Problem 1

(a)

According to Lecture 38, we know that it's always that $\theta_{\text{inc}} = \theta_{\text{refl}}$. For the θ_{tran}, according to the Snell's law, we have $\sin\theta_{\text{tran}} n_2 = \sin\theta_{\text{inc}} n_1$ and therefore

$$\theta_{\text{tran}} = \arcsin\left(\frac{n_1}{n_2}\sin\theta_{\text{inc}}\right)$$

However, if we just want $\cos\theta_{\text{tran}}$, then it's

$$\cos\theta_{\text{tran}} = \sqrt{1-\sin^2\theta_{\text{tran}}} = \sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_{\text{inc}}\right)^2}$$

> For (b), (c), (d), and (e), (f). This is similar to what Lecture 38 did in the case 1. We just follow the similar process. Denote the income electric field E_i, reflected electric field E_r, transmitted electric field E_t. Same for the magnetic field.

(b)

Call the space outside the material *space 1*, call the space inside the material *space 2*. We know from the fact that **D-normal** is continuous is:

$$\begin{aligned}
\vec{D}_1\cdot\hat{n} &= \vec{D}_2\cdot\hat{n} \\
\epsilon_0\kappa_1(\vec{E}_i+\vec{E}_r)\cdot\hat{n} &= \epsilon_0\kappa_2\vec{E}_t\cdot\hat{n} \\
-\epsilon_0\kappa_1(E_i\sin\theta_{\text{inc}}+E_r\sin\theta_{\text{inc}}) &= -\epsilon_0\kappa_2(E_t\sin\theta_{\text{tran}}) \\
n_1\cdot n_1\sin\theta_{\text{inc}}(E_i+E_r) &= n_2\cdot n_2\sin\theta_{\text{tran}}E_t
\end{aligned}$$

since from Snell's Law, we know that $n_1\cdot\sin\theta_{\text{inc}} = n_2\cdot\sin\theta_2$ therefore

$$n_1(E_i+E_r) = n_2\cdot E_t$$

(c)

We know from the fact that $E_\parallel$ is continuous:

$$\begin{aligned}
\vec{E}_{\parallel,1} &= \vec{E}_{\parallel,2} \\
E_i\cos\theta_{\text{inc}} - E_r\cos\theta_{\text{inc}} &= E_t\cos\theta_{\text{tran}}
\end{aligned}$$

and therefore we get

$$(E_i-E_r)\cos\theta_{\text{inc}} = E_t\cos\theta_{\text{tran}}$$

(d)

Since we know from the fact that $H_\parallel$ is continuous:

$$\begin{aligned}
\vec{H}_{\parallel,1} &= \vec{H}_{\parallel,2} \\
\vec{B}_i+\vec{B}_r &= \vec{B}_t \\
\frac{E_i}{c}n_1+\frac{E_r}{c}n_1 &= \frac{E_t}{c}n_2
\end{aligned}$$

so that

$$n_1(E_i + E_r) = n_2 E_t$$

(e)

we see that the equation we get from (b) and (d) is actually the same (so that they are degenerate). To solve the equation, we have

$$\begin{aligned}(E_i - E_r)\cos\theta_{\text{inc}} &= E_t \cos\theta_{\text{tran}} \\ n_1(E_i + E_r) &= n_2 E_t\end{aligned}$$

We see that

$$\begin{aligned}1 - \frac{E_r}{E_i} &= \frac{E_t}{E_i}\frac{\cos\theta_{\text{tran}}}{\cos\theta_{\text{inc}}} \\ 1 + \frac{E_r}{E_i} &= \frac{n_2 E_t}{n_1 E_i}\end{aligned}$$

Therefore we have

$$\frac{E_t}{E_i} = \frac{2n_1\cos\theta_{\text{inc}}}{n_2\cos\theta_{\text{inc}} + n_1\cos\theta_{\text{tran}}} = \frac{2n_1 n_2\cos\theta_{\text{inc}}}{n_2^2\cos\theta_{\text{inc}} + n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_{\text{inc}}}}$$

and (with some messy derivation listed in the slides),

$$\frac{E_r}{E_i} = \frac{n_2^2\cos\theta_{\text{inc}} - n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_{\text{inc}}}}{n_2^2\cos\theta_{\text{inc}} + n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_{\text{inc}}}}$$

