Problem 1

(a)

In cylindrical coordinate, we have

$$\nabla^2 V(\vec{r}) = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial f}{\partial s}) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

(b)

We write $V(\vec{r}) = V(s, \theta) = S(s)\Theta(\theta)$, and then we have

$$\nabla^2 V(\vec{r}) = \frac{1}{s}\Theta(\theta)\frac{\partial}{\partial s}(s\frac{\partial S}{\partial s}) + \frac{1}{s^2}S(s)\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

divide both side by $S(s)\Theta(\theta)$, we get

$$\frac{1}{s \cdot S(s)}\frac{\partial}{\partial s}(s\frac{\partial S}{\partial s}) + \frac{1}{s^2\Theta(\theta)}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

we could multiply both side by s^2,

$$\frac{s}{S(s)}\frac{\partial}{\partial s}(s\frac{\partial S}{\partial s}) + \frac{1}{\Theta(\theta)}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

and thus we get the equation of s part and θ part:

$$\frac{s}{S(s)}\frac{\partial}{\partial s}(s\frac{\partial S}{\partial s}) \quad \text{and} \quad \frac{1}{\Theta(\theta)}\frac{\partial^2 \Theta}{\partial \theta^2}$$

We set the s part to positive the attenuation of force from the sphere surface, and thus the θ should be negative. Let's set the s part constant be m^2 and θ part constant to be $-m^2$.

(c)

We transform the part into

$$s\frac{\partial S}{\partial s} + s^2\frac{\partial^2 S}{\partial s^2} = m^2 S(s)$$

this is

$$s^2 S'' + sS' - m^2 S = 0$$

Try solution $S = C_m s^m$. Then, we get

$$C_m s^m(m(m-1) + m - m^2) = 0$$
$$s^m \cdot 0 = 0$$

which satisfy the equation. So, we get:

$$S_m = C_m s^m$$

(d)

We solve the θ part first:

$$\frac{\partial^2 \Theta}{\partial \theta^2} = -m^2 \Theta(\theta)$$

and we have

$$\Theta(\theta) = P_m \cos m\theta + Q_m \sin m\theta$$

we need to impose the boundary condition $\Theta(\theta = 0) = \Theta(\theta = 2n\pi)$. This means

$$\Theta(0) = \Theta(2n\pi) = P_m$$

and we find that $\cos m \cdot 2n\pi = 1$, and this indicates that $m \cdot 2n\pi$ needs to be a multiple of 2π for all n. That means m must also be an integer. This means the solution in (c) will always have integer exponent (the r^m). This also means the number of solution (with respect to different m) is countably infinite, and we could express them using a summation.

(e)

Combine the solution we have

$$V(s, \theta) = \sum_{m=-\infty}^{\infty} s^m (A_m \cos m\theta + B_m \sin m\theta)$$

where $A_m = P_m C_m$ and $B_m = Q_m C_m$, we merge them together for two constant is already enough. We notice that outside the cylinder, we should get $V(\infty, \theta) = 0$, and that means any $m > 0$ term needs to be 0. Therefore, we get

$$V(s, \theta) = \sum_{m=-\infty}^{0} s^m (A_m \cos m\theta + B_m \sin m\theta)$$

(f)

Similarly, in this case we are inside the cylinder, we expect the V is close to the boundary condition as r increase, so all $m < 0$ term needs to be 0. We get

$$V(s, \theta) = \sum_{m=0}^{\infty} s^m (A_m \cos m\theta + B_m \sin m\theta)$$

(g)

we consider the boundary condition when the $s = R$, and $V(\theta) = V_0(\sin^2(\theta) - \cos^2(\theta))$. Then, we get

$$V_{\text{in}} = \sum_{m=0}^{\infty} R^m (A_m \cos m\theta + B_m \sin m\theta)$$

and we notice that this is similar to Fourier series (only be a constant). Then we could do

$$
\begin{aligned}
A_m &= \frac{1}{R^m}\frac{1}{\pi}\int_{-\pi}^{\pi} V_{\text{in}}\cos(m\theta)\mathrm{d}\theta \\
B_m &= \frac{1}{R^m}\frac{1}{\pi}\int_{-\pi}^{\pi} V_{\text{in}}\sin(m\theta)\mathrm{d}\theta
\end{aligned}
$$

