Question 1 Particle on a spring

> A particle of mass m is constrained to move along a line in the x direction. It's attached to a spring with constant k that is connected to a point at a distance L perpendicular to the line. The natural length of the spring is $L_0 = L/2$.

(a)

> Determine the system's potential energy U in terms of the particle's position $x = x(t)$

The potential energy is

$$U = \frac{1}{2}k(\Delta L)^2 = \frac{1}{2}k\left(\sqrt{L^2+x^2} - \frac{L}{2}\right)^2$$

(b)

> Find the position $x = x_{min}$ where the potential energy has a minimum

From the intuition, the minimum potential energy happens when $x = 0$ (the spring is least stretched). The calculation also agrees on that:

$$U' = \frac{1}{2}k \cdot 2\left(\sqrt{L^2+x^2} - \frac{L}{2}\right) \cdot \frac{1}{2}(L^2+x^2)^{-\frac{1}{2}} \cdot 2x = kx \cdot \left(1 - \frac{L}{2\sqrt{L^2+x^2}}\right)$$

is equal to zero only when $\boxed{x = 0}$, and that

$$U'' = k(1 - \frac{L}{2\sqrt{L^2+x^2}}) + kx(-\frac{L}{2} \cdot -\frac{1}{2}(L^2+x^2)^{-3/2})$$

with $U''(0) = \frac{1}{2}k > 0$ show that the point is indeed the minimum.

(c)

> Determine the effective spring constant $k_{\text{eff}} = U''(x = x_{\text{min}})$

From the previous calculation, we know that $\boxed{k_{\text{eff}} = U''(0) = \frac{1}{2}k}$

(d)

> Determine the system's oscillation frequency

The oscillation frequency $\boxed{w = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{k}{2m}}}$

Question 2 Simple Pendulum

> (A pendulum, with a massless rod of length L and a point bob of mass m, swings in a gravitational field $\vec{g}$ as shown. Assume the pendulum to be near the Earth's surface such that $g = \text{const}$. The pendulum has energy E, that is not necessarily small. We choose the reference frame such that the potential energy vanishes at $\theta = 0$, i.e., $U(\theta = 0) = 0$

(a)

> Determine the pendulum maximum angular speed $\dot{\theta}$, in terms of E, L, m and g

The total energy E is equal to

$$E = U + T = mgL(1 - \cos\theta) + \frac{1}{2}mL^2\dot{\theta}^2$$

Thus

$$\dot{\theta} = \sqrt{\frac{2E}{mL^2} - \frac{2g}{L}(1 - \cos\theta)}$$

and maximum is achieved when setting $\cos\theta = 1$.

$$\boxed{\dot{\theta} = \sqrt{\frac{2E}{mL^2}}} \tag{1}$$

(b)

> Determine the maximum amplitude, $\theta_{\max}$, of the oscillation in terms of E, L, m, g

From the question (a), we could rearrange the equation to

$$\theta = \arccos(1 - \frac{E}{mgL} - \frac{L}{2g}\dot{\theta}^2)$$

and the maximum is achieved when setting $\dot{\theta} = 0$.

$$\boxed{\theta = \arccos(1 - \frac{E}{mgL})} \tag{2}$$

(c)

> How large must the energy E be such that the pendulum whirls around rather than just oscillates?

For pendulum to whirls around, the maximum θ must be at least π (see the graph). Thus

$$\pi = \arccos(1 - \frac{E}{mgL})$$

$$-1 = 1 - \frac{E}{mgL}$$

$$\boxed{E = 2mgL} \tag{3}$$

Question 3 Compound pendulum

> A compound pendulum is formed of a rigid rod of length $2a$, with one mass $m_1 = m$ at its tip and another mass $m_2 = 2m$ at its midpoint as shown.
>
> Determine the pendulum's period for small oscillations, i.e. , $\theta(t)$ small.
>
> Hint: one approach is start by constructing the formulas for its kinetic energy as a function of $\dot{\theta}$ and for its potential energy as a function of θ

