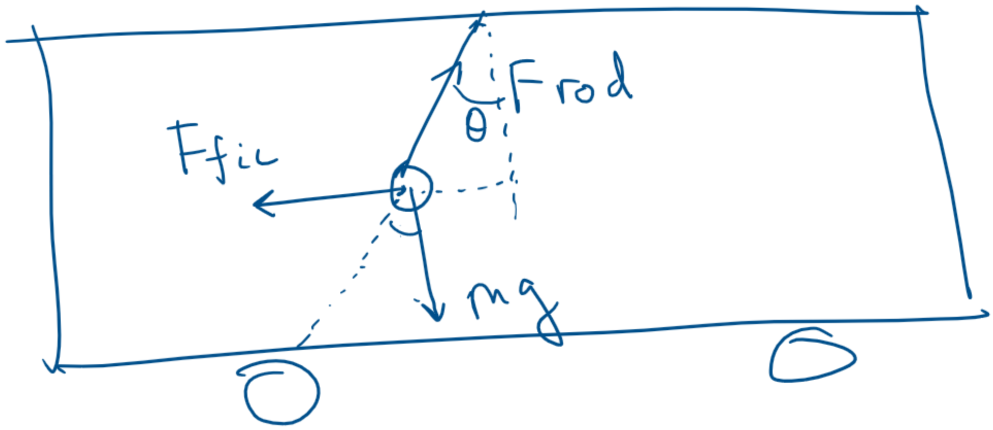
Problem 1

> A simple pendulum with a point mass m on the end of a massless rod of length L is suspended from the roof of a truck that is smoothly accelerating at a rate a in x-direction. Ignore the rotation of the Earth.

(a)

> Construct a free-body diagram showing the (true and fictitious) forces acting on the mass. At what angle θ away from the vertical will the pendulum hang when it is at equilibrium.



From the free-body diagram, we know that

$$
\begin{aligned}
F_{\text{rod}} \sin\theta = F_{\text{fic}} = ma \\
F_{\text{rod}} \cos\theta = mg
\end{aligned}
$$

and thus

$$\theta = \arctan\left(\frac{ma}{mg}\right) = \arctan\left(\frac{a}{g}\right)$$

(b)

> What is the tension in the rod as it is in equilibrium?

From the free-body diagram, we could see that

$$F_{\text{rod}} = m\sqrt{a^2 + g^2} = m(a\sin\theta + g\cos\theta)$$

(c)

> What is the normal frequency w_n of free oscillations of the pendulum around that equilibrium.

From the free-body diagram, we could see that the torque is

$$I\ddot{\theta} = F_{\text{fic}}L\cos\theta - mgL\sin\theta$$

$$mL^2\ddot{\theta} = maL\cos\theta - mgL\sin\theta$$

$$L\ddot{\theta} = a\cos\theta - g\sin\theta$$

$$L\Delta\ddot{\theta} = a\cos(\theta_{\text{eq}} + \Delta\theta) - g\sin(\theta_{\text{eq}} + \Delta\theta)$$

$$L\Delta\ddot{\theta} = a(\cos\theta_{\text{eq}}\cos\Delta\theta - \sin\theta_{\text{eq}}\sin\Delta\theta) - g(\sin\theta_{\text{eq}}\cos\Delta\theta + \cos\theta_{\text{eq}}\sin\Delta\theta)$$

We know that

$$a\cos\theta_{\text{eq}} - g\sin\theta_{\text{eq}} = 0$$

and thus

$$L\Delta\ddot{\theta} = -a\sin\theta_{\text{eq}}\sin\Delta\theta - g\cos\theta_{\text{eq}}\sin\Delta\theta$$

$$L\Delta\ddot{\theta} + (a\sin\theta_{\text{eq}} + g\cos\theta_{\text{eq}})\sin\Delta\theta = 0$$

