Question 1

> Consider an ensemble in which 25% of the systems are known to be prepared in the state
>
> $$|\phi\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle$$
>
> and 75% of the systems are prepared in the state
>
> $$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

(a)

> Find the density operator for each of these state, and show they are pure states, If measurements are made on systems in each of these states, what are the probabilities they are found to be in state $|0\rangle$, $|1\rangle$, respectively?

$$\rho_1 = \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} \quad \text{and} \quad \rho_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

and we could find that

$$\begin{aligned} \operatorname{Tr}(\rho_1^2) &= \operatorname{Tr}\begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} = 1 \\ \operatorname{Tr}(\rho_2^2) &= \operatorname{Tr}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = 1 \end{aligned}$$

which means they are pure state. The probablity they found is

$$\begin{aligned} \mathrm{Pr}_{0,|0\rangle} &= \operatorname{Tr}(P_0\rho_0) = \operatorname{Tr}\begin{pmatrix} 4/5 & 2/5 \\ 0 & 0 \end{pmatrix} = 4/5 \\ \mathrm{Pr}_{0,|1\rangle} &= \operatorname{Tr}(P_1\rho_0) = \operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 2/5 & 1/5 \end{pmatrix} = 1/5 \\ \mathrm{Pr}_{1,|0\rangle} &= \operatorname{Tr}(P_0\rho_0) = \operatorname{Tr}\begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix} = 1/2 \\ \mathrm{Pr}_{1,|1\rangle} &= \operatorname{Tr}(P_0\rho_0) = \operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix} = 1/2 \end{aligned}$$

(b)

> Determine the density operator of the ensamble.

$$\rho = \frac{1}{4}\rho_1 + \frac{1}{3}\rho_2 = \begin{pmatrix} 23/40 & 19/40 \\ 19/40 & 17/40 \end{pmatrix}$$

(c)

> Show that $\operatorname{Tr}(\rho) = 1$

$$\operatorname{Tr}(\rho) = 23/40 + 17/40 = 1$$

(d)

> A measurement of Z is made on a memeber drawn from the ensemble. What are the probabilities it is found to be in state $|0\rangle$ and $|1\rangle$, respectively

$$\mathrm{Pr}_0 = \mathrm{Tr}(P_0\rho) = \mathrm{Tr}\begin{pmatrix} 23/40 & 19/40 \\ 0 & 0 \end{pmatrix} = 23/40$$

$$\mathrm{Pr}_1 = \mathrm{Tr}(P_1\rho) = \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 19/40 & 17/40 \end{pmatrix} = 17/40$$

Problem 2

> Consider the ensemble described in McMahon 5.7. Show that the density matrix for the ensemble is Hermitian and that its eigenvalues are all positive.

$$\rho^\dagger = \begin{pmatrix} 23/40 & 19/40 \\ 19/40 & 17/40 \end{pmatrix} = \rho$$

the eigenvalues are

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In[3]:= rho = {{23 / 40, 19 / 40}, {19 / 40, 17 / 40}}
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Out[3]= $\left\{\left\{\frac{23}{40}, \frac{19}{40}\right\}, \left\{\frac{19}{40}, \frac{17}{40}\right\}\right\}$

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In[4]:= Eigenvalues[rho]
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Out[4]= $\left\{\frac{1}{40}\left(20+\sqrt{370}\right), \frac{1}{40}\left(20-\sqrt{370}\right)\right\}$

they are positive

Problem 3

> Again considering the ensemble from McMahon 5.7, and using McMahon Equations (5.9) and (5.11), determine the probability of measuring the $|0\rangle$ state (i.e., using the appropriate projection operator) for the ensemble. Determine the density matrix for the ensemble following the measurement.

$$\mathrm{Pr}_0 = \mathrm{Tr}(P_0\rho) = \mathrm{Tr}\begin{pmatrix} 23/40 & 19/40 \\ 0 & 0 \end{pmatrix} = 23/40$$

$$\rho = \frac{P_0\rho P_0}{\mathrm{Tr}(P_0\rho)} = \frac{40}{23}\begin{pmatrix} 23/40 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$$

