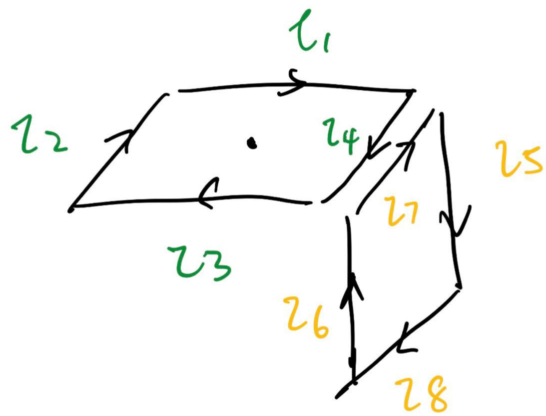
Problem 1

(a)



We could divide this current loop into two smaller loops as the **Example 5.13** in textbook does. Therefore, this two smaller loops gives the magnetic dipole moment of

$$\vec{m} = -IL^2\hat{y} - IL^2\hat{z}$$

(b)

we know that

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{1}{|r - r'|}\mathrm{d}\vec{l}'$$

At the point $a = (0, L/2, L)$, we see that if we split the current as shown above (we introduced l_5 and l_7 but they cancel each other so it's okay), the l_1, l_2, l_3, l_4 will cancel each other due to symmetry. Also, l_6 and l_5 will also cancel each other due to symmetry. We see that in this case only l_7 and l_8 contribute to the final $\vec{A}$. Therefore

$$\vec{A} = \frac{\mu_0 I}{4\pi}(\int_{-L/2}^{L/2} \frac{1}{\sqrt{\frac{L^2}{4} + L^2 + x^2}} - \frac{1}{\sqrt{\frac{L^2}{4} + x^2}}\mathrm{d}x)\hat{x}$$

We find that

$$\begin{aligned}\vec{A} &= \frac{\mu_0 I}{4\pi}\left(\operatorname{arcsinh}(\frac{2x}{\sqrt{5}L}) - \operatorname{arcsinh}(\frac{2x}{L})\right)\Bigg|_{-L/2}^{L/2}\hat{x} \\ &= \frac{\mu_0 I}{2\pi}\left(\operatorname{arcsinh}(\frac{1}{\sqrt{5}}) - \operatorname{arcsinh}(1)\right)\hat{x}\end{aligned}$$

(c)

Similar to (b), except this time we find that all $l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8$ cancelled each other due to symmetry. So

$$\vec{A} = \vec{0}$$

(d)

Point $c = (0, L, L/2)$. Similar to (b), except this time only l_2 and l_4 contribute to final $\vec{A}$. We see that (it's almost identical to (b) except the sign in the integral)

$$\vec{A} = \frac{\mu_0 I}{4\pi}(\int_{-L/2}^{L/2} -\frac{1}{\sqrt{\frac{L^2}{4} + L^2 + x^2}} + \frac{1}{\sqrt{\frac{L^2}{4} + x^2}} \mathrm{d}x)\hat{x}$$

and we get:

$$\begin{aligned}\vec{A} &= \frac{\mu_0 I}{4\pi}\left(-\mathrm{arcsinh}(\frac{2x}{\sqrt{5}L}) + \mathrm{arcsinh}(\frac{2x}{L})\right)\Bigg|_{-L/2}^{L/2} \hat{x} \\ &= \frac{\mu_0 I}{2\pi}\left(\mathrm{arcsinh}(1) - \mathrm{arcsinh}(\frac{1}{\sqrt{5}})\right)\hat{x}\end{aligned}$$

(e)

We could therefore calculate the vector potential $\vec{A}$ (given $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $z \to \infty$, and $z \gg x, z \gg y$)

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi}\frac{(-IL^2\hat{y} - IL^2\hat{z}) \times (x\hat{x} + y\hat{y} + z\hat{z})}{(x^2 + y^2 + z^2)^{3/2}} \\ &= \frac{\mu_0 I}{4\pi}\frac{L^2}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{z} - x\hat{y} + (y - z)\hat{x})\end{aligned}$$

and since $z \gg x, z \gg y$, we could do the simplification of the $r^3 = z^3$ instead of $r^3 = (x^2 + y^2 + z^2)^{3/2}$, and it becomes

$$\vec{A} = \frac{\mu_0 I}{4\pi}\frac{L^2}{z^3}(x\hat{z} - x\hat{y} + (y - z)\hat{x})$$

