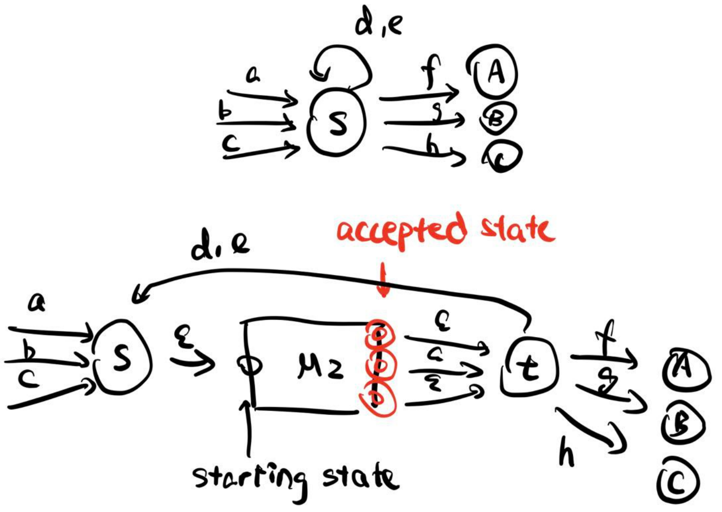
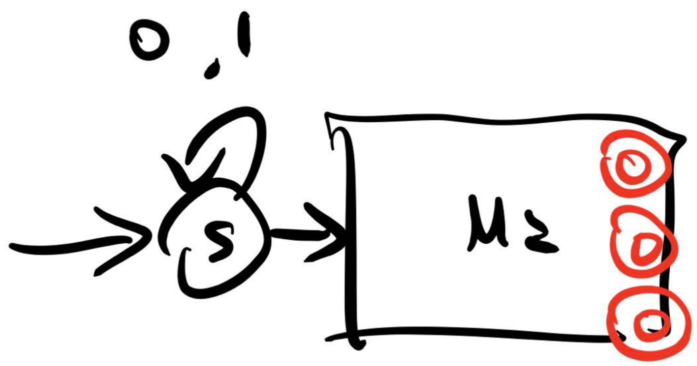
Problem 2

(a)

We could do product construction of two machine, we modify the M_1 in the following way:



that is, the every transition in the M_1 will be inserted with a M_2 so that the string in M_2 could be plugged into the machine. This, however, does not constrain the number of the string in M_2 that could appear. We design another machine that accept the string with the substring that in M_2 only appeared once.



The illustration show above is a machine (A) that accept the string if any of its substring is in M_2. We could negate the machine to have a machine that doesn't accept any string if it has a substring in M_2. We could do contatenation with these two machine, then the machine will accept the string with the substring that in M_2 only appeared once. Do production construction of these two machines (the modified M_1 and this one) we will get the NFA we wanted.

(b)

> In this second part you will give a different proof. Let r_1 be a regular expression for L_1 and r_2 be a regular expression for L_2, that is $L_1 = L(r_1)$ and $L_2 = L(r_2)$. We will develop a recursive algorithm that given r_1 and r_2 constructs a regular expression r' such that $L(r') = \text{insert}(L_1, L_2)$. No correctness proof is required but a brief explanation of the derivation would help you get partial credit in case of mistakes

(i)

> For each of base cases when $r_1 = \emptyset, \epsilon, a$ ($a \in \Sigma$), describe a regular expression for $\text{insert}(L(r_1), L(r_2))$ in terms of r_2 and the letters in Σ.