Question 4

> Suppose that we have an ensemble with 60% of the states prepared as
>
> $$|a\rangle = \frac{\sqrt{2}}{\sqrt{5}}|+\rangle - \frac{\sqrt{3}}{\sqrt{5}}|-\rangle$$
>
> and 40% of the states are prepared as
>
> $$|b\rangle = \frac{\sqrt{5}}{\sqrt{8}}|+\rangle + \frac{\sqrt{3}}{\sqrt{8}}|-\rangle$$
>
> A member is drawn from the ensemble. What is the probability that measurement finds it in the $|0\rangle$ state?

$$\rho_\pm = \frac{3}{5}\begin{pmatrix} 2/5 & -\sqrt{6}/5 \\ -\sqrt{6}/5 & 3/5 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 5/8 & \sqrt{15}/8 \\ \sqrt{15}/8 & 3/8 \end{pmatrix} = \begin{pmatrix} 49/100 & \frac{\sqrt{15}}{20} - \frac{3\sqrt{6}}{25} \\ \frac{\sqrt{15}}{20} - \frac{3\sqrt{6}}{25} & 51/100 \end{pmatrix}$$

Using mathematica, we could calculate the:

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In[35]:= P.H.rho.H
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$$\text{Out[35]=} \left\{\left\{\frac{\frac{49}{100\sqrt{2}} + \frac{\frac{\sqrt{\frac{3}{5}}}{4} - \frac{3\sqrt{6}}{25}}{\sqrt{2}}}{\sqrt{2}} + \frac{\frac{51}{100\sqrt{2}} + \frac{\frac{\sqrt{\frac{3}{5}}}{4} - \frac{3\sqrt{6}}{25}}{\sqrt{2}}}{\sqrt{2}},\ \frac{\frac{49}{100\sqrt{2}} + \frac{\frac{\sqrt{\frac{3}{5}}}{4} - \frac{3\sqrt{6}}{25}}{\sqrt{2}}}{\sqrt{2}} - \frac{\frac{51}{100\sqrt{2}} + \frac{\frac{\sqrt{\frac{3}{5}}}{4} - \frac{3\sqrt{6}}{25}}{\sqrt{2}}}{\sqrt{2}}\right\}, \{0, 0\}\right\}$$

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In[34]:= Simplify[Tr[P.H.rho.H]]
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$$\text{Out[34]=} \frac{1}{100}\left(50 - 12\sqrt{6} + 5\sqrt{15}\right)$$

$$\mathrm{Pr}_0 = \mathrm{Tr}(P_0 H \rho_\pm H) = \frac{1}{2} + \frac{\sqrt{15}}{20} - \frac{3\sqrt{6}}{25}$$

Problem 5

> Consider a simple ensemble of two-state system $(\langle a|b\rangle = \delta_{ab})$
>
> $$|A\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle) \quad \text{and} \quad |B\rangle = \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle)$$
>
> each contributing 50% to the whole. Calculate the density matrix ρ for this ensemble. What is $\mathrm{Tr}(\rho)$? What is special about this density matrix

$$\begin{aligned} \rho &= \frac{1}{2}|A\rangle\langle A| + \frac{1}{2}|B\rangle\langle B| \\ &= \frac{1}{4}(|a\rangle\langle a| + |a\rangle\langle b| + |b\rangle\langle a| + |b\rangle\langle b|) + \frac{1}{4}(|a\rangle\langle a| - |a\rangle\langle b| - |b\rangle\langle a| + |b\rangle\langle b|) \\ &= \frac{1}{2}|a\rangle\langle a| + |b\rangle\langle b| \end{aligned}$$

$$\mathrm{Tr}(\rho) = 1$$

We could actually find that

$$\rho = \frac{1}{2}I$$

which means the ensemble is a completely mixed state.