and we get

$$
\begin{aligned}
A_m &= \frac{1}{\pi}\frac{V_0}{R^m}\int_{-\pi}^{\pi}(\sin^2(\theta)-\cos^2(\theta))\cos(m\theta)\mathrm{d}\theta = -\frac{1}{\pi}\frac{V_0}{R^m}\frac{2m\sin(\pi m)}{m^2-4} \\
B_m &= \frac{1}{\pi}\frac{V_0}{R^m}\int_{-\pi}^{\pi}(\sin^2(\theta)-\cos^2(\theta))\sin(m\theta)\mathrm{d}\theta = 0
\end{aligned}
$$

and thus

$$
V_{\text{in}}(s,\theta) = -\frac{V_0}{\pi}\sum_{m=0}^{\infty}\frac{s^m}{R^m}\frac{2m\sin(\pi m)}{m^2-4}\cos m\theta
$$

do the same for V_{out}:

$$
V_{\text{out}} = \sum_{m=-\infty}^{0} R^m(A_m\cos m\theta + B_m\sin m\theta)
$$

It should have the same coefficient, and yield the similar result (since only the summation start and end is different):

$$
V_{\text{out}} = -\frac{V_0}{\pi}\sum_{m=-\infty}^{0}\frac{s^m}{R^m}\frac{2m\sin(\pi m)}{m^2-4}\cos(m\theta)
$$

We could do some simplification by let $n=-m$. Then $\cos(n\theta)=\cos(m\theta)$ and $\sin(n\pi) = -\sin(m\pi)$.

$$
\begin{aligned}
V_{\text{out}} &= -\frac{V_0}{\pi}\sum_{n=0}^{\infty}\frac{R^n}{s^n}\frac{2(-n)\cdot(-\sin(n\pi))}{n^2-4}\cos(n\theta) \\
&= -\frac{V_0}{\pi}\sum_{m=0}^{\infty}\frac{R^m}{s^m}\frac{2m\sin(m\pi)}{m^2-4}\cos(m\theta)
\end{aligned}
$$

Problem 2

We know that the V has the form of (when not depend on ϕ)

$$
V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta)
$$

Inside

Since the term $\frac{B_l}{r^{l+1}}$ vanish as r increase, this term could not be present when considering V insider the sphere. Therefore, set $B_l = 0$, and we get

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

We know that

$$\int_{-1}^{1} \mathrm{d}x P_l(x) P_l'(x) = \frac{2}{2l+1}\delta_{ll'}$$

Let $x = \cos\theta$, then $\mathrm{d}x = -\sin\theta \mathrm{d}\theta$, and thus

$$\int_{-1}^{1} \mathrm{d}x P_l(x) P_l'(x) = \int_{\pi}^{0} -P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta \mathrm{d}\theta = \int_{0}^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta \mathrm{d}\theta$$

and thus

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\int_0^{\pi} \mathrm{d}\theta \sin\theta P_m(\cos\theta) V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l \int_0^{\pi} \mathrm{d}\theta P_m(\cos\theta) P_l(\cos\theta)\sin\theta$$

$$\int_0^{\pi} \mathrm{d}\theta \sin\theta P_m(\cos\theta) V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l \frac{2}{2l+1}\delta_{lm}$$

$$A_m r^m \frac{2}{2m+1} = \int_0^{\pi} \mathrm{d}\theta \sin\theta P_m(\cos\theta) V(r,\theta)$$

since this happened on the boundary of the sphere, that is $r = R$, then

$$A_m = \frac{2m+1}{2R^m}\int_0^{\pi} V(\theta) P_m(\cos\theta)\sin\theta \mathrm{d}\theta$$

Plug in the $V(\theta)$, we get

$$A_m = \frac{2m+1}{2R^m}\int_0^{\pi} V_0\cos^3(\theta) P_m(\cos\theta)\sin\theta \mathrm{d}\theta$$

