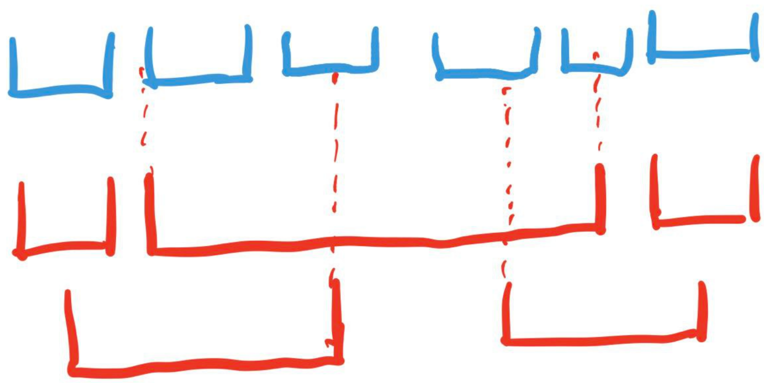
(a)

There is one counterexample:



> This is just a rough illustration, the top left red interval covers the first blue interval, the top middle red interval covers the second, third, fourth, and fifth blue intervals, the top right red interval covers the sixth blue interval. The bottom left red interval covers the first, second, third intervals. The bottom right red interval covers fourth, fifth, and sixth intervals. The exact number / coordinate doesn't matter here.

The wrong greedy algorithm will choose the top three red intervals. (Or the top middle, bottom left and bottom right. Or the top middle, bottom left, top right. Or the top middle, top left, bottom right. No matter which one it choses though, it always needed to choose 3 intervals.) However, the optimal solution is the two intervals in the bottom. The optimal solution is different from the greedy solution. (Greedy solution is worse).

(b)

Before we run the algorithm, we do:

- Sort the R in the increasing order of each interval's $a(I)$, and sort the B in the increasing order of each interval's $b(J)$. Initialize the solution O as empty.

The greedy algorithm is below (It receives three parameters: the red intervals R, the blue intervals B, and the solution O).

See if B is empty.

- If it's not, get the first interval J_0 in the sorted B. Iterate the intervals I_i in R that has $a(I_i) \leq b(J_0)$ and $b(I_i) \geq b(J_0)$ (this means they have overlap). $\boxed{1}$
  - If such intervals could not be found, immediately raise `Exception` to indicate that there isn't a solution. $\boxed{5}$

- Otherwise, find the interval that has maximum $b(I_i)$ among these intervals $\boxed{2}$ (this could be done along with iterating the I_i). Record this interval as I_{right} and add this interval to the solution O. Try to iterate to find the first interval that doesn't satisfy $a(I_i) \leq b(J_0)$ the $I_{\text{next start}}$. $\boxed{4}$ (this $I_{\text{next start}}$ could be undefined, this will just cause the recursive call to raise `Exception` if the algorithm choose to call it).

  Start from start from J_1, iterate intervals J_j in B until either:

  - Find an interval $J_{\text{next start}}$ such that $a(J_{\text{next start}}) > b(I_{\text{right}})$ $\boxed{3}$, then call the algorithm recursively with parameters:
    - red intervals equal to intervals starting $I_{\text{next start}}$ to the end of R, inclusive.
    - blue intervals equal to intervals starting from $J_{\text{next start}}$ to the end of B, inclusive.
    - solution just equal to the current solution O.

    and return the O after the recursive call finishes.
  - Reach the end of the B, in this case just return solution O.
- Otherwise just immediately return.

Then we just call this greedy algorithm with the parameter:

- Red Intervals $= R$
- Blue Intervals $= B$
- Solution $= O$

and either the algorithm will raise an exception telling that solution doesn't exist. Or the solution will be store in the O.

We first analyze the time complexity of the algorithm. We see that the first two sort takes $O(n \log n)$ and $O(m \log m)$, respectively. Then, we first iterate through from I_0 to $I_{\text{next start}}$. Since we only just check the $a(I_i)$ and calculate an maximum of $b(I_i)$ for these intervals. Each iteration takes $O(1)$ to complete. Also, we iterate through J_1 to $J_{\text{next start}}$, since each iteration we only check $a(J_j)$. Each iteration takes $O(1)$ to complete. Further notice that although we call the algorithm recursively, we only iterate every interval at most once for the entire program. (This is because we only iterate forward, and we only pass the part of the R and B that is not iterated yet to the recursive call). That means the runtime for the algorithm besides the sorting takes $O(n+m)$.

So, in the end, we see that the total runtime for the entire program takes $O(n \log n + m \log m)$. This is dominated by the sorting at the start of the program. This runtime is with in the require bound of $O((n+m)\log(n+m))$.

