = 0.25$

Third Approximation:

Since $f(2.75) = (2.75)\log_{10}^{2.75} - 1.2 = 1.2081 - 1.2 = 0.0081 > 0$

Thus, the root lies between $x_1 = 2.5$ and $x_2 = 2.75$.

Therefore the second approximation to the root is: $x_3 = \dfrac{x_1 + x_2}{2} = \dfrac{2.5 + 2.75}{2} = 2.625$

Absolute error of the second approximation $\varepsilon_a = |2.625 - 2.75| = 0.125$

Proceeding in this manner we have the following tables of approximations with the corresponding error.

iterations	a	b	x_m	$f(a)$	$f(b)$	$f(x_m)$	**Error (ε_a)**
1	2	3	2.5	-0.598	0.2313	-0.2052	0.5000
2	2.5	3	2.75	-0.2052	0.2313	0.0081	0.2500
3	2.5	2.75	2.625	-0.2052	0.0081	− 0.0999	0.1250
4	2.75	2.625	2.6875	0.0081	− 0.0999	– 0.0463	0.0625
5	2.75	2.6875	2.7188	0.0081	– 0.0463	– 0.019	0.0312
6	2.75	2.7188	2.7344	0.0081	– 0.019	– 0.0058	0.0156
7	2.75	2.7344	2.742	0.0081	– 0.0058	0.0012	0.0078
8	2.7344	2.742	2.738	– 0.0058	0.0012	– 0.0023	0.0039

At the 8th iteration, the absolute error $\varepsilon_a = |x_8 - x_7| = 0.004 < TOL = 0.005$

Hence we can stop the iteration and the root of $f(x) = x^3 - x - 1 = 0$ up to two decimal places is x = 2.74, which is of the desired accuracy.

Advantages of Bisection Method

a) The bisection method is simple to use.

b) The bisection method is always convergent. Since the method brackets the root, the method is guaranteed to converge.

c) As iterations are conducted, the interval gets halved. So one can guarantee the error in the solution of the equation. That is good estimate of maximum error $|E_{\max}| \le \left|\dfrac{x_\ell - x_u}{2}\right|$

Drawbacks of bisection method

a) The convergence of the bisection method is slow as it is simply based on halving the interval.

b) If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.

c) If a function $f(x)$ is such that it just touches the x-axis (Figure 2.7) such as $f(x) = x^2 = 0$ it will be unable to find the lower guess, x_ℓ, and upper guess, x_u, such that $f(x_\ell)f(x_u) < 0$

d) For functions $f(x)$ where there is a singularity and it reverses sign at the singularity, the bisection method may converge on the singularity (Figure 2.8).An example includes

$f(x) = \frac{1}{x}$ where $x_\ell = -2$, $x_u = 3$ are valid initial guesses which satisfy $f(x_\ell)\, f(x_u) < 0$

However, the function is not continuous and the theorem that a root exists is also not applicable.

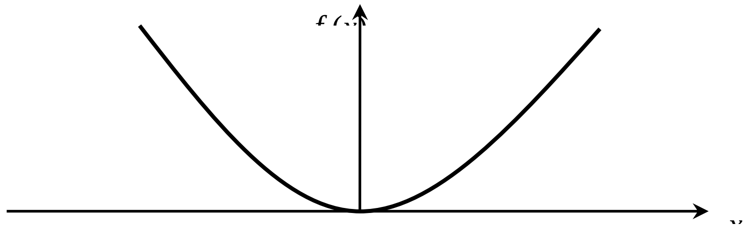

Figure 2.7: The equation $f(x) = x^2 = 0$ has a single root at $x = 0$ that cannot be bracketed.