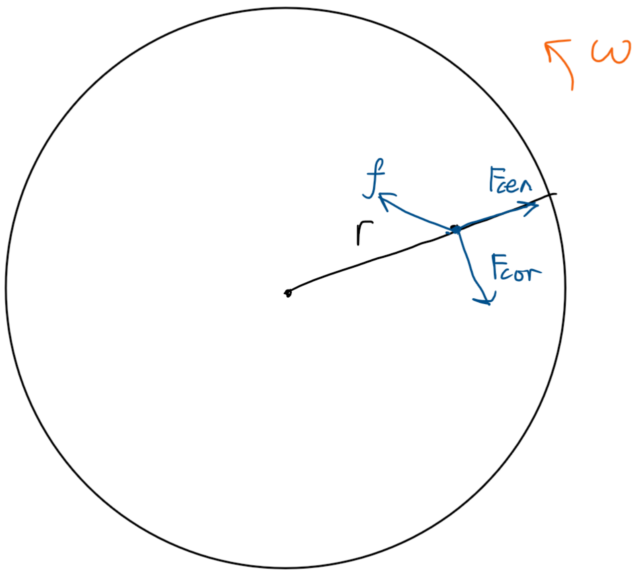
using small angle approximation $\sin\Delta\theta \approx \Delta\theta$.

$$L\Delta\ddot{\theta} + (a\sin\theta_{\text{eq}} + g\cos\theta_{\text{eq}})\Delta\theta = 0$$

and thus the ω_n is

$$\omega_n = \sqrt{\frac{a\sin\theta_{\text{eq}} + g\cos\theta_{\text{eq}}}{L}} = \sqrt{\frac{F_{\text{rod}}}{mL}}$$

Problem 2



The graph should roughly look like above. Using the cylindrical coordinate

$$\vec{\omega} = \omega\hat{z}$$

$$\vec{v}_{\text{rel}} = v_r\hat{r}$$

$$\vec{r}_{O'A} = r\hat{r}$$

and thus

$$\vec{F}_{\text{centrifugal}} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}_{O'A}) = -m\omega\hat{z}\times(\omega\hat{z}\times r\hat{r}) = -m\omega^2 r(-\hat{r}) = m\omega^2 r\hat{r}$$

$$\vec{F}_{\text{Coriolis}} = -2m\vec{\omega}\times\vec{v}_{\text{rel}} = -2m\omega\hat{z}\times v_r\hat{r} = -2m\omega v_r\hat{\theta}$$

and thus when the bug slips,

$$f = \mu mg < \sqrt{(m\omega^2 r)^2 + (-2m\omega v_r)^2} = \sqrt{m^2\omega^4 r^2 + 4m^2\omega^2 v_r^2} = m\omega\sqrt{\omega^2 r^2 + 4v_r^2}$$

which means

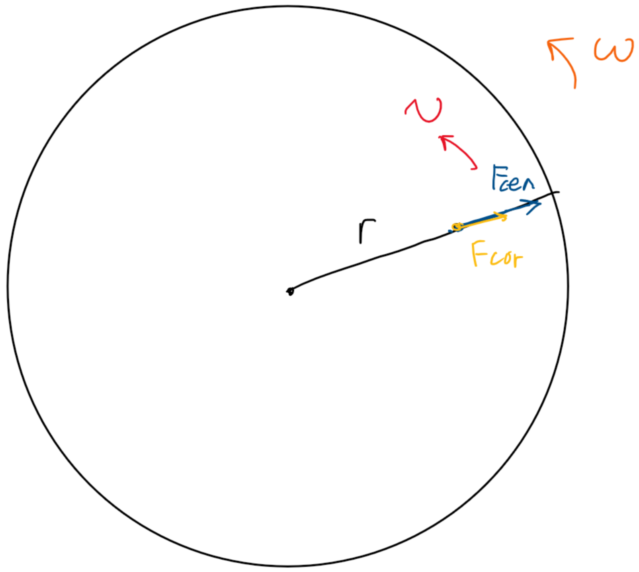
$$\mu g < \omega\sqrt{\omega^2 r^2 + 4v_r^2}$$
$$r > \sqrt{\left(\frac{\mu g}{\omega^2}\right)^2 - 4\left(\frac{v_r}{\omega}\right)^2}$$

(b)

> The bug crawls with a constant speed v_r relative to the turntable in a *circular* path with a radius b. The circular path is concentric with the center of the turntable. For what value of v_r (expressed in terms of ω, μ, b, and g) will the bug start to slip if it crawls

(i)

> In the direction of rotation?



