Chapter Four: Interpolation

4.1 Introduction

What is Interpolation?

Often, data is known only at certain **discrete points** (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n).

If we want the value of y at some value of x **between** the given data points, we must estimate it. To do this, we construct a **continuous function** $f(x)$ that passes through all the given data points. Then any value $f(x)$ for intermediate x-values can be computed**.**

This process is called **interpolation**.

- If the desired x-value lies **within** the range of given data → **Interpolation**
- If it lies **outside** the data range → **Extrapolation**

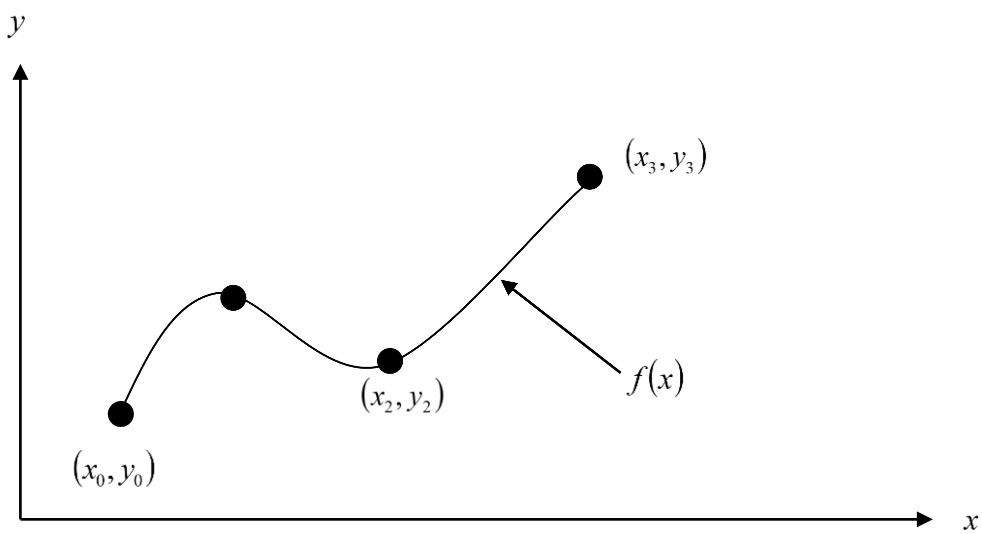


Figure 1: Interpolation of discrete data.

Why use polynomials?

Polynomials are commonly used as interpolating functions because they are:

- Simple to compute
- Easy to differentiate and integrate
- Flexible enough to fit many types of data

Polynomial Interpolation

Polynomial interpolation finds a polynomial of degree n that passes **exactly** through the $n+1$ given data points.

The most widely used methods include:

1. **Newton's Forward Interpolation Formula** – used when x-values are equally spaced and interpolation is near the beginning of the table

2. **Newton's Backward Interpolation Formula** – used when x-values are equally spaced and interpolation is near the end of the table
3. **Lagrange Interpolation Formula** – works for any distinct data points and does not require equal spacing
4. **Newton's Divided Difference Method** – suitable for unequally spaced data and allows easy updating when new points are added

4.2 Finite Differences

Assume we are given a table of values $(\boldsymbol{x_i}, \boldsymbol{y_i}), \boldsymbol{i = 0, 1, 2, \ldots, n}$ for some function $y = f(x)$. The x-values are **equally spaced**, so each point can be written as

$$\boldsymbol{x_i = x_0 + ih, i = 0, 1, 2, \ldots, n}, \quad \text{where } h \text{ is the constant spacing.}$$

Often, we need to determine the value of $f(x)$ at some point **between** the tabulated values, or we may need to approximate the derivative of $f(x)$ at a point within the interval $\boldsymbol{x_0 \le x \le x_n}$.

To do this, we use methods based on **finite differences**, which provide a systematic way to construct interpolation formulas for equally spaced data.

Before developing these interpolation methods, we first define the different types of **differences** of a function.

4.1.1. Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denotes a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the differences of y. Denoting these differences by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0,$$

$$\Delta y_1 = y_2 - y_1,$$

$$\ldots$$

$$\Delta y_{n-1} = y_n - y_{n-1}.$$

Where Δ is called the forward difference operator and $\Delta y_0, \Delta y_1, \Delta y_2, \ldots,$ are called first forward differences. The differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0$, $\Delta^2 y_1, \ldots$. Similarly, one can define third forward differences, fourth forward differences, etc. Thus,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

It is therefore clear that any higher-order differences can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients. Table 4.1 shows how the forward differences of all orders can be formed:

Table 4.1 Forward Difference Table

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Where $x_0 + h = x_1$, $x_0 + 2h = x_2$, ..., $x_0 + nh = x_n$.

