Problem 1

(a)

We know that $\vec{J}_B = \vec{\nabla} \times \vec{M}$ and $\vec{M} = M_0\hat{z}$. Therefore,

$$\vec{J}_B = \vec{\nabla} \times \vec{M} = 0$$

(b)

We know that $\vec{K}_B = \vec{M} \times \hat{n}$. We see that on the sphere, $\hat{n} = \hat{r}$ and therefore

$$\vec{K}_B = \vec{M} \times \hat{n} = M_0(\cos\theta\hat{r} - \sin\theta\hat{\theta}) \times \hat{r} = M_0 \sin\theta\hat{\phi}$$

(c)

We see such a sphere, when rotating, must satisfy that $\vec{J}_B = 0$ and $\vec{K}_B = M_0 \sin\theta(\cos\phi\hat{y} - \sin\phi\hat{x})$. This means there should only be surface charge density (otherwise the $\vec{J}_B$ will not equal to 0). We know that at point (R, θ, ϕ) on the surface of the sphere:

$$\vec{K}_B = \sigma\vec{v} = \sigma R \sin\theta\omega\hat{\phi}$$

and therefore the relationship that

$$\sigma = \frac{M_0}{\omega R}$$

(d)

We know that $\vec{J}_B = \vec{\nabla} \times \vec{M}$ and $\vec{M} = M_0\hat{z}$. Therefore,

$$\vec{J}_B = \vec{\nabla} \times \vec{M} = 0$$

Same as (a).

Also, for the side of the cylinder, we see that $\hat{n} = \hat{s}$ and therefore

$$\vec{K}_{B,\text{side}} = \vec{M} \times \hat{n} = M_0\hat{z} \times \hat{s} = M_0\hat{\phi}$$

and for the top and bottom of the cylinder, we see that $\hat{n} = \pm\hat{z}$ and therefore

$$\vec{K}_{B,\text{tb}} = \vec{M} \times \hat{n} = M_0\hat{z} \times (\pm\hat{z}) = 0$$

(e)

We see that it should only have surface charge density otherwise it would product non-zero $\vec{J}_B$. We see that on the side of the

$$\vec{K}_{B,\text{side}} = \sigma_{\text{side}}\vec{v} = \sigma_{\text{side}}R\omega\hat{\phi}$$

and therefore

$$\sigma_{\text{side}} = \frac{M_0}{\omega R}$$

and on the top / bottom of the cylinder

$$\vec{K}_{B,\text{tb}} = \sigma_{\text{tb}} \vec{v} = \sigma_{\text{tb}} r \omega \hat{\phi}$$

we see that

$$\sigma_{\text{tb}} r \omega = 0$$

we see that ω is not zero, and $0 \leq r \leq R$. That means we must make the $\sigma_{\text{tb}} = 0$. That means $\sigma_{\text{tb}} = 0$. (There is only charge density on the side, but not on the top and bottom)

(f)

We know that $\vec{J}_B = \vec{\nabla} \times \vec{M}$ and $\vec{M} = M_0 \hat{z}$. Therefore,

$$\vec{J}_B = \vec{\nabla} \times \vec{M} = 0$$

This is also same as (a)

We see that on the six side of the cube, the normal vector is:

$$\hat{n}_{\text{side}} = \hat{s}$$
$$\hat{n}_{\text{top}} = \hat{z} \qquad \hat{n}_{\text{bottom}} = -\hat{z}$$

and therefore

$$\begin{aligned} \vec{K}_{B,\text{side}} &= M_0 \hat{z} \times \hat{s} = M_0 \hat{\phi} \\ \vec{K}_{B,\text{top}} &= M_0 \hat{z} \times \hat{z} = 0 \\ \vec{K}_{B,\text{bottom}} &= M_0 \hat{z} \times (-\hat{z}) = 0 \end{aligned}$$

we see that on the side of the cube, the distance to the z-axis is

$$\begin{aligned} s_{\text{left}}(\phi) &= -a \sec \phi \\ s_{\text{right}}(\phi) &= a \sec \phi \\ s_{\text{front}}(\phi) &= -a \csc \phi \\ s_{\text{back}}(\phi) &= a \csc \phi \end{aligned}$$