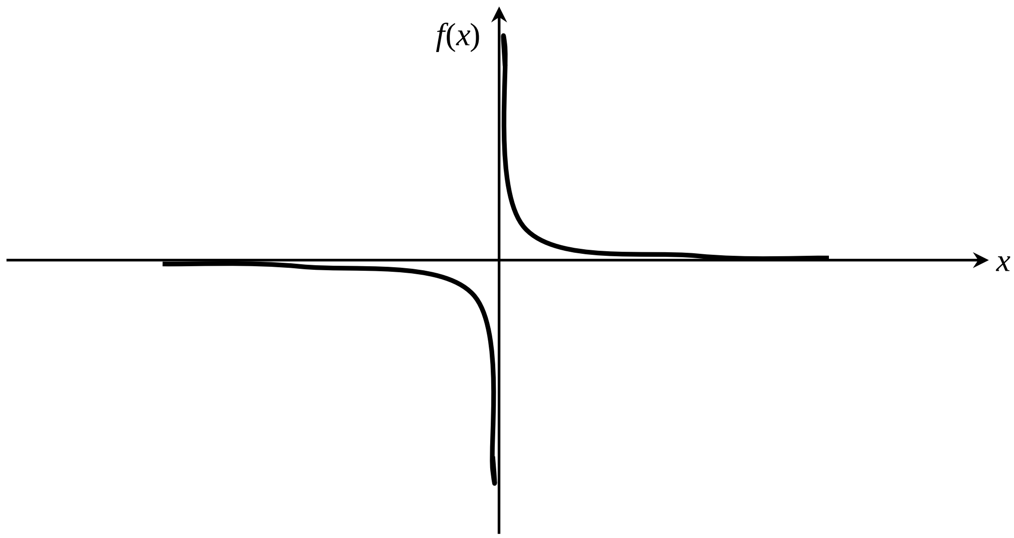

Figure 2.8: The equation $f(x) = \frac{1}{x} = 0$ has no root but changes sign.

e) The other disadvantage of the bisection method is it requires initial interval around the root.

Note: we use the following techniques to choose the initial guesses of the root.

- Use graph of function,
- incremental search, or
- trial& error.

2.3. Fixed-Point Iteration

-A number α is called a fixed point of a function $g(x)$ if $g(\alpha) = \alpha$. The mathematical problem of finding values of x which satisfy the equation $x = g(x)$ is called the **fixed point problem**. We introduced the concept of a fixed point for the purpose of using it in solving root-finding problems.

-To find the root of the equation $f(x) = 0$ by fixed point successive approximations, we rewrite the given equation in the form $x = g(x)$. Now, first we assume the initial estimate of the root be x_0. Then to find the first approximate of the root x_1, we substitute the initial estimate, x_0 in $g(x)$. Hence it follows that: $x_1 = g(x_0)$.

Similarly, the second approximation x_2 is given by: $x_2 = g(x_1)$

In general, the n[th] approximation of the root is given by: $\boldsymbol{x_{n+1}} = g(x_n)$ which is called the **fixed point iteration formula.**

Procedure for Iteration Method to Find the Root of the Equation $f(x) = 0$

Step-1: Take an initial approximation as x_0.

Step-2: Find the next (first) approximation x_1 by using $x_1 = g(x_0)$.

Step-3: Follow the above procedure to find the successive approximations x_{n+1} by using

$x_{n+1} = g(x_n)$, $n = 1, 2, 3...$.

Step-4: Stop the evaluation where absolute error ≤ TOL, where TOL is the prescribed accuracy or tolerance.

Example 2.3:

Solve the equation $x^3 - x - 1 = 0$ by using fixed point iteration up to four decimal places accuracy.

Solution:

(i) Re-arranging $x^3 - x - 1 = 0$ as $x = x^3 - 1$ gives the iterative formula $x_{n+1} = x_n^3 - 1$

Using the initial value $x_0 = 1$ gives the sequence of approximations:

$x_1 = 0$, $x_2 = -1$, $x_3 = -2$, $x_4 = -9$, $x_5 = -730$, $x_6 = -389{,}017{,}007$. This sequence is clearly diverging towards negative infinity, so the re-arrangement $x = x^3 - 1$ with $x_0 = 1$ clearly fails to solve the original equation $x^3 - x - 1 = 0$.

(ii) Re-arranging $x^3 - x - 1 = 0$ as $x^3 = x + 1$ so that $x = \sqrt[3]{x+1}$ gives the iterative formula

$x_{r+1} = \sqrt[3]{x_r + 1}$.

Using the initial value $x_0 = 1$ gives the sequence of approximations:

$x_1 = 1.25992105$, $x_4 = 1.324268745$, $x_7 = 1.324714878$,

$x_2 = 1.312293837$, $x_5 = 1.324632625$, $x_8 = 1.324717372$,

$x_3 = 1.322353819$, $x_6 = 1.324701749$, $x_9 = 1.324717846$.