Example 4.1:

Given $f(0) = 3$, $f(1) = 12$, $f(2) = 81$, $f(3) = 200$, $f(4) = 100$ and $f(5) = 8$. Construct the forward difference table and find $\Delta^5 f(0)$.

Solution:

The difference table for the given data is as follows:

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	3					
		9				
1	12		60			
		69		-10		
2	81		50		-259	
		119		-269		755
3	200		-219		496	
		-100		227		
4	100		8			
		-92				
5	8					

Hence, $\Delta^5 f(0) = 755$.

4.2.2. Backward Differences

The differences $y_1 - y_0$, $y_2 - y_1$, … ,$y_n - y_{n-1}$ are called first backward differences if they are denoted by ∇y_1, ∇y_2, …, ∇y_n respectively, so that

$$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_2, \ \ldots, \nabla y_n = y_n - y_{n-1},$$

where ∇ is called the backward difference operator. In a similar way, one can define backward differences of higher orders. Thus we obtain:

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0 \text{ and so on.}$$

Similarly the n[th] order backward differences can be defined as: $\nabla^n y_n = \nabla^{n-1} y_n - \nabla^{n-1} y_{n-1}$

Table 4.2. Backward Difference Table

x	$y = f(x)$	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_0				
x_1	y_1		$\nabla^2 y_0$			
		∇y_1		$\nabla^3 y_0$		
x_2	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$	
		∇y_2		$\nabla^3 y_1$		$\nabla^5 y_0$
x_3	y_3		$\nabla^2 y_2$		$\nabla^4 y_1$	
		∇y_3		$\nabla^3 y_2$		
x_4	y_4		$\nabla^2 y_3$			
		∇y_4				
x_5	y_5					

Example 4.2:

Construct the backward difference table for y = log x given below:

X	10	20	30	40	50
Y	1	1.3010	1.4771	1.6021	1.6990

Find the values of $\nabla^3 \log 40$ and $\nabla^4 \log 50$.

Solution: The backward difference table is constructed as under:

x	$y = f(x)$	∇	∇^2	∇^3	∇^4
10	1				
		0.3010			
20	1.3010		-0.1249		
		0.1761		0.0738	
30	1.4771		-0.0511		-0.0508
		0.1250		0.0230	
40	1.6021		-0.0281		
		0.0969			
50	1.6990				

Hence, $\nabla^3 \log 40 = 0.0738$ and $\nabla^4 \log 50 = -0.0508$

4.2.3. Central Differences

The central difference operator δ is defined by the relations:

$$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2}) \text{ which is equivalent to } \delta y_x = y_{x+\frac{1}{2}} - y_{x-\frac{1}{2}}.$$

The first order central differences are given by:

$$\delta y_{\frac{1}{2}} = y_1 - y_0,\ \delta y_{\frac{3}{2}} = y_2 - y_1,\ \delta y_{\frac{5}{2}} = y_3 - y_2,\ \ldots,\ \delta y_{n-\frac{1}{2}} = y_n - y_{n-1}.$$

Similarly, higher-order central differences can be defined as:

$$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}},\ \delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1, \delta^4 y_2 = \delta^3 y_{\frac{5}{2}} - \delta^3 y_{\frac{3}{2}}, \text{ and so on.}$$

With the values of x and y as in the preceding two tables, a central difference table can be formed:

Table 4.3: Central Difference Table

x	$y = f(x)$	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
		$\delta y_{\frac{1}{2}}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$		$\delta^5 y_{\frac{5}{2}}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{\frac{7}{2}}$		$\delta^3 y_{\frac{7}{2}}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{\frac{9}{2}}$				
x_5	y_5					

It is clear from the three tables that in a definite numerical case, the same numbers occur in the same positions whether we use forward, backward or central differences. Thus we obtain $\Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$, $\Delta^3 y_2 = \nabla^3 y_3 = \delta^3 y_{\frac{7}{2}}$, and so on.