(f)

Then we just want that the numerator of E_r/E_i to be zero, so

$$\begin{aligned}n_2^2\cos\theta_{\text{inc}} &= n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_{\text{inc}}} \\ n_2^4\cos^2\theta_{\text{inc}} &= n_1^2(n_2^2 - n_1^2\sin^2\theta_{\text{inc}}) \\ (n_2^4 - n_1^2 n_2^2)\cos^2\theta_{\text{inc}} &= (n_1^2 n_2^2 - n_1^4)\sin^2\theta_{\text{inc}} \\ n_2^2(n_2^2 - n_1^2)\cos^2\theta_{\text{inc}} &= n_1^2(n_2^2 - n_1^2)\sin^2\theta_{\text{inc}} \\ n_2\cos\theta_{\text{inc}} &= n_1\sin\theta_{\text{inc}}\end{aligned}$$

Therefore, we find that

$$\frac{n_2}{n_1} = \frac{\sin\theta_{\text{inc}}}{\cos\theta_{\text{inc}}}$$

when

$$\theta_{\text{inc}} = \arctan\left(\frac{n_2}{n_1}\right)$$

Then the reflected wave vanishes.

(g)

$$R=\left(\frac{E_r}{E_i}\right)^2=\left(\frac{n_2^2\cos\theta_{\text{inc}}-n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}{n_2^2\cos\theta_{\text{inc}}+n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}\right)^2$$

It's still that when

$$\theta_{\text{inc}}=\arctan\left(\frac{n_2}{n_1}\right)$$

the reflected wave vanishes.

(h)

Then,

$$T=\frac{n_2}{n_1}\frac{\cos\theta_{\text{tran}}}{\cos\theta_{\text{inc}}}\left(\frac{2n_1n_2\cos\theta_{\text{inc}}}{n_2^2\cos\theta_{\text{inc}}+n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}\right)^2$$

we find that

we see that

$$n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}=n_1n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_{\text{inc}}\right)^2}=n_1n_2\cos\theta$$

therefore,

$$\begin{aligned}R+T&=\left(\frac{n_2^2\cos\theta_{\text{inc}}-n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}{n_2^2\cos\theta_{\text{inc}}+n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}\right)^2+\frac{n_2}{n_1}\frac{\cos\theta_{\text{tran}}}{\cos\theta_{\text{inc}}}\left(\frac{2n_1n_2\cos\theta_{\text{inc}}}{n_2^2\cos\theta_{\text{inc}}+n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}\right)^2\\&=\frac{4n_1n_2^3\cos\theta_{\text{tran}}\cos\theta_{\text{inc}}+\left(n_2^2\cos\theta_{\text{inc}}-n_1n_2\cos\theta_{\text{inc}}\right)^2}{\left(n_2^2\cos\theta_{\text{inc}}+n_1n_2\cos\theta\right)^2}\end{aligned}$$

we see that let $a=n_2^2\cos\theta_{\text{inc}}$ and $b=n_1n_2\cos\theta_{\text{tran}}$, the formula becomes

$$R+T=\frac{4ab+(a-b)^2}{(a+b)^2}=1$$

this is true because energy is conserved $(S_{\text{inc}}=S_{\text{ref}}+S_{\text{tran}})$. (Similar to the case in Lecture 37 Page 10)

Problem 2

(a)

$E_{\parallel}$, $B_{\parallel}$, $B_{\perp}$

(b)

Call the space outside the material *space 1*, call the space inside the material *space 2*. We know from the fact that **B-normal** is continuous is:

$$\begin{aligned}\vec{B}_1 \cdot \hat{n} &= \vec{B}_2 \cdot \hat{n} \\ B_i \sin\theta_{\text{inc}} + B_r \sin\theta_{\text{inc}} &= B_t \sin\theta_{\text{tran}} \\ \frac{E_i}{c} n_1 \sin\theta_{\text{inc}} + \frac{E_r}{c} n_1 \sin\theta_{\text{inc}} &= \frac{E_t}{c} n_2 \sin\theta_{\text{tran}} \\ n_1 \sin\theta_{\text{inc}} (E_i + E_r) &= E_t n_2 \sin\theta_{\text{tran}}\end{aligned}$$

since from Snell's Law, we know that $n_1 \cdot \sin\theta_{\text{inc}} = n_2 \cdot \sin\theta_2$ therefore

$$E_i + E_r = E_t$$

(c)

