Problem 1

(a)

From the graph, we could see that

$$\kappa(z) = \frac{\kappa_1 - 1}{d} z + 1 = \frac{(\kappa_1 - 1)z + d}{d}$$

(b)

we know that for a linear dielectric, we have

$$\vec{E}(z) = -\frac{\sigma}{\kappa(z)\epsilon_0}\hat{z}$$

and the displacement field is just

$$\vec{D}(z) = \epsilon_0 \kappa(z) \vec{E} = -\sigma \hat{z}$$

the **magnitude** of $\vec{E}(z)$, which is $E(z) = \dfrac{\sigma}{\kappa(z)\epsilon_0}$, depend on the $\kappa(z)$, and we find now that $\kappa(z)$ is smallest at 0, and biggest at d. Therefore, the **magnitude** $E(z)$ is biggest at 0, and smallest at d. (that is to say, the magnitude that $\vec{E}(z)$ pointing to left is the maximum at 0, and smallest at d).

(c)

Since we know that

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

It's true that

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \left(1 - \frac{1}{\kappa(z)}\right)\sigma\hat{z} = -\frac{\kappa - 1}{\kappa}\sigma\hat{z}$$

It's **magnitude** is biggest at d, and smallest at 0. When you compare this with the fact that the **magnitude** $E(z)$ is biggest at 0, and smallest at d, this is expected because we know that

$$\vec{P} = \epsilon_0 \chi \vec{E}$$

we see that

$$\chi = \kappa - 1 = \frac{\kappa_1 - 1}{d} z$$

and this increases from 0 to d. The effect of χ is greater than $\vec{E}$ and that's why the maximum and minimum (the increasing / decreasing trend) is same as χ but not $\vec{E}$.

(d)

Integrate E on z, we get

$$\begin{aligned}\Delta V_{0\to d} &= -\int_0^d -\frac{\sigma}{\kappa(z)\epsilon_0}\mathrm{d}z\\ &= \frac{\sigma}{\epsilon_0}\int_0^d \frac{1}{\kappa(z)}\mathrm{d}z\\ &= \frac{\sigma d}{\epsilon_0}\int_0^d \frac{1}{(\kappa_1-1)z+d}\mathrm{d}z\\ &= \frac{\sigma d}{\epsilon_0}\ln\left(\frac{(\kappa_1-1)z+d}{k-1}\right)\Bigg|_0^d\\ &= \frac{\sigma d}{\epsilon_0(\kappa_1-1)}\ln\kappa_1\end{aligned}$$

(e)

We know that $C \equiv \frac{Q}{V}$ and therefore

$$C = \frac{Q}{V} = \frac{\sigma A\cdot\epsilon_0(\kappa_1-1)}{\sigma d\ln\kappa_1} = \frac{\epsilon_0 A}{d}\frac{\kappa_1-1}{\ln\kappa_1}$$

(f)

We know that

$$\sigma_B = -\vec{P}\cdot\hat{n}$$

and when on the left plate

$$\sigma_B = \vec{P}\cdot\hat{z} = -\frac{\kappa-1}{\kappa}\sigma = 0$$

when on the right plate

$$\sigma_B = -\vec{P}\cdot\hat{z} = \frac{\kappa-1}{\kappa}\sigma = \frac{\kappa_1-1}{\kappa_1}\sigma$$

We also know that

$$\begin{aligned}\rho_B = -\vec{\nabla}\cdot\vec{P} &= \vec{\nabla}\cdot\frac{\kappa-1}{\kappa}\sigma\hat{z}\\ &= \frac{\mathrm{d}}{\mathrm{d}z}(\frac{\kappa-1}{\kappa}\sigma)\\ &= \sigma\frac{\mathrm{d}}{\mathrm{d}z}(1-\frac{d}{(\kappa_1-1)z+d})\\ &= -d\sigma\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(\kappa_1-1)z+d}\\ &= \frac{d\sigma(\kappa_1-1)}{((\kappa_1-1)z+d)^2}\end{aligned}$$

Integrate over z,

$$
\begin{aligned}
\sigma_{B'} &= \int_0^d \frac{d\sigma(\kappa_1 - 1)}{((\kappa_1 - 1)z + d)^2}\mathrm{d}z \\
&= d(\kappa_1 - 1)\sigma \int_0^d ((\kappa_1 - 1)z + d)^{-2}\mathrm{d}z \\
&= d(\kappa_1 - 1)\sigma\left(\frac{1}{(\kappa_1 - 1)((\kappa_1 - 1)z + d)}\right)\Bigg|_0^d \\
&= d(k-1)\sigma\frac{1}{\kappa_1 d} \\
&= \frac{\kappa_1}{\kappa_1 - 1}\sigma
\end{aligned}
$$

We see that they are actually the same, $\sigma_B = \sigma_{B'}$. That's why it's common sense.

Problem 2

(a)

This is just similar as the one we did in HW4. (So the derivation will be ignored here). It's

$$
\begin{aligned}
s\frac{\partial}{\partial s}(s\frac{\partial S}{\partial s}) &= m^2 S(s) \\
\frac{\partial^2 \Theta}{\partial \theta^2} &= -m^2\Theta(\theta)
\end{aligned}
$$

The $m^2 > 0$. We choose the negative coefficient for $\Theta(\theta)$ because we want the sinusoidal solution. It's reasonable because we need to make sure the solution is periodical.

