Question 1

(a)

$$
\begin{aligned}
\vec{v}(t) &= \frac{d\vec{r}(t) - \vec{r}'(t)}{dt} \\
&= \frac{d}{dt}((6\alpha_1 t^2 + 4\alpha_2 t)\vec{e}_x - 3\alpha_2 t^3 \vec{e}_y + 6\alpha_3 \vec{e}_z) - (6\alpha_1 t^2 \vec{e}_x - (3\alpha_2 t^3 - 7\alpha_3)\vec{e}_y + 4\alpha_3 \vec{e}_z) \\
&= \frac{d}{dt}(\alpha_2 t \vec{e}_x + 7\alpha_3 \vec{e}_y + 2\alpha_3 \vec{e}_z) \\
&= \alpha_2 \vec{e}_x
\end{aligned}
$$

(b)

$$
\begin{aligned}
\vec{a}(t) = \dot{v}(t) = \ddot{r}(t) = 12\alpha_1 \vec{e}_x - 18\alpha_2 t \vec{e}_y \\
\vec{a}'(t) = \dot{v}'(t) = \ddot{r}'(t) = 12\alpha_1 \vec{e}_x - 18\alpha_2 t \vec{e}_y
\end{aligned}
$$

(c)

Yes, since $\vec{a}(t) = \vec{a}'(t)$, that means the S' is not accerating, and the change in the frame is a Galilean transformations. So, since Newton's law applies in the S frame for it's a intertial frame, it also applies in the S' frame. Therefore, S' is also a inertial frame.

Question 2

(a)

$$
\begin{aligned}
\vec{F}_r = -\frac{GMm}{r^2} = m\dot{v}(t) \\
vm\frac{dv}{dt} = -\frac{GMm}{r^2}\frac{dr}{dt} \\
\int_{v_0}^{v(r)} vmdv = \int_R^r -\frac{GMm}{r^2} dr \\
\frac{1}{2}mv^2\Big|_{v_0}^{v(r)} = \frac{GMm}{r}\Big|_R^r \\
\frac{1}{2}m(v^2 - v_0^2) = \frac{GMm}{r} - \frac{GMm}{R} \\
\boxed{v = \sqrt{2(\frac{GM}{r} - \frac{GM}{R}) + v_0^2}}
\end{aligned}
$$

(b)

When the rocket have minimum velocity, it will reach $v = 0$ after escape the Mars ($r = \infty$). That is

$$\lim_{r\to\infty} v = 0$$

$$\sqrt{\frac{-2GM}{R} + v_0^2} = 0$$

$$v_0^2 = \frac{2GM}{R}$$

$$v_0 = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2 \cdot 6.674 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \cdot 6.39 \cdot 10^{23} kg}{3396 \cdot 1000m}} \approx 5011.58m$$

Question 3

(a)

$$F = -\frac{dU(x)}{dx} = -2Ax - 3Bx^2$$

(b)

The total energy of the particle is

$$E_0 = U(0) + T(0) = \frac{1}{2} m v_0^2$$

Thus, the kinetic energy at given point $T(x)$ is

$$T = E_0 - U(x)$$

The potential function $U(x)$ have extrema points

$$\frac{dU(x)}{dx} = 0 = 2Ax + 3Bx^2$$

$$x = 0, -\frac{2A}{3B}$$

$$\frac{d^2U(x)}{dx^2} = 2A + 6Bx$$

$$\frac{dU}{dx}(-\frac{2A}{3B}) = 2A - 6B\frac{2A}{3B} = -2A < 0$$

$x = -\frac{2A}{3B}$ is a local maximum.

When $B > 0, U(x)$ increase and goes unbounded as $x \to \infty$, while as $x \to -\infty$, it first approaches $x = -\frac{2A}{3B}$ the local maximum, and the decrease to $-\infty$

When $B < 0, U(x)$ increase and goes unbounded as $x \to -\infty$ while as $x \to \infty$, it first approaches $x = -\frac{2A}{3B}$ the local maximum, and the decrease to $-\infty$

When, $B = 0, U(x)$ increase as $|x| \to \infty$, and approaches ∞.

Thus, the particles must have at least $U(-\frac{2A}{3B})$ total energy to pass through the point $x = -\frac{2A}{3B}$, that is

$$E_0 \geq U(-\frac{2A}{3B}) = A(-\frac{2A}{3B})^2 + B(-\frac{2A}{3B})^3 = \frac{4A^3}{9B^2} - \frac{8A^3}{27B^2} = \frac{4A^3}{27B^2}$$

and thus

$$\frac{1}{2}mv_0^2 \geq \frac{4A^3}{27B^2}$$
$$|v_0| \geq \sqrt{\frac{8A^3}{m \cdot 27B^2}}$$
$$v_c = \sqrt{\frac{8A^3}{m \cdot 27B^2}}$$

Question 4

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In[29]:= -D[A*x^2 + B*x^3, x]

Out[29]= -2 A x - 3 B x^2
```

Same results as in the 3(a)

Question 5

Let

$$\frac{dU}{dx} = -2ax^{-3} + 2bx = 0$$
$$ax^{-3} = bx$$
$$x^4 = \frac{a}{b}$$
$$x = \sqrt[4]{\frac{a}{b}}$$
$$\frac{dU}{dx} = 6ax^{-4} + 2b = 6a\frac{b}{a} + 2b = 8b > 0$$

Thus, $x_0 = \sqrt[4]{\frac{a}{b}}$.

$$U(x) \approx U(x_0) + \frac{dU}{dx}(x_0)(x - x_0) + \frac{1}{2}\frac{d^2U}{dx^2}(x_0)(x - x_0)^2 + O(x^3)\cdots$$

The linear term is 0, thus

$$U(x) \approx U(\sqrt[4]{\frac{a}{b}}) + \frac{1}{2}8b(x - x_0)^2 = 2\sqrt{ab} + \frac{1}{2}8bz^2$$

and $k = 8b$

(c)

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8b}{m}}$

and the period of oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{8b}}$

Question 6

(a)

A: distance divided by time (like m/s)

β: inverse of distance m^{-1}

(b)

$$F = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m\dot{v}v$$

$$F = m \cdot \frac{1}{2}A(\beta e^{\beta x} - \beta e^{\beta x}) \cdot \frac{1}{2}A(e^{\beta x} + e^{-\beta x})$$

$$F = \frac{1}{4}mA^2\beta(e^{2\beta x} - e^{-2\beta x})$$

(c)

A^2 : distance squared divided by time squared

m: mass (kg)

$(e^{2\beta x} - e^{-2\beta x})$: dimensionless

F: distance squared divided by time squared multiply by inverse of distance and mass ($kg \cdot m^2/s^2 \cdot m^{-1} = kg \cdot m/s^2$) is **distance times mass divided by time squared**, which is correct unit for force, the dimension is consistant.