Question 1 Bumpy Road

> Consider a unicycle fitted with a damped, shock-absorbing spring with spring constant k and damping coefficient c. When a rider of mass $m = 80\text{kg}$ sits on the unicycle. the spring compresses and the seat sags by 0.02m. Recall $g = 9.81\text{m/s}^2$
>
> The unicycle (including the rider) goes over a crack that sets the spring in motion. It takes $\tau = 5\text{s}$ for the amplitude of the underdamped spring oscillations to decay by a factor of $1/e$
> .
>
> Hint: recall the equation for a free, damped spring
>
> $$m\ddot{x} + c\dot{x} + kx = 0, \quad \text{or, equivalently} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$
>
> with $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2m\omega_n}$. When underdamped, this has the general (homogeneous) solution $x(t) = e^{-\zeta\omega_n t}\left(A\exp\left(i\omega_n\sqrt{1-\zeta^2}t\right) + B\exp\left(-i\omega_n\sqrt{1-\zeta^2}t\right)\right)$.

(a)

> Find the dimensionless damping parameter ζ. Justify calling the system "underdamped".

Call the distance seat sags $\Delta h = 0.02\text{m}$, thus

$$mg = k\Delta h$$
$$k = \frac{mg}{\Delta h}$$

we know t2hat after $\tau = 5\text{s}$, the oscillation decay by a factor of $1/e$. Thus

$$e^{-\zeta\omega_n\tau} = e^{-1}$$

and that

$$\zeta = \frac{1}{w_n\tau} = \tau^{-1}\sqrt{\frac{m}{k}} = \tau^{-1}\sqrt{\frac{\Delta h}{g}}$$

$$\boxed{\zeta = 0.2 \cdot \sqrt{\frac{0.02}{9.81}} \approx 0.00903}$$

which is less than 1, and this shows that the system is indeed "underdamped".

(b)

> Find the damping coefficient c

$$c = \zeta \cdot 2m\omega_n = 2\zeta m\sqrt{\frac{k}{m}} = 2\tau^{-1}\sqrt{\frac{\Delta h}{g}mk} = 2\tau^{-1}m$$

and thus

$$\boxed{c = 2\frac{m}{\tau} = 32\text{kg/s}}$$

(c)

> Now suppose that the unicycle is ridden over a "washboard" road with bump spaced $1\mathrm{m}$ apart. Treat the surface of the road as sinusoidal. At what speed must the unicycle go for the spring to be forced into resonance?

Then the frequency of bumping ω_b should be equal to natural frequency ω_n. Thus

$$w_b = w_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{m\Delta h}} = \sqrt{\frac{g}{\Delta h}}$$

and thus

$$\boxed{v = \omega_b d = \sqrt{\frac{9.81}{0.02}} \cdot 1 \approx 22.15\mathrm{m/s}}$$

(d)

> Calculate the steady-state phase difference between the spring oscillations and the driving force caused by the bumps in the road near the resonance.

we know that $\omega_b = \omega_n$, and thus from lecture notes 15,

$$\tilde{G}(\omega) = \frac{1}{k}\frac{1}{2i\zeta} = -\frac{1}{2k\zeta}i$$

and thus

$$\phi(\omega) = -\arg\tilde{G}(\omega) = -\frac{3}{2}\pi \overset{\text{equiv}}{=} \frac{1}{2}\pi$$

Question 2 Wave Rectifier

> A full-wave rectifier acting on a sinusoidal function of the form $F(t) = F_0\sin(\omega t)$, with $F_0 > 0$ constant, will produce a rectified function $F_{\text{rec}} = |F_0\sin(\omega t)|$

(a)

> Find the Fourier series representation of $F_{\text{rec}} = |F_0\sin(\omega t)|$

The period of the function is $T = \frac{\pi}{\omega}$, and for $t \in [0, \pi]$, the $F_{\text{rec}} = F_0\sin(\omega t)$. For the function, we could see that is purely even, and thus $b_n = 0$. For a_n, they are

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^{\pi/\omega} F_0\sin(\omega t)\cos(n\omega t)\mathrm{d}t \\ &= \frac{\omega}{\pi}F_0\left(\frac{1}{(2n+1)\omega}\cos((2n+1)\omega t) + \frac{1}{(2n-1)\omega}\cos((2n-1)\omega t)\right)\Bigg|_0^{\pi/\omega} \\ &= \frac{-4F_0}{\pi(4n^2-1)} \end{aligned}$$

and thus

$$F_{\text{rec}} = \sum_{n=0}^{\infty}\frac{-4F_0}{\pi(4n^2-1)}\sin(n\omega t)$$

(b)