This sequence is clearly converging so we can say that, after 9 iterations, the sequence has converged to $x = 1.32472$ to 5 decimal places.

Theorem:

Let $g(x)$ is continuous over $[a,b]$. Suppose, in addition, that $g'(x)$ exists on (a,b) and that $|g'(x)| < 1$, for all $x \in (a,b)$. Then, for any number x_0 in $[a,b]$, the sequence defined by $x_{n+1} = g(x_n)$, $n \geq 0$, converges to the unique fixed point p in $[a,b]$.

That is simply to say the fixed point iteration method $x = g(x)$ is convergent if $|g'(x)| < 1$ in an interval containing the initial guess x_0 of the root.

Example 2.4:

Find the real root of the equation cos x = 3x – 1 correct to three decimal places, using fixed point iteration method.

Solution: Here, we have $f(x) = \cos x - 3x + 1$. And $f(0) = 2 > 0$ and $f(\frac{\pi}{2}) = -3(\frac{\pi}{2}) + 1 < 0$

Therefore the root lies between 0 and $\frac{\pi}{2}$.

Now, the given equation can be re-written as: $x = \frac{1}{3}(\cos x + 1) = g(x)$

Then, we have $g'(x) = -\frac{\sin x}{3}$ and so $|g'(x)| < 1$ in $(0, \frac{\pi}{2})$. Hence if we take the above function g(x), the fixed point iteration method will converge to the solution.

Taking the first approximation $x_0 = 0$, we can find the successive approximations as:

$$x_1 = g(x_0) = \frac{1}{3}[\cos 0 + 1] = 0.667$$

$$x_2 = g(x_1) = \frac{1}{3}[\cos(0.667) + 1] = 0.5953$$

$$x_3 = g(x_2) = \frac{1}{3}[\cos(0.5953) + 1] = 0.6093$$

$$x_4 = g(x_3) = \frac{1}{3}[\cos(0.6093) + 1] = 0.6067$$

$$x_5 = g(x_4) = \frac{1}{3}[\cos(0.6067) + 1] = 0.6072$$

$$x_6 = g(x_5) = \frac{1}{3}[\cos(0.6072) + 1] = 0.6071$$

$$x_7 = g(x_6) = \frac{1}{3}[\cos(0.6071) + 1] = 0.6071$$

Now, x_6 and x_7 being the same, the required root is given by x = 0.6071.

2.4. Newton-Raphson Method

The Newton Raphson method is by far the most popular numerical method for approximating roots of functions. The method assumes that the function $f(x)$ is differentiable in the neighborhood of the root and that the derivative is not zero in anywhere in that neighborhood.

Derivation of the formula

The Newton-Raphson method is based on the principle that if the initial guess of the root of $f(x)=0$ is at x_0, then if one draws the tangent to the curve at $(x_0, f(x_0))$, the point x_1 where the tangent crosses the x-axis is an improved estimate of the root .

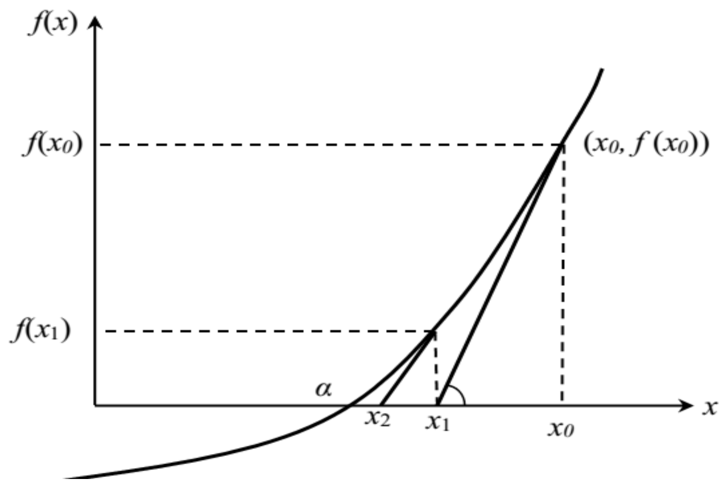

Figure 2.8 Geometrical illustration of the Newton-Raphson method.

