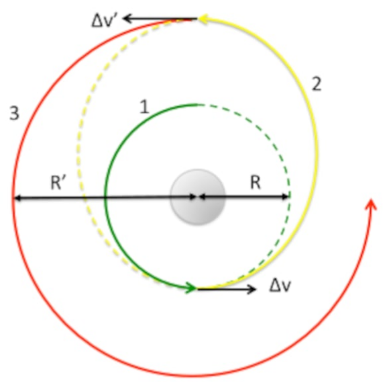
Question 1: Satellite orbiting Earth

> A satellite is in a circular orbit (label 1) around the Earth (mass M) at a radius R and with speed $v_{\text{initial}} = \sqrt{GM/R}$. We wish to bring the satellite to a higher orbit (label 3) with radius $R' > R$ where it has the speed $v_{\text{final}} = \sqrt{GM/R'}$. We will do this via an elliptical *Hohmann* transfer orbit (label 2). Therefore, the satellite applies an impulse (giving a change in the speed, Δv) at $t = 0$ which puts it into the transfer orbit (label 2). When the satellite arrives on its new circular orbit with radius R', it is at the apogee of the elliptical transfer orbit. It then applies an additional impulse $\sim \Delta v'$, increasing its speed again, to enter the final circular orbit (label 3).
>
> All answers should in terms M, G, R, R'



(a)

> What is the angular momentum per unit mass, ℓ, and the energy per unit mass e, of the transfer orbit?
>
> Hint: recall the energy at the perigee and apogee of an ellipse.

From Lecture notes 9, we know that

$$a = \frac{(r_{\max} + r_{\min})}{2} = \frac{GM}{2|e|}$$

thus

$$R' + R = \frac{GM}{|e|}$$

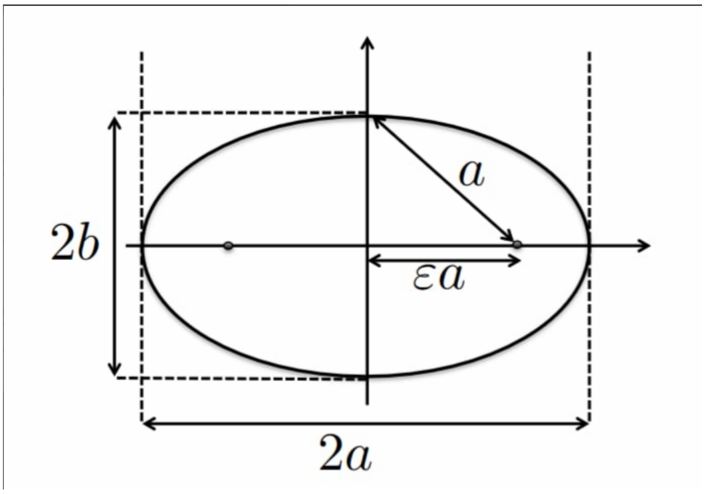
$$|e| = \frac{GM}{R' + R}$$

and since this is an elliptic orbit, we know that $e < 0$, thus

$$\boxed{e = -\frac{GM}{R' + R}} \tag{1}$$

we also know that (using some geometry)

$$b = a\sqrt{(1 - \varepsilon^2)} = \frac{\ell}{\sqrt{2|e|}}$$



Since the planet is one focus of the ellipse, we thus now that $\varepsilon a = \frac{R' - R}{2}$

and thus

$$\varepsilon \frac{R' + R}{2} = \frac{R' - R}{2}$$

$$\varepsilon = \frac{R' - R}{R' + R}$$

and

$$
\begin{aligned}
\ell &= a\sqrt{2|e|(1-\varepsilon^2)} \\
&= \frac{GM}{2|e|}\sqrt{2|e|(1-\varepsilon^2)} \\
&= GM\sqrt{\frac{1}{2}\frac{(1-\varepsilon^2)}{|e|}} \\
&= GM\sqrt{\frac{1}{2}\frac{R'+R}{GM}\cdot\left(1-\left(\frac{R'-R}{R'+R}\right)^2\right)} \\
&= GM\sqrt{\frac{1}{2}\frac{R'+R}{GM}\cdot\frac{(R'+R)^2-(R'-R)^2}{(R'+R)^2}} \\
&= GM\sqrt{\frac{1}{2GM}\cdot\frac{4R'R}{R'+R}}
\end{aligned}
$$

that is

$$
\boxed{\ell = \sqrt{GM\frac{2R'R}{R'+R}}} \tag{2}
$$

(b)

