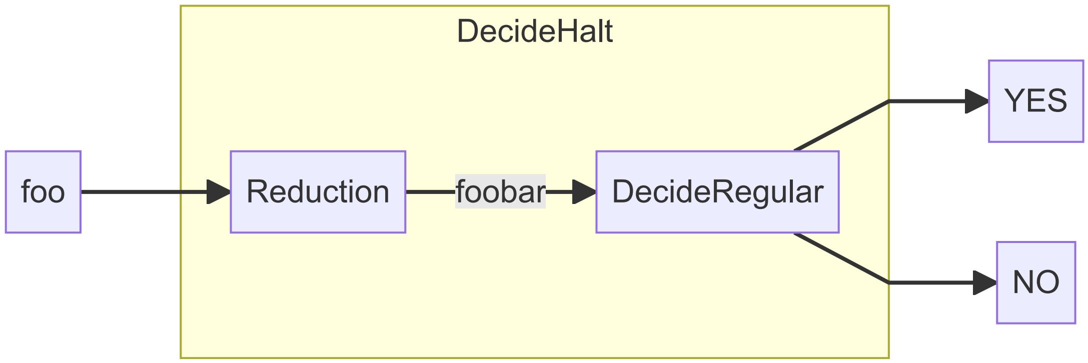
(a)



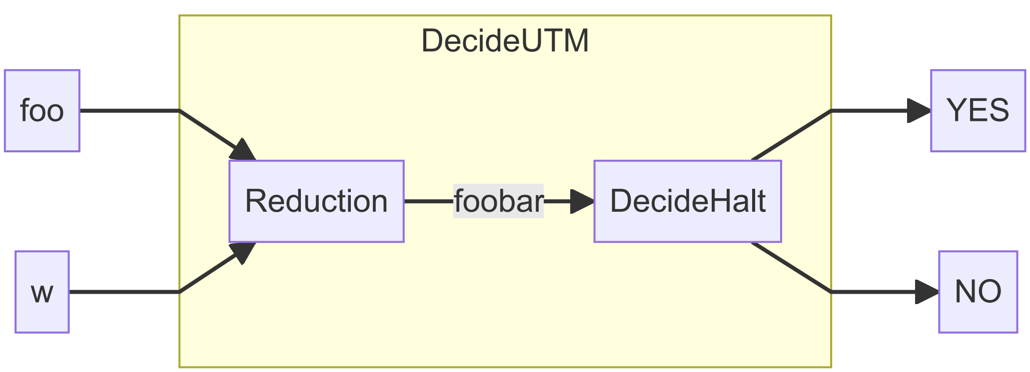
The reduction looks like above. We are trying to write a reduction so that $L_{\text{HALT}} \leq L_{\text{regular}}$. The reduction is

- The reduction receives arbitrary program, call it `foo`.
- It produce a new program, call it `foobar`. This program look like below (it takes an string `w` and return a boolean to decide whether it's regular) :

```
foobar(w):
    if w is palindrome:
        return true;
    else:
        foo()
        return true;
```

We see that if `foo` halts, then the language of `foobar` is Σ^* (which is regular language). If `foo` doesn't halt, then the language of `foobar` is all the palindrome (which is not regular language). Follow the same logic, we could see that the reverse is true. Then we see that `foobar` in L_{regular} if and only if `foo` in L_{HALT}. That means if L_{regular} is decidable then L_{HALT} is decidable. Since we know that L_{HALT} is undecidable that means L_{regular} is not decidable.

(b)



The reduction looks like above. We are trying to write a reduction so that $L_u \leq L_{\text{HALT}}$. The reduction is

- The reduction receives arbitrary program, call it `foo`, and a string called `w`. The `foo` is a program (function) that will take an input string and output a boolean to indicates whether it's accepted or not.
- It produce a new program, call it `foobar`. This program look like below:

  ```
  foobar():
      if foo(w):
          return
      else:
          loop infinitely
  ```

We see that when `w` is accepted by `foo`, the `foobar` halts. Otherwise, `foobar` will never halt. Following the same logic, the reverse is also true. Then we see that `foobar` halts if and only if `w` is accepted by `foo`. That means `foobar` is in L_{HALT} if and only if `(foo, w)` is in L_u and our reduction if valid. Therefore, $L_u \leq L_{\text{HALT}}$

(c)

Adapted from `https://stackoverflow.com/questions/46140969/how-is-turing-machine-which-accepts-nothing-is-not-recursively-enumerable`.

First we know that from Lecture slides:

> If L and $\bar{L}$ are recursively enumerable, then L is recursive. (1)

Then we know that

> $L_{\text{emptylang}} = \{\langle M\rangle | L(M) = \emptyset\}$ is not recursive. (2)

This is due to Rice's theorem. The property "accept nothing" is true to some languages (consider a language with only one valid string which is the encoding of a program that `return "no"` always, this language has this "accept nothing" property, but for a language with only one valid string which is the encoding of a program that `return "yes"` always, this language doesn't have this "accept nothing" property), and it's a semantic property of the language itself. Therefore, the $L_{\text{emptylang}}$ is not decidable / recursive.

> $L_{\text{somelang}} = \{\langle M\rangle | L(M) \neq \emptyset\}$ is recursively enumerable. (3)

note that L_{somelang} is the complement of $L_{\text{emptylang}}$. We know that for all possible input string $s \in \Sigma^*$, it is countable. So, for a given M if we want to know if it accept something, the set of (M, s_i) that iterates through all possible input string in Σ^* is countable. Then, we could use our UTM to run the simulation of the all the input (M, s_i) in parallel. (Just like what a single cpu core, which is equivalent to Turing machine, doing multiple things in parallel by switching between them) If one of the input (M, s_i) finishes / halts in accepted state, the UTM halts with accepted states. Otherwise the UTM will continue to run. We see the language of this UTM is L_{somelang}

> $L_{\text{emptylang}} = \{\langle M\rangle | L(M) = \emptyset\}$ is not recursively enumerable.

Suppose it's recursively enumerable. Then (3) said that it's complement is recursively enumerable, and that (1) indicates both language are recursive. However, (2) shows it's not recursive. This is a contradiction, that means our assumption is false, and that $L_{\text{emptylang}} = \{\langle M\rangle | L(M) = \emptyset\}$ has to be not recursively enumerable.