4、高斯型低通滤波器在频域中的传递函数是

$$H(u,v)=Ae^{-(u^2+v^2)/2\sigma^2}$$

根据二维傅里叶性质，证明空间域的相应滤波器形式为

$$h(x,y)=A2\pi\sigma^2e^{-2\pi^2\sigma^2(x^2+y^2)}$$

（这些闭合形式只适用于连续变量情况。）

在证明中假设已经知道如下结论：函数$e^{-\pi(x^2+y^2)}$的傅立叶变换为$e^{-\pi(u^2+v^2)}$

$$h(x,y)=\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty}H(u,v)\,e^{j2\pi(ux+vy)}dudv$$

$$=\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty}Ae^{-(u^2+v^2)/2\sigma^2}\,e^{j2\pi(ux+vy)}dudv$$

$$=\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty}Ae^{-(u^2+v^2)/2\sigma^2+j2\pi(ux+vy)}\,dudv$$

$$=\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty}Ae^{-\pi((\frac{u}{\sqrt{2\pi}\sigma})^2-j2ux)}e^{-\pi((\frac{v}{\sqrt{2\pi}\sigma})^2-j2vy)}\,dudv$$

$$=\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty}Ae^{-\pi((\frac{u}{\sqrt{2\pi}\sigma})^2-j2ux-2\pi\sigma^2x^2+2\pi\sigma^2x^2)}e^{-\pi((\frac{v}{\sqrt{2\pi}\sigma})^2-j2vy-2\pi\sigma^2y^2+2\pi\sigma^2y^2)}\,dudv$$

$$=\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty}Ae^{-\pi((\frac{u}{\sqrt{2\pi}\sigma}-j\sqrt{2\pi}\sigma x)^2+2\pi\sigma^2x^2)}e^{-\pi((\frac{v}{\sqrt{2\pi}\sigma}-j\sqrt{2\pi}\sigma y)^2+2\pi\sigma^2y^2)}\,dudv$$

设 $m=\frac{u}{\sqrt{2\pi}\sigma}-j\sqrt{2\pi}\sigma x$，$n=\frac{v}{\sqrt{2\pi}\sigma}-j\sqrt{2\pi}\sigma y$

所以 $du=\sqrt{2\pi}\sigma dm,dv=\sqrt{2\pi}\sigma dn$,

原积分式子转换为

$$\int_{m=-\infty}^{\infty}\int_{n=-\infty}^{\infty}A2\pi\sigma^2e^{-\pi(m^2+2\pi\sigma^2x^2)}e^{-\pi(n^2+2\pi\sigma^2y^2)}\,dmdn$$

$$=A2\pi\sigma^2e^{-2\pi^2\sigma^2(x^2+y^2)}\int_{m=-\infty}^{\infty}\int_{n=-\infty}^{\infty}e^{-\pi m^2}e^{-\pi n^2}\,dmdn$$

$$=A2\pi\sigma^2e^{-2\pi^2\sigma^2(x^2+y^2)}\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}e^{-\pi r^2}r\,d\theta dr$$

因为 $\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}e^{-\pi r^2}r\,d\theta dr=1$

所以

$$A2\pi\sigma^2e^{-2\pi^2\sigma^2(x^2+y^2)}\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}e^{-\pi r^2}r\,d\theta dr=A2\pi\sigma^2e^{-2\pi^2\sigma^2(x^2+y^2)}$$

得证。