Question 1: Particle motion in 1D

> A particle moves along the x direction under the influence of a force $F = F(v,t)$ that varies with both time and velocity given by
>
> $$F(v,t) = -ktv^2$$
>
> where $k > 0$ is a constant

(a)

> At initial time $t = 0$ the particle is located at $x(t=0) = v_0$. Determine the particle's velocity $v(t)$ at a later time.

Set up the equation:

$$\begin{aligned}
F = m\frac{dv}{dt} &= -ktv^2 \\
m\frac{dv}{v^2} &= -ktdt \\
\int_{v_0}^{v(t)} m\frac{dv}{v^2} &= \int_0^t -kudu \\
-m\left(\frac{1}{v}\right)\Bigg|_{v_0}^{v(t)} &= -k\frac{1}{2}t^2 \\
\frac{1}{v(t)} - \frac{1}{v_0} &= \frac{k}{2m}t^2 \\
\frac{1}{v(t)} = \frac{1}{v_0} + \frac{k}{2m}t^2 &= \frac{2m + kv_0t^2}{2mv_0} \\
\boxed{v(t) = \frac{2mv_0}{2m + kv_0t^2}}
\end{aligned}$$

(b)

> How much time, t_f, does it take the particle to stop?

$\lim_{t\to\infty} v(t) = 0$. As the time goes to infinity, the particle tends to stop. $t_f = \infty$

(c)

> Determine the distance $x_f = x(t_f)$ at which the particles stops. You may leave the final expression in the form of an integral.

Simply do the integral:

$$dx = v(t)dt$$

$$\int_{x(0)=0}^{x(t)} dx = \int_{t=0}^{t_f=\infty} \frac{2mv_0}{2m + kv_0t^2} dt$$

$$\boxed{x(t) = \int_{t_0=0}^{t_f=\infty} \frac{2mv_0}{2m + kv_0t^2} dt}$$

Question 2: Bead on a rope

> A bead slides out along a frictionless rod of length R that is fixed at point O and rotating in a horizontal plane at a constant rate ω as shown below. At time $t = t_0$ and angle $\phi = 90°$, the bead flies off the rod (i.e. when $r = R$). Just before the lead slides off the rod it has $\dot{r}(t = t_0) = \frac{3}{4}\omega R$. After the bead slides off the rod it experiences a friction force with $\vec{F}_f = -\kappa\vec{v}$, where κ is a constant.
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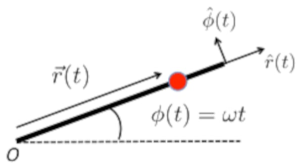
> 
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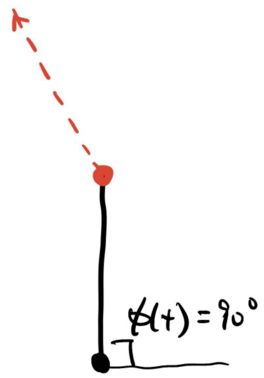
(a)

> What is the bead's trajectory after it flies off the rod? Draw a sketch and be as accurate as you can.



(b)

> Determine the velocity $\vec{v}(t)$ of the bead as the function of time, after it flies off the rod. How long will it take the bead to stop?

it basically becomes a 1D problem. Set up the equation

$$F_f = -\kappa v = m\frac{dv}{dt}$$

$$-\frac{\kappa}{m}dt = \frac{dv}{v}$$

$$\int_0^t -\frac{\kappa}{m}dt = \int_{v_i}^{v(t)} \frac{dv}{v}$$

$$-\frac{\kappa}{m}t = \ln(v)|_{v_i}^{v(t)} = \ln(v(t)) - \ln(v_i) = \ln(\frac{v(t)}{v_i})$$

$$v(t) = v_i e^{\frac{-\kappa}{m}t}$$

When the bead flies off the bead $\hat{r} = \hat{y}$, $\hat{\phi} = -\hat{x}$, and thus

$$\vec{v} = \dot{r}(t)\hat{r} + r(t)\dot{\phi}\hat{\phi} = (\frac{3}{4}\omega R)\hat{r} + (R\omega)\hat{\phi} = (\frac{3}{4}\omega R)\hat{y} - (R\omega)\hat{x}$$

and thus, the speed of the bead when it flies off:

$$v_i = |\vec{v}| = \sqrt{(\frac{3}{4}\omega R)^2 + (\omega R)^2} = \frac{5}{4}\omega R$$

So, the velocity $v(t)$

$$\boxed{v(t) = \frac{5}{4}\omega R e^{\frac{-\kappa}{m}t}}$$

$\lim_{t\to\infty} v_f = 0$. It takes infinite amount of time $t = \infty$ to stop.

Question 3: Curvilinear coordinates

> Denote Cartesian coordinates (x, y, z), and introduce parabolic cylindrical coordinates (u, v, z) determined by the transformation:
>
> $$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z$$

(a)

> In Cartesian coordinates we are given points $P_1 = (1, 0, 1) = \vec{e}_x + \vec{e}_z$, $P_2 = (0, 1, -1)$ and $P_3 = (0, 0, 3)$. What are their components in parabolic cylindrical coordinates (u, v, z) ?

