Question 1: Air force practice

The zero order motion is

$$\ddot{x}_0(t) = 0$$
$$\ddot{y}_0(t) = 0$$
$$\ddot{z}_0(t) = -g$$

and with initial condition

$$x(0) = 0$$
$$y(0) = 0$$
$$z(0) = h$$
$$\dot{x}(0) = \frac{u}{\sqrt{2}}$$
$$\dot{y}(0) = -\frac{u}{\sqrt{2}}$$
$$\dot{z}(0) = 0$$

and thus

$$x_0(t) = \frac{u}{\sqrt{2}}t$$
$$y_0(t) = -\frac{u}{\sqrt{2}}t$$
$$z_0(t) = h - \frac{1}{2}gt^2$$

and for the first order

$$\ddot{x}_1 = 2\omega(-\frac{u}{\sqrt{2}})\sin\theta - 2\omega\cdot 0\cos\theta = -\sqrt{2}\omega u\sin\theta$$
$$\ddot{y}_1 = -2\omega(\frac{u}{\sqrt{2}})\sin\theta = -\sqrt{2}\omega u\sin\theta$$
$$\ddot{z}_1 = -g + 2\omega(\frac{u}{\sqrt{2}})\cos\theta = \sqrt{2}\omega u\cos\theta$$

and thus

$$x_1 = -\frac{\sqrt{2}}{2}\omega u\sin\theta t^2$$
$$y_1 = -\frac{\sqrt{2}}{2}\omega u\sin\theta t^2$$
$$z_1 = \frac{\sqrt{2}}{2}\omega u\cos\theta t^2$$

and thus, combining the terms, we got

$$x(t) = \frac{u}{\sqrt{2}}t - \frac{\sqrt{2}}{2}\omega u\sin\theta t^2$$
$$y(t) = -\frac{u}{\sqrt{2}}t - \frac{\sqrt{2}}{2}\omega u\sin\theta t^2$$
$$z(t) = h - \frac{1}{2}gt^2 + \frac{\sqrt{2}}{2}\omega u\cos\theta t^2$$

thus, the time when Humvee lands is

$$z(t) = 0$$
$$(\frac{1}{2}g - \frac{\sqrt{2}}{2}\omega u\cos\theta)t^2 = h$$
$$t^* = \sqrt{\frac{2h}{g - \sqrt{2}\omega u\cos\theta}}$$

and it thus lands on

$$x(t^*) = \frac{u}{\sqrt{2}}\sqrt{\frac{2h}{g-\sqrt{2}\omega u\cos\theta}} - \frac{\sqrt{2}}{2}\omega u\sin\theta\frac{2h}{g-\sqrt{2}\omega u\cos\theta}$$

$$y(t^*) = -\frac{u}{\sqrt{2}}\sqrt{\frac{2h}{g-\sqrt{2}\omega u\cos\theta}} - \frac{\sqrt{2}}{2}\omega u\sin\theta\frac{2h}{g-\sqrt{2}\omega u\cos\theta}$$

and, trying to simplify a bit

$$x(t^*) = \sqrt{\frac{u^2h}{g-\sqrt{2}\omega u\cos\theta}} - \frac{\sqrt{2}h\omega u\sin\theta}{g-\sqrt{2}\omega u\cos\theta}$$

$$y(t^*) = -\sqrt{\frac{u^2h}{g-\sqrt{2}\omega u\cos\theta}} - \frac{\sqrt{2}h\omega u\sin\theta}{g-\sqrt{2}\omega u\cos\theta}$$

Question 2: Golf on an alien planet

The period T is $2 * 3600 = 7200\mathrm{s}$, and thus
$\omega = \frac{2\pi}{T} = \frac{\pi}{3600} \approx 0.000873\mathrm{rad/s}$

We could just use the formula $R = \frac{|v|}{2\omega\sin\theta}$, and thus

$$R = \frac{v}{2\cdot\omega\sin 45^\circ} = \frac{v}{\sqrt{2}\omega}$$

notice that since the ground is frictionless, so the golf ball will continue to move on the ground, and since the Coriolis force is $F = -2m\vec{\omega}\times\vec{v}$, it's always perpendicular to the motion (thus doesn't change it's speed). Therefore

$$T = \frac{2\pi R}{v} = \frac{2\pi}{\sqrt{2}\omega} = \frac{\sqrt{2}\pi}{\omega} \approx 5091.17\mathrm{s}$$

Question 3: Great circles

> Show that the shortest distance between two points on a sphere is a great circle. Start with the expression for the line element in spherical polar coordinates (r, θ, ϕ), given by
>
> $$\mathrm{d}l^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$
>
> Consider two points on the sphere for which $r = R = \mathrm{const}$ (and $\mathrm{d}r = 0$), specified by $P_1 = (\theta_1, \phi_1)$

We know that line elements is $\mathrm{d}l^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$. Since $r = R = \mathrm{const}$

$$\mathrm{d}l^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

and thus

$$\frac{\mathrm{d}l}{\mathrm{d}\theta} = \sqrt{R^2 \left(\frac{\mathrm{d}\theta^2}{\mathrm{d}\theta^2}\right) + \sin^2\theta \left(\frac{\mathrm{d}\phi^2}{\mathrm{d}\theta^2}\right)}$$

$$\mathrm{d}l = R\sqrt{1 + \sin^2\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