(f)

We want to the magnetic field at the points along the z-axis, so

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \vec{A} \\ &= \frac{\mu_0 I}{4\pi}L^2 \cdot \vec{\nabla} \times (\frac{y - z}{z^3}\hat{x} - \frac{x}{z^3}\hat{y} + \frac{x}{z^3}\hat{z}) \\ &= \frac{\mu_0 I}{4\pi}L^2(-\frac{3x}{z^4}\hat{x} + (\frac{2z - 3y}{z^4} - \frac{1}{z^3})\hat{y} + (-\frac{1}{z^3} - \frac{1}{z^3})\hat{z})\end{aligned}$$

Now set $x = y = 0$, we get

$$\vec{B} = \frac{\mu_0 I}{4\pi}L^2(\frac{1}{z^3}\hat{y} - \frac{2}{z^3}\hat{z})$$

Problem 2

(a)

From Wikipedia, the Bohr magneton is $\mu_B = 9.274 \cdot 10^{-24} J \cdot T^{-1}$, since the electron spin points to +z direction, the $m_s = +1/2$ and therefore

$$\begin{aligned}\vec{m} &= -2 \cdot 9.274 \cdot 10^{-24} \cdot \frac{1}{2}\hat{z} \\ &= -9.274 \cdot 10^{-24}\hat{z} = -\mu_B\hat{z}\end{aligned}$$

(b)

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{-9.274 \cdot 10^{-24}\hat{z} \times 0.1\hat{z}}{r^3} = \vec{0}$$

(c)

It's similar to part (b), vector potential at 0.2 nm and 0.5 nm is also $\vec{0}$.

(d)

we want to find the vector potential near the z-axis, so $z \gg x, z \gg y$ and we could use z^3 to replace r^3. Then,

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{-\mu_B\hat{z} \times (x\hat{x} + y\hat{y} + z\hat{z})}{z^3} = \frac{\mu_0}{4\pi}\frac{-\mu_B x\hat{y} + \mu_B y\hat{x}}{z^3}$$

Then,

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \vec{A} \\ &= \frac{\mu_0}{4\pi}\mu_B \cdot \vec{\nabla} \times \left(-\frac{x}{z^3}\hat{y} + \frac{y}{z^3}\hat{x}\right) \\ &= \frac{\mu_0}{4\pi}\mu_B \cdot (-3\frac{x}{z^4}\hat{x} - 3\frac{y}{z^4}\hat{y} + (-\frac{1}{z^3} - \frac{1}{z^3})\hat{z})\end{aligned}$$

set $x = y = 0$ so we get the B on z-axis

$$\vec{B} = -\frac{\mu_0}{2\pi}\frac{\mu_B}{z^3}\hat{z}$$

We see that

$$\begin{aligned}\vec{B}_{0.1} &= -2 \cdot 10^{-7}\frac{9.274 \cdot 10^{-24}}{10^{-30}}\hat{z} = -1.855T\hat{z} \\ \vec{B}_{0.2} &= -2 \cdot 10^{-7}\frac{9.274 \cdot 10^{-24}}{4 \cdot 10^{-30}}\hat{z} = 4.637 \cdot 10^{-1}T\hat{z} \\ \vec{B}_{0.5} &= -2 \cdot 10^{-7}\frac{9.274 \cdot 10^{-24}}{25 \cdot 10^{-30}}\hat{z} = 7.419 \cdot 10^{-2}T\hat{z}\end{aligned}$$

(e)

