Problem 1

(a)

We know that the $\vec{M} = n\vec{m}$, and we know the volume of the magnet $V = \pi R^2 t$, and therefore, the $\vec{m} = V\vec{M} = \pi R^2 t M_0 \hat{z}$

(b)

we see that $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$. We know that the $\vec{\nabla} \cdot \vec{M} = 0$ inside the magnet, therefore the $\vec{\nabla} \cdot \vec{H} = 0$, since we know that $\vec{\nabla} \times \vec{H} = 0$, that means the $\vec{H}_{\text{in}} = 0$, and therefore, $\vec{B}_{\text{in}} = \mu_0(\vec{H} + \vec{M}) = \mu_0\vec{M}$. Since that $\vec{\nabla} \cdot \vec{B} = 0$ always, so the magnetic field just outside the magnet is the same as the magnetic field just below the magnet surface, $\vec{B}_{\text{in}} = \vec{B}_{\text{out}}$. Since outside the magnet, $\vec{H}_{\text{out}} = \vec{B}_{\text{out}}/\mu_0$, therefore, $\vec{B}_{\text{out}} = \vec{B}_{\text{in}} = \mu_0\vec{M} = \mu\vec{H}_{\text{out}}$.

(c)

We know that

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}(\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3})$$

since we only approximate the magnetic field around the magnet, we could use the $\vec{m}$ we calculate from part (a), and therefore we get (let $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, and suppose the magnet is centered at the origin)

$$\begin{aligned}\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}(\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}) \\ &= \frac{\mu_0}{4\pi}(\frac{3(\pi R^2 t M_0 z)\vec{r}}{r^5} - \frac{\pi R^2 t M_0 \hat{z}}{r^3})\end{aligned}$$

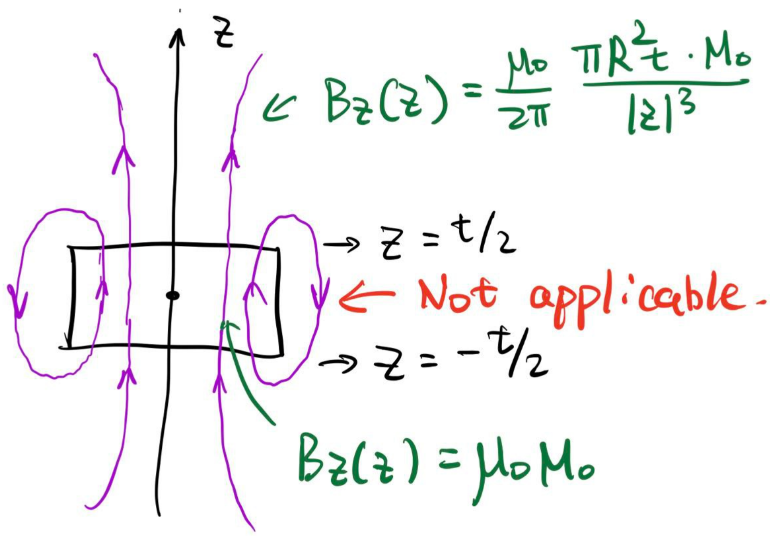
since we want to find the $\vec{B}_Z(z)$ on the z-axis, we set $x = y = 0$, and therefore

$$\begin{aligned}B_Z(z) &= \frac{\mu_0}{4\pi}(\frac{3\pi R^2 t M_0 z^2}{|z|^5} - \frac{\pi R^2 t M_0}{|z|^3}) \\ &= \frac{\mu_0}{4\pi}(\frac{3\pi R^2 t M_0}{|z|^3} - \frac{\pi R^2 t M_0}{|z|^3}) \\ &= \frac{\mu_0}{2\pi}(\frac{\pi R^2 t M_0}{|z|^3})\end{aligned}$$

this is only valid outside the magnet, suppose that magnet is in $z = \pm t/2$, then we see that

$$B_Z(z) = \begin{cases} \frac{\mu_0}{2\pi}\frac{\pi R^2 t M_0}{|z|^3} & z > t/2 \text{ or } z < t/2 \\ \mu_0 M_0 & \text{inside magnet } -t/2 \leq z \leq t/2 \end{cases}$$

note that this approximation only valid for the place near the z-axis, and this doesn't work for the place near the side of the magnet (notice that $B_Z(z)$ is always bigger than 0, this is obviously not the case for the place that is marked red)



