Question 1: Delta Distribution

> Evaluate the following integrals

(a)

> $$\int_0^{\infty} \delta(2t^{1/3} - 54)e^{2t}\mathrm{d}t$$
>
> Hint: Make a variable substitution $u = 2t^{1/3}$ to bring the Delta function into a standard form.

Let $u = 2t^{1/3}$ and thus $t = \frac{1}{8}u^3$, and thus

$$\mathrm{d}t = \frac{3}{8}u^2\mathrm{d}u$$

and thus

$$\int_0^{\infty} \delta(u-54)e^{\frac{1}{4}u^3}\frac{3}{8}u^2\mathrm{d}u = \frac{2187}{2}e^{39366}$$

(b)

> $$\int_{-\infty}^{\infty} \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta(t-4)\right)\sin\left(\frac{t^2}{4}\right)\mathrm{d}t$$
>
> Hint: you may integrate by parts

We could actually find a more general rule, notice that $\delta^{(n)}(\pm\infty) = 0$, and that means

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)\mathrm{d}x = f(x)\delta^{(n)}(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}f(x)\delta^{(n-1)}\mathrm{d}x = -\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}f(x)\delta^{(n-1)}\mathrm{d}x$$

and thus

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)\mathrm{d}x = (-1)^n\int_{-\infty}^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(x)\delta(x)\mathrm{d}x$$

and thus

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta(t-4)\right)\sin\left(\frac{t^2}{4}\right)\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\delta(t-4)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sin\left(\frac{t^2}{4}\right)\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\delta(t-4)\left(\frac{1}{2}\cos\left(\frac{t^2}{4}\right) - \frac{1}{4}t^2\sin\left(\frac{t^2}{4}\right)\right)\mathrm{d}t \\
&= \frac{1}{2}\cos(4) - 4\sin(4)
\end{aligned}$$

Question 2: Impulse response

> Consider a damped oscillator with impulse response, i.e., Green's function, given by
>
> $$G(t) = \begin{cases} 2e^{-t}\sin(4t) & t > 0 \\ 0 & t < 0 \end{cases}$$
>
> and subjected to a force

> $$F(t) = \begin{cases} 0 & t < 1 \\ 8 & 1 < t < 2 \\ 0 & 2 < t < 3 \\ 6(t-3) & t > 3 \end{cases}$$

(a)

> Sketch the force

By Mathematica

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In[11]:= g[t_] =
  Piecewise[{{0, t < 1}, {8, 1 < t < 2}, {0, 2 < t < 3 },
    {6 * (t - 3), t > 3}}]

         ⎧ 0          t < 1
         ⎪ 8          1 < t < 2
Out[11]= ⎨ 0          2 < t < 3
         ⎪ 6 (-3 + t) t > 3
         ⎩ 0          True

In[12]:= Plot[g[t], {t, 0, 5}]
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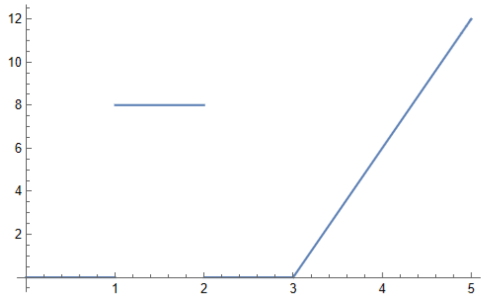
Out[12]=



(b)

> Construct an integral convolution expression for the response that is valid at all times. You do not need to evaluate the integrals, but they should be of a form that one could look up in a table of integrals.

the convolution have the form of

$$x(t) = \int_{-\infty}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau$$

for $t < 1$, the $F(\tau) = 0$ in above expression, and thus

$$x(t) = \int_{-\infty}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau = 0$$

for $1 < t < 2$,

$$x(t) = \int_{-\infty}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau = \int_{1}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau = \int_{1}^{t} 8 \cdot 2e^{-(t-\tau)} \sin(4(t-\tau))\mathrm{d}\tau$$

for $2 < t < 3$, the $F(\tau) = 0$ for $2 < \tau < 3$

$$x(t) = \int_{-\infty}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau = \int_{1}^{2} 8 \cdot 2e^{-(t-\tau)} \sin(4(t-\tau))\mathrm{d}\tau$$

for $t > 3$,

$$x(t) = \int_{-\infty}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau = \int_{1}^{2} 8 \cdot 2e^{-(t-\tau)} \sin(4(t-\tau))\mathrm{d}\tau + \int_{3}^{t} 6(\tau-3)2e^{-(t-\tau)} \sin(4(t-\tau))\mathrm{d}\tau$$

and the integral could be evaluated.

