(a)

There are few cases:

- Yulie drives to one shop $x \in X$, and then goes to Aaron's place t directly, no need to add fuel.
- Yulie first drives to gas station $y \in Y$, and then goes to one shop $x \in X$, and finally goes to Aaron's place t.
- Yulie first drives to one shop $x \in X$, and then goes to the gas station $y \in Y$, and finally goes to Aaron's place t.

we first construct a new graph $G' = (V', E')$ such that $V' = V \times \{F, T\}$. The $\{F, T\}$ stores the information whether we visited the shop yet, F means we haven't visited it yet, T means we have visited it. For every edge $e = (v, u) \in E$, we have

- if $u \in X$, then there is edge $e' = (v, g) \to (u, T)$ for $g \in \{F, T\}$.
- Otherwise, then there is edge $e' = (v, g) \to (u, g)$ for $g \in \{F, T\}$.

We could then first run Dijkstra on G', record all the shortest distance from the (s, F) to all other nodes in V'. Then we get the reverse of G', the G'_{rev}. We run another Dijkstra on G'_{rev}, and we could record the shortest distance from (t, T) to all other nodes in V' in graph G'_{rev}. Notice this is equivalent to the shortest distance from all other nodes in V' to (t, T) in graph G'. Then we check:

- If the minimum distance from (s, F) to (t, T), the $\text{dist}((s, F), (t, T)) \leq D$, then that means we could by the gift and directly go to Aaron's house, no need to add oil. (of course, we could pass a gas station, but this doesn't necessarily we have to add oil there). If this is true, we set $\text{dist}_1 = \text{dist}((s, F), (t, T))$, otherwise we set $\text{dist}_1 = \infty$
- For all the $(y, T) \in V'$ that $y \in Y$ and also that $\text{dist}((s, F), (y, T)) \leq D$, we find the $\text{dist}_{y,T} = \text{dist}((s, F), (y, T)) + \text{dist}((y, T), (t, T))$. We take the minimum of all $\text{dist}_{y,T}$ and call it dist_2. (If there is not such (y, T), we set $\text{dist}_2 = \infty$) This is the case when Yulie first drives to one shop $x \in X$, and then goes to the gas station $y \in Y$, and finally goes to Aaron's place t.
- For all the $(y, F) \in V'$ that $y \in Y$ and also that $\text{dist}((s, F), (y, F)) \leq D$, we find the $\text{dist}_{y,F} = \text{dist}((s, F), (y, F)) + \text{dist}((y, F), (t, T))$. We take the minimum of all $\text{dist}_{y,F}$ and call it dist_3 (If there is not such (y, F), we set $\text{dist}_3 = \infty$). This is the case when Yulie first drives to gas station $y \in Y$, and then goes to one shop $x \in X$, and finally goes to Aaron's place t.

The final result is $\text{dist} = \min(\text{dist}_1, \text{dist}_2, \text{dist}_3)$, the minimum of three possible cases. We see that we first uses two Dijkstra to find all the distance from s to each node, and the distance from each node to t. This takes $2 \cdot \text{Dijkstra}(V', E')$. Then we see that for the second and third case, when we iterate every possible cases in Y, we have $O(V')$ time complexity. We notice that $|V'| = 2|V|$ and that $|E'| = 2|E'|$.

So, in the end, the total time complexity is just $2 \cdot \text{Dijkstra}(V', E') + 2O(V') = O(V \log V + E)$

(b)

We first develop a short-hand notation $U = \{s, t\} \cup Y$. Then we construct a new vertex set $V' = U \times \{F, T\}$. The $\{F, T\}$ stores the information whether we visited the shop yet, F means we haven't visited it yet, T means we have visited it.

Then for the new edge set E' that connects vertices in V', it includes the following:

- For every vertex $v' \in U$, we run a Dijkstra in G starting from v'. Then for every $u' \in U' \backslash v'$, our Dijkstra could tell us the $\operatorname{dist}(v', u')$ and the corresponding path p goes from v' to u' that has minimum distance. if $\operatorname{dist}(v', u') \leq R$, there are two cases:
  - If there is a vertex x in the path p (notice this considers the start and the end) such that $x \in X$, then we add new edge $e' = (v', g) \to (u', T)$, where $v', u' \in V'$ and $g \in \{F, T\}$.
  - Otherwise, we add new edge $e' = (v', g) \to (u', g)$, where $v', u' \in V'$ and $g \in \{F, T\}$.

Since for every vertex in the V', we need to run Dijkstra in G, this will result in $O(V \cdot \operatorname{Dijkstra}(V, E))$ time complexity to construct the E'.

Then let the new graph be $G' = (V', E')$. The original problem on G then is equivalent of finding the minimum distance from the (s, F) to (t, T). This could be done by one Dijkstra on the new G'. We know that the new graph has $|V'| = 2(|Y| + 2) = O(V)$ and that the upper bound for the $|E'|$ is $O(V^2)$

So, in the end, the time complexity is just $O(V \cdot \operatorname{Dijkstra}(V, E)) + \operatorname{Dijkstra}(V', E')$. We know that $\operatorname{Dijkstra}(V, E)$ is $O(V \log V + E)$ so the time complexity is just $O(V^2 \log V + VE) + O(V' \log V' + E')$ = $O(V^2 \log V + VE)$. (The last $\operatorname{Dijkstra}(V', E')$ is not the dominate term in the final time complexity)