Using the definition of the slope of the tangent line through a point $(x_0, f(x_0))$ on the curve y = *f*(x), we have:

$$slope = f'(x_0) = \frac{f(x_0)-0}{x_0 - x_1} \text{ which gives } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The value x_1 is then accepted as a new approximation to the root. The point $(x_1, f(x_1))$ can be taken as a new point to draw a tangent through. Its intersection with the x - axis, given by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ is also accepted as a new approximation to the root. This process can be repeated over and over, leading to the iteration method given by the Newton-Raphson formula:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots}$$

which is called the Newton-Raphson formula for solving nonlinear equations of the form $f(x)=0$.

Each iteration using the Newton Raphson method requires one function evaluation and one first derivative evaluation. Compared to the other numerical methods, the Newton Raphson method converges very rapidly to the root. One can repeat this process until one finds the root within a desirable tolerance.

The procedure of the Newton-Raphson method

The steps of the Newton-Raphson method to find the root of an equation $f(x) = 0$ are:

1. Evaluate $f'(x)$ symbolically

2. Use an initial guess of the root, x_i, to estimate the new value of the root, x_{i+1}, as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3. Find the absolute error $|\in_a|$ as: $|\in_a| = |x_{i+1} - x_i|$

4. Compare the absolute error with the pre-specified error tolerance, TOL. If $|\in_a|$>TOL, then go to Step 2, else stop the algorithm. Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the iteration and notify the user.

■ **Example 1.8** Find the root of the equation $x^3 - x - 4 = 0$ correct to three places of decimal using NR-method.

Solution: Let $x_0 = 2$. Then

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 1.81818, \text{ and } f(x_1) = 0.192336 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.796613, \text{ and } f(x_2) = 0.002527 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.796322, \text{ and } f(x_3) = 0.000000457
\end{aligned}$$

We have

$$|x_3 - x_2| = |1.796322 - 1.796613| = 0.000291 < 0.5 \times 10^{-3}$$

Thus, the required root is $x = 1.796322$ correct to three decimal place.

■ **Example 1.9** compute $17^{1/3}$ correct to four decimal places using NR-method, assuming t initial approximation as $x_0 = 2$.

Solution: Let $x = 17^{1/3} \Rightarrow x^3 - 17 = 0$. $f(x) = x^3 - 17 \Rightarrow f'(x) = 3x^2$. Then

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 2.75, \text{ and } f(x_1) = 3.7969 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2.5826, \text{ and } f(x_2) = 0.2264 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 2.5713, \text{ and } f(x_3) = 0.00099 \\
x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 2.57128, \text{ and } f(x_4) = 0.000000019
\end{aligned}$$

We have

$$|x_3 - x_2| = |2.57128 - 2.5713| = 0.00002 < 0.5 \times 10^{-4}$$

Thus, the required root is $x = 2.571281592$ correct to four decimal place.

Example 2.5:

Use the Newton-Raphson iteration method to estimate the root of the equation cos(x) = 2x up to 5 decimal places accuracy.

Solution:

This is equivalent to solving f(x) = 0 where f(x) = cos(x) − 2x. [Note: make sure your calculator is in radian mode].

The iteration formula is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{/}(x_n)}$$, where *f(x)* = cos(x) −2x and $f'(x) = -\sin(x) - 2$.

With an initial guess of $x_0 = 0.5$, we obtain the following successive approximations.

$x_0 = 0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f^{/}(x_0)} = 0.5 - \frac{f(0.5)}{f^{/}(0.5)} = 0.45063$$

$$x_2 = x_1 - \frac{f(x_1)}{f^{/}(x_1)} = 0.45063 - \frac{f(0.45063)}{f^{/}(0.45063)} = 0.45018$$

$$x_3 = x_2 - \frac{f(x_2)}{f^{/}(x_2)} = 0.45018 - \frac{f(0.45018)}{f^{/}(0.45018)} = 0.45018$$

Clearly the second and third approximations are identical to five decimal places. Therefore, to this degree of accuracy, the root is x = 0.45018.