The potential energy is

$$
\begin{aligned}
U &= U_{\text{small}} + U_{\text{big}} \\
&= m_1 g \cdot 2a(1-\cos\theta) + m_2 g \cdot a(1-\cos\theta) \\
&= 4mga(1-\cos\theta)
\end{aligned}
$$

and the kinetic energy is

$$
\begin{aligned}
T &= T_{\text{small}} + T_{\text{big}} \\
&= \frac{1}{2} m_1 (2a\dot{\theta})^2 + \frac{1}{2} m_2 (a\dot{\theta})^2 \\
&= \frac{1}{2} m \cdot 4a^2\dot{\theta}^2 + \frac{1}{2} \cdot 2m \cdot a^2\dot{\theta}^2 \\
&= 3ma^2\dot{\theta}^2
\end{aligned}
$$

so, the total energy is

$$
E = U + T = 4mga(1-\cos\theta) + 3ma^2\dot{\theta}^2
$$

and since the total energy is conserved

$$
\frac{\mathrm{d}E}{\mathrm{d}t} = 4mga\sin\theta \cdot \dot{\theta} + 3ma^2 \cdot 2\dot{\theta} \cdot \ddot{\theta} = 0
$$

and thus

$$
4mga\sin\theta + 6ma^2\ddot{\theta} = 0
$$

since θ is small, we use the approximation that $\sin\theta \approx \theta$

$$
4mga\theta + 6ma^2\ddot{\theta} = 0
$$

which we recognize that

$$
\begin{aligned}
M_{\text{eff}} &= 6ma^2 \\
K_{\text{eff}} &= 4mga
\end{aligned}
$$

and thus the

$$
\omega = \sqrt{\frac{K_{\text{eff}}}{M_{\text{eff}}}} = \sqrt{\frac{4mga}{6ma^2}} = \sqrt{\frac{2g}{3a}}
$$

and thus the period is

$$
\boxed{P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3a}{2g}}} \tag{4}
$$

Question 4

> A pendulum, with a massless rod of length L and a point bob of mass m at its tip, oscillates in an oil bath. The drag force acting on the bob is $\vec{F} = -c\vec{v}$ and is directed opposite to the bob's velocity vector $\vec{v}$. You may ignore buoyancy and assume motion near the Earth's surface such that $g = \text{const}$

(a)

> Derive the differential equation describing the bob's motion $\theta(t)$ for an arbitrary amplitude θ_{max} (but such that the bob remains submerged in the oil).

Doing some forces analysis:

$$I\ddot{\theta} = -mgL\sin\theta - cL(L\dot{\theta})$$

that is

$$\boxed{mL^2\ddot{\theta} + cL^2\dot{\theta} + mgL\sin\theta = 0}$$

(b)

> Linearize the differential equation that you obtained in (a) and check it for obvious errors: is the damping positive? Its the effective stiffness (or "effective spring constant") positive? What is the damped frequency of the vibration?
>
> Assume that the system is underdamped.

Assume that θ is small, so $\sin\theta \approx \theta$. So, the equation is

$$mL^2\ddot{\theta} + cL^2\dot{\theta} + mgL\theta = 0$$

which is equivalent to solving

$$\boxed{\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\theta = 0} \tag{5}$$

The $M_{\text{eff}} = 1$, $C_{\text{eff}} = \frac{c}{m}$, $K_{\text{eff}} = \frac{g}{L}$. and thus

$$\omega_n = \sqrt{\frac{K_{\text{eff}}}{M_{\text{eff}}}} = \sqrt{\frac{g}{L}}$$

and thus the damping zeta ζ is

$$\zeta = \frac{C_{\text{eff}}}{2M_{\text{eff}}\omega_n} = \frac{c}{2m}\sqrt{\frac{L}{g}}$$

by assumption, $c > 0$. and obviously $L > 0$ and $m > 0$. Thus, the $\zeta > 0$.

The effective spring constant is just $K_{\text{eff}} = g/L > 0$, (or mgL for original equation) is obviously greater than zero.

By assumption of that system is underdamped, the damped frequency ω_d is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{g}{L}(1 - \frac{c^2}{4m^2}\cdot\frac{L}{g})} = \sqrt{\frac{g}{L} - \frac{c^2}{4m^2}}$$

(c)

> Setting the initial conditions $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = \Omega$, find the trajectory $\theta(t)$

By the Lecture note 13, Equation (43), the solution has the form of

$$\theta(t) = \exp(-\zeta\omega_n t)(A\cos(\omega_d t) + B\sin(\omega_d t))$$

It's derivative is

$$\dot{\theta}(t) = -\zeta\omega_n \exp(-\zeta\omega_n t)(A\cos(\omega_d t) + B\sin(w_d t)) + \exp(-\zeta\omega_n t)(-A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t))$$

