## **Problem 1**

(a)

We know that  $ec{J}_B = ec{
abla} imes ec{M}$  and  $ec{M} = M_0 \hat{z}$ . Therefore,

$$ec{J}_{B}=ec{
abla} imesec{M}=0$$

(b)

We know that  $ec{K}_B = ec{M} imes \hat{n}.$  We see that on the sphere,  $\hat{n} = \hat{r}$  and therefore

$$ec{K}_B = ec{M} imes \hat{n} = M_0 (\cos heta \hat{r} - \sin heta \hat{ heta}) imes \hat{r} = M_0 \sin heta \hat{\phi}$$

(c)

We see such a sphere, when rotating, must satisfy that  $\vec{J}_B=0$  and  $\vec{K}_B=M_0\sin\theta(\cos\phi\hat{y}-\sin\phi\hat{x})$ . This means there should only be surface charge density (otherwise the  $\vec{J}_B$  will not equal to 0). We know that at point  $(R,\theta,\phi)$  on the surface of the sphere:

$$ec{K}_B = \sigma ec{v} = \sigma R \sin heta \omega \hat{\phi}$$

and therefore the relationship that

$$\sigma = rac{M_0}{\omega R}$$

(d)

We know that  $ec{J}_B = ec{
abla} imes ec{M}$  and  $ec{M} = M_0 \hat{z}$ . Therefore,

$$ec{J}_B = ec{
abla} imes ec{M} = 0$$

Same as (a).

Also, for the side of the cylinder, we see that  $\hat{n}=\hat{s}$  and therefore

$$ec{K}_{B, ext{side}} = ec{M} imes \hat{n} = M_0 \hat{z} imes \hat{s} = M_0 \hat{\phi}$$

and for the top and bottom of the cylinder, we see that  $\hat{n}=\pm\hat{z}$  and therefore

$$ec{K}_{B, ext{tb}} = ec{M} imes \hat{n} = M_0 \hat{z} imes (\pm \hat{z}) = 0$$

(e)

We see that it should only have surface charge density otherwise it would product non-zero  $\vec{J}_B$ . We see that on the side of the

$$ec{K}_{B. ext{side}} = \sigma_{ ext{side}} ec{v} = \sigma_{ ext{side}} R \omega \hat{\phi}$$

and therefore

$$\sigma_{
m side} = rac{M_0}{\omega R}$$

and on the top / bottom of the cylinder

$$ec{K}_{B,\mathrm{tb}} = \sigma_{\mathrm{tb}}ec{v} = \sigma_{\mathrm{tb}}r\omega\hat{\phi}$$

we see that

$$\sigma_{
m th}r\omega=0$$

we see that  $\omega$  is not zero, and  $0 \le r \le R$ . That means we must make the  $\sigma_{\rm tb}=0$ . That means  $\sigma_{\rm tb}=0$ . (There is only charge density on the side, but not on the top and bottom)

(f)

We know that  $ec{J}_B = ec{
abla} imes ec{M}$  and  $ec{M} = M_0 \hat{z}$ . Therefore,

$$ec{J}_B = ec{
abla} imes ec{M} = 0$$

This is also same as (a)

We see that on the six side of the cube, the normal vector is:

$$\hat{n}_{
m side} = \hat{s}$$
  $\hat{n}_{
m top} = \hat{z}$   $\hat{n}_{
m bottom} = -\hat{z}$ 

and therefore

$$egin{aligned} ec{K}_{B, ext{side}} &= M_0 \hat{z} imes \hat{s} = M_0 \hat{\phi} \ ec{K}_{B, ext{top}} &= M_0 \hat{z} imes \hat{z} = 0 \ ec{K}_{B, ext{bottom}} &= M_0 \hat{z} imes (-\hat{z}) = 0 \end{aligned}$$

we see that on the side of the cube, the distance to the z-axis is

$$egin{aligned} s_{ ext{left}}(\phi) &= -a\sec\phi \ s_{ ext{right}}(\phi) &= a\sec\phi \ s_{ ext{front}}(\phi) &= -a\csc\phi \ s_{ ext{back}}(\phi) &= a\csc\phi \end{aligned}$$