Example 2.6

Use the Newton-Raphson iteration method to estimate the root of the equation $e^{-x} - x = 0$ employing an initial guess of $x_0 = 0$.

Solution:

Let $f(x) = e^{-x} - x$ and $f'(x) = -e^{-x} - 1$. The initial guess is $x_0 = 0$.

The Newton-Raphson iteration formula is given by: $x_{n+1} = x_n - \dfrac{f(x_n)}{f^{/}(x_n)}$.

Therefore successive approximations of the root are calculated as follows.

$x_0 = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f^{/}(x_0)} = 0 - \frac{e^{-(0)} - 0}{-e^{-(0)} - 1} = 0 - \frac{1}{-2} = 0.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f^{/}(x_1)} = 0.5 - \frac{e^{-(0.5)} - 0.5}{-e^{-(0.5)} - 1} = 0.5 - \frac{0.1065}{-1.6065} = 0.5663$$

$$x_3 = x_2 - \frac{f(x_2)}{f^{/}(x_2)} = 0.5663 - \frac{e^{-(0.5663)} - 0.5663}{-e^{-(0.5663)} - 1} = 0.5663 - \frac{0.001322}{-1.567622} = 0.5671$$

$$x_4 = x_3 - \frac{f(x_3)}{f^{/}(x_3)} = 0.5671 - \frac{e^{-(0.5671)} - 0.5671}{-e^{-(0.5671)} - 1} = 0.5671 - \frac{0.00006784}{-1.56716784} = 0.5671$$

Thus, the approach rapidly converges on the true root of x = 0.5671 to four decimal places.

Hence the root is x = 0.5671.

Drawbacks of the Newton-Raphson Method

1. Difficulty in evaluation of the derivative

A potential problem in utilizing Newton-Raphson method is the evaluation of the derivative.

Although this is not true for polynomials and many other functions, there are certain functions whose derivatives may be extremely difficult or inconvenient to evaluate.

2. Divergence at inflection points

If the selection of the initial guess or an iterated value of the root turns out to be close to the inflection point of the function $f(x)$ in the equation $f(x) = 0$, Newton-Raphson method may start diverging away from the root. It may then start converging back to the root.

3. Division by zero

For the equation $f(x) = x^3 - 0.03x^2 + 2.4 \times 10^{-6} = 0$ the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4 \times 10^{-6}}{3x_i^2 - 0.06x_i}$$

For $x_0 = 0$ or $x_0 = 0.02$, division by zero occurs. For an initial guess close to 0.02 such as $x_0 = 0.01999$, one may avoid division by zero, but then the denominator in the formula is a small number. For this case, even after 9 iterations, the Newton-Raphson method does not converge.

4. Root jumping

In some case where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root. For example for solving the equation $\sin x = 0$ if you choose $x_0 = 2.4\pi = (7.539822)$ as an initial guess, it converges to the root of $x = 0$. However, one may have chosen this as an initial guess to converge to $x = 2\pi = 6.2831853$.

2.5. False Position method and Secant Method

The method is also called linear interpolation method or chord method or regula-falsi method. At the start of all iterations of the method, we require the interval in which the root lies. Let the root of the equation $f(x) = 0$, lie in the interval (x_l, x_u), that is, $f(x_l)\, f(x_u) < 0$. Rather than bisecting the interval (a, b), it locates the root by joining $f(x_l,)$ and $f(x_u)$ with a straight line. The intersection of this line with the x-axis represents an improved estimate of the root.

Using similar triangles the intersection of the straight line with the x axis can be estimated as