> What are the needed changes in speeds, Δv and $\Delta v'$

when changing from v_{initial} to v, the total energy of the elliptic orbit is

$$
\begin{aligned}
e &= \Phi + t \\
-\frac{GM}{R'+R} &= -\frac{GM}{R} + \frac{1}{2}v_i^2 \\
v_i &= \sqrt{2GM(\frac{1}{R} - \frac{1}{R'+R})}
\end{aligned}
$$

so

$$
\Delta v = v_i - v_{\text{initial}} = \sqrt{2GM(\frac{1}{R} - \frac{1}{R'+R})} - \sqrt{\frac{GM}{R}}
$$

and when changing form v to v_{final}, then total energy of the elliptic orbit is

$$e = \Phi + t$$
$$-\frac{GM}{R' + R} = -\frac{GM}{R'} + \frac{1}{2}v_f^2$$
$$v_f = \sqrt{2GM(\frac{1}{R'} - \frac{1}{R' + R})}$$

so

$$\Delta v' = v_{\text{final}} - v_f = \sqrt{\frac{GM}{R'}} - \sqrt{2GM(\frac{1}{R'} - \frac{1}{R' + R})}$$

Question 2: Elliptic orbits

> According to Kepler's first law, the orbit of a small mass m around a larger mass $M \gg m$ is described by planar ellipses, with M being at a focus point, with M being at a focus point of the ellipse. Their motion can be described by the conservation of energy and the conservation of angular momentum. Introducing the energy per unit mass, $e = E/m$, and the angular momentum per unit mass $\vec{\ell} = \vec{L}/m$, the conservation laws are
>
> $$\vec{\ell} = \vec{r} \times \vec{v} \qquad e = \frac{1}{2}v^2 - \frac{GM}{r}$$
>
> The apogee r_a and perigee r_p, corresponding to the maximum and minimum distance from the origin at M, are given by
>
> $$r_{\text{p,a}} = \frac{\ell}{v_{\text{p,a}}} \quad \text{with} \quad v_{\text{p,a}} = \frac{GM}{\ell} \pm \sqrt{\left(\frac{GM}{\ell}\right)^2 + 2e}$$

(a)

> Use the conservation laws to explain whether the following statements are true, false, or not determinable. Take the "midway" point to be at either midpoint of the orbit between apogee and perigee

we notice that energy and angular momentum is conserved, and thus

- $e_{\text{a}} = e_{\text{p}}$, correct
- $\ell_a > \ell_{\text{midway}}$, incorrect

we further notice that this is an elliptic orbit, and thus

- $e < 0$, correct

we could rewrite the energy per mass equation

$$v = \sqrt{2\left(e + \frac{GM}{r}\right)}$$

and thus

- $v_{\mathrm{p}} > v_{\mathrm{a}}$, correct, since $r_{\mathrm{p}} < r_{\mathrm{a}}$
- $v_{\mathrm{p}} > v_{\mathrm{midway}}$, correct, since $r_{\mathrm{p}} > r_{\mathrm{midway}}$

Also, at pedigree and apogee, the $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = 0$, and thus $\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} = r\dot{\phi}\hat{\phi}$. The $\vec{r} \cdot \vec{v} = 0$, and they are perpendicular, and thus the angle θ between $\vec{r}$ and $\vec{v}$ is $\frac{1}{2}\pi$. Thus