4.2.4. The Shift Operator E

The operator E is called shift operator or displacement or translation operator. It shows the operation of increasing the argument value *x* by its interval of differencing *h* so that:

$Ef(x) = f(x+h)$ in the case of a continuous variable x, and $Ey_x = y_{x+1}$ in the case of a discrete variable. Similarly, $Ef(x + h) = f(x + 2h)$

Powers of the operator (positive or negative) are defined in a similar manner: $E^n f(x) = f(x+nh)$; $\quad E^n y_x = y_{x+nh}$

In the same manner, $E^{-1}f(x) = f(x - h)$. Also, $E^{-2}f(x) = f(x - 2h)$, $\quad E^{-n} f(x) = f(x - nh)$

4.3 Interpolation with Equally Spaced Points

4.3.1 Newton's Forward Interpolation Formula

Given the set of (n+1) values, viz., (x_0, y_0), (x_1, y_1), (x_2, y_2), …, (x_n, y_n), of x and y, it is required to find $P_n(x)$, a polynomial of n^{th} degree such that y and $P_n(x)$ agree at the tabulated points. Let the values of x be equidistant, i.e. $x_i = x_0 + ih$, i =0, 1, 2, …, n.

Since $P_n(x)$ is a polynomial of the n^{th} degree, it may be written as:

$$\begin{aligned} P_n(x) = a_0 &+ a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots \\ &+ a_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \end{aligned} \tag{1}$$

Imposing now the conditions that y and $P_n(x)$ should agree at the set of tabulated points, that is putting $x = x_0$, x_1, x_2, …,x_n successively in (4.1), we obtain:

$$a_0 = y_0; \quad a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}; \quad a_2 = \frac{\Delta^2 y_0}{h^2 2!}; \quad a_3 = \frac{\Delta^3 y_0}{h^3 3!}; \ \cdots; \quad a_n = \frac{\Delta^n y_0}{h^n n!}.$$

Setting $x = x_0 + ph$ and substituting for a_0, a_1, a_2, …, a_n, equation (3.9) gives:

$$\begin{aligned} P_n(x) = y_0 &+ p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \\ &+ \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\Delta^n y_0 \end{aligned} \tag{2}$$

where $p = \frac{(x-x_0)}{h}$. This is Newton's forward difference interpolation formula and is useful for interpolation near the beginning of a set of tabular values.

4.3.2 Newton's Backward Interpolation Formula

Instead of assuming $P_n(x)$ as in (3.9) if we choose it in the form

$$P_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \cdots$$
$$+ a_n(x-x_n)(x-x_{n-1})(x-x_{n-2})\cdots(x-x_1) \tag{3}$$

and then impose the condition that y and $P_n(x)$ should agree at the tabulated points x_n, x_{n-1}, …, x_2 , x_1, x_0, we obtain (after some simplification)

$$P_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$
$$+ \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n \tag{4}$$

where $p = \frac{(x-x_n)}{h}$ .This is Newton's backward difference interpolation formula and it uses tabular values to the left of y_n. This formula is therefore useful for interpolation near the end of the tabular values.

Example 4.3:

The population of a town in the census was as given below.

Estimate the population for the years a) 1895 b) 1925

Year, x	1891	1901	1911	1921	1931
Population, y (in thousands)	46	66	81	93	101

Solution: First we construct the difference table:

Year, x	Population, y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

a) Since interpolation is desired at the beginning of the table, we use Newton's forward difference interpolation formula.

Here x_0 = 1891, y_0 = 46, h = 10 and x = 1895, and so $p = \frac{x-x_0}{h} = \frac{1895-1891}{10} = 0.4$.

Hence Newton's forward difference interpolation formula gives:

$$P(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$P(1895) = 46 + 0.4(20) + \frac{0.4(0.4-1)}{2}(-5) + \frac{0.4(0.4-1)(0.4-2)}{6}(2) + $$
$$+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$
$= 54.8528$ thousands.

Hence the population of the town in the year 1895 is 54.853.

b) Since interpolation is desired at the end of the table, we use Newton's backward difference interpolation formula.

Here $x_n = 1931$, $y_n = 101$, $h = 10$ and $x = 1925$, and so

$$p = \frac{x - x_n}{h} = \frac{1925 - 1931}{10} = -0.6$$

Hence Newton's backward difference interpolation formula gives:

$$P(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$P(1925) = 101 + (-0.6)(8) + \frac{(-0.6)(-0.6+1)}{2}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-1) +$$
$$\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$
$= 96.8368$ thousands.

Hence the population of the town in the year 1925 is 96.837.

