Problem 1

(a)

Using method of images, we know for every charge that is above the plane (x, y, z), there is a image charge below the plane $(x, y, -z)$. Therefore, the green function is

$$G(\vec{r}, \vec{r}') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}$$

(b)

Just taking the derivative

$$\frac{\partial G(\vec{r}, \vec{r}')}{\partial z'} = (z-z')((x-x')^2 + (y-y')^2 + (z-z')^2)^{-3/2} + (z+z')((x-x')^2 + (y-y')^2 + (z+z')^2)^{-3/2}$$

(c)

If $z' = 0$, then

$$\frac{\partial G(\vec{r}, \vec{r}')}{\partial z'} = 2z((x-x')^2 + (y-y')^2 + z^2)^{-3/2}$$

(d)

We know that

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int_{\mathrm{V}} G(\vec{r}, \vec{r}')\rho(\vec{r}')\mathrm{d}^3r' + \frac{1}{4\pi} \int_{\partial \mathrm{V}} \hat{m}' \cdot \boldsymbol{\nabla}_{r'} G(\vec{r}, \vec{r}') V(\vec{r}')\mathrm{d}a'$$

Since we are only considering the potential due to volatage biased disk, then

$$V(r) = \frac{1}{4\pi} \int_{\partial \mathrm{V}} \hat{m}' \cdot \boldsymbol{\nabla}_{r'} G(\vec{r}, \vec{r}') V(\vec{r}')\mathrm{d}a'$$

We notice that the $\hat{m}'$ is just $\hat{z}$ in this case (the normal vector on the boundary), and therefore the $\hat{m}' \cdot \boldsymbol{\nabla}_{r'} G(\vec{r}, \vec{r}')$ is just $\frac{\partial G(\vec{r}, \vec{r}')}{\partial z'}$. On the boundary it's when $z = 0$, so that's just what we got at in part (c). Thus,

$$V(r) = \frac{1}{4\pi} \int_{\partial \mathrm{V}} \left. \frac{\partial G(\vec{r}, \vec{r}')}{\partial z'} \right|_{z'=0} V(\vec{r}')\mathrm{d}a'$$

The $V(\vec{r}')$ is only V when it's in the disk, therefore

$$V(r) = \frac{V}{4\pi} \int_{\text{sphere}} 2z((x-x')^2 + (y-y')^2 + z^2)^{-3/2}\mathrm{d}a'$$

(e)

Since we only care about the $V(r)$ along x axis, so that $x = 0$ and $y = 0$. Using the cylindrical coordinate, we could write $x'^2 + y'^2 = s'^2$

$$
\begin{aligned}
V(r) &= \frac{zV}{2\pi}\int_0^R\int_0^{2\pi}\frac{s'}{(s'^2+z^2)^{3/2}}\mathrm{d}\theta\mathrm{d}s \\
&= zV\int_0^R\frac{s'}{(s'^2+z^2)^{3/2}}\mathrm{d}s \\
&= zV(\frac{1}{z}-\frac{1}{\sqrt{z^2+R^2}}) \\
&= V(1-\frac{z}{\sqrt{z^2+R^2}})
\end{aligned}
$$

If $z \ll R$, (i.e., just above origin) then

$$
V(r) \approx V(1-\frac{0}{\sqrt{0^2+R^2}}) = V
$$

It matches the intuition, because as the voltage gradually "propagates" to the space above the plane, its potential decrease gradually to 0 (when reaches infinity). If it's close to the disk, its potential should be close to potential of the disk (that is, just V approximately). Or, if we want to be more accurate on our approxiamation, we could do a taylor expansion and see

$$
V(r) = V(1-\frac{z}{R}+\frac{z^2}{2R^3}+\cdots)
$$

since $z \ll R$, we only keep the linear team and it is

$$
V(r) = V(1-\frac{z}{R})
$$

this could be understand as: when the $z \ll R$, all the influence from the $V=0$ conductor outside this charged biased circle could be ignored, and this situation is similar to the top/bottom plate of the capacitor. (where the potential linearly decreased)

