Exercise 2.1.1

> Prove that multiplication of complex numbers is associative. More precisely, let $z = a + bi$, $w = c + di$, and $v = g + hi$, and prove that $z(wv) = (zw)v$.

Proof:

$$\begin{aligned} z(wv) &= (a+bi)((c+di)(g+hi)) \\ &= (a+bi)(cg-dh+(ch+dg)i) \\ &= acg - adh + achi + adgi + bcgi - bdhi - bch - bdg \end{aligned}$$

$$\begin{aligned} (zw)v &= ((a+bi)(c+di))(g+hi) \\ &= (ac-bd+(ad+bc)i)(g+hi) \\ &= acg + achi - bdg - bdhi + adgi + bcgi - adh - bch \end{aligned}$$

The result is the same, and that means $z(wv) = (zw)v$

Exercise 2.1.2

> Let $z = a + bi, w = c + di \in \mathbb{C}$ and prove each of the following statements:

(i)

> $z + \overline{z}$ is real and $z - \overline{z}$ is imaginary.

Proof :

$\bar{z} = a - bi$

$z + \overline{z} = 2a \in \mathbb{R}$, $z - \overline{z} = 2bi \in \mathbb{C}$

(ii)

> $\overline{z+w} = \overline{z} + \overline{w}$

Proof:

$z + w = (a + c) + (b + d)i$

$\overline{z+w} = (a + c) - (b + d)i$

$\overline{w} = c - di$

$\overline{z} + \overline{w} = (a + c) - (b + d)i$

and thus $\overline{z+w} = \overline{z} + \overline{w}$

(iii)

> $\overline{zw} = \overline{z}\,\overline{w}$

$zw = ac - bd + (ad + bc)i$

$\overline{zw} = ac - bd - (ad + bc)i$

$\overline{z}\,\overline{w} = ac - adi - bci - bd = ac - bd - (ad + bc)i$

and thus $\overline{zw} = \overline{z}\,\overline{w}$

Exercise 2.2.1

> Prove that if $\mathbb{F}$ is a field and $a, b \in \mathbb{F}$ with $ab = 0$, then either $a = 0$ or $b = 0$.

If $a = 0$, then $ab = 0$, and we are done. So, suppose $a \neq 0$. Then there exist its multiplicative inverse a^{-1}. According to **Proposition 2.2.2 (ii)**

$$ab = 0$$
$$a^{-1}ab = a^{-1}0 = 0$$
$$1 \cdot b = 0$$
$$b = 0$$

Thus, it's either $a = 0$ or $b = 0$, if $a, b \in \mathbb{F}$, where $\mathbb{F}$ is a field.

Exercise 2.2.2

> Prove that $\mathbb{Q}(\sqrt{2})$ is a field.

$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \in \mathbb{R} | a, b \in \mathbb{R}\}$

Thus, suppose $x = a + b\sqrt{2}, y = c + d\sqrt{2}, z = e + f\sqrt{2} \in \mathbb{Q}(\sqrt{2})$.

Since $\sqrt{2} \notin \mathbb{Q}$. define

$x + y = (a + c) + (b + d)\sqrt{2}$ and $x \cdot y = ac + 2bd + (ad + bc)\sqrt{2}$

Note, below all use that $\sqrt{2} \notin \mathbb{Q}$, thus, the term $p\sqrt{2}$ is completely separate from $q, p, q \in \mathbb{Q}$

Addition is commutive:

$y + x = (c + a) + (b + d)\sqrt{2} = (a + c) + (b + d)\sqrt{2} = x + y$

and $(x + y) + z = ((a + c) + (b + d)\sqrt{2}) + e + f\sqrt{2} = (a + c + e) + (b + d + f)\sqrt{2}$

Addition is associative:

$x + (y + z) = a + b\sqrt{2} + ((c + e) + (d + f)\sqrt{2}) = (a + c + e) + (b + d + f)\sqrt{2}$

and $(x + y) + z = x + (y + z)$.