Using mathematica

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In[18]:= DSolve[m*x''[t] + k*x[t] == a*Sin[O*t], x[t], t]
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$$\text{Out[18]=}\ \left\{\left\{x[t] \to c_1 \operatorname{Cos}\left[\frac{\sqrt{k}\,t}{\sqrt{m}}\right] + c_2 \operatorname{Sin}\left[\frac{\sqrt{k}\,t}{\sqrt{m}}\right] + \frac{\operatorname{Cos}\left[\frac{\sqrt{k}\,t}{\sqrt{m}}\right]^2 \operatorname{Sin}[O\,t] + \operatorname{Sin}\left[\frac{\sqrt{k}\,t}{\sqrt{m}}\right]^2 \operatorname{Sin}[O\,t]}{k - m\,O^2}\right\}\right\}$$

the first and then

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Simplify[x[t]]
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$$c_1 \operatorname{Cos}\left[\frac{\sqrt{k}\,t}{\sqrt{m}}\right] + c_2 \operatorname{Sin}\left[\frac{\sqrt{k}\,t}{\sqrt{m}}\right] + \frac{\operatorname{Sin}[O\,t]}{k - m\,O^2}$$

two parts is just the homogeneous solution, we use the third part, and thus the particular solution should be

$$x_p(t) = \sum_{n=0}^{\infty} \frac{1}{k - m\omega^2} \sin(\omega t) = \infty$$

this makes sense because there is no damping, and external force is in resonance with motion.

Question 3

(a)

The frequency $\Omega = \frac{2\pi}{T} = 2\pi$. Note the function is odd from the graph, and the average value should be 0, and thus

$$F(t) = \sum_{n=1}^{\infty} b_n \sin(n\Omega t)$$

we know that

$$b_n = \frac{2}{T} \int_{-\frac{1}{2}}^{\frac{1}{2}} 2F_0 t \sin(2\pi n t) = \frac{2}{1} \cdot \frac{F_0}{\pi}$$

and thus

$$F(t) = \frac{2F_0}{\pi} \sum_{n=1}^{\infty} \sin(2\pi n t)$$

(b)

the particular solution will have the form of

$$x_p(t) = \sum_{n=1}^{\infty} c_n \sin(n\Omega t)$$

and thus

$$\sum_{n=1}^{\infty} c_n(-mn^2\Omega^2 + cn\Omega + k)\sin(n\Omega t) = \sum_{n=1}^{\infty} \frac{2F_0}{\pi}\sin(n\Omega t)$$

and thus

$$c_n = \frac{2F_0}{(-mn^2\Omega^2 + cn\Omega + k)\pi}$$

and we could see the c_n decrease as n increase. That means c_1 term $c_1 \sin(\Omega t)$ (with frequency Ω) dominate the response.

Question 4

(a)

> What is the period T of the force?

From the graph, we could see it's $T = 2\mathrm{s}$

(b)

From the plot, we could see that the function is even. Thus, $F_{\text{odd}} = 0$, and all the b_n should thus vanish.

(c)

From (b), we know that the function reduced to

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right)$$

and thus a solution of

$$x_p(t) = c + d_n \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi t}{T}\right) + \sum_{n=1}^{\infty} e_n \sin\left(\frac{2n\pi t}{T}\right)$$

the frequency is just $\Omega = \frac{2\pi}{T}$. For shorthand, just denote $\cos_n = \cos(n\Omega t)$ and $\sin_n = \sin(n\Omega t)$

and thus

$$\begin{aligned} x_p''(t) &= -d_n(n\Omega)^2 \cos_n - e_n(n\Omega)^2 \sin_n \\ x_p'(t) &= -d_n(n\Omega)\sin_n + e_n(n\Omega)\cos_n \end{aligned}$$

and thus

$$(-md_n(n\Omega)^2 + ce_n(n\Omega) + kd_n)\cos_n + (-me_n(n\Omega)^2 - cd_n(n\Omega) + ke_n)\sin_n + kc = \frac{a_0}{2} + a_n \cos_n$$

$$\begin{cases} c = \frac{a_0}{2k} \\ a_n = -md_nA^2 + ce_nA + kd_n \\ 0 = -me_nA^2 - cd_nA + ke_n \end{cases}$$

with result

$$c = \frac{a_0}{2k}$$
$$d_n = \frac{(k - mA^2)a_n}{(k - mA^2)^2 + (cA)^2}$$
$$e_n = \frac{cA \cdot a_n}{(k - mA^2)^2 + (cA)^2}$$

(d)

For a long time, the oscillator will result in steady-state $x_p(t)$. Therefore, we could just integrate over one period T for $x_p(t)$ and divide the time T to get the average position. Do notice that

$$\int_{\text{period}} \sin(\omega t + \phi)\mathrm{d}t = \int_{\text{period}} \cos(\omega t + \phi)\mathrm{d}t = 0$$

where ω and ϕ could be arbitrary constant, and thus

$$\frac{1}{T}\int_{\text{period}} x_p(t)\mathrm{d}t = \frac{cT}{T} = \frac{a_0}{2k}$$