(f)

Set $V(r)$ to kV, define $l = 1-k$

$$
\begin{aligned}
V(r) = V(1-\frac{z}{\sqrt{z^2+R^2}}) &= kV \\
\frac{z}{\sqrt{z^2+R^2}} &= l \\
\left(\frac{z}{l}\right)^2 &= z^2+R^2 \\
z &= R\sqrt{\frac{l^2}{1-l^2}}
\end{aligned}
$$

therefore, set $k = 1/2, 1/4, 1/16$. therefore, $l = 1/2, 3/4, 15/16$, we get

$$
\begin{aligned}
z_{1/2} &= \frac{1}{\sqrt{3}}R \\
z_{1/4} &= \frac{3}{\sqrt{7}}R \\
z_{1/16} &= \frac{15}{\sqrt{31}}R
\end{aligned}
$$

(g)

we see that

$$\begin{aligned}
V(r) &= V(1-\frac{z}{\sqrt{z^2+R^2}}) \\
&= V\left(1-(1-\frac{R^2}{z^2})^{-1/2}\right) \\
&\approx V(1-1+\frac{1}{2}\frac{R^2}{z^2}) \\
&= \frac{VR^2}{2z^2}
\end{aligned}$$

We see that the $V \sim \frac{1}{z^2}$

Problem 2

(a)

We know that the monopole moment is $Q_{\text{total}} = 0$. The dipole moment is

$$\vec{p} = Q((a\hat{x}+a\hat{y})+(-a\hat{x}-a\hat{y})) - Q((-a\hat{x}+a\hat{y})+(a\hat{x}-a\hat{y})) = \vec{0}$$

is a zero vector.

(b)

We know that

$$\begin{aligned}
Q_{xx} &= \frac{1}{2}\int \rho(r')(2x'^2-y'^2-z'^2)\mathrm{d}^3r' \\
&= \frac{1}{2}\cdot Q\iiint_{\text{cartisan}} (\delta(x'-a)\delta(y'-a)\delta(z')+\delta(x'+a)\delta(y'+a)\delta(z'))(2x'^2-y'^2-z'^2)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&+\frac{1}{2}\cdot(-Q)\iiint_{\text{cartisan}} (\delta(x'+a)\delta(y'-a)\delta(z')+\delta(x'-a)\delta(y'+a)\delta(z'))(2x'^2-y'^2-z'^2)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&= \frac{1}{2}Q(2a^2-a^2+2a^2-a^2) - \frac{1}{2}Q(2a^2-a^2+2a^2-a^2) = 0
\end{aligned}$$

for Q_{yy}, we have

$$Q_{yy} = \frac{1}{2}Q(2a^2-a^2+2a^2-a^2) - \frac{1}{2}Q(2a^2-a^2+2a^2-a^2) = 0$$

and for Q_{zz}, we have

$$Q_{zz} = \frac{1}{2}Q(-a^2-a^2-a^2-a^2) - \frac{1}{2}Q(-a^2-a^2-a^2-a^2) = 0$$

(c)

From the symmatrical property of Q, we know that $Q_{xy} = Q_{yx}$. So, we only solve for Q_{xy}:

$$
\begin{aligned}
Q_{xy} &= \frac{1}{2}\int \rho(r')(3x'y')\mathrm{d}^3r' \\
&= \frac{1}{2}\cdot Q \iiint_{\text{cartisan}} (\delta(x'-a)\delta(y'-a)\delta(z')+\delta(x'+a)\delta(y'+a)\delta(z'))(3x'y')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&+ \frac{1}{2}\cdot(-Q) \iiint_{\text{cartisan}} (\delta(x'+a)\delta(y'-a)\delta(z')+\delta(x'-a)\delta(y'+a)\delta(z'))(3x'y')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&= \frac{1}{2}Q(3a^2+3a^2)-\frac{1}{2}Q(-3a^2-3a^2) = 6Qa^2
\end{aligned}
$$