(d)

we draw a loop on the outer layer of the pipe, we see that the area $A = \pi(R+d)^2$, therefore the magnetic flux is

$$\Phi_B(z) = A \cdot B_Z(z) = \frac{\mu_0}{2} \frac{tR^2 M_0}{|z|^3} \cdot \pi(R+d)^2$$

and we see that the EMF is different for $z > t/2$ and $z < -t/2$, and in the $z > t/2$ case

$$\mathcal{E}(z) = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = \frac{3}{2}\mu_0 \frac{tR^2 M_0}{z^4} \pi(R+d)^2 v$$

and in the case $z < t/2$ case,

$$\mathcal{E}(z) = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{3}{2}\mu_0 \frac{tR^2 M_0}{z^4} \pi(R+d)^2 v$$

we see that the EMF $\mathcal{E}$ is positive above the magnet, and negative below the magnet. Their magnitude, however, only depends on the z-distance to the origin.

(e)

We see that the wall of the pipe has a thickness of d, and that means the current flowing having a surface density of

$$\mathcal{E} = \int_L \vec{E} \cdot \mathrm{d}l$$

We see that our loop has length of $2\pi(R+d)$, and we will do a rough estimation of the current flowing through one "loop" in the pipe (in reality, the $\mathcal{E}$ is different for the loop with different radius chosen in pipe, and this will yield an $\vec{K}$ that is different with respect to distance to the z-axis). We see that

$$\mathrm{d}I = \frac{\mathcal{E}}{\mathbf{R}} = \frac{\mathcal{E}\sigma A}{L} = \frac{\mathcal{E}\sigma d \cdot \mathrm{d}z}{2\pi(R+d)}$$

and therefore

$$K = \frac{\mathrm{d}I}{\mathrm{d}z} = \frac{\mathcal{E}\sigma d}{2\pi(R+d)}$$

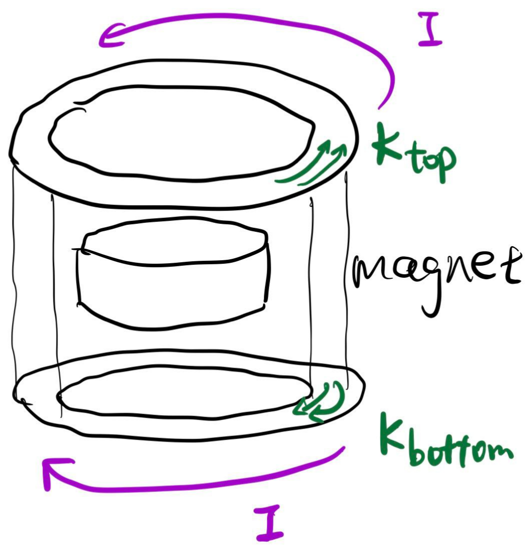
and for the case $z \geq t/2$

$$K = \frac{3}{4}\sigma \cdot \mu_0 M_0 \cdot tR^2 d(R+d)\frac{v}{z^4}$$

and for the case $z \leq -t/2$

$$K = -\frac{3}{4}\sigma \cdot \mu_0 M_0 \cdot tR^2 d(R+d)\frac{v}{z^4}$$

In both case, $\vec{K}$ has direction of $\vec{\phi}$



(f)

From the graph above, we see that the above the magnet, since $\vec{m} = I\vec{a}$, the $\vec{m}_{\text{induced}}$ is pointing upwards in z-direction, since I is flowing counter-clockwise, but $\vec{m}_{\text{induced}}$ is pointing downwards in z-direction, since I is flowing clock-wise.