and we see that in this case, on the side of the cube, we have

$$\vec{K}_{B,\text{left}} = \sigma_{\text{left}} \vec{v} = \sigma_{\text{left}} s_{\text{left}}(\phi) \omega \hat{\phi}$$

and thus we have

$$\sigma_{\text{left}}(\phi) = \frac{M_0}{s_{\text{left}}(\phi)\omega} = -\frac{M_0}{a \sec(\phi)\omega}$$

and similarly that

$$\sigma_{\text{right}}(\phi) = \frac{M_0}{s_{\text{right}}(\phi)\omega} = \frac{M_0}{a \sec(\phi)\omega}$$

$$\sigma_{\text{front}}(\phi) = \frac{M_0}{s_{\text{front}}(\phi)\omega} = -\frac{M_0}{a\csc(\phi)\omega}$$

$$\sigma_{\text{back}}(\phi) = \frac{M_0}{s_{\text{back}}(\phi)\omega} = \frac{M_0}{a\csc(\phi)\omega}$$

the intuition here is that the part where it's farer from the z-axis (it rotate faster), the charge density is smaller.

and we see that the on the top and bottom side of the cube, the

$$\sigma_{\text{tb}} s_{\text{tb}}(\phi)\omega = 0$$

and therefore $\sigma_{\text{tb}} = 0$

Problem 2

(a)

We know that in the case where paramagnet is placed in an external potential field, there is no free current.

We know that

$$\vec{H}_{\text{out}} = \frac{\vec{B}_{\text{out}}}{\mu_0} = \frac{B_0}{\mu_0}(\cos\theta\hat{x} + \sin\theta\hat{z})$$

Therefore, we know that $\vec{H}^{\parallel}_{\text{out}} = \vec{H}^{\parallel}_{\text{in}} = \frac{B_0}{\mu_0}\cos\theta\hat{x}$

We know that at boundary, the $\vec{B}^{\perp}_{\text{in}} = \vec{B}^{\perp}_{\text{out}} = B_0\sin\theta\hat{z}$.

Since we know that inside the paramagnet the $\vec{B}_{\text{in}} = \mu_0(1+\chi_m)\vec{H}_{\text{in}}$. That means

$$\vec{B}^{\parallel}_{\text{in}} = \mu_0(1+\chi_m)\vec{H}^{\parallel}_{\text{in}} = \mu_0(1+\chi_m)\frac{B_0}{\mu_0}\cos\theta\hat{x} = B_0(1+\chi_m)\cos\theta\hat{x}$$

and therefore

$$\vec{B}_{\text{in}} = B_0((1+\chi_m)\cos\theta\hat{x} + \sin\theta\hat{z})$$

(b)

We use the formula that

$$\vec{B}_{\text{in}} = \mu_0(1+\chi_m)\vec{H}_{\text{in}}$$

and therefore

$$\vec{H} = \frac{B_0}{\mu_0}(\cos\theta\hat{x} + \frac{\sin\theta}{1+\chi_m}\hat{z})$$

(c)

We know that in this case the $\vec{H}$ in the slab is

$$\vec{H}_{\text{out}} = \frac{B_0\hat{z}}{\mu_0(1+\chi_m)}$$

Using the method mentioned in Lecture 28. We define a $\vec{H} = -\vec{\nabla}\phi_m(\vec{r})$, and we could therefore solve the equation

$$\phi_m(\vec{r}) = \sum_{i=0}^{l}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta)$$

So, in the outside of the sphere, we see as $r \to \infty$, the

$$\phi_m(\vec{r}) = -\frac{B_0}{\mu_0(1+\chi_m)}z$$

therefore, to satisfy the boundary condition, only $l = 1$ term remains.