(d)

We know that

$$
\begin{aligned}
Q_{xx} &= \frac{1}{2}\int \rho(r')(2x'^2-y'^2-z'^2)\mathrm{d}^3r' \\
&= \frac{1}{2}\cdot Q \iiint_{\text{cartisan}} (\delta(x'-\sqrt{2}a)\delta(y')\delta(z')+\delta(x'+\sqrt{2}a)\delta(y')\delta(z'))(2x'^2-y'^2-z'^2)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&+ \frac{1}{2}\cdot(-Q) \iiint_{\text{cartisan}} (\delta(x')\delta(y'-\sqrt{2}a)\delta(z')+\delta(x')\delta(y'+\sqrt{2}a)\delta(z'))(2x'^2-y'^2-z'^2)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&= \frac{1}{2}Q(4a^2+4a^2)-\frac{1}{2}Q(-2a^2-2a^2) = 6Qa^2
\end{aligned}
$$

Also

$$
\begin{aligned}
Q_{yy} &= \frac{1}{2}\int \rho(r')(2y'^2-x'^2-z'^2)\mathrm{d}^3r' \\
&= \frac{1}{2}\cdot Q \iiint_{\text{cartisan}} (\delta(x'-\sqrt{2}a)\delta(y')\delta(z')+\delta(x'+\sqrt{2}a)\delta(y')\delta(z'))(2y'^2-x'^2-z'^2)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&+ \frac{1}{2}\cdot(-Q) \iiint_{\text{cartisan}} (\delta(x')\delta(y'-\sqrt{2}a)\delta(z')+\delta(x')\delta(y'+\sqrt{2}a)\delta(z'))(2y'^2-x'^2-z'^2)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&= \frac{1}{2}Q(-2a^2+-2a^2)-\frac{1}{2}Q(4a^2+4a^2) = -6Qa^2
\end{aligned}
$$

From the symmatrical property of Q, we know that $Q_{xy}=Q_{yx}$. So, we only solve for Q_{xy}:

$$
\begin{aligned}
Q_{xy} &= \frac{1}{2}\int \rho(r')(3x'y')\mathrm{d}^3r' \\
&= \frac{1}{2}\cdot Q \iiint_{\text{cartisan}} (\delta(x'-\sqrt{2}a)\delta(y')\delta(z')+\delta(x'+\sqrt{2}a)\delta(y')\delta(z'))(3x'y')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&+ \frac{1}{2}\cdot(-Q) \iiint_{\text{cartisan}} (\delta(x')\delta(y'-\sqrt{2}a)\delta(z')+\delta(x')\delta(y'+\sqrt{2}a)\delta(z'))(3x'y')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\
&= \frac{1}{2}Q\cdot 0-\frac{1}{2}Q\cdot 0 = 0
\end{aligned}
$$

(e)

The original tensor is

$$
Q = \begin{pmatrix} 0 & 6Qa^2 \\ 6Qa^2 & 0 \end{pmatrix}
$$

and the rotated one is

$$
Q' = \begin{pmatrix} 6Qa^2 & 0 \\ 0 & -6Qa^2 \end{pmatrix}
$$

the corresponding $2\text{x}2$ matrix is actually

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and we could apply this on the tensor we have to see it actually works

$$RQ = \begin{pmatrix} \cos -\frac{\pi}{2} & -\sin -\frac{\pi}{2} \\ \sin -\frac{\pi}{2} & \cos -\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} 0 & 6Qa^2 \\ 6Qa^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 6Qa^2 \\ 6Qa^2 & 0 \end{pmatrix} = \begin{pmatrix} 6Qa^2 & 0 \\ 0 & -6Qa^2 \end{pmatrix} = Q'$$

So it indeed works.

For the more general 3d rotational case. The rotation matrix is

$$R = R_z(\alpha)R_y(\beta)R_x(\gamma) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$

where each part is yaw, pitch, roll. The 2d rotation case only use the first matrix. (That is, $\beta = 0$ and $\gamma = 0$)