and we see that in this case, on the side of the cube, we have

$$ec{K}_{B, ext{left}} = \sigma_{ ext{left}} ec{v} = \sigma_{ ext{left}} s_{ ext{left}}(\phi) \omega \hat{\phi}$$

and thus we have

$$\sigma_{
m left}(\phi) = rac{M_0}{s_{
m left}(\phi)\omega} = -rac{M_0}{a\sec(\phi)\omega}$$

and similarly that

$$\sigma_{
m right}(\phi) = rac{M_0}{s_{
m right}(\phi)\omega} = rac{M_0}{a\sec(\phi)\omega}$$

- -

$$\sigma_{
m front}(\phi) = rac{M_0}{s_{
m front}(\phi)\omega} = -rac{M_0}{a\csc(\phi)\omega}$$

$$\sigma_{
m back}(\phi) = rac{M_0}{s_{
m back}(\phi)\omega} = rac{M_0}{a\csc(\phi)\omega}$$

the intuition here is that the part where it's farer from the z-axis (it rotate faster), the charge density is smaller.

and we see that the on the top and bottom side of the cube, the

$$\sigma_{
m tb} s_{
m tb}(\phi) \omega = 0$$

and therefore  $\sigma_{\mathrm{tb}}=0$ 

## **Problem 2**

(a)

We know that in the case where paramagnet is placed in an external potential field, there is no free current.

We know that

$$ec{H}_{
m out} = rac{ec{B}_{
m out}}{\mu_0} = rac{B_0}{\mu_0}(\cos heta\hat{x} + \sin heta\hat{z})$$

Therefore, we know that  $ec{H}_{
m out}^{\parallel}=ec{H}_{
m in}^{\parallel}=rac{B_0}{\mu_0}\cos heta\hat{x}$ 

We know that at boundary, the  $ec{B}_{ ext{in}}^{\perp} = ec{B}_{ ext{out}}^{\perp} = B_0 \sin heta \hat{z}.$ 

Since we know that inside the paramagnet the  $ec{B}_{
m in}=\mu_0(1+\chi_m)ec{H}_{
m in}.$  That means

$$ec{B}_{ ext{in}}^{\parallel} = \mu_0 (1 + \chi_m) ec{H}_{ ext{in}}^{\parallel} = \mu_0 (1 + \chi_m) rac{B_0}{\mu_0} \cos heta \hat{x} = B_0 (1 + \chi_m) \cos heta \hat{x}$$

and therefore

$$ec{B}_{ ext{in}} = B_0((1+\chi_m)\cos heta\hat{x} + \sin heta\hat{z})$$

(b)

We use the formula that

$$ec{B}_{
m in} = \mu_0 (1 + \chi_m) ec{H}_{
m in}$$

and therefore

$$ec{H} = rac{B_0}{\mu_0}(\cos heta\hat{x} + rac{\sin heta}{1+\chi_m}\hat{z})$$

We know that in this case the  $\vec{H}$  in the slab is

$$ec{H}_{
m out} = rac{B_0 \hat{z}}{\mu_0 (1+\chi_m)}$$

Using the method mentioned in Lecture 28. We define a  $\vec{H}=-\vec{\nabla}\phi_m(\vec{r})$ , and we could therefore solve the equation

$$\phi_m(ec{r}) = \sum_{i=0}^l (A_l r^l + rac{B_l}{r^{l+1}}) P_l(\cos heta)$$

So, in the outside of the sphere, we see as  $r o \infty$ , the

$$\phi_m(ec{r}) = -rac{B_0}{\mu_0(1+\chi_m)}z$$

therefore, to satisfy the boundary condition, only l=1 term remains.