the motion roughly looks like above. Again, using cylindrical coordinate, we get

$$\vec{\omega} = \omega\hat{z}$$
$$\vec{v}_{\text{rel}} = v_r\hat{\theta}$$
$$\vec{r}_{O'A} = b\hat{r}$$

and thus

$$\vec{F}_{\text{centrifugal}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}_{O'A}) = -m\omega\hat{z} \times (\omega\hat{z} \times b\hat{r}) = -m\omega^2 r(-\hat{r}) = m\omega^2 b\hat{r}$$
$$\vec{F}_{\text{Coriolis}} = -2m\vec{\omega} \times \vec{v}_{\text{rel}} = -2m\omega\hat{z} \times v_r\hat{\theta} = 2m\omega v_r\hat{r}$$

both centrifugal and Coriolis force are pointing outward from the origin. Thus, when bug begin to slip,

$$f = \mu mg < m\omega^2 b + 2m\omega v_r$$

and thus

$$\mu g < \omega^2 b + 2\omega v_r$$
$$v_r > \frac{1}{2}\left(\frac{\mu g}{\omega} - \omega b\right)$$

(ii)

The opposite is quite similar, we have the Coriolis force pointing inward to the origin now (it's in the opposite direction of centrifugal direction).

There are two situations, one is that the centrifugal force is stronger, and the Coriolis force is weaker. In this case, the static friction force is in the same direction of Coriolis force, and thus when the bugs begin to slip

$$f = \mu mg < m\omega^2 b - 2m\omega v_r$$

$$\begin{aligned} \mu g &< \omega^2 b - 2\omega v_r \\ v_r &< \frac{1}{2}\left(\omega b - \frac{\mu g}{\omega}\right) \end{aligned}$$

another case is that the Coriolis force is stronger, and centrifugal force is weaker. In this case, the static friction force is in the same direction of centrifugal force, and thus when the bugs begin to slip.

$$\begin{aligned} f = \mu mg &< 2m\omega v_r - m\omega^2 b \\ \mu g &< 2\omega v_r - \omega^2 b \\ v_r &> \frac{1}{2}\left(\frac{\mu g}{\omega} + \omega b\right) \end{aligned}$$

Question 3: Space Ship

There is a better way to think about it. It's the same that there is acceleration pointing backward as that there is gravity on earth. Then

(a)

> What happens to the balloon?

The balloon just flies toward the front of the ship, just like a helium balloon flies upward in earth

(b)

> What happens instead if the balloon is filled with air?

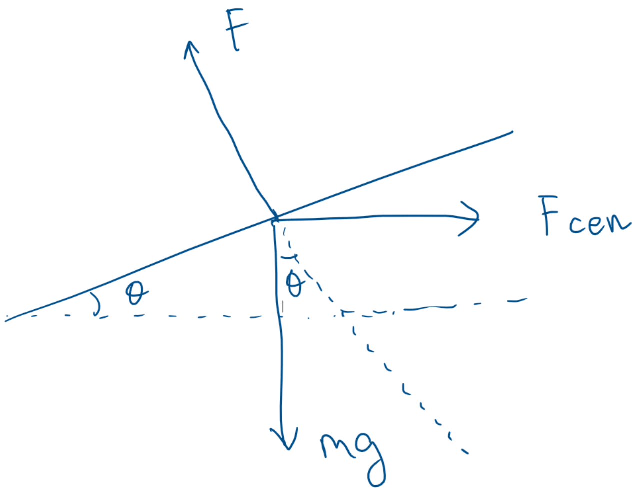
The balloon will just stay where it is (if we don't consider the mass of the plastic that forms the balloon, otherwise it will slowly move back to the bottom of the ship). It's just like the balloon filled with air will stay where it is (again, if we ignore the mass of the plastic that forms the balloon, otherwise it will just fall on the ground).

(c)

> Is the answer for the helium balloon any different if there was no air inside the ship?

Realistically, if there is no air, the pressure difference will just cause the balloon to explode. So, suppose the balloon is tight enough to just hold helium gas in it, then the helium balloon will just fall towards the back of the ship just like a helium balloon (if it's tight enough) will fall into the ground if it's in a vacuum chamber on earth.