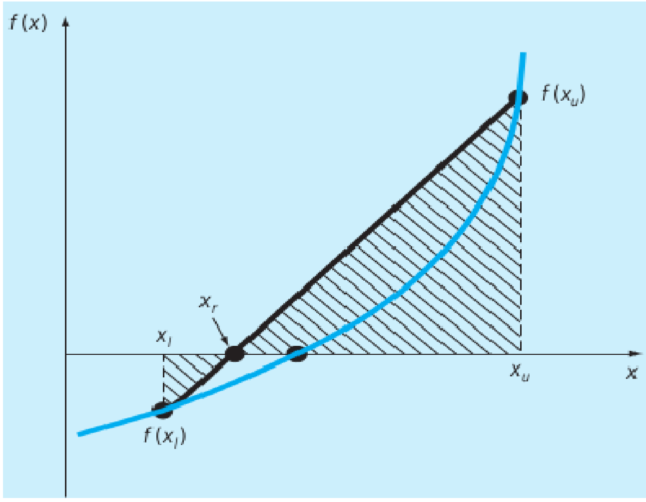

Figure 1.4: False-Position Method

$$\frac{f(x_u)-0}{x_u-x_r}=\frac{0-f(x_l)}{x_r-x_l}$$

which can be solved for

$$x_r=\frac{x_l f(x_u)-x_u f(x_l)}{f(x_u)-f(x_l)}$$

Procedure for the False Position Method to Find the Root of the Equation $f(x)=0$

Step 1: Choose two initial guess values (approximations) x_0 and x_1 (where $x_1 > x_0$) such that $f(x_0).f(x_1)<0$.

Step 2: Find the next approximation x_2 using the formula

$$x_2=\frac{x_0 f(x_1)-x_u f(x_0)}{f(x_1)-f(x_0)}$$

Step 3: If $f(x_2)f(x_1)<0$, then go to the next step. If not, rename x_0 as x_1 and then go to the next step.

Step 4: Evaluate successive approximations using the formula

$$x_{n+1}=\frac{x_{n-1}f(x_n)-x_n f(x_{n-1})}{f(x_n)-f(x_{n-1})}, \text{ where n= 2,3,...}$$

But before applying the formula for x_{n+1}, ensure whether $f(x_{n-1}).f(x_n)<0$; if not, rename x_{n-2} as x_{n-1} and proceed.

Step 5: Stop the evaluation when $|x_n-x_{n-1}|<\varepsilon$, where ε is the prescribed accuracy (tolerance error).

■ **Example 1.6** Using the False Position method, find a root of the function $f(x)=e^x-3x^2$ correct to three decimal place. The root is known to lie between 0.5 and 1.0. ■

Solution: Let $x_0=0.5, x_1=1$. then we have, $f(0.5)=0.8987, f(1)=-0.2817$. Thus,

$$x_2=\frac{x_0 f(x_1)-x_u f(x_0)}{f(x_1)-f(x_0)}=0.88067$$

$f(x_2)=0.0858$ Since $f(x_1)f(x_2)<0$, the root lies in the interval (0.88067, 1), then next approximation is,

$$x_3 = \frac{x_1 f(x_2) - x_u f(x_1)}{f(x_2) - f(x_1)} = 0.90852$$

$f(x_3) = 0.00441$. Since $f(1)f(0.90852) < 0$, the root lies in the interval (0.90852, 1), then next approximation is,

$$x_4 = \frac{x_2 f(x_3) - x_u f(x_2)}{f(x_3) - f(x_2)} = 0.90993$$

$f(x_4) = 0.00022$. Since $f(1)f(0.90993) < 0$, the root lies in the interval (0.90993, 1), then next approximation is,

$$x_5 = \frac{x_3 f(x_4) - x_u f(x_3)}{f(x_4) - f(x_3)} = 0.91000$$

$f(x_5) = 0.00001$. Since $f(1)f(0.91000) < 0$, the root lies in the interval (0.91000, 1), then next approximation is,

$$x_6 = \frac{x_4 f(x_5) - x_u f(x_4)}{f(x_5) - f(x_4)} = 0.91001$$

$f(x_6) = 0.00000$

Therefore, the root is 0.91 correct to four decimal places.

The Seant Method

The Newton-Raphson method of solving a nonlinear equation $f(x) = 0$ is given by the iterative formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

One of the drawbacks of the Newton-Raphson method is that you have to evaluate the derivative of the function. To overcome these drawbacks, the derivative of the function, $f'(x)$ is approximated as:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Substituting Equation (2) in Equation (1) gives

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

The secant method can also be derived from geometry, as shown in Figure 2.10. Taking two initial guesses, x_{i-1} and x_i, one draws a straight line between $f(x_i)$ and $f(x_{i-1})$ passing through the x -axis at x_{i+1}. Δ *ABE* and Δ *DCE* are similar triangles.

