1

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two different complex numbers, then prove algebraically that

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1 z_2^*)$$

and that

$$|z_1 z_2| = |z_1||z_2|$$

Proof:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(z_1 + z_2)^* \\ &= z_1 z_1^* + z_2 z_2^* + z_1^* z_2 + z_1 z_2^* \\ &= |z_1|^2 + |z_2|^2 + (z_1 z_2^*)^* + (z_1 z_2^*) \end{aligned}$$

Suppose $z = x + iy$

$$z + z^* = x + iy + x - iy = 2x = 2Re(z)$$

So

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + (z_1 z_2^*)^* + (z_1 z_2^*) = |z_1|^2 + |z_2|^2 + 2Re(z_1 z_2^*)$$

Also, because commutativity of complex number multiplication.

$$|z_1 z_2|^2 = z_1 z_2 z_1^* z_2^* = z_1 z_1^* z_2 z_2^* = |z_1|^2 |z_2|^2$$
$$|z_1 z_2| = |z_1||z_2|$$

2

For the following pair of complex numbers, find:

1. Their polar form
2. Their moduli
3. The product of the two
4. The quotient (i.e. z_1/z_2)

$$z_1 = \frac{1+i}{\sqrt{2}} \quad z_2 = \frac{3+4i}{3-4i}$$

$$|z_1|^2 = z_1 z_1^* = \frac{(1+i)(1-i)}{\sqrt{2} \cdot \sqrt{2}} = 1$$

$$\boxed{|z_1| = \sqrt{|z_1|^2} = 1}$$

$$\theta_{z_1} = \arctan\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = \arctan(1) = \frac{\pi}{4}$$

$$\boxed{z_1 = e^{\frac{\pi}{4} i}}$$

$$z_2 = \frac{(3+4i)(3+4i)}{(3-4i)(3+4i)} = \frac{-7+24i}{25}$$

$$|z_2|^2 = z_2 z_2^* = \frac{-7+24i}{25}\frac{-7-24i}{25} = \frac{49+576}{625} = 1$$

$$\boxed{|z_2| = \sqrt{|z_2|^2} = 1}$$

$$\theta_{z_2} = \arctan(\frac{\frac{24}{25}}{-\frac{7}{25}}) = \arctan(\frac{-24}{7}) \approx 1.8546$$

$$\boxed{z_2 = e^{1.8546i}}$$

$$\boxed{z_1 \cdot z_2 = \frac{1+i}{\sqrt{2}} \cdot \frac{-7+24i}{25} = \frac{-31+17i}{25\sqrt{2}}}$$

$$\boxed{\frac{z_1}{z_2} = \frac{\frac{1+i}{\sqrt{2}}}{\frac{3+4i}{3-4i}} = \frac{(1+i)(3-4i)}{\sqrt{2}(3+4i)} = \frac{(7-i)(3-4i)}{\sqrt{2}(3+4i)(3-4i)} = \frac{17-31i}{25\sqrt{2}}}$$

3

Two quantum states are given by

$$|a\rangle = \begin{pmatrix} -4i \\ 2 \end{pmatrix}, \quad |b\rangle = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

a) Find $|a+b\rangle$

$$|a\rangle + |b\rangle = \begin{pmatrix} 1-4i \\ 1+i \end{pmatrix}$$

b) Calculate $3|a\rangle - 2|b\rangle$

$$3|a\rangle - 2|b\rangle = \begin{pmatrix} -12i-2 \\ 8-2i \end{pmatrix}$$

c) Normalize $|a\rangle$**,** $|b\rangle$

$$|e_a\rangle = \frac{|a\rangle}{||a\rangle|^2} = \begin{pmatrix} \frac{-4i}{\sqrt{20}} \\ \frac{2}{\sqrt{20}} \end{pmatrix}$$

$$|e_b\rangle = \frac{|b\rangle}{||b\rangle|^2} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1+i}{\sqrt{3}} \end{pmatrix}$$

d) $\langle a|b\rangle$ **and verify that** $\langle a|b\rangle = \langle b|a\rangle^*$

$$(4i,\ 2)\begin{pmatrix} 1 \\ -1+i \end{pmatrix} = 4i - 2 + 2i = 6i - 2$$

$$\left((1, -1-i)\begin{pmatrix} -4i \\ 2 \end{pmatrix}\right)^* = (-4i-2-2i)^* = (-6i-2)^* = 6i-2$$

It's indeed that $\langle a|b\rangle = \langle b|a\rangle^*$

4.

A quantum system is in the state

$$|\psi\rangle = \frac{3i|0\rangle + 4|1\rangle}{5}$$

a) Is the state normalized?

$$\langle\psi|\psi\rangle = \frac{-3i\langle 0| + 4\langle 1|}{5} \cdot \frac{3i|0\rangle + 4|1\rangle}{5} = 1$$

Yes, it is.

b) Express the state in the $|+\rangle$, $|-\rangle$ basis

$$\begin{aligned} |\psi\rangle &= \frac{3i|0\rangle + 4|1\rangle}{5} \\ &= \frac{1}{5}(3i \cdot \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + 4 \cdot \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)) \\ &= \frac{(3i+4)|+\rangle + (3i-4)|-\rangle}{5\sqrt{2}} \end{aligned}$$