Similarly, we have

$$\vec{A}=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}=\frac{\mu_0}{4\pi}\frac{-\mu_B\hat{z}\times x\hat{x}}{x^3}=\frac{\mu_0}{4\pi}\frac{-\mu_B\hat{y}}{x^2}$$

and therefore

$$\vec{A}_{0.1}=10^{-7}\frac{-9.274\cdot 10^{-24}\hat{y}}{10^{-20}}=-9.274\cdot 10^{-11}Tm\hat{y}$$
$$\vec{A}_{0.2}=10^{-7}\frac{-9.274\cdot 10^{-24}\hat{y}}{4\cdot 10^{-20}}=-2.319\cdot 10^{-11}Tm\hat{y}$$
$$\vec{A}_{0.5}=10^{-7}\frac{-9.274\cdot 10^{-24}\hat{y}}{4\cdot 10^{-20}}=-3.710\cdot 10^{-12}Tm\hat{y}$$

(f)

$$\vec{B}=\frac{\mu_0}{4\pi}(\frac{3x\hat{x}(-\mu_B\hat{z}\cdot x\hat{x})}{x^5}-\frac{-\mu_B\hat{z}}{x^3})=\frac{\mu_0}{4\pi}\frac{\mu_B}{x^3}\hat{z}$$

and we see that

$$\vec{B}_{0.1}=10^{-7}\frac{9.274\cdot 10^{-24}}{10^{-30}}\hat{z}=9.274\cdot 10^{-1}T\hat{z}$$
$$\vec{B}_{0.2}=10^{-7}\frac{9.274\cdot 10^{-24}}{4\cdot 10^{-30}}\hat{z}=2.319\cdot 10^{-1}T\hat{z}$$
$$\vec{B}_{0.5}=10^{-7}\frac{9.274\cdot 10^{-24}}{25\cdot 10^{-30}}\hat{z}=3.710\cdot 10^{-2}T\hat{z}$$

(g)

When it's on the z-axis, we see that

$$U=-\vec{m}\cdot\vec{B}=-(-\mu_B\hat{z})\cdot-\frac{\mu_0}{2\pi}\frac{\mu_B}{z^3}\hat{z}=-\frac{\mu_0}{2\pi}\frac{\mu_B^2}{z^3}$$

Then we see

$$U_{0.1}=-2\cdot 10^{-7}\frac{(9.274\cdot 10^{-24})^2}{10^{-30}}\hat{z}=-1.720\cdot 10^{-23}T\hat{z}$$
$$U_{0.2}=-2\cdot 10^{-7}\frac{(9.274\cdot 10^{-24})^2}{4\cdot 10^{-30}}\hat{z}=-4.300\cdot 10^{-24}T\hat{z}$$
$$U_{0.5}=-2\cdot 10^{-7}\frac{(9.274\cdot 10^{-24})^2}{25\cdot 10^{-30}}\hat{z}=-6.881\cdot 10^{-25}T\hat{z}$$

and when we change the spin to -z-direction, we see that $\vec{m}=\mu_B\hat{z}$ now, and $\vec{B}=\frac{\mu_0}{2\pi}\frac{\mu_B}{z^3}\hat{z}$. Since they both change their sign, the potential energy doesn't change.

When it's on the x-axis, we see that

$$U=-\vec{m}\cdot\vec{B}=-(-\mu_B\hat{z})\cdot\frac{\mu_0}{4\pi}\frac{\mu_B}{x^3}\hat{z}=\frac{\mu_0}{4\pi}\frac{\mu_B^2}{x^3}$$

$$U_{0.1} = 10^{-7} \frac{(9.274 \cdot 10^{-24})^2}{10^{-30}} \hat{z} = 8.601 \cdot 10^{-23} T \hat{z}$$
$$U_{0.2} = 10^{-7} \frac{(9.274 \cdot 10^{-24})^2}{4 \cdot 10^{-30}} \hat{z} = 2.150 \cdot 10^{-24} T \hat{z}$$
$$U_{0.5} = 10^{-7} \frac{(9.274 \cdot 10^{-24})^2}{25 \cdot 10^{-30}} \hat{z} = 3.440 \cdot 10^{-25} T \hat{z}$$

and when we change the spin to -z-direction, we see that $\vec{m} = \mu_B \hat{z}$ now, and $\vec{B} = -\frac{\mu_0}{4\pi} \frac{\mu_B}{x^3} \hat{z}$. Since they both change their sign, the potential energy doesn't change.