Question 1

> Consider the following state vector:
>
> $$|\psi\rangle = \sqrt{\frac{5}{6}}\,|0\rangle + \frac{1}{\sqrt{6}}|1\rangle$$

(a)

> Is the state normalized?

$$\left(\sqrt{\frac{5}{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = 1$$

The state is normalized.

(b)

> What is the probability that the system is found to be in state $|0\rangle$, if Z is measured?

The probability is

$$\mathrm{Pr}_0 = \left(\sqrt{\frac{5}{6}}\right)^2 = \frac{5}{6}$$

(c)

> Write down the density operator

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= \left(\frac{\sqrt{5}}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle\right)\left(\frac{\sqrt{5}}{\sqrt{6}}\langle 0| + \frac{1}{\sqrt{6}}\langle 1|\right) \\ &= \frac{5}{6}|0\rangle\langle 0| + \frac{\sqrt{5}}{6}|0\rangle\langle 1| + \frac{\sqrt{5}}{6}|1\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|\end{aligned}$$

(d)

> Find the density matrix in the $\{|0\rangle, |1\rangle\}$ basis, and show that $\mathrm{Tr}(\rho) = 1$

From (c), the density matrix is just

$$\rho = \begin{pmatrix} 5/6 & \sqrt{5}/6 \\ \sqrt{5}/6 & 1/6 \end{pmatrix}$$

and $\mathrm{Tr}(\rho) = 5/6 + 1/6 = 1$

Question 2

> Consider the state
>
> $$|\psi\rangle = \begin{pmatrix} \cos\theta \\ i\sin\theta \end{pmatrix}$$
>
> Is this state normalized? Is $\rho = |\psi\rangle\langle\psi|$ a density operator?

The state is normalized since

$$\langle\psi|\psi\rangle = \cos^2\theta + \sin\theta^2 = 1$$

and

$$\rho = |\psi\rangle\langle\psi| = (\cos\theta\,|0\rangle + i\sin\theta\,|1\rangle)(\cos\theta\,\langle 0| - i\sin\theta\,\langle 1|) = \begin{pmatrix} \cos^2\theta & -i\sin\theta\cos\theta \\ i\sin\theta\cos\theta & \sin\theta^2 \end{pmatrix}$$

and

- $\mathrm{Tr}(\rho) = \cos^2\theta + \sin^2\theta = 1$
- $\rho^\dagger = \begin{pmatrix} \cos^2\theta & -i\sin\theta\cos\theta \\ -(-i\sin\theta\cos\theta) & \sin\theta^2 \end{pmatrix} = \rho$
- the eigenvalue of ρ is $\lambda = 0, 1$, and thus the ρ is positive semi-definite, and thus it's a positive operator.

thus it's a density operator.

Question 3

> Let
>
> $$|\psi\rangle = \sqrt{\frac{3}{7}}\,|0\rangle + \frac{2}{\sqrt{7}}|1\rangle$$

(a)

> Write down the density matrix in the $\{|0\rangle, |1\rangle\}$ basis.

$$\rho = \frac{1}{7}\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix}$$

(b)

> Determine whether or not this is a pure state.

we could check that

$$\mathrm{Tr}\big(\rho^2\big) = \mathrm{Tr}\left(\frac{1}{49}\begin{pmatrix} 21 & 14\sqrt{3} \\ 14\sqrt{3} & 28 \end{pmatrix}\right) = 1$$

which means that this is a pure state.

(c)

> Write down the density matrix in the $\{|+\rangle, |-\rangle\}$ basis, show that $\mathrm{Tr}(\rho) = 1$ still holds, and determine is you still obtain the same result as in part (b)

$$\rho_H = H\rho H = \frac{1}{14}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 7+4\sqrt{3} & -1 \\ -1 & 7-4\sqrt{3} \end{pmatrix}$$

and thus

$$\mathrm{Tr}(\rho_H) = 1$$

and we could also find that

$$\mathrm{Tr}\left(\rho^2\right) = \mathrm{Tr}\left(\frac{1}{14^2}\begin{pmatrix} 98+56\sqrt{3} & -14 \\ -14 & 98-56\sqrt{3} \end{pmatrix}\right) = 1$$

means it's still a pure state (same as (b))

Question 4

> Suppose that a system is in the state
>
> $$|\psi\rangle = \sqrt{\frac{2}{3}}\,|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

(a)

> Compute $\mathrm{Tr}(\rho)$ and $\mathrm{Tr}(\rho^2)$. Is this a mixed state?

$$\rho = \begin{pmatrix} 2/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 1/3 \end{pmatrix}$$

and thus

$$\mathrm{Tr}(\rho) = 1$$

and

$$\mathrm{Tr}\left(\rho^2\right) = \mathrm{Tr}\left(\frac{1}{9}\begin{pmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{pmatrix}\right) = 1$$

This is not a mixed state

(b)

> Find $\langle X\rangle$ for this state

$$\langle X\rangle = \mathrm{Tr}(\rho X) = \mathrm{Tr}\left(\frac{1}{3}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{3}\mathrm{Tr}\begin{pmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{pmatrix} = \frac{2\sqrt{2}}{3}$$

Question 5

> Suppose that
>
> $$\rho = \begin{pmatrix} 1/3 & i/4 \\ -i/4 & 2/3 \end{pmatrix}$$

(a)

> Is this a valid density matrix? If not, why not?

We check that

- $\mathrm{Tr}(\rho) = 1$
- $\rho^\dagger = \begin{pmatrix} 1/3 & -(-i/4) \\ -i/4 & 2/3 \end{pmatrix} = \rho$
- The eigenvalues for ρ is $\frac{1}{12}(6+\sqrt{13})$, $\frac{1}{12}(6-\sqrt{13})$. Since all its eigenvalues are bigger than 0. ρ is positive definite.

Thus it's a valid density matrix.

(b)

> If this is a valid density matrix, does it represent a pure state or a mixed state?

$$\rho^2 = \begin{pmatrix} 25/144 & i/4 \\ -i/4 & 73/144 \end{pmatrix}$$

$$\mathrm{Tr}\left(\rho^2\right) = \frac{49}{72} \neq 1$$

It's a mixed state.

Question 6

> For the density matrix given by
>
> $$\rho = \frac{1}{5}\begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix}$$

(a)

> Is this a mixed state?

$$\mathrm{Tr}\left(\rho^2\right) = \mathrm{Tr}\left(\frac{1}{5}\begin{pmatrix} 11/5 & 1-i \\ 1+i & 6/5 \end{pmatrix}\right) \neq 1$$

It's a mixed state.

(b)

> Find $\langle X\rangle$, $\langle Y\rangle$, and $\langle Z\rangle$ for this state

$$\langle X\rangle = \mathrm{Tr}(\rho X) = \mathrm{Tr}\left(\frac{1}{5}\begin{pmatrix} 1-i & 3 \\ 2 & 1+i \end{pmatrix}\right) = \frac{2}{5}$$

$$\langle Y\rangle = \mathrm{Tr}(\rho Y) = \mathrm{Tr}\left(\frac{1}{5}\begin{pmatrix} 1+i & -3i \\ 2i & 1-i \end{pmatrix}\right) = \frac{2}{5}$$

$$\langle Z\rangle = \mathrm{Tr}(\rho Z) = \mathrm{Tr}\left(\frac{1}{5}\begin{pmatrix} 3 & -1+i \\ 1+i & -2 \end{pmatrix}\right) = \frac{1}{5}$$