By the initial conditions

$$\theta(0) = 1(A \cdot 1 + B \cdot 0) = \theta_0$$
$$\dot{\theta}(0) = -A\zeta\omega_n + B\omega_d = \Omega$$

and thus

$$\boxed{A = \theta_0 \quad B = \frac{\Omega + \theta_0\zeta\omega_n}{\omega_d}}$$

and the solution is thus

$$\boxed{\theta(t) = \exp(-\zeta\omega_n t)(\theta_0 \cos(\omega_d t) - \frac{\Omega}{\zeta\omega_n\omega_d}\sin(\omega_d t))} \tag{6}$$

where the ω_n, ω_d, and ζ is calculated in (b)

Question 5

> Computer Algebra programs like mathematica are useful tools to do lengthy computations. Here you will practice working with mathematica using the problem in Question 4, In particular,

(a)

> Verify that your solution in $(4c)$ does indeed solve the differential equation of $(4b)$ by inserting your solution

We define some constants first

```
In[1]:= Meff = 1; Ceff = c / m; Keff = g / L
Out[1]= g/L

In[2]:= Wn = Sqrt[Keff / Meff]
Out[2]= Sqrt[g/L]

In[3]:= Ze = Ceff / (2 * Meff * Wn)
Out[3]= c / (2 Sqrt[g/L] m)

In[4]:= Wd = Wn * Sqrt[1 - Ze ^ 2]
Out[4]= Sqrt[g/L] Sqrt[1 - (c^2 L)/(4 g m^2)]

In[5]:= A = ThetaZero
Out[5]= ThetaZero

In[6]:= B = (Omega + ThetaZero * Ze * Wn) / Wd
Out[6]= (Omega + (c ThetaZero)/(2 m)) / (Sqrt[g/L] Sqrt[1 - (c^2 L)/(4 g m^2)])
```

and write the $\theta(t)$ function, put it back into (5)

```
In[7]:= Theta[t_] = Exp[-Ze*Wn*t]*(A*Cos[Wd*t] + B*Sin[Wd*t])
```

Out[7]= $e^{-\frac{c\,t}{2\,m}}\left(\text{ThetaZero}\,\text{Cos}\left[\sqrt{\frac{g}{L}}\sqrt{1-\frac{c^2\,L}{4\,g\,m^2}}\;t\right]+\frac{\left(\text{Omega}+\frac{c\,\text{ThetaZero}}{2\,m}\right)\text{Sin}\left[\sqrt{\frac{g}{L}}\sqrt{1-\frac{c^2\,L}{4\,g\,m^2}}\;t\right]}{\sqrt{\frac{g}{L}}\sqrt{1-\frac{c^2\,L}{4\,g\,m^2}}}\right)$

```
In[8]:= Simplify[Theta''[t] + c/m*Theta'[t] + g/L*Theta[t]]

Out[8]= 0
```

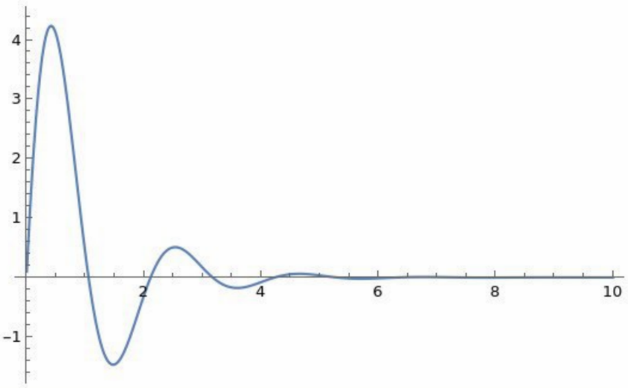
we get 0 as expected, which shows that our solution is correct.

(b)

> Plot the motion $\theta(t)$ in the interval $t = 0\mathrm{s}, \ldots, 10\mathrm{s}$. Set the parameters $\theta_0 = 0.1$, $L = 1m$, $m = 0.5\mathrm{kg}$, $c = 1\mathrm{kg/s}$, $g = 9.81\mathrm{m/s}^2$,$\Omega = 20\mathrm{Hz}$
>
> Hint: it is useful to introduce the dimensionless time parameter $\bar{t} = t/s$, and plot $\theta(\bar{t})$

```
In[16]:= ThetaZero = 0.1; L = 1; m = 0.5; c = 1; g = 9.81; Omega = 20

Out[16]= 20

In[17]:= Theta[t]

Out[17]= e^(-1. t) (0.1 Cos[2.96816 t] + 6.77186 Sin[2.96816 t])

In[21]:= Plot[Theta[t], {t, 0, 10}, PlotRange → All]
```

Out[21]=



theme... frame... labels... axes ▼ more...

Giving value to the constants, and plot the $\theta(\bar{t})$. As we could see, this is indeed a damped oscillation.