Example 4.4:

The table below gives the value of tan x for $0.10 \leq x \leq 0.30$

X	0.10	0.15	0.20	0.25	0.30
tan (x)	0.1003	0.1511	0.2027	0.2553	0.3093

Find (a) tan 0.12 (b) tan 0.26 (c) tan 0.50

Solution:

First we construct the difference table:

x	y = tan x	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

a) Since interpolation is desired at the beginning of the table, we use Newton's forward difference interpolation formula.

Here $x_0 = 0.10$, $y_0 = 0.1003$, $h = 0.05$ and $x = 0.12$, and so $p = \frac{x - x_0}{h} = \frac{0.12 - 0.10}{0.05} = 0.4$.

Hence Newton's forward difference interpolation formula gives:

$$P(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$P(0.12) = 0.1003 + 0.4(0.0508) + \frac{0.4(0.4-1)}{2}(0.0008) + \frac{0.4(0.4-1)(0.4-2)}{6}(0.0002) +$$

$$+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(0.0002)$$

$$= 0.1205$$

Hence tan(0.12) ≅ P(0.12) = 0.1205.

b) Since interpolation is desired at the end of the table, we use Newton's backward difference interpolation formula.

Here $x_n = 0.30$, $y_n = 0.3093$, $h = 0.05$ and $x = 0.26$, and so

$$p = \frac{x - x_n}{h} = \frac{0.26 - 0.30}{0.05} = -0.8$$

Hence Newton's backward difference interpolation formula gives:

$$P(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$P(0.26) = 0.3093 + (-0.8)(0.0540) + \frac{(-0.8)(-0.8+1)}{2}(0.0014) + \frac{(-0.8)(-0.8+1)(-0.8+2)}{6!}(0.0004) +$$

$$\frac{(-0.8)(-0.8+1)(-0.8+2)(-0.8+3)}{24}(0.0002)$$

Hence

$$= 0.2662$$

tan(0.26) ≅ P(0.26) = 0.2662

4.4 Interpolation with unequally spaced points

4.4.1 Newton's Divided Difference Interpolation

We will discuss Newton's divided difference polynomial method in this section.

To illustrate the general form, cubic interpolation is shown in Figure 1.

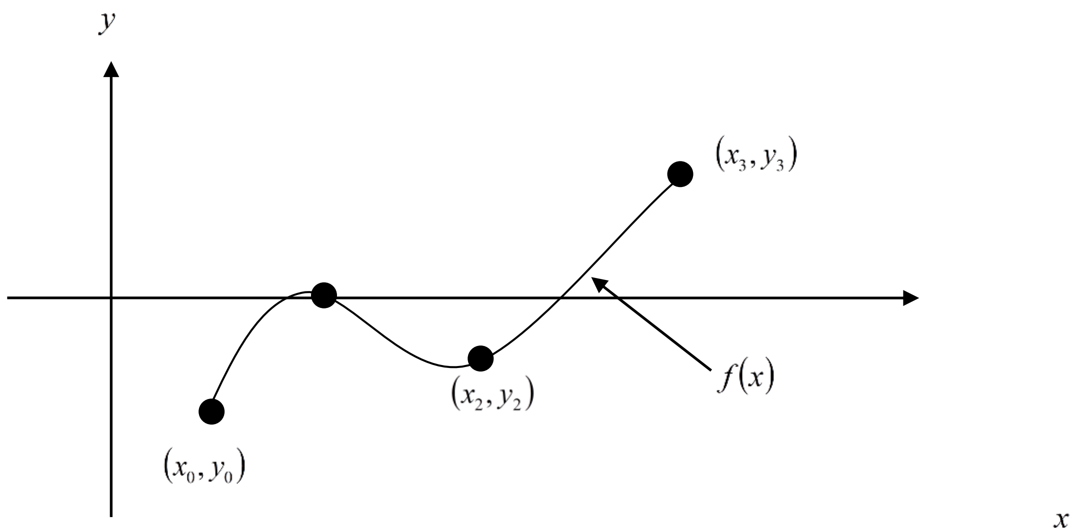


Figure 2 Interpolation of discrete data.

Linear Interpolation

Given (x_0, y_0) and (x_1, y_1), fit a linear interpolant through the data. Noting $y = f(x)$ and $y_1 = f(x_1)$, assume the linear interpolant $f_1(x)$ is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$

$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So $b_0 = f(x_0)$ $\qquad$ $b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1: Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using first order polynomial interpolation by Newton's divided difference polynomial method.