- If $r_1 = \epsilon$, that actually means $L(r_1) = \{\epsilon\}$. So, $\text{insert}(L(r_1), L(r_2))$ means insert the string in L_2 into the ϵ. Then the regular expression is just $\boxed{r' = \epsilon r_2 \epsilon = r_2}$.
- If $r_1 = a$, then to insert the string in L_2 into either before or after a. That is, the regular expression is

$$\boxed{r' = \epsilon a r_2 + r_2 a \epsilon = a r_2 + r_2 a}$$

- If $r_1 = \emptyset$, then $L(r_1) = \emptyset$. There is no string in language $L(r_1)$ and we thus couldn't insert any string in L_2 into L_1. Thus $\boxed{r' = \emptyset}$

(ii)

> Suppose $r_1 = s + t$ where s and t are regular expressions. Moreover let s' be a regular expression for $\text{insert}(L(s), L(r_2))$ and t' be a regular expression for $\text{insert}(L(t), L(r_2))$. Describe a regular expression for the language $\text{insert}(L(r_1), L(r_2))$ using r_2, s, t, s', t'.

The $L(r_1)$ either satisfies the regular expression s or t. In each cases, we already have the regular expression that tells us how to insert string of $L(r_2)$ into it (namely, s' and t'). Therefore, $r' = s' + t'$.

(iii)

> Same as previous part but now consider $r_1 = st$.

Using the assumption as **(ii)**, that is, define $s' = \text{insert}(L(t), L(r_2))$ and $t' = \text{insert}(L(t), L(r_2))$. $r_1 = st$ means a string that first satisfies regular expression s and then t. We could either insert the string in L_2 into the s part or t part. (we could also insert the string into the "space" between s and t / or before s / after t, these cases are already handled by s' and t', so we doesn't to consider them additionally). Therefore, the regular expression looks like

$$\boxed{r' = s't + st'}$$

which corresponding to insertion into either s part or t part.

(iv)

> Same as previous part but now consider $r_1 = (s)^*$.

Using the assumption as **(ii)**, that is, define $s' = \text{insert}(L(t), L(r_2))$. We could rewrite the r_1 as $r_1 = \epsilon + s(s)^*$ (this explicitly distinguishes the ϵ case). We know that from part **(i)** that $r_\epsilon = \epsilon$ has $r'_\epsilon = r_2$. So we only need to consider the case of $r_{\text{other}} = s(s)^*$. This regular expression represents all the string that have 1 or unlimited pattern of s. (namely $s+$). Then we could pick any one of s from the r_{other}, and insert the string represented by r_2 into it. The resulted regular expression thus looks like $r'_{\text{other}} = (s)^*s'(s)^*$. Combine all cases, we get

$$\boxed{r' = r_2 + (s)^*s'(s)^*}$$

(v)

We use some shorthand for clarity: for a regular expression s, define $L(s') = \text{insert}(L(s), L_2)$. We first calculate some pattern that will used later:

$$0' = 0(101) + (101)0$$
$$1' = 1(101) + (101)1$$
$$\begin{aligned}(01)' &= 0'1 + 01' \\ &= 0(101)1 + (101)01 + 01(101) + 0(101)1 \\ &= 01011 + 10101 + 01101\end{aligned}$$

and then

$$(0^*)' = 0^*0'0^* + 101 = 0^*(0101 + 1010)0^* + 101$$
$$(1^*)' = 1^*1'1^* + 101 = 1^*(1101 + 1011)1^* + 101$$

We consider the part 011^*0 first (it is complicated):

$$\begin{aligned}(011^*0)' &= (011^*)'0 + (011^*)0' = ((01)'1^* + 01(1^*)')0 + (011^*)(0101 + 1010) \\ &= (01011 + 10101 + 01101)1^*0 + 01(1^*(1101 + 1011)1^* + 101) + (011^*)(0101 + 1010)\end{aligned}$$

We then used the algorithm described above for r_1:

$$\begin{aligned}(r_1)' &= (0^* + (01)^* + 011^*0)' \\ &= (0^*)' + ((01)^*)' + (011^*0)' \\ &= 0^*(0101 + 1010)0^* + (01)^*(01)'(01)^* + 101 + (011^*0)' \\ &= 0^*(0101 + 1010)0^* + (01)^*(01011 + 10101 + 01101)(01)^* + \\ &\quad 101 + (01011 + 10101 + 01101)1^*0 + 01(1^*(1101 + 1011)1^* + 101) + (011^*)(0101 + 1010)\end{aligned}$$

it's complicated.