Multiplication is commutive:

$y \cdot x = ca + 2db + (cb + da)\sqrt{2} = ac + 2bd + (ad + bc)\sqrt{2} = x \cdot y$

Mutliplication is associative:

$(x \cdot y) \cdot z = (ac + 2bd)e + 2(ad + bc)f + ((ac + 2bd)f + e(ad + bc))\sqrt{2}$

$x \cdot (y \cdot z) = a(ce + 2df) + 2b(cf + de) + ((ce + 2df)b + a(cf + de))\sqrt{2}$

and, simplify it, it's true that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Multiplication distributes over addition:

$x(y + z) = a(c + e) + 2b(d + f) + (a(d + f) + b(c + e))\sqrt{2}$

$xy + xz = ac + 2bd + (ad + bc)\sqrt{2} + (ae + 2bf) + (af + be)\sqrt{2} = a(c + e) + 2b(d + f) + (ad + af + bc + be)\sqrt{2}$

and thus $x(y + z) = xy + xz$

There exist an additive identity denote $\mathbf{0} = 0 + 0\sqrt{2}$

and thus $x + \mathbf{0} = (0 + a) + (0 + b)\sqrt{2} = a + b\sqrt{2} = x$

There exist an an additive inverse $-a$ for all $a \in \mathbb{Q}(\sqrt{2})$

with out loss of generosity, use as x as example, $-x = -a - b\sqrt{2}$ since

$x + (-x) = (a - a) + (b - b)\sqrt{2} = 0 + 0\sqrt{2} = \mathbf{0}$

There exists a multiplicative identity denote $\mathbf{1} = 1 + 0\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ with $\mathbf{1} \neq 0$, and $\mathbf{1}a = a$ for all $a \in Q(\sqrt{2})$ since

$\mathbf{1}x = (1a) + 2(0b) + (0a + 1b)\sqrt{2} = a + b\sqrt{2} = x$

There exists a multiplicative inverse for all a in $a \in \mathbb{Q}(\sqrt{2}) - \{0\}$, called a^{-1}

suppose $u = a + b\sqrt{2} \neq \mathbf{0}$. $u^{-1} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}$ exist since $a \neq 0$ and $b \neq 0$ and is the mulitplicative inverse of u since $u^{-1}u = \frac{(a - b\sqrt{2})(a + b\sqrt{2})}{a^2 - 2b^2} = \frac{a^2 - 2b^2}{a^2 - 2b^2} = 1$

Thus, with above proved, $Q(\sqrt{2})$ is indeed a field.

Exercise 2.3.1

> Prove parts (iii) - (v) of Proposition 2.3.2:

Suppose $f \in \mathbb{F}[x]$,

$$f = \sum_{k}^{n} a_k x^k$$

(iii)

> $0 \in \mathbb{F}$ is an additive identity in $\mathbb{F}[x]$: $f + 0 = f$ for all $f \in \mathbb{F}[x]$

Notice $0 = \sum_k^n 0x^k$

$$f + 0 = \sum_{n}^{k} (a_k + 0)x^k = f$$

0 is indeed additive identity.

(iv)

> Every $f \in \mathbb{F}[x]$ has an additive inverse given by $-f = (-1)f$ with $f + (-f) = 0$

So, the $-f = (-1)f = -1\sum_k^n a_k x^k$ and $f = \sum_k^n a_k x^k$ and it's trivial that $f + (-f) = 0$. So the additive inverse is indeed $-f$.

(v)

> $1 \in \mathbb{F}$ is the multiplicative identity in $\mathbb{F}[x]$: $1f = f$ for all $f \in \mathbb{F}[x]$

$1f = 1 \cdot \sum_k^n a_k x^k = \sum_k^n 1 \cdot a_k x^k = \sum_k^n a_k x^k = f$. So, the $1 \in \mathbb{F}$ is indeed the multiplicative identity in $\mathbb{F}[x]$