Solution

For linear interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0)$$

Since we want to find the velocity at $t = 16$, and we are using a first order polynomial, we need to choose the two data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t = 15$ and $t = 20$.

Then

$$t_0 = 15, \; v(t_0) = 362.78$$

$$t_1 = 20, \; v(t_1) = 517.35$$

gives

$$b_0 = v(t_0) \quad = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{517.35 - 362.78}{20 - 15} = 30.914$$

Hence

$$\begin{aligned} v(t) &= b_0 + b_1(t - t_0) \\ &= 362.78 + 30.914(t - 15), \qquad 15 \le t \le 20 \end{aligned}$$

At $t = 16,$

$$\begin{aligned} v(16) &= 362.78 + 30.914(16 - 15) \\ &= 393.69 \text{ m/s} \end{aligned}$$

If we expand

$$v(t) = 362.78 + 30.914(t - 15), \qquad 15 \le t \le 20$$

we get

$$v(t) = -100.93 + 30.914t, \qquad 15 \le t \le 20$$

and this is the same expression as obtained in the direct method.

Quadratic Interpolation

Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data. Noting $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$, assume the quadratic interpolant $f_2(x)$ is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$,

$$f_2(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1)$$
$$= b_0$$
$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$
$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$
$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$
$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

Example 2

The upward velocity of a rocket is given as a function of time in Table

Table 2 :Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using second order polynomial interpolation using Newton's divided difference polynomial method.

Solution

For quadratic interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

Since we want to find the velocity at $t = 16,$ and we are using a second order polynomial, we need to choose the three data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The three points are $t_0 = 10,$ $t_1 = 15,$ and $t_2 = 20$.

Then

$$t_0 = 10, \; v(t_0) = 227.04$$

$$t_1 = 15, \; v(t_1) = 362.78$$

$$t_2 = 20, \; v(t_2) = 517.35$$

gives

$$b_0 = v(t_0) \quad = 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} \quad = \frac{362.78 - 227.04}{15 - 10} = 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0}$$

$$= \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10} = \frac{30.914 - 27.148}{10} \quad = 0.37660$$

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16,$

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15)$$

$$= 392.19 \text{ m/s}$$

If we expand

$$v(t) = 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \; 10 \le t \le 20$$

we get

$$v(t) = 12.05 + 17.733t + 0.37660t^2, \quad 10 \le t \le 20$$

This is the same expression obtained by the direct method.

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that $b_0, b_1,$ and b_2 are finite divided differences. $b_0, b_1,$ and b_2 are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, and $f[x_2, x_1, x_0]$ are called bracketed functions of their variables enclosed in square brackets.

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots, x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots, x_0]$$

where the definition of the m^{th} divided difference is

$$b_m = f[x_m, \ldots\ldots, x_0]$$

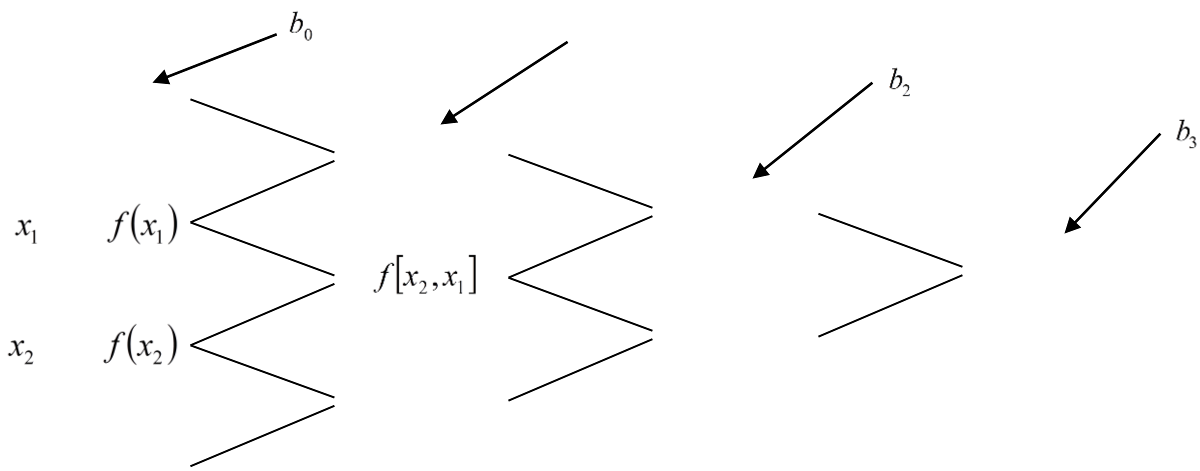
$$= \frac{f[x_m, \ldots\ldots, x_1] - f[x_{m-1}, \ldots\ldots, x_0]}{x_m - x_0}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3),

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

Table 4.4: Table of divided differences for a cubic polynomial.