We see that the only case when there is no solution is there is some blue intervals, but there is no red intervals that overlaps with it (the fact that it explore all the possible intervals that could be overlapping with the blue intervals is detailed in $(\star)$ (this is in later part)). This is detected exactly by $\boxed{5}$. So, our algorithm behaves properly when there is no solution.

Otherwise, we will prove the correctness by contradiction. Given the greedy solution G, We first notice few key properties

- If I_i is selected as the solution. Then there exists a certain interval J_j, I_i is the interval that has overlap with I_i. We call all the intervals that have the overlap with J_j the set I^j, and I_i has maximum $b(I_i)$ among I^j.

  This is true because we see that in the algorithm that the only way a I_i could be added into the solution is when there exists a J_j that overlaps with I_i, and I_i has the maximum $b(I_i)$ (this is shown by $\boxed{2}$) among all the intervals considered at $\boxed{1}$. This proves the existence of such J_j, and the maximum $b(I_i)$.

  Then, we need to show that $\boxed{1}$ actually considers all the intervals that overlaps with J_j. This is true because: (1) in the case where the algorithm is not called recursively, the algorithm will just iterate from the beginning of the R, so every possible intervals that overlaps with I_i is considered. $(\star)$ **(2)** in the case where the algorithm is called recursively, although we only passed the from $I^{\text{prev}}_{\text{next start}}$ to the end of R from the caller, this is sufficient, since we know that all the intervals $b(I^{\text{prev}}_i)$ before the $I^{\text{prev}}_{\text{next start}}$ is strictly less than than the $b(I_{i-1})$ according to $\boxed{2}$, and that we also know that $\boxed{3}$ guarantees that $a(J_j) = a(J^{\text{prev}}_{\text{next start}}) > b(I_{i-1})$, that means all the intervals before $I^{\text{prev}}_{\text{next start}}$ will have have $b(I^{\text{prev}}_i) < a(J_j)$ which means that they are not in overlap with J_j, so ignoring them will still make sure that that $\boxed{1}$ actually considers all the intervals that overlaps with J_j.

- We also see that suppose I_a and I_b is both selected as solution. Then we see could find the corresponding J_c and J_d according to the property above, respectively. We could also find the I^c and I^d. Then, it's true that I^c doesn't have one interval that overlaps the J_d and I^d doesn't have one interval that overlaps the J_c.

  we see that I_a and I_b must be inserted to the solution during different recursion call (since in each call, it's only possible to add one interval to solution). Without loss of generality, let's assume I_a is added to the solution first. (That means, the I_a along with J_c is considered by the algorithm first, and this function call that considers I_a and J_c the initiates a recursive call which will consider the I_b and J_d eventually).

  The interval associate with the I_a is the J_c. Suppose there is one interval in the I^d that overlaps J_c, then in $\boxed{1}$, this interval will be already be considered and is thus put in I^c, and it will be thus already iterated $\boxed{4}$ and therefore won't be passed into the recursive call (and thus never appear in I^d). This leads to a contradiction. That means there is no interval in I^d that overlaps J_c.

  In the same way, suppose there is one interval I_i in the I^c that overlaps J_d, then that means $b(I_i) \geq a(J_d)$ and $a(I_i) \leq b(J_d)$. Since by property 1, we see that $b(I_a) \geq b(I_i)$, that means $b(I_a) \geq a(J_d)$. Since by the order we sort the B, since J_d is considered later than J_c, that means $b(J_d) \geq b(J_c)$. However, since I_c overlaps with J_c, that means $a(I_a) \leq b(J_c)$. This results in $a(I_a) \leq b(J_d)$ So, that means I_a overlaps with J_d, according to $\boxed{3}$, that means such J_d will be iterated and skipped. It won't be passed to the recursive function call. This means that later function call won't even consider the J_d that associate with I_b. This contradicts with the assumption. Therefore, that means this is no interval in the I^c that overlaps J_d.

  We therefore concludes that it's true that I^c doesn't have one interval that overlaps the J_d and I^d doesn't have one interval that overlaps the J_c.

Now suppose that our greedy solution G is worse than the optimal solution O, that is, we select n intervals while the optimal solution select less than n intervals. Then we could see that for the n red intervals $I'_0, I'_1 \cdots I'_n$ selected by the G, it corresponds to n blue intervals $J'_0, J'_1 \cdots J'_n$ defined by properties. Foreach J'_j, its overlapping set is the I^j. Then we see that the optimal must include one interval in the I^j otherwise the J_j' will not be covered. However, choosing anything in I^j will not help you to cover any other $I^{j'}$ where $j' \neq j$ due to the property 2. This means that the optimal solution has to choose a interval from every I^j. Yet, this will result in choosing at least n intervals, and this contradicts with our assumption. So, the greedy solution G is at least equally better than the optimal solution O. That means it's one optimal solution.

Q. E. D.