Hence $\frac{AB}{AE} = \frac{DC}{DE}$ That is $\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$

On rearranging, the secant method is given as:

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

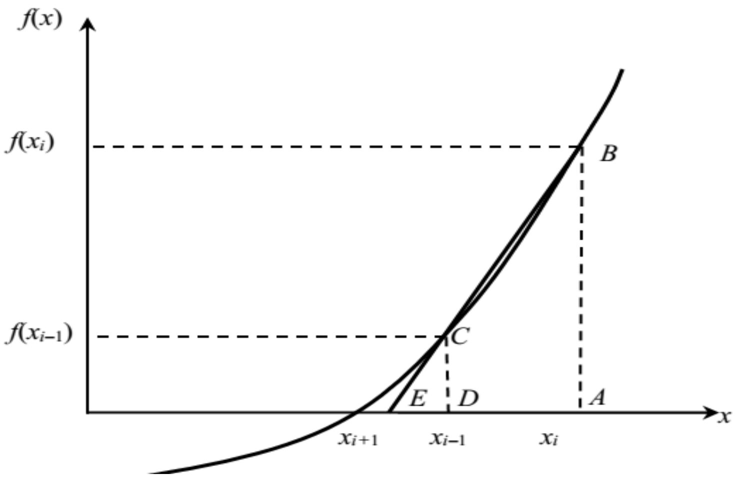

Figure 2.10: Geometrical representation of the secant method.

Note: The secant method always uses the latest two approximations without the requirement that they bracket the root. As a consequence, the secant method can sometime diverge.

Procedure for Secant Method to find the Root of f(x) = 0

Step-1: Choose the interval $[x_0, x_1]$ in which $f(x) = 0$ has a root, where $x_1 > x_0$.

Step-2: Find the next approximation x_2 of the required root using the formula:

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1) \qquad x$$

Step-3: Find the successive approximations of the required root using the formula:

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] f(x_n), \text{ n } = 1,\ 2,\ 3,\ \ldots$$

Step-4: Stop the process when the prescribed accuracy is obtained.

Example 2.7:

Apply Secant method to find the root of the equation $x^3 - 5x^2 - 17x + 20 = 0$ up to five decimal places.

Solution:

Taking initial approximations as, $x_0 = 0$, $x_1 = 1$ and $f(x_0) = 20$, $f(x_1) = -1$, then by secant method the next approximation is given by:

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1) = 1 - \left[\frac{1-0}{-1-20}\right](-1) = 0.95238$$

Now using $x_1 = 1$, $x_2 = 0.95238$, $f(x_1) = -1$ and $f(x_2) = 0.13824$, the next approximation can be obtained as:

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] f(x_2) = 0.95238 \ - \left[\frac{0.95238 - 1}{0.13824 - (-1)}\right](0.13824) = 0.95816$$

Similarly, other approximations can be obtained by using two recent approximations in secant method.

$x_4 = 0.95818$,

$x_5 = 0.95818$.

Thus the approximate root can be taken as x = 0.95818, which is correct up to five decimal places.

Example 2.8:

Find the root of the equation f(x) = 4 sin x + x^2 = 0 by Secant method up to five decimal places.

Solution:

In secant method we neglect to check the condition $f(x_0)\, f(x_1) < 0$.

So initially, we take $x_0 = -1$, $x_1 = -2$ and $f(x_0) = -2.36588$, $f(x_1) = 0.36281$.

Therefore the first approximation to the root by secant method is given by:

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1) = (-2) - \left[\frac{-2-(-1)}{0.36281 - (-2.36588)}\right](0.36281) = -1.86704$$

Now using $x_1 = -2$, $x_2 = -1.86704$, $f(x_1) = 0.36281$ and $f(x_2) = -0.33992$, the next approximation can be obtained as:

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] f(x_2) = (-1.86704) - \left[\frac{-1.86704 - (-2)}{-0.33992 - 0.36281}\right](-0.33992) = -1.93135$$

Continuing this process and using two recent approximations, to get next approximation, in secant method, we get:

$x_4 = -1.93384$,

$x_5 = -1.93375$,

$x_6 = -1.93375$.

Thus, the approximation value to the root is x = –1.93375, correct up to five decimal places.