- $v_{\mathrm{a}} r_{\mathrm{a}} = v_{\mathrm{p}} r_{\mathrm{p}}$, correct, since at pedigree and apogee, $\ell_{\mathrm{a}} = v_{\mathrm{a}} r_{\mathrm{a}} \sin\theta = v_{\mathrm{a}} r_{\mathrm{a}}$, and $\ell_{\mathrm{p}} = v_{\mathrm{p}} r_{\mathrm{p}} \sin\theta = v_{\mathrm{p}} r_{\mathrm{p}}$ and that angular momentum is conserved.
- $v_{\mathrm{a}} r_{\mathrm{a}} > v_{\mathrm{midway}} r_{\mathrm{midway}}$, incorrect, since at some midway point,

$$\ell_{\mathrm{midway}} = \ell_{\mathrm{a}}$$
$$v_{\mathrm{midway}} r_{\mathrm{midway}} \sin\theta_{\mathrm{midway}} = v_{\mathrm{a}} r_{\mathrm{a}}$$

and only when at pedigree and apogee, the $\vec{r} \cdot \vec{v} = 0$, and the $\theta = \frac{1}{2}\pi$, thus

$$\theta_{\mathrm{midway}} \neq \frac{1}{2}\pi$$
$$\sin\theta_{\mathrm{midway}} < 1$$
$$v_{\mathrm{midway}} r_{\mathrm{midway}} \sin\theta_{\mathrm{midway}} < v_{\mathrm{midway}} r_{\mathrm{midway}}$$
$$v_a r_a < v_{\mathrm{midway}} r_{\mathrm{midway}}$$

Last

- $\frac{v_a^2}{r_a} = \frac{GM}{r_a^2}$ happens only when the orbit is circular. Well, it depends on our definition of a "elliptic orbit", if we consider "elliptic orbit" is a general case of "circular orbit", then there is a chance (not determinable) for the statements to be true. If "elliptic orbit" excludes the "circular orbit", then the statement is incorrect. To show that the statement is true iff the orbit is circular, we have

$$\frac{v_\mathrm{a}^2}{r_\mathrm{a}} = \frac{GM}{r_\mathrm{a}^2}$$
$$v_\mathrm{a} = \sqrt{\frac{GM}{r_\mathrm{a}}}$$
$$e = \frac{1}{2}\frac{GM}{r_\mathrm{a}} - \frac{GM}{r_\mathrm{a}}$$
$$e = -\frac{1}{2}\frac{GM}{r_\mathrm{a}}$$

  from e, we could see the orbit will be circular if it satisfy the statement.

(b)

> What speed of the test particle would be required for the orbit to be circular with a radius R

as we solved in previous question, when $v = \sqrt{\frac{GM}{r}}$, the $e = -\frac{GM}{2r}$ and the orbit will become circular.

Question 3: Gravitational field of a star

> Consider a star of mass M whose density $\rho(r)$ is a function of the distance r from the center of the star and it is given by
>
> $$\rho = \frac{Ma^2}{2\pi r(r^2+a^2)^2}, \quad \text{with} \quad 0 \le r \le \infty \quad \text{and} \quad a = \text{const}$$

(a)

> Show that the gravitational potential of the star is
>
> $$\Phi = -\frac{GM}{2a}\left(\pi - 2\arctan\left(\frac{r}{a}\right)\right)$$

According to the formula (28) in the Lecture notes 7, the gravitational potential follows the equation: (given that the density is symmetric in ϕ and θ)

$$\mathrm{d}\Phi = \begin{cases} -4\pi G\frac{\rho}{r}u^2\mathrm{d}u & \text{for } r > u \\ -4\pi G\rho u\mathrm{d}u & \text{for } r < u \end{cases}$$

Thus, the total potential should be

$$\begin{aligned}\Phi(r) &= -4\pi G\left(\int_0^r \frac{\rho}{r}u^2\mathrm{d}u + \int_r^\infty \rho u\mathrm{d}u\right)\\ &= -4\pi G\left(\int_0^r \frac{Mua^2}{2\pi r(u^2+a^2)^2}\mathrm{d}u + \int_r^\infty \frac{Ma^2}{2\pi(u^2+a^2)^2}\mathrm{d}u\right)\\ &= -4\pi G\cdot\frac{Ma^2}{2\pi}\left(\int_0^r \frac{u}{r(u^2+a^2)^2}\mathrm{d}u + \int_r^\infty \frac{1}{(u^2+a^2)^2}\mathrm{d}u\right)\end{aligned}$$