Exercise 1

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds with third order polynomial interpolation using Newton's divided difference polynomial method.

b) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\text{s}$ to $t = 16\text{s}$.

c) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\text{s}$

4.4.2 Lagrange's interpolation method

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0),(x_1, y_1),......,(x_{n-1}, y_{n-1}),(x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Table 4. Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using a first order Lagrange polynomial. **Solution**

For first order polynomial interpolation (also called linear interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{1} L_i(t) v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1)$$

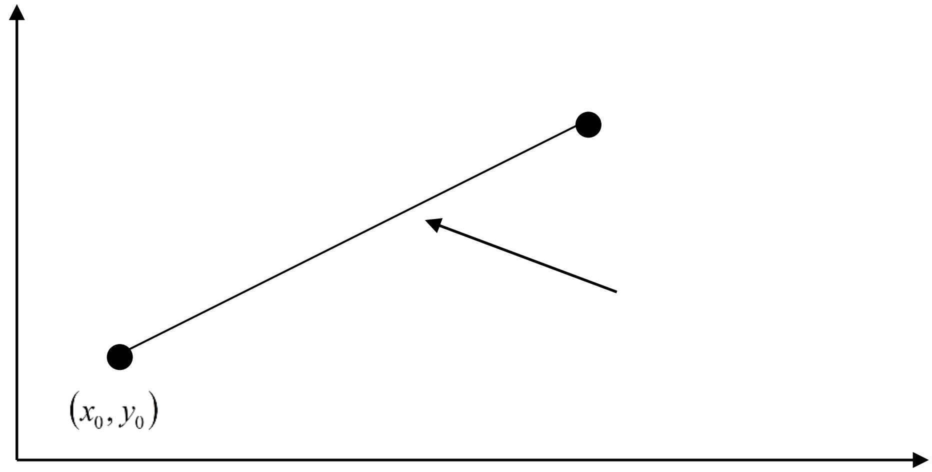


Figure 3. Linear interpolation.

Since we want to find the velocity at $t = 16$, and we are using a first order polynomial, we need to choose the two data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t_0 = 15$ and $t_1 = 20$.

Then

$$t_0 = 15,\ v(t_0) = 362.78$$

$$t_1 = 20,\ v(t_1) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{t - t_j}{t_0 - t_j}$$

$$= \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{t - t_j}{t_1 - t_j}$$

$$= \frac{t - t_0}{t_1 - t_0}$$

Hence

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1)$$

$$= \frac{t-20}{15-20}(362.78) + \frac{t-15}{20-15}(517.35), \quad 15 \le t \le 20$$

$$v(16) = \frac{16-20}{15-20}(362.78) + \frac{16-15}{20-15}(517.35)$$

$$= 0.8(362.78) + 0.2(517.35)$$

$$= 393.69 \text{ m/s}$$

You can see that $L_0(t) = 0.8$ and $L_1(t) = 0.2$ are like weightages given to the velocities at $t = 15$ and $t = 20$ to calculate the velocity at $t = 16$.

Quadratic Interpolation

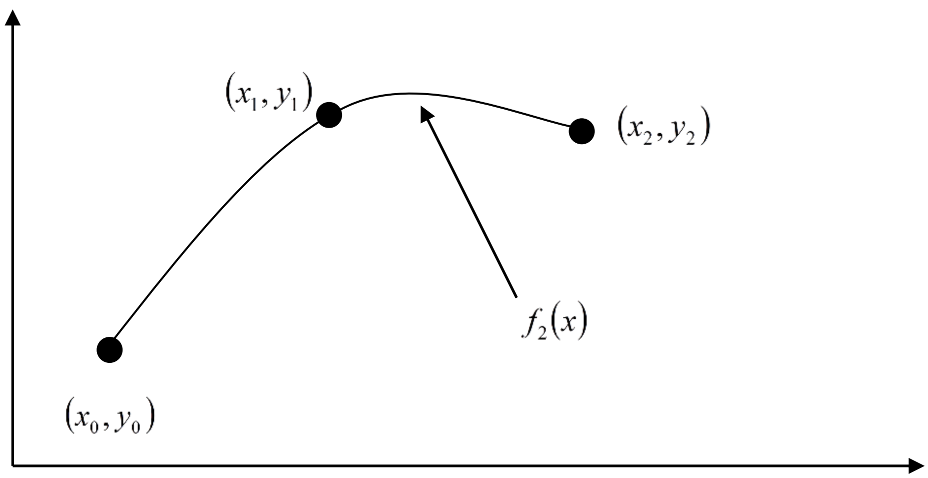


Figure 4: Quadratic interpolations**.**

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 5 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds with second order polynomial interpolation using Lagrangian polynomial interpolation.