Question 3: Piecewise driving force

> A force acts on an initial quiescent ($x_0 = x(t=0) = 0, v_0 = v(t=0) = 0$), undamped mass-spring system described by the equation of motion
>
> $$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = F(t)$$
>
> where the driving force $F(t)$ is piece-wise analytic
>
> $$F(t) = \begin{cases} 0 & t < 0 \\ F_0\frac{t^2}{T^2} & 0 < t < T \\ F_0 & T < t \end{cases}$$
>
> Like $F(t)$, the response $x(t)$ will be piece-wise analytic, i.e., it has different analytic form depending on whether t is smaller or larger than T.
>
> In both cases, you do not need to evaluate the integral, but you should conclude with an expression that one could look up in a table of integrals.

From (41) of Lecture 16, we know that since the equation is undamped

$$G(t) = \begin{cases} \frac{\sin(\omega_d t)}{m\omega_d} & t > 0 \\ 0 & t < 0 \end{cases}$$

(a)

> Use the convolution to compose an expression for the response $x(t)$ at times $0 < t < T$.

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau \\ &= \int_{0}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau \\ &= \int_{0}^{t} F_0\frac{t^2}{T^2}\frac{\sin(\omega_d(t-\tau))}{m\omega_d}\mathrm{d}\tau \\ &= \frac{F_0}{m\omega_d}\frac{t^2}{T^2}\int_{0}^{t}\sin(\omega_d(t-\tau))\mathrm{d}\tau \end{aligned}$$

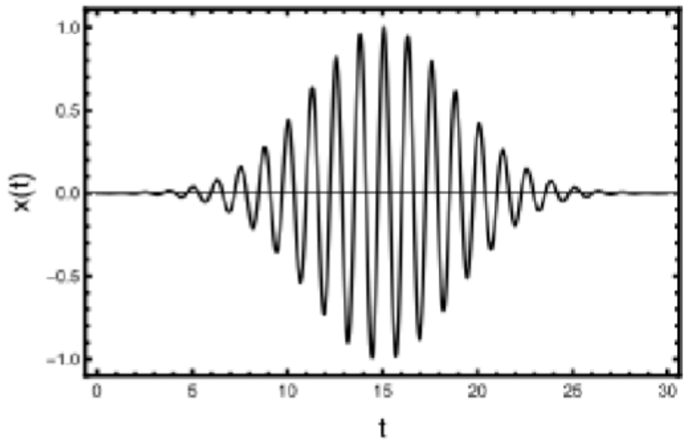
(b)

> Use the convolution to compose an expression for the response $x(t)$ valid at times $t > T$

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau \\ &= \int_{0}^{T} F(\tau)G(t-\tau)\mathrm{d}\tau + \int_{T}^{t} F(\tau)G(t-\tau)\mathrm{d}\tau \\ &= \int_{0}^{T} F_0\frac{t^2}{T^2}\frac{\sin(\omega_d(t-\tau))}{m\omega_d}\mathrm{d}\tau + F_0\int_{T}^{t}\frac{\sin(\omega_d(t-\tau))}{m\omega_d}\mathrm{d}\tau \\ &= \frac{F_0}{m\omega_d}\left(\int_{0}^{T}\frac{t^2}{T^2}\sin(\omega_d(t-\tau))\mathrm{d}\tau + \int_{T}^{t}\sin(\omega_d(t-\tau))\mathrm{d}\tau\right) \end{aligned}$$

Question 4: Gaussian tone burst

> The complex Gaussian tone burst
>
> $$x(t) = e^{i\alpha t}\exp\left(-\frac{(t-t_0)^2}{T^2}\right)$$

> is oscillatory with nominal frequency α, under a Gaussian envelop of width T and centered on time t_0. The figure plots its real part for the case $\alpha = 5, T = \sqrt{30}$ and $t_0 = 15$. If this was a plot of pressure on your ear versus time, you would hear a short beep, assuming that the frequency $f = \frac{\alpha}{2\pi}$ is within the hearing range.
>
> 
>

(a)

> Show that
>
> $$|\tilde{x}(\omega)| = \sqrt{\pi} T \exp\left(-\alpha \frac{(\alpha - \omega)^2 T^2}{4}\right)$$
>
> Hint: you may use
>
> $$\int_{-\infty}^{\infty} \exp(i\beta x) \exp(-x^2) \mathrm{d}x = \sqrt{\pi} \exp\left(-\frac{\beta^2}{4}\right)$$