we have

$$\phi_{m, ext{out}}(ec{r}) = lpha_{ ext{out}} r \cos heta + rac{eta_{ ext{out}}}{r^2} \cos heta$$

and the following must be true

$$lpha_{
m out} r \cos heta = -rac{B_0}{\mu_0 (1+\chi_m)} z$$

we see that  $r\cos\theta=z$  and therefore

$$lpha_{
m out} = -rac{B_0}{\mu_0(1+\chi_m)}$$

Inside the sphere we find that

$$\phi_{m, ext{in}}(ec{r}) = lpha_{ ext{in}} r \cos heta + rac{eta_{ ext{in}}}{r^2} \cos heta$$

and since  $\phi_{m, ext{in}}(ec{r})$  should be a finite value as r o 0 , we see  $eta_{ ext{in}}=0$ 

From the lecture 28 notes, we see that

$$ec{B}_{
m out} \cdot \hat{n} = ec{B}_{
m in} \cdot \hat{n}$$

and

$$\mu_0(1+\chi_m)\vec{H}_{\mathrm{out}}\cdot\hat{n}=\mu_0\vec{H}_{\mathrm{in}}\cdot\hat{n}$$

then

$$egin{aligned} (1+\chi_m) rac{\partial \phi_{m, ext{out}}(ec{r})}{\partial r} &= rac{\partial \phi_{m, ext{in}}(ec{r});}{\partial r} \ (1+\chi_m) rac{\partial}{\partial r} igg(lpha_{ ext{out}}r + rac{eta_{ ext{out}}}{r^2}igg) &= rac{\partial}{\partial r} (lpha_{ ext{in}}r) \ (1+\chi_m) (lpha_{ ext{out}} - rac{2eta_{ ext{out}}}{r^3}) &= lpha_{ ext{in}} \end{aligned}$$

evaluated at r=R

$$(1+\chi_m)(lpha_{
m out}-rac{2eta_{
m out}}{R^3})=lpha_{
m in}$$

and then

$$-rac{B_0}{\mu_0(1+\chi_m)}-rac{2eta_{
m out}}{R^3}=rac{lpha_{
m in}}{1+\chi_m}$$

the other boundary condition is that  $ec{H}_{ ext{in},\parallel} = ec{H}_{ ext{out},\parallel}.$  We see that

$$-\frac{1}{R}\frac{\partial}{\partial \theta}\phi_{m,\mathrm{in}}(\vec{r}) = -\frac{1}{R}\frac{\partial}{\partial \theta}\phi_{m,\mathrm{out}}(\vec{r})$$

this giving us

$$-rac{1}{R}rac{\partial}{\partial heta}igg(lpha_{
m out}r+rac{eta_{
m out}}{r^2}igg)\cos heta=-rac{1}{R}rac{\partial}{\partial heta}lpha_{
m in}r\cos heta$$

therefore

$$lpha_{
m out} r + rac{eta_{
m out}}{r^2} = lpha_{
m in} r$$

and therefore

$$lpha_{
m out} + rac{eta_{
m out}}{r^3} = lpha_{
m in}$$

and thus at r=R

$$-rac{B_0}{\mu_0(1+\chi_m)}+rac{eta_{
m out}}{R^3}=lpha_{
m in}$$

and therefore

$$-rac{3B_0}{\mu_0(3+2\chi_m)}=lpha_{
m in}$$

and therefore

$$egin{align} \phi_{m, ext{in}} &= lpha_{ ext{in}} r \cos heta &= -rac{3B_0}{\mu_0 (3+2\chi_m)} z \ ec{H}_{ ext{in}} &= -ec{
abla} \phi_{m, ext{in}} &= \hat{z} rac{B_0}{\mu_0 (1+rac{2}{3}\chi_m)} \ ec{B}_{ ext{in}} &= \mu_0 ec{H}_{ ext{in}} &= \hat{z} rac{B_0}{(1+rac{2}{3}\chi_m)} \ \end{aligned}$$

When  $\chi_m \to 0$ , the  $\vec{B}_{\rm in}=B_0\hat{z}$ . If  $\chi_m \to \infty$ ,  $\vec{B}_{\rm in}=0$ . The  $\chi_m$  is the magnetic susceptibility of the slab. If it's large, that means the paramagnet slab will generate a huge M for the external B. (it's easy to get magnetized)