b) Find the absolute relative approximate error for the second order polynomial approximation.

Solution

a) For second order polynomial interpolation (also called quadratic interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$

Since we want to find the velocity at $t = 16$, and we are using a second order polynomial, we need to choose the three data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The three points are $t_0 = 10,\ t_1 = 15,$ and $t_2 = 20$.

Then

$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{t - t_j}{t_0 - t_j}$$

$$= \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{t - t_j}{t_1 - t_j}$$

$$= \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{t - t_j}{t_2 - t_j}$$

$$= \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)$$

Hence

$$v(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)v(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)v(t_1) + \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)v(t_2),\ t_0 \le t \le t_2$$

$$v(16) = \frac{(16-15)(16-20)}{(10-15)(10-20)}(227.04) + \frac{(16-10)(16-20)}{(15-10)(15-20)}(362.78)$$

$$+\frac{(16-10)(16-15)}{(20-10)(20-15)}(517.35)$$

$$= (-0.08)(227.04) + (0.96)(362.78) + (0.12)(517.35)$$

$$= 392.19\ \text{m/s}$$

b) The absolute relative approximate error $|\in_a|$ for the second order polynomial is calculated by considering the result of the first order polynomial (Example 1) as the previous approximation.

$$|\in_a| = \left|\frac{392.19-393.69}{392.19}\right| \times 100$$

$$= 0.38410\%$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 6: Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds using third order Lagrangian polynomial interpolation.

b) Find the absolute relative approximate error for the third order polynomial approximation.

c) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\text{s}$ to $t = 16\text{s}$.

d) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\text{s}$.

Solution

a) For third order polynomial interpolation (also called cubic interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{3} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)$$

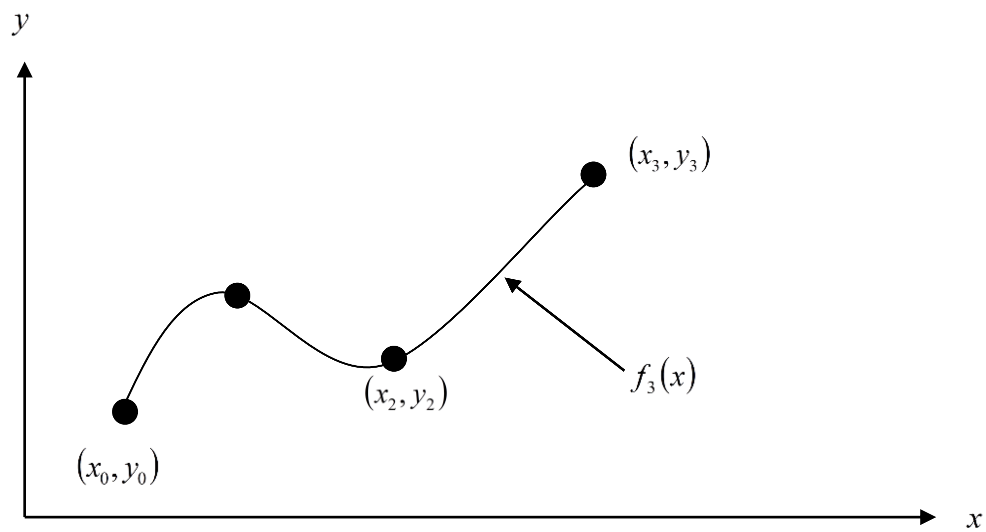


Figure 5: Cubic interpolation.

Since we want to find the velocity at $t = 16$, and we are using a third order polynomial, we need to choose the four data points closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The four points are $t_0 = 10,\ t_1 = 15,\ t_2 = 20$ and $t_3 = 22.5$.