We know from the fact that $E_{\|}$ is continuous:

$$\begin{aligned}\vec{E}_{\|,1} &= \vec{E}_{\|,2} \\ E_i + E_r &= E_t\end{aligned}$$

(d)

Since we know from the fact that $H_{\|}$ is continuous:

$$\begin{aligned}\vec{H}_{\|,1} &= \vec{H}_{\|,2} \\ \vec{B}_i + \vec{B}_r &= \vec{B}_t \\ B_i \cos\theta_{\text{inc}} - B_r \cos\theta_{\text{inc}} &= B_t \cos\theta_{\text{tran}} \\ \frac{E_i}{c} n_1 \cos\theta_{\text{inc}} - \frac{E_r}{c} n_1 \cos\theta_{\text{inc}} &= \frac{E_t}{c} n_2 \cos\theta_{\text{tran}}\end{aligned}$$

so that

$$n_1 (E_i - E_r) \cos\theta_{\text{inc}} = n_2 E_t \cos\theta_{\text{tran}}$$

(e)

e see that the equation we get from (b) and (c) is actually the same (so that they are degenerate). To solve the equation, we have

$$\begin{aligned}E_i + E_r &= E_t \\ n_1 (E_i - E_r) \cos\theta_{\text{inc}} &= n_2 E_t \cos\theta_{\text{tran}}\end{aligned}$$

That means

$$\begin{aligned}1 + \frac{E_r}{E_i} &= \frac{E_t}{E_i} \\ 1 - \frac{E_r}{E_i} &= \frac{E_t}{E_i} \frac{n_2 \cos\theta_{\text{tran}}}{n_1 \cos\theta_{\text{inc}}}\end{aligned}$$

and therefore

$$\left(1+\frac{n_2\cos\theta_{\text{tran}}}{n_1\cos\theta_{\text{inc}}}\right)\frac{E_t}{E_i}=2$$

$$\frac{E_t}{E_i}=\frac{2n_1\cos\theta_{\text{inc}}}{n_1\cos\theta_{\text{inc}}+n_2\cos\theta_{\text{tran}}}=\frac{2n_1\cos\theta_{\text{inc}}}{n_1\cos\theta_{\text{inc}}+\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}$$

and we find that

$$\frac{E_r}{E_i}=\frac{E_t}{E_i}-1=\frac{n_1\cos\theta_{\text{inc}}-\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}{n_1\cos\theta_{\text{inc}}+\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}$$

(f)

we see that

$$R=\left(\frac{n_1\cos\theta_{\text{inc}}-\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}{n_1\cos\theta_{\text{inc}}+\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}\right)^2$$

(g)

$$T=\frac{n_2}{n_1}\frac{\cos\theta_{\text{tran}}}{\cos\theta_{\text{inc}}}\left(\frac{2n_1\cos\theta_{\text{inc}}}{n_1\cos\theta_{\text{inc}}+\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}}\right)^2$$

We see that

$$T=\frac{4n_1n_2\cos\theta_{\text{tran}}\cos\theta_{\text{inc}}}{\left(n_1\cos\theta_{\text{inc}}+\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}\right)^2}$$

Let $a=n_1\cos\theta_{\text{inc}}$. Let $b=\sqrt{n_2^2-n_1^2\sin^2\theta_{\text{inc}}}$, we see that $b=n_2\cos\theta_{\text{tran}}$.

Therefore

$$R+T=\left(\frac{a-b}{a+b}\right)^2+\frac{4ab}{(a+b)^2}=1$$

this is true because energy is conserved ($S_{\text{inc}}=S_{\text{ref}}+S_{\text{tran}}$). (Similar to the case in Lecture 37 Page 10)