we have

$$\phi_{m,\text{out}}(\vec{r}) = \alpha_{\text{out}} r\cos\theta + \frac{\beta_{\text{out}}}{r^2}\cos\theta$$

and the following must be true

$$\alpha_{\text{out}} r\cos\theta = -\frac{B_0}{\mu_0(1+\chi_m)}z$$

we see that $r\cos\theta = z$ and therefore

$$\alpha_{\text{out}} = -\frac{B_0}{\mu_0(1+\chi_m)}$$

Inside the sphere we find that

$$\phi_{m,\text{in}}(\vec{r}) = \alpha_{\text{in}} r\cos\theta + \frac{\beta_{\text{in}}}{r^2}\cos\theta$$

and since $\phi_{m,\text{in}}(\vec{r})$ should be a finite value as $r \to 0$, we see $\beta_{\text{in}} = 0$

From the lecture 28 notes, we see that

$$\vec{B}_{\text{out}} \cdot \hat{n} = \vec{B}_{\text{in}} \cdot \hat{n}$$

and

$$\mu_0(1+\chi_m)\vec{H}_{\text{out}} \cdot \hat{n} = \mu_0\vec{H}_{\text{in}} \cdot \hat{n}$$

then

$$(1+\chi_m)\frac{\partial\phi_{m,\text{out}}(\vec{r})}{\partial r} = \frac{\partial\phi_{m,\text{in}}(\vec{r});}{\partial r}$$

$$(1+\chi_m)\frac{\partial}{\partial r}\left(\alpha_{\text{out}}r + \frac{\beta_{\text{out}}}{r^2}\right) = \frac{\partial}{\partial r}(\alpha_{\text{in}}r)$$

$$(1+\chi_m)(\alpha_{\text{out}} - \frac{2\beta_{\text{out}}}{r^3}) = \alpha_{\text{in}}$$

evaluated at $r = R$

$$(1+\chi_m)(\alpha_{\text{out}} - \frac{2\beta_{\text{out}}}{R^3}) = \alpha_{\text{in}}$$

and then

$$-\frac{B_0}{\mu_0(1+\chi_m)} - \frac{2\beta_{\text{out}}}{R^3} = \frac{\alpha_{\text{in}}}{1+\chi_m}$$

the other boundary condition is that $\vec{H}_{\text{in},\parallel} = \vec{H}_{\text{out},\parallel}$. We see that

$$-\frac{1}{R}\frac{\partial}{\partial\theta}\phi_{m,\text{in}}(\vec{r}) = -\frac{1}{R}\frac{\partial}{\partial\theta}\phi_{m,\text{out}}(\vec{r})$$

this giving us

$$-\frac{1}{R}\frac{\partial}{\partial\theta}\left(\alpha_{\text{out}}r + \frac{\beta_{\text{out}}}{r^2}\right)\cos\theta = -\frac{1}{R}\frac{\partial}{\partial\theta}\alpha_{\text{in}}r\cos\theta$$

therefore

$$\alpha_{\text{out}}r + \frac{\beta_{\text{out}}}{r^2} = \alpha_{\text{in}}r$$

and therefore

$$\alpha_{\text{out}} + \frac{\beta_{\text{out}}}{r^3} = \alpha_{\text{in}}$$

and thus at $r = R$

$$-\frac{B_0}{\mu_0(1+\chi_m)} + \frac{\beta_{\text{out}}}{R^3} = \alpha_{\text{in}}$$

and therefore

$$-\frac{3B_0}{\mu_0(3+2\chi_m)} = \alpha_{\text{in}}$$

and therefore

$$\phi_{m,\text{in}} = \alpha_{\text{in}}r\cos\theta = -\frac{3B_0}{\mu_0(3+2\chi_m)}z$$

$$\vec{H}_{\text{in}} = -\vec{\nabla}\phi_{m,\text{in}} = \hat{z}\frac{B_0}{\mu_0(1+\frac{2}{3}\chi_m)}$$

$$\vec{B}_{\text{in}} = \mu_0\vec{H}_{\text{in}} = \hat{z}\frac{B_0}{(1+\frac{2}{3}\chi_m)}$$

(d)

When $\chi_m \to 0$, the $\vec{B}_{\text{in}} = B_0\hat{z}$. If $\chi_m \to \infty$, $\vec{B}_{\text{in}} = 0$. The χ_m is the magnetic susceptibility of the slab. If it's large, that means the paramagnet slab will generate a huge M for the external B. (it's easy to get magnetized)