(g)

we see that we want to calculate the magnetic dipole moment for each "loop" in the pipe:

$$\mathrm{d}\vec{m} = \mathrm{d}I \cdot \vec{A} = \mathrm{d}I \cdot \pi(R+d)^2\hat{z}$$

this dipole should centered on the z-axis, and we have that the force between two dipoles

$$\mathrm{d}F = \frac{3\mu_0}{4\pi r^5}((\mathrm{d}\vec{m}\cdot\vec{r})\cdot\vec{m} + (\vec{m}\cdot\vec{r})d\vec{m} + (\vec{m}\cdot d\vec{m})\vec{r} - \frac{5(\mathrm{d}\vec{m}\cdot\vec{r})(\vec{m}\cdot\vec{r})}{r^2}\vec{r})$$

and we see that the distance between two dipoles is just $\vec{r} = -z\hat{z}$

$$\mathrm{d}F_Z = \frac{3\mu_0}{2\pi z^4} m_0 \mathrm{d}m$$
$$= \frac{3\mu_0}{2\pi z^4} \cdot \pi R^2 t M_0 \cdot \pi(R+d)^2 \cdot \frac{3}{4}\sigma \cdot \mu_0 M_0 \cdot t R^2 d(R+d) \frac{v}{z^4} \mathrm{d}z$$
$$= \frac{9}{8}\sigma \cdot \frac{\mu_0^2}{\pi} \cdot (\pi R^2 t M_0)^2 \cdot d(R+d)^3 \frac{v}{z^8} \mathrm{d}z$$

(this is true both for above and below magnet)

and we could integrate this with respect to z from $-\infty \to -t/2$ and $t/2 \to \infty$, and we get

$$\vec{F} = \int_{-\infty}^{-t/2} \mathrm{d}F_Z + \int_{t/2}^{\infty} \mathrm{d}F_Z$$
$$= \frac{9}{8}\sigma \cdot \frac{\mu_0^2}{\pi} \cdot (\pi R^2 t M_0)^2 \cdot d(R+d)^3 v \cdot \frac{256}{7t^7}$$
$$= \frac{9 \cdot 32}{7} \cdot \sigma \cdot \frac{\mu_0^2}{\pi} \cdot (\pi R^2 M_0)^2 \cdot d(R+d)^3 \frac{1}{t^5} v$$

set $\vec{F} = m_M g$ and thus

$$v_T = \frac{7\pi t^5 \cdot m_M g}{9 \cdot 32 \cdot \sigma \cdot \mu_0^2 \cdot (\pi R^2 M_0)^2 \cdot d(R+d)^3}$$

(h)

We see that it's linearly velocity is linearly proportional to the magnet's mass, and $\propto 1/\sigma$, and $\propto 1/d^3$. This makes sense. Its terminal velocity will be faster if it's heavier, and will be slower if the pipe could produce more induced current (this is indicated by σ and d)

Problem 2

(a)

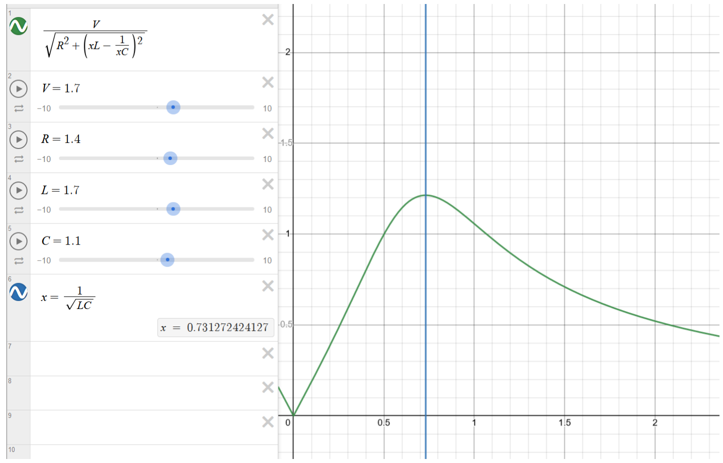
since they are in series, the voltage get distributed to R, L, and C, so we see that since V_R is in the phase of V, and the voltage for inductor is $\pi/2$ ahead, and the capacitor is $-\pi/2$ behind, therefore, the relationship between these voltage is