Question 4: Water bucket



Consider the surface of the water. When it's at static equilibrium in the rotating frame, the Coriolis force is zero, and there should only be gravity, centrifugal force and normal force provided to the surface of the water. Then

$$F_{\text{cen}} = m\omega^2 r$$

$$mg\sin\theta = m\omega^2 r\cos\theta$$

$$\tan\theta = \frac{\omega^2 r}{g}$$

and from hint, we know that $\tan\theta = \frac{\mathrm{d}z}{\mathrm{d}r}$, thus

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{\omega^2 r}{g}$$

$$z = \frac{\omega^2}{2g}r^2 + z_0$$

where z_0 is the lowest water level at the center of the bucket.

Question 5: Coriolis force

(a)

We know that

$$\vec{a}(t) = \ddot{x}\hat{e} + \ddot{y}\hat{n} + \ddot{z}\hat{u}$$

$$\vec{\omega} = u\sin\theta\hat{u} + \omega\cos\theta\hat{n}$$

and thus

$$m\vec{a}(t) = -mg\hat{u} - 2m\vec{\omega}\times\vec{v}_{\text{rel}}$$

$$m(\ddot{x}\hat{e} + \ddot{y}\hat{n} + \ddot{z}\hat{u}) = -mg\hat{u} - 2m(\omega\sin\theta\hat{u} + \omega\cos\theta\hat{n})\times(\dot{x}\hat{e} + \dot{y}\hat{n} + \dot{z}\hat{u})$$

and thus

$$m(\ddot{x}\hat{e} + \ddot{y}\hat{n} + \ddot{z}\hat{u}) = -mg\hat{u} - 2m(\omega\sin\theta\hat{u} + \omega\cos\theta\hat{n})\times(\dot{x}\hat{e} + \dot{y}\hat{n} + \dot{z}\hat{u})$$

$$(\ddot{x}\hat{e} + \ddot{y}\hat{n} + \ddot{z}\hat{u}) = -g\hat{u} + 2\omega(-\sin\theta\dot{x}\hat{n} + \sin\theta\dot{y}\hat{e} + \cos\theta\dot{x}\hat{u} - \cos\theta\dot{z}\hat{e})$$

and thus

$$\ddot{x} = -2\omega\dot{z}\cos\theta + 2\omega\dot{y}\sin\theta$$
$$\ddot{y} = -2\omega\dot{x}\sin\theta$$
$$\ddot{z} = -g + 2\omega\dot{x}\cos\theta$$

(b)

At zero order, we have

$$\ddot{x}_0(t) = 0$$
$$\ddot{y}_0(t) = 0$$
$$\ddot{z}_0(t) = -g$$

Applying initial condition

$$\begin{aligned} \dot{x}_0(t) &= 0 & x_0(t) &= 0 \\ \dot{y}_0(t) &= 0 & y_0(t) &= 0 \\ \dot{z}_0(t) &= v - gt & z_0(t) &= vt - \frac{1}{2}gt^2 \end{aligned}$$

putting the result into first order differential equation

$$\ddot{x}_1 = -2\omega\dot{z}_0\cos\theta + 2\omega\dot{y}_0\sin\theta = -2\omega(v - gt)\cos\theta$$
$$\ddot{y}_1 = -2\omega\dot{x}_0\sin\theta = 0$$
$$\ddot{z}_1 = 2\omega\dot{x}_0\cos\theta = 0$$

and thus

$$x_1 = -2\omega(\frac{1}{2}vt^2 - \frac{1}{6}gt^3)\cos\theta$$
$$y_1 = 0$$
$$z_1 = 0$$

and thus

$$x = x_0 + x_1 = -2\omega(\frac{1}{2}vt^2 - \frac{1}{6}gt^3)\cos\theta$$
$$y = y_0 + y_1 = 0$$
$$z = z_0 + z_1 = vt - \frac{1}{2}gt^2$$

when ball lands ($t = \frac{2v}{g}$), the ball lands on

$$x(t) = -2\omega(\frac{1}{2}v\frac{4v^2}{g^2} - \frac{1}{6}g\frac{8v^3}{g^3})\cos\theta = -\frac{4}{3}\frac{\omega v^3}{g^2}\cos\theta$$
$$y(t) = 0$$
$$z(t) = 0$$

that is, a bit west to where it starts.