Then

$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$

$$t_3 = 22.5,\ v(t_3) = 602.97$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{3} \frac{t - t_j}{t_0 - t_j}$$

$$= \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)\left(\frac{t - t_3}{t_0 - t_3}\right)$$

$$L_1(t)=\prod_{\substack{j=0\\j\neq 1}}^{3}\frac{t-t_j}{t_1-t_j}$$

$$=\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)$$

$$L_2(t)=\prod_{\substack{j=0\\j\neq 2}}^{3}\frac{t-t_j}{t_2-t_j}$$

$$=\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)$$

$$L_3(t)=\prod_{\substack{j=0\\j\neq 3}}^{3}\frac{t-t_j}{t_3-t_j}$$

$$=\left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)$$

Hence

$$v(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)v(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)v(t_1)$$
$$+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)v(t_2)+\left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)v(t_3),\ t_0\le t\le t_3$$

$$v(16)=\frac{(16-15)(16-20)(16-22.5)}{(10-15)(10-20)(10-22.5)}(227.04)+\frac{(16-10)(16-20)(16-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+\frac{(16-10)(16-15)(16-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+\frac{(16-10)(16-15)(16-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97)$$
$$=(-0.0416)(227.04)+(0.832)(362.78)+(0.312)(517.35)+(-0.1024)(602.97)$$
$$=392.06\ \text{m/s}$$

b) The absolute percentage relative approximate error, $|\in_a|$ for the value obtained for $v(16)$ can be obtained by comparing the result with that obtained using the second order polynomial (Example 2)

$$|\in_a|=\left|\frac{392.06-392.19}{392.06}\right|\times 100$$
$$=0.033269\%$$

c) The distance covered by the rocket between $t=11\text{s}$ to $t=16\text{s}$ can be calculated from the interpolating polynomial as

$$v(t)=\frac{(t-15)(t-20)(t-22.5)}{(10-15)(10-20)(10-22.5)}(227.04)+\frac{(t-10)(t-20)(t-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+\frac{(t-10)(t-15)(t-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+\frac{(t-10)(t-15)(t-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97), 10\le t\le 22.5$$

$$=\frac{(t^2-35t+300)(t-22.5)}{(-5)(-10)(-12.5)}(227.04)+\frac{(t^2-30t+200)(t-22.5)}{(5)(-5)(-7.5)}(362.78)$$
$$+\frac{(t^2-25t+150)(t-22.5)}{(10)(5)(-2.5)}(517.35)+\frac{(t^2-25t+150)(t-20)}{(12.5)(7.5)(2.5)}(602.97)$$

$$=(t^3-57.5t^2+1087.5t-6750)(-0.36326)+(t^3-52.5t^2+875t-4500)(1.9348)$$
$$+(t^3-47.5t^2+712.5t-3375)(-4.1388)+(t^3-45t^2+650t-3000)(2.5727)$$

$$=-4.245+21.265t+0.13195t^2+0.00544t^3,\ 10\le t\le 22.5$$

Note that the polynomial is valid between $t=10$ and $t=22.5$ and hence includes the limits of $t=11$ and $t=16$.

So

$$s(16)-s(11)=\int_{11}^{16}v(t)dt$$

$$=\int_{11}^{16}(-4.245+21.265t+0.13195t^2+0.00544t^3)dt$$

$$=\left[-4.245t+21.265\frac{t^2}{2}+0.13195\frac{t^3}{3}+0.00544\frac{t^4}{4}\right]_{11}^{16}$$

$$=1605\text{ m}$$

d) The acceleration at $t=16$ is given by

$$a(16)=\frac{d}{dt}v(t)\Big|_{t=16}$$

Given that

$$v(t)=-4.245+21.265t+0.13195t^2+0.00544t^3,\ 10\le t\le 22.5$$

$$a(t)=\frac{d}{dt}v(t)$$

$$=\frac{d}{dt}\left(-4.245+21.265t+0.13195t^2+0.00544t^3\right)$$

$$=21.265+0.26390t+0.01632t^2,\ 10\le t\le 22.5$$

$$a(16) = 21.265 + 0.26390(16) + 0.01632(16)^2$$

$$= 29.665 \text{ m/s}^2$$

Note: There is no need to get the simplified third order polynomial expression to conduct the differentiation. An expression of the form

$$L_0(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)$$

gives the derivative without expansion as

$$\frac{d}{dt}\left(L_0(t)\right) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right) + \left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right) + \left(\frac{t-t_3}{t_0-t_3}\right)\left(\frac{t-t_1}{t_0-t_1}\right)$$