$$V^2 = V_R^2 + (V_L - V_C)^2 = I_0^2 R^2 + (I_0 X_L - I_0 X_C)^2$$
$$= I_0^2 (R^2 + (X_L - X_C)^2)$$

and therefore

$$I_0 = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$$

where $X_L = \omega L$ and $X_C = \frac{1}{\omega C}$



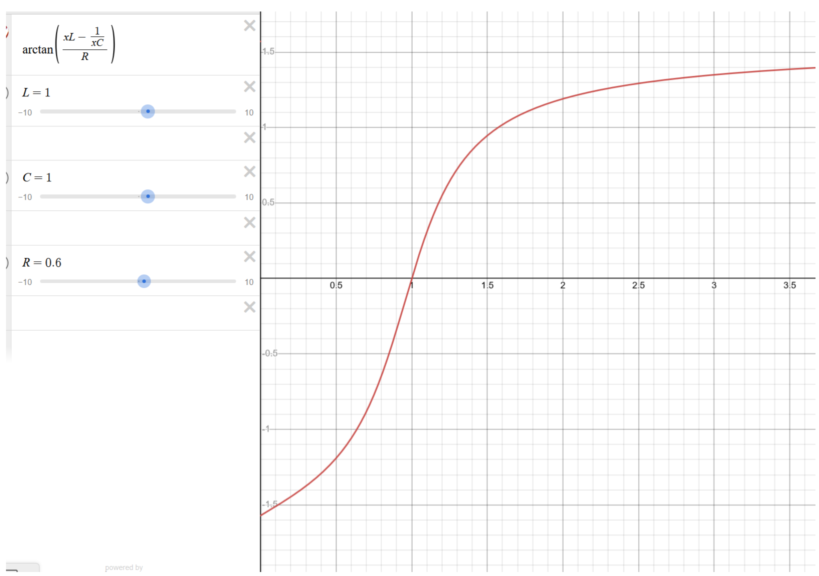
(b)

we know that

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{Z_L - Z_C}{R}$$

and I_0 it's ϕ phase ahead

$$\phi = \arctan\left(\frac{Z_L - Z_C}{R}\right)$$



(c)

we could use this circuit as a resonant circuit, the resonant angular frequency is where $\omega L = \frac{1}{\omega C}$. and we find that $\omega = \frac{1}{\sqrt{LC}}$, and then the I_0 will have the biggest current, this could be detected by other components.

(d)

We see that the actual frequency is $f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$

Since we want the resonant frequency be within 0.6 - 2 MHz therefore, we see that

$$0.6 \leq \frac{1}{2\pi\sqrt{LC}} \leq 2$$

and therefore we see that

$$\frac{1}{1.6\pi^2} \leq C \leq \frac{1}{0.144\pi^2}$$

so the capacitor should be tunable within this range.

(e)

The impedance for NC is

$$Z_{NC} = \frac{1}{i\omega NC}$$

and the impedance for an inductor and a capacitor

$$Z_L + Z_C = i\omega L + \frac{1}{i\omega C}$$

we see that to make them equal

$$i\omega L + \frac{1}{i\omega C} = \frac{1}{i\omega NC}$$

and

$$-\omega^2 L + \frac{1}{C} = \frac{1}{NC}$$

and

$$L = \frac{1}{\omega^2}\frac{N-1}{NC}$$

for ω, we could use this L and C to have the same impedance as a single NC could have.