For P_1:

$$1 = \frac{1}{2}(u^2 - v^2)$$
$$0 = uv$$
$$z = 1$$

$u = \sqrt{2}, v = 0, z = 1.$

For P_2:

$$0 = \frac{1}{2}(u^2 - v^2)$$
$$0 = uv$$
$$z = 3$$

$u = 0, v = 0, z = 3$

For P_2:

$$0 = \frac{1}{2}(u^2 - v^2)$$
$$1 = uv$$
$$z = -1$$

$u = 1, v = 1, z = -1$ or $u = -1, v = -1, z = -1$

(b)

> Determine the basis vectors $(\vec{e}_u, \vec{e}_v, \vec{e}_z)$ in terms of Cartesian ones. Draw the coordinate lines, i.e., lines of u = constant, and lines of v = constant, in the u, v plane with $z = 0$

Find the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{u} = J \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = u\vec{e}_x - v\vec{e}_y$$

$$\vec{v} = J \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = v\vec{e}_x + u\vec{e}_y$$

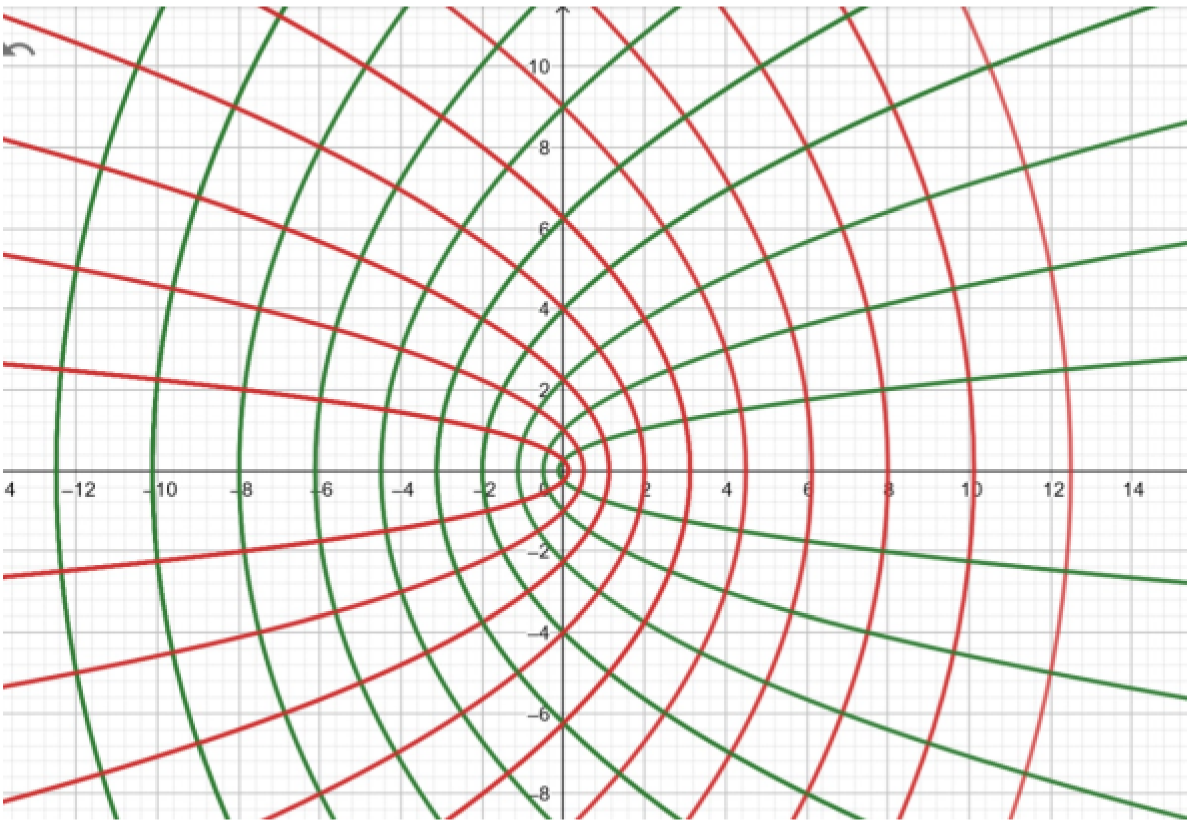
$$\vec{z} = J \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \vec{e}_z$$

$$\vec{e}_u = \frac{u\vec{e}_x - v\vec{e}_y}{\sqrt{u^2 + v^2}}$$

$$\vec{e}_v = \frac{v\vec{e}_x + u\vec{e}_y}{\sqrt{u^2 + v^2}}$$

$$\vec{e}_z = \vec{e}_z$$

the red line is when u = constant, the greed line is when v = constant.