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In[29]:= Simplify[-4*Pi*G*M*a^2/(2*Pi)
           (Integrate[u/(r*(u^2+a^2)^2), {u, 0, r}] +
             Integrate[1/(u^2+a^2)^2, {u, r, Infinity}])]
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Out[29]= $-\frac{1}{2}\sqrt{\frac{1}{a^2}}\,G\,M\,\pi + \frac{G\,M\,\mathrm{ArcTan}\left[\frac{r}{a}\right]}{a}$ if condition

which is just

$$\Phi = -\frac{GM}{2a}\left(\pi - 2\arctan\left(\frac{r}{a}\right)\right)$$

(b)

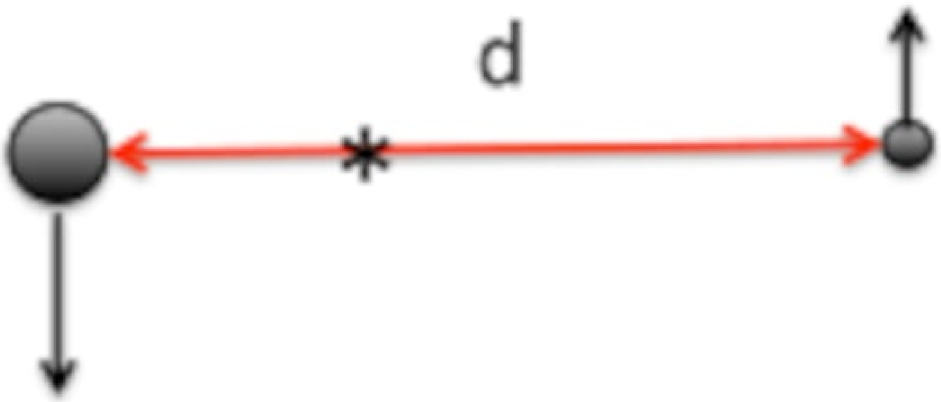
> Find the gravitational field $\vec{g}(r)$ of the stars

Since we know the potential, the $\vec{g}(r)$ is simply

$$\vec{g}(r) = -\nabla\Phi = -\frac{GM}{a}\frac{a^2}{r^2+a^2}\cdot\frac{1}{a}\hat{r} = -\frac{GM}{r^2+a^2}\hat{r}$$

Question 4: Orbiting black holes

> Two black holes of masses $m_1 = 10M_{\text{solar}}$ and $m_2 = 5M_{\text{solar}}$ are in a circular orbit around each other. Their separation is $d = 10000\text{km}$, They each have a speed relative to the joint center of mass (indicated in the figure by a $*$).
>
> Note: a solar mass is $M_{\text{solar}} = 2 \cdot 10^{30}\text{kg}$ and Newton's gravitational constant is $G = 6.67 \cdot 10^{-11}\text{Nm}^2/\text{kg}^2$



(a)

> What is the ratio of the black holes' speed relative to the center of mass? Which one is moving faster?
>
> Hint: consider the total momentum of the system

The total momentum in the system $L = 0$, and since they are in a circular orbit around each other:

$$m_1 v_1 = m_2 v_2$$
$$\frac{v_1}{v_2} = \frac{m_2}{m_1} = \frac{1}{2}$$

The $v_2 > v_1$ and that the smaller black hole is moving faster.

(b)

> What is the period P of their orbit in seconds?

The position of the center of the mass is

$$\vec{C} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{2}{3}\vec{r}_1 + \frac{1}{3}\vec{r}_2 = \vec{r}_1 + \frac{1}{3}(\vec{r}_2 - \vec{r}_1)$$

That is, the distance between center of mass and m_1 is $d_1 = \frac{1}{3}d$ and distance between center of mass and m_2 is $d_2 = \frac{2}{3}d$.