(b)

From HW4 we see that the general solution is that

$$
V(s, \phi) = \sum_{m=-\infty}^{\infty} s^m(A_m \cos m\phi + B_m \sin m\phi)
$$

From symmetry, the solution should be an even function of ϕ, We first consider the case when it's outside the cylinder:

From $E = E_0\hat{x}$, we see that this

$$
\begin{aligned}
V(\infty, 0) &= -E_0 s \\
V(\infty, \pi) &= E_0 s
\end{aligned}
$$

also we know that

$$
V(\infty, \frac{\pi}{2}) = V(\infty, \frac{3\pi}{2}) = 0
$$

using symmetry, that means all the $B_m = 0$, and therefore

$$
V_{\text{out}}(s, \phi) = -sE_0 \cos\phi + \sum_{m=0}^{\infty} s^{-m} A_{\text{m,out}} \cos m\phi
$$

Now consider the situation in the cylinder. Using symmetry, that means all the $B_m = 0$. We also see that we shouldn't approach infinity as $s \to 0$. Therefore

$$V_{\text{in}}(s,\phi) = \sum_{m=0}^{\infty} s^m A_{\text{m,in}} \cos m\phi$$

In fact, only $m = 1$ is possible (this is discussed in part (c), so it is

$$V_{\text{out}}(s,\phi) = -sE_0 \cos\phi + \frac{1}{s} A_{1,\text{ out}} \cos\phi$$
$$V_{\text{in}}(s,\phi) = sA_{1,\text{in}} \cos\phi$$

(c)

we can write

$$\vec{D} = -\epsilon_0 \kappa \vec{\nabla} V(s,\theta)$$

and we see that

$$\hat{n} \cdot \vec{D}_{\text{out}} = -\epsilon \hat{s} \cdot \vec{\nabla} V_{\text{out}}\,(s,\phi)$$
$$\hat{n} \cdot \vec{D}_{\text{in}} = -\epsilon\kappa \hat{s} \cdot \vec{\nabla} V_{\text{in}}\,(s,\phi)$$

and we know that $\hat{s} \cdot \vec{\nabla} = \frac{\partial}{\partial s}$.

$$\frac{\partial}{\partial s}(-sE_0 \cos\phi + \sum_{m=0}^{\infty} s^{-m} A_{\text{m,out}} \cos m\phi) = \frac{\partial}{\partial s} \kappa \sum_{m=0}^{\infty} s^m A_{\text{m,in}} \cos m\phi$$

we see that

$$-E_0 \cos\phi - ms^{-m-1} A_{\text{m, out}} \cos m\phi = \kappa m s^{m-1} A_{\text{m, in}} \cos m\phi$$

replace s with R and we see

$$-E_0 \cos\phi - mR^{-m-1} A_{\text{m, out}} \cos m\phi = \kappa m R^{m-1} A_{\text{m, in}} \cos m\phi$$

and we see that this must be true for every ϕ, which means that m could only be 1. Therefore

$$-E_0 \cos\phi - \frac{A_{1,\text{ out}}}{R^2} \cos\phi = \kappa A_{\text{m, in}} \cos\phi$$

which is just

$$-E_0 - \frac{A_{1,\text{ out}}}{R^2} = \kappa A_{\text{m, in}}$$

We also know that

$$E_{\parallel} = -\hat{\phi} \cdot \vec{\nabla} V(s,\phi)$$

we know that

$$\hat{\phi} \cdot (\frac{\partial V}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}) = \frac{1}{s}\frac{\partial V}{\partial \phi}$$

therefore

$$E_{\parallel} = -\frac{1}{s}\frac{\partial V}{\partial \phi}$$

we know that

$$E_{\parallel,\text{out}} = E_{\parallel,\text{in}}$$
$$-\frac{1}{s}\frac{\partial}{\partial \phi}(-sE_0\cos\phi + \frac{1}{s}A_{1,\text{ out}}\cos\phi) = -\frac{1}{s}\frac{\partial V}{\partial \phi}(sA_{1,\text{in}}\cos\phi)$$
$$-sE_0 + \frac{1}{s}A_{1,\text{ out}} = sA_{1,\text{in}}$$

Let $s = R$, we see that

$$-E_0 + \frac{A_{1,\text{ out}}}{R^2} = A_{1,\text{ in}}$$

Now, we have

$$-E_0 - \frac{A_{1,\text{ out}}}{R^2} = \kappa A_{1,\text{ in}}$$
$$-E_0 + \frac{A_{1,\text{ out}}}{R^2} = A_{1,\text{ in}}$$

we see that

$$-E_0 - \frac{A_{1,\text{ out}}}{R^2} = \kappa(-E_0 + \frac{A_{1,\text{ out}}}{R^2})$$
$$(\kappa-1)E_0 = (1+\kappa)\frac{A_{1,\text{ out}}}{R^2}$$
$$A_{1,\text{ out}} = \frac{\kappa-1}{\kappa+1}E_0R^2$$

and

$$A_{1,\text{ in}} = -\frac{2}{\kappa+1}E_0$$

Therefore

$$V_{\text{out}}(s,\phi) = -sE_0\cos\phi + \frac{1}{s}\frac{\kappa-1}{\kappa+1}E_0R^2\cos\phi$$
$$V_{\text{in}}(s,\phi) = -s\frac{2}{\kappa+1}E_0\cos\phi$$

(d)

we see that

$$\vec{E}_{\text{in}} = -\vec{\nabla}(-s\frac{2}{\kappa+1}E_0\cos\phi) = \frac{2}{\kappa+1}E_0\hat{x}$$

therefore

$$\epsilon_0(\chi + 2)E_{\text{in}} = 2\epsilon_0 E_0$$
$$\epsilon_0 E_{\text{in}} = \epsilon_0 E_0 - \frac{1}{2}P$$

and we see that it's indeed $1/2$

(e)

Imagine when the cylinder is parallel to the electric field direction, then actually there won't be any induced P and E. and the formula

$$\epsilon_0 E_{\text{in}} = \epsilon_0 E_0 {-} ?P$$

and in this case the $?$ in above formula will just be 0, as expected. So, the depolarizing coefficient is just 0.