Let $x = \cos\theta$, and $\mathrm{d}x = -\sin\theta \mathrm{d}\theta$, and thus

$$A_m = \frac{2m+1}{2R^m} V_0 \int_{-1}^{1} P_m(x) x^3 \mathrm{d}x$$

and thus we have

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(\frac{2l+1}{2R^l} V_0 \int_{-1}^{1} P_l(x) x^3 \mathrm{d}x\right) r^l P_l(\cos\theta)$$

Outside

Since we know that $V(\infty) = 0$, so the term $A_l r^l$ must be 0, and thus we set $A_l = 0$. We get

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Similarly, following the same process in (a), we have

$$\int_0^{\pi} \mathrm{d}\theta \sin\theta P_m(\cos\theta) V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} \frac{2}{2l+1} \delta_{lm}$$

and thus

$$\frac{B_m}{r^{m+1}} \frac{2}{2m+1} = \int_0^{\pi} \mathrm{d}\theta \sin\theta P_m(\cos\theta) V(r,\theta)$$

and thus

$$B_m = \frac{2m+1}{2} R^{m+1} V_0 \int_{-1}^{1} P_m(x) x^3 \mathrm{d}x$$

therefore

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \left(\frac{2l+1}{2} R^{l+1} V_0 \int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta)$$

Surface Charge Density

We know that

$$[E_{\text{out}}^{\perp}(r,\theta) - E_{\text{in}}^{\perp}(r,\theta)]_{r=R} = \frac{\sigma(\theta)}{\epsilon_0}$$

and in this case it becomes

$$\left[\frac{\partial V_{\text{out}}(r,\theta)}{\partial r} - \frac{\partial V_{\text{in}}(r,\theta)}{\partial r} \right]_{r=R} = -\frac{\sigma(\theta)}{\epsilon_0}$$

and the derivative for V_{out} and V_{in}

$$\begin{aligned}
V_{\text{out}}(r,\theta) &= \sum_{l=0}^{\infty} \left(\frac{2l+1}{2R^l} V_0 \int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) r^l P_l(\cos\theta) \\
\frac{\partial V_{\text{out}}(r,\theta)}{\partial r} &= \sum_{l=1}^{\infty} l \left(\frac{2l+1}{2R^l} V_0 \int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) r^{l-1} P_l(\cos\theta) \\
\left. \frac{\partial V(r,\theta)_{\text{out}}}{\partial r} \right|_{r=R} &= \sum_{l=1}^{\infty} (l(2l+1)) \frac{V_0}{2R} \left(\int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta) \\
&= \sum_{l=0}^{\infty} (l(2l+1)) \frac{V_0}{2R} \left(\int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta)
\end{aligned}$$

notice the change of start of summation ($l = 0$ / $l = 1$).

and that

$$V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \left(\frac{2l+1}{2} R^{l+1} V_0 \int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta)$$

$$\frac{\partial V_{\text{in}}(r,\theta)}{\partial r} = \sum_{l=0}^{\infty} -(l+1) \frac{1}{r^{l+2}} \left(\frac{2l+1}{2} R^{l+1} V_0 \int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta)$$

$$\left. \frac{\partial V(r,\theta)_{\text{in}}}{\partial r} \right|_{r=R} = \sum_{l=0}^{\infty} -(l+1)(2l+1) \frac{V_0}{2R} \left(\int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta)$$

and thus we get

$$\left[\frac{\partial V_{\text{out}}(r,\theta)}{\partial r} - \frac{\partial V_{\text{in}}(r,\theta)}{\partial r} \right]_{r=R} = \sum_{l=0}^{\infty} (-l(2l+1) - (l+1)(2l+1)) \frac{V_0}{2R} \left(\int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta)$$

$$= \sum_{l=0}^{\infty} -(2l+1)^2 \frac{V_0}{2R} \left(\int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta)$$

which means

$$\sigma(\theta) = \epsilon_0 \sum_{l=0}^{\infty} (2l+1)^2 \frac{V_0}{2R} \left(\int_{-1}^{1} P_l(x) x^3 \mathrm{d}x \right) P_l(\cos\theta)$$