(c)

> Derive the gradient of a scalar function, $\vec{\nabla} f(u, v, z)$ in parabolic cylindrical coordinates (u, v, z). Hint: recall the Nabla operator in Cartesian coordinates $\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$

the line element is

$$d\vec{l} = \frac{d\mathbf{r}}{du}du + \frac{d\mathbf{r}}{dv}dv + \frac{d\mathbf{r}}{dz}dz = \sqrt{u^2+v^2}\vec{e}_u du + \sqrt{u^2+v^2}\vec{e}_v dv + \vec{e}_z dz$$

So, the corresponding gradient is:

$$\vec{\nabla} f = \frac{1}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial u}\vec{e}_u + \frac{1}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial v}\vec{e}_v + \frac{\partial f}{\partial z}\vec{e}_z$$

Question 4: Conservative forces

(a)

$\vec{F} = (\alpha_1 y^2 z^3 - 6\alpha_2 x z^2)\vec{e}_x + 2\alpha_1 x y z^3 \vec{e}_y + (3\alpha_1 x y^2 z^2 - 6\alpha_2 x^2 z)\vec{e}_z$

$$\frac{\partial F_x}{\partial y} = 2\alpha_1 y z^3$$
$$\frac{\partial F_y}{\partial x} = 2\alpha_1 y z^3 = \frac{\partial F_y}{\partial x}$$
$$\frac{\partial F_x}{\partial z} = 3\alpha_1 y^2 z^2 - 12\alpha_2 x z$$
$$\frac{\partial F_z}{\partial x} = 3\alpha_1 y^2 z^2 - 12\alpha_2 x z = \frac{\partial F_x}{\partial z}$$
$$\frac{\partial F_y}{\partial z} = 6\alpha_1 x y z^2$$
$$\frac{\partial F_z}{\partial y} = 6\alpha_1 x y z^2 = \frac{\partial F_z}{\partial y}$$

Thus $\vec{F}(\vec{r})$ is conservative.

(b)

It does. Suppose $\vec{F} = \nabla V$

$$V = \int (\alpha_1 y^2 z^3 - 6\alpha_2 x z^2)dx = \alpha_1 x y^2 z^3 - 3\alpha_2 x^2 z^2 + \phi_1(yz) + \phi_2(y) + \phi_3(z) + C$$

$$\frac{\partial V}{\partial y} = 2\alpha_1 x y z^3 + \phi_1'(yz) + \phi_2'(y) = 2\alpha_1 x y z^3$$
$$\phi_1'(yz) + \phi_2'(y) = 0$$
$$\phi_1(yz) + \phi_2(y) = D$$

$$\frac{\partial V}{\partial z} = 3\alpha_1 x y^2 z^2 - 6\alpha_2 x^2 z + \phi_1'(yz) + \phi_3'(z) = 3\alpha_1 x y^2 z^2 - 6\alpha_2 x^2 z$$
$$\phi_1'(yz) + \phi_3'(z) = 0$$
$$\phi_1(yz) + \phi_3(y) = E$$

$$\boxed{V = \alpha_1 x y^2 z^3 - 3\alpha_2 x^2 z^2 + C_0}$$

Question 5: Variable mass and drag

(a)

$$V = \frac{4}{3}\pi R^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3R^2 \frac{dR}{dt} = 4\pi\gamma R^2$$

$$4\pi R^2 \frac{dR}{dt} = 4\pi\gamma R^2$$

$$\frac{dR}{dt} = \gamma$$

$$R(t) = \gamma t + R_0$$

(b)

$$\frac{dm}{dt} = \rho\frac{dV}{dt} = 4\rho\pi\gamma R^2 = 4\rho\pi\gamma(\gamma t + R_0)^2$$

(c)

$$m = \frac{4}{3}\rho\pi(\gamma t + R_0)^3 = \frac{4}{3}\rho\pi R^3$$

$$F_f = mg - \alpha R^2 v = \frac{dp}{dt} = \frac{dm}{dt}v + \frac{dv}{dt}m = (4\rho\pi\gamma R^2)v + m\frac{dv}{dt}$$

$$mg - (4\rho\pi\gamma + \alpha)R^2 v = m\frac{dv}{dt}$$

$$g - \frac{(4\rho\pi\gamma + \alpha)}{\frac{4}{3}\rho\pi R}v = \frac{dv}{dR}\frac{dR}{dt}$$

$$g - \frac{3(4\rho\pi\gamma + \alpha)}{4\rho\pi R}v = \frac{dv}{dR}\gamma$$

$$\frac{g}{\gamma} - \frac{3(4\rho\pi\gamma + \alpha)}{4\rho\pi\gamma R}v = \frac{dv}{dR}$$

let $c = \frac{g}{\gamma}$, $a = 3(4\rho\pi\gamma + \alpha), b = 4\rho\pi\gamma$.

$$c = \frac{a}{bR}v + v'$$

$$v = \frac{bcR}{a+b} + kR^{-\frac{a}{b}}$$

let $v(0) = 0$, $R(0) = R_0$

$$0 = \frac{bcR_0}{a+b} + kR_0^{-a/b}$$

$$-\frac{bcR_0^{\frac{a}{b}+1}}{a+b} = k$$

Thus:

$$v = \frac{bcR}{a+b} - \frac{bcR_0^{\frac{a}{b}+1}}{a+b}R^{-\frac{a}{b}}$$