If $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ are two different complex numbers, then prove algebraically that

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1z_2^*)$$

and that

$$|z_1 z_2| = |z_1||z_2|$$

Proof:

$$egin{aligned} z_1+z_2&=(x_1+x_2)+i(y_1+y_2)\ |z_1+z_2|^2&=(z_1+z_2)(z_1+z_2)^*\ &=z_1z_1^*+z_2z_2^*+z_1^*z_2+z_1z_2^*\ &=|z_1|^2+|z_2|^2+(z_1z_2^*)^*+(z_1z_2^*) \end{aligned}$$

Suppose z = x + iy

$$z + z^* = x + iy + x - iy = 2x = 2Re(z)$$

So

$$|z_1+z_2|^2=|z_1|^2+|z_2|^2+(z_1z_2^*)^*+(z_1z_2^*)=|z_1|^2+|z_2|^2+2Re(z_1z_2^*)$$

Also, because commutativity of complex number multiplication.

$$|z_1 z_2|^2 = z_1 z_2 z_1^* z_2^* = z_1 z_1^* z_2 z_2^* = |z_1|^2 |z_2|^2 \ |z_1 z_2| = |z_1||z_2|$$

2

For the following pair of complex numbers, find:

- 1. Their polar form
- 2. Their moduli
- 3. The product of the two
- 4. The quotient (i.e. z_1/z_2)

$$z_1 = rac{1+i}{\sqrt{2}} \quad z_2 = rac{3+4i}{3-4i} \ |z_1|^2 = z_1 z_1^* = rac{(1+i)(1-i)}{\sqrt{2} \cdot \sqrt{2}} = 1 \ |z_1| = \sqrt{|z_1|^2} = 1 \ ext{}$$
 $heta_{z_1} = rctan(rac{rac{1}{\sqrt{2}}}{rac{1}{\sqrt{2}}}) = rctan(1) = rac{\pi}{4} \ |z_1 = e^{rac{\pi}{4}i}|$

$$z_2 = rac{(3+4i)(3+4i)}{(3-4i)(3+4i)} = rac{-7+24i}{25}$$
 $|z_2|^2 = z_2 z_2^* = rac{-7+24i}{25} rac{-7-24i}{25} = rac{49+576}{625} = 1$
 $|z_2|^2 = \sqrt{|z_2|^2} = 1$
 $heta_{z_2} = \arctan(rac{rac{24}{25}}{-rac{7}{25}}) = \arctan(rac{-24}{7}) pprox 1.8546$
 $ag{z_2 = e^{1.8546i}}$
 $ag{z_1 \cdot z_2 = rac{1+i}{\sqrt{2}} \cdot rac{-7+24i}{25} = rac{-31+17i}{25\sqrt{2}}}$

$$\boxed{\frac{z_1}{z_2} = \frac{\frac{1+i}{\sqrt{2}}}{\frac{3+4i}{3-4i}} = \frac{(1+i)(3-4i)}{\sqrt{2}(3+4i)} = \frac{(7-i)(3-4i)}{\sqrt{2}(3+4i)(3-4i)} = \frac{17-31i}{25\sqrt{2}}}$$

3

Two quantum states are given by

$$|a
angle = inom{-4i}{2}, \;\; |b
angle = inom{1}{-1+i}$$

a) Find |a+b
angle

$$|a
angle + |b
angle = inom{1-4i}{1+i}$$

b) Calculate $3|a\rangle-2|b\rangle$

$$3|a
angle - 2|b
angle = inom{-12i-2}{8-2i}$$

c) Normalize $|a\rangle$, $|b\rangle$

$$egin{align} |e_a
angle &= rac{|a
angle}{||a
angle|^2} = egin{pmatrix} rac{-4i}{\sqrt{20}} \ rac{2}{\sqrt{20}} \end{pmatrix} \ |e_b
angle &= rac{|b
angle}{||b
angle|^2} = egin{pmatrix} rac{1}{\sqrt{3}} \ rac{-1+i}{\sqrt{3}} \end{pmatrix} \end{aligned}$$

d) $\langle a|b
angle$ and verify that $\langle a|b
angle = \langle b|a
angle^*$

$$(4i,\ 2)inom{1}{-1+i}=4i-2+2i=6i-2$$
 $igg((1,-1-i)inom{-4i}{2}igg)^*=(-4i-2-2i)^*=(-6i-2)^*=6i-2$

It's indeed that $\langle a|b
angle = \langle b|a
angle^*$

A quantum system is in the state

$$|\psi
angle = rac{3i|0
angle + 4|1
angle}{5}$$

a) Is the state normalized?

$$\langle \psi | \psi
angle = rac{-3i \langle 0 | + 4 \langle 1 |}{5} \cdot rac{3i | 0
angle + 4 | 1
angle}{5} = 1$$

Yes, it is.

b) Express the state in the |+
angle, |angle basis

$$egin{aligned} |\psi
angle &= rac{3i|0
angle + 4|1
angle}{5} \ &= rac{1}{5}(3i\cdotrac{1}{\sqrt{2}}(|+
angle + |-
angle) + 4\cdotrac{1}{\sqrt{2}}(|+
angle - |-
angle)) \ &= rac{(3i+4)|+
angle + (3i-4)|-
angle}{5\sqrt{2}} \end{aligned}$$