The orbit is circular, and that means

$$\frac{v^2}{r} = \frac{F}{m}$$

Thus

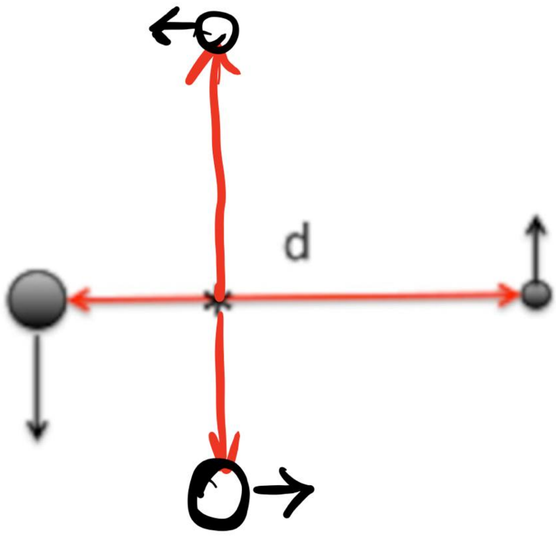
$$v_1 = \sqrt{\frac{Gm_1m_2d_1}{m_1d^2}} = \sqrt{\frac{5}{3}\frac{GM_{\text{solar}}}{d}} \approx 4.715 \cdot 10^6 \text{m/s}$$
$$v_2 = \sqrt{\frac{Gm_1m_2d_2}{m_2d^2}} = \sqrt{\frac{20}{3}\frac{GM_{\text{solar}}}{d}} \approx 9.430 \cdot 10^6 \text{m/s}$$

and thus the period P is

$$P_1 = \frac{2\pi d_1}{v_1} = \frac{2\pi}{3}\frac{d}{v_1} \approx 4.442\text{s}$$
$$P_2 = \frac{2\pi d_2}{v_2} = \frac{4\pi}{3}\frac{d}{v_2} \approx 4.442\text{s}$$

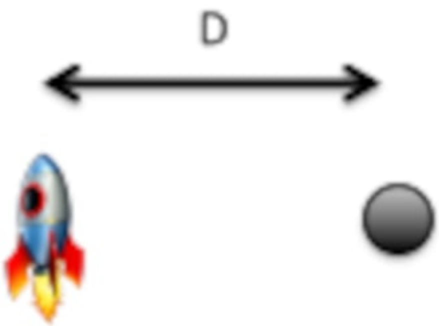
(c)

> Where is each black hole after $t = \frac{1}{4}P$ later than the time is the figure, Indicate these positions in the figure.



Question 5: Rocket near a black hole

> A spaceship is in a circular orbit around a black hole of mass M. The orbit has a radius D.
>
> Note: a solar mass is $M_{\text{solar}} = 2 \cdot 10^{30}\text{kg}$ and Newton's gravitational constant is $G = 6.67 \cdot 10^{-11}\text{Nm}^2/\text{kg}^2$



(a)

> Determine the spaceship's acceleration in terms of G, M, and D.

$$a = \frac{F}{m} = \frac{GMm}{D^2 m} = \frac{GM}{D^2}$$

The direction is towards the center of the black hole.

(b)

> What is the period P of the spaceship's orbit?

Since it's a circular orbit,

$$\frac{v^2}{D} = a$$

$$v = \sqrt{\frac{GM}{D}}$$

and

$$P = \frac{2\pi D}{v} = 2\pi\sqrt{\frac{D^3}{GM}}$$

(c)

> Evaluate the period P numerically for $M = 5M_{\text{solar}}$ and $D = 5000\text{km}$

$$P = 2\pi\sqrt{\frac{D^3}{GM}} = 2 \cdot 3.1415 \cdot \sqrt{\frac{(5000000)^3}{6.67 \cdot 10^{-11} \cdot 5 \cdot 2 \cdot 10^{30}}} \approx 2.720\text{s}$$

(d)

> What is the tidal effect on a particle floating in the spaceship? That is, given a particle at a distance $h \ll D$ to the right of the spaceship's center of mass, what is the particle's acceleration relative to the ship's center of mass? Evaluate this numerically for $D = 5000\text{km}$ and $h = 1\text{m}$

According the to the formula (8) of Lecture Notes 10

$$hg' = \frac{2GMh}{D^3} \approx 10.672\text{m/s}^2$$

which is little bigger than a g, not very big effect.