Then, the path between two points is

$$L = \int_{\theta_1}^{\theta_2} \mathrm{d}l = R\int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

etaand sietance the Euler-Lagrange equation in this case is

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} = 0$$

because L is not dependent directly on ϕ. Thus

$$\frac{\partial L}{\partial \dot{\phi}} = c$$

$$\frac{1}{2}(1+\sin^2\theta\dot{\phi}^2)^{-1/2}\cdot 2\sin^2\theta\dot{\phi} = c$$

$$\frac{\sin^2\theta\dot{\phi}}{\sqrt{1+\sin^2\theta\dot{\phi}^2}} = c$$

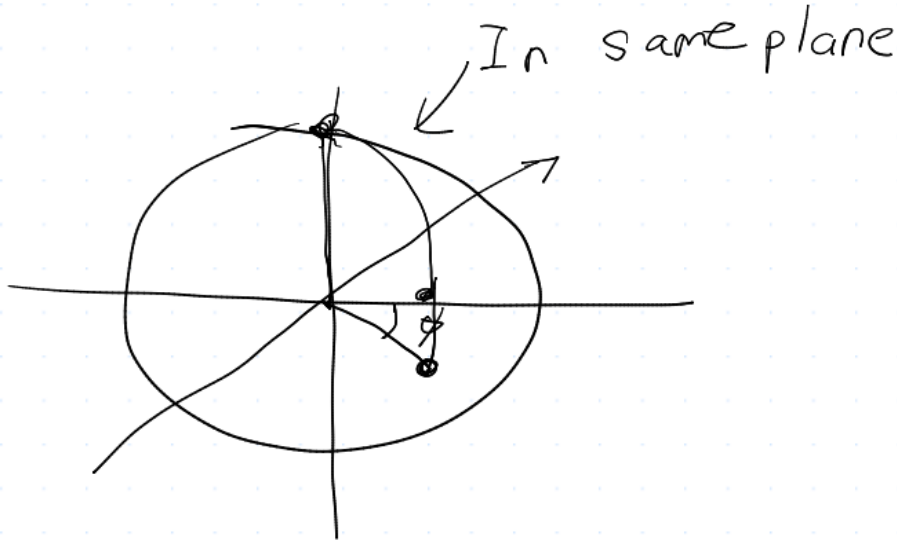
Without loss of generality, we could see set z axis to pass the $P_1 = (\theta_1, \phi_1)$ where $\theta_1 = 0$, and thus

$$\frac{\sin^2 0\dot{\phi}}{\sqrt{1+\sin^2 0\dot{\phi}^2}} = c$$

and thus it is evident that $c = 0$, and thus

$$\sin\theta = 0 \quad \text{or} \quad \dot{\phi} = 0$$

If $\theta = 0$, then both points are on same location, it is trivially the great circle. If $\dot{\phi} = 0$, then two points only different in θ, which should roughly looks like this:



so they are indeed in the same great circle.

Question 4 Variational derivatives

Using the same technique in Lecture Notes 21

$$\delta F=\int_0^1 \frac{\partial f(x,y,y',y'')}{\partial y''}\delta y''\mathrm{d}x=\int_0^1 2y''\delta y''\mathrm{d}x$$

We find that

$$\begin{aligned}2y''\delta y'' &= \frac{\mathrm{d}}{\mathrm{d}x}(2y''\delta y')-2y'''\delta y'\\ 2y'''\delta y' &= \frac{\mathrm{d}}{\mathrm{d}x}(2y'''\delta y)-2y''''\delta y\end{aligned}$$

and thus

$$\delta F=\int_0^1 \mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}(2y''\delta y'-2y'''\delta y)+\int_0^1 2y''''\delta y\mathrm{d}x$$

and thus

$$\delta F=\int_0^1 2y''''\delta y\mathrm{d}x+2y''\delta y'\Big|_0^1-2y'''\delta y\Big|_0^1$$

As question indicate, we must satisfy that

$$\eta'(0)y''(0)=0,\quad \eta'y''(1)=0\quad \eta(0)y'''(0)=0,\quad \eta(1)y'''(1)=0$$
$$\eta(0)=\eta(1)=\eta'(0)=0$$

and $\eta'(1)$ doesn't necessarily to be 0. This means $y''(1)$ must be 0.

and thus we have the equation and the boundary conditions

$$\begin{gathered}y''''=0\\ y(0)=y(1)=y'(0)=y''(1)=0\end{gathered}$$

and thus

$$\begin{gathered}y=Ax^3+Bx^2+Cx+D\\ y'=3Ax^2+2Bx+C\\ y''=6Ax+2B\end{gathered}$$

and applying boundary condition, we know

$$
\begin{gathered}
D = 0 \\
A + B + C = 0 \\
C = 0 \\
6A + 2B = 0
\end{gathered}
$$

and thus

$$A = B = C = D = 0$$

and thus

$$\boxed{y = 0}$$

which turns out to be really simple.