$$\begin{aligned}
\tilde{x}(\omega) &= \int_{-\infty}^{\infty} x(t) \exp(-i\omega t) \mathrm{d}t = \int_{-\infty}^{\infty} e^{i\alpha t} \exp\left(-\frac{(t-t_0)^2}{T^2}\right) \exp(-i\omega t) \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \exp(i(\alpha - \omega)t) \exp\left(-\left(\frac{t - t_0}{T}\right)^2\right) \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \exp(i(\alpha - \omega)(T\tau + t_0)) \exp(-\tau^2) T \mathrm{d}\tau \\
&= T \exp(i(\alpha - \omega)t_0) \int_{-\infty}^{\infty} \exp(i(\alpha - \omega)T\tau) \exp(-\tau^2) \mathrm{d}\tau \\
&= T \exp(i(\alpha - \omega)t_0) \sqrt{\pi} \exp\left(-\frac{(\alpha - \omega)^2 T^2}{4}\right)
\end{aligned}$$

$$\boxed{|\tilde{x}(\omega)| = T\sqrt{\pi} \exp\left(-\frac{(\alpha - \omega)^2 T^2}{4}\right)}$$

(b)

> Plot the absolute value $|\tilde{x}(\omega)|$ for two cases (i) $T = 30, \alpha = 5, t_0 = 15$, and (ii) $T = \sqrt{3}, \alpha = 5, t_0 = 15$

when $T = \sqrt{30}, \alpha = 5, t_0 = 15$

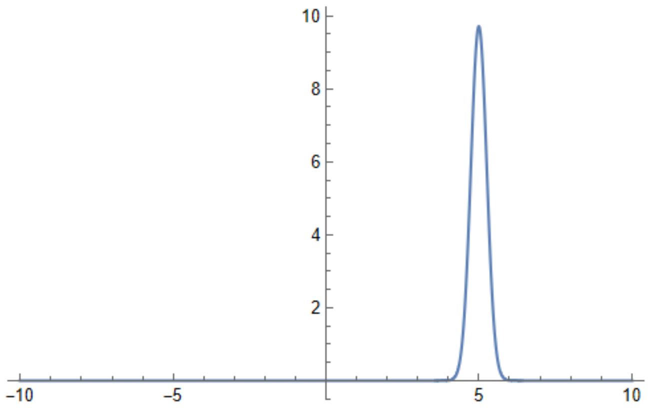
```
In[27]:= x[o_] = Sqrt[Pi] * T * Exp[- (a - o) ^2 * T^2 / 4]
```

Out[27]= $e^{-\frac{15}{2}(5-o)^2}\sqrt{30\pi}$

```
In[34]:= Plot[x[o], {o, -10, 10}, PlotRange → All]
```

General: Exp[-1687.41] is too small to represent as a normalized machine number; precision may be lost.

Out[34]=



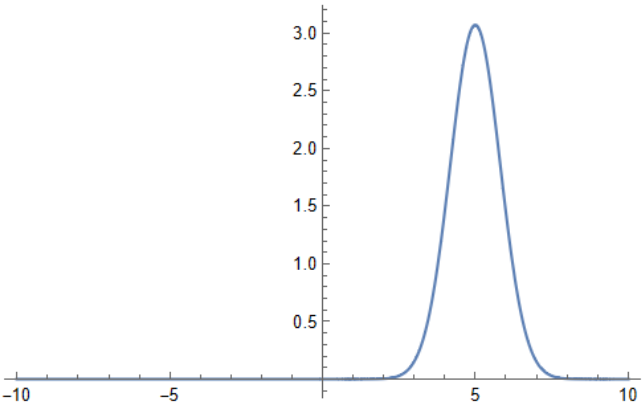
when $T = \sqrt{3}, \alpha = 5, t_0 = 15,$

```
In[36]:= x[o_] = Sqrt[Pi] * T * Exp[- (a - o) ^2 * T^2 / 4]
```

Out[36]= $e^{-\frac{3}{4}(5-o)^2}\sqrt{3\pi}$

```
In[37]:= Plot[x[o], {o, -10, 10}, PlotRange → All]
```

Out[37]=



(c)

> Show that $|\tilde{x}(\omega)|$ peaks at the nominal frequency $\omega = \alpha$. Discuss how your result illustrates that large T (i.e., being well spread out in time) corresponds to a Fourier transform that is highly compact in frequency.

From the graph, we could indeed see that it peek at $\alpha = 5$. The reason is simply because $(a - \omega)^2$ has its minimum value at $\alpha = \omega$, and thus $\exp\left(-\frac{(\alpha-\omega)^2 T^2}{4}\right)$ has maximum value and thus $|\tilde{x}(\omega)|$.

The graph illustrate that larger T result in a more compact peak. This is expected since with higher T, $-\frac{(\alpha-\omega)^2 T^2}{4}$ decrease more rapidly when away from the $\omega = \alpha$, and thus the $|\tilde{x}(\omega)|$. A more intuitive explanation comes from signal processing. If you have a sound at certain frequency for a longer period of time, you are more certain about the frequency of that sound (which, when illustrate on the plot of $|\tilde{x}(\omega)|$, is a more compact peak).