Question 1

> Find the eigenvalues of the X operator. Show that the X, Y, Z operators (the Pauli matrices) are traceless.

Since $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\det(X - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$$

Thus $\lambda = \pm 1$

$$\begin{aligned} \operatorname{tr} X &= 0 + 0 = 0 \\ \operatorname{tr} Y &= 0 + 0 = 0 \\ \operatorname{tr} Z &= 1 + (-1) = 0 \end{aligned}$$

Indeed, all their traces are zero (traceless)

Question 2

> Find the eigenvalues of $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$

$$\det(B - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 4 \\ 1 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)(2-\lambda) - 2(3-\lambda) = -\lambda(\lambda-3)^2 = 0$$

So, $\lambda = 0, 3$

Question 3

> Using the matrix representation of the Hadamard gate
>
> $$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
>
> write down the matrix $H \otimes H$ and find $(H \otimes H)(|0\rangle \otimes |1\rangle)$. Show that this is equivalent to
>
> $$|\phi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$H \otimes H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and thus

$$(H \otimes H)(|0\rangle \otimes |1\rangle) = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

which is equal to

$$|\phi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{1}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle) = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Question 4

> The beam splitter gate has a matrix representation given by
>
> $$B = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$
>
> Show that B generates superposition states out of the computational basis states $|0\rangle$ and $|1\rangle$. In particular,show that
>
> $$B \otimes B\,|00\rangle = \left(\frac{i\,|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{i\,|0\rangle + |1\rangle}{\sqrt{2}}\right)$$
>
> Show that two applications of the beam splitter gate on the same state, namely that $B(B\,|\psi\rangle)$ act analogously to the NOT gate, giving the same probabilities of finding $|0\rangle$ and $|1\rangle$

B indeed generates superposition states out of $|0\rangle$ and $|1\rangle$

$$\begin{aligned} B\,|0\rangle &= \frac{1}{\sqrt{2}}(i\,|0\rangle + |1\rangle) \\ B\,|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \end{aligned}$$

with $|0\rangle$ and $|1\rangle$ share equal probability ($|1|^2 = |i|^2$)

and

$$B \otimes B = \frac{1}{2}\begin{pmatrix} -1 & i & i & 1 \\ i & -1 & 1 & i \\ i & 1 & -1 & i \\ 1 & i & i & -1 \end{pmatrix}$$

$$(B \otimes B)\,|00\rangle = \frac{1}{2}(-\,|00\rangle + i\,|01\rangle + i\,|10\rangle + |11\rangle) = \left(\frac{i\,|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{i\,|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

and

$$\begin{aligned} B(B\,|\phi\rangle) &= B(B(\alpha\,|0\rangle + \beta\,|1\rangle)) \\ &= B(\frac{1}{\sqrt{2}}((\alpha i + \beta)\,|0\rangle + (\alpha + \beta i)\,|1\rangle)) \\ &= \frac{1}{2}((\alpha i + \beta)(i\,|0\rangle + |1\rangle) + (\alpha + \beta i)(|0\rangle + i\,|1\rangle)) \\ &= \frac{1}{2}(-\alpha\,|0\rangle + \alpha i\,|1\rangle + \beta i\,|0\rangle + \beta\,|1\rangle + \alpha\,|0\rangle + \beta i\,|0\rangle + \alpha i\,|1\rangle - \beta\,|1\rangle) \\ &= \frac{1}{2}(2\beta i\,|0\rangle + 2\alpha i\,|1\rangle) \\ &= \beta i\,|0\rangle + \alpha i\,|1\rangle \end{aligned}$$

That is, B^2 acting quite similar to X, flip the $|0\rangle$ and $|1\rangle$. Except that it add a global phase of $i = e^{i\pi/4}$

Question 5

(a)

> Show that the matrix representation of $HP(\theta)HP(\phi)$ is given by
> $$HP(\theta)HP(\phi) = e^{i\theta/2}\begin{pmatrix}\cos\frac{\theta}{2} & -ie^{i\phi}\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix}$$

$$\begin{aligned}
HP(\theta)HP(\phi) &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\phi}\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & e^{i\theta}\\ 1 & -e^{i\theta}\end{pmatrix}\begin{pmatrix}1 & e^{i\phi}\\ 1 & -e^{i\phi}\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1+e^{i\theta} & e^{i\phi}-e^{i\theta}e^{i\phi}\\ 1-e^{i\theta} & e^{i\phi}+e^{i\theta}e^{i\phi}\end{pmatrix}\\
&= e^{i\theta/2}\begin{pmatrix}\frac{e^{-i\theta/2}+e^{i\theta/2}}{2} & -ie^{i\phi}\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}\\ -i\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i} & e^{i\phi}\frac{e^{-i\theta/2}+e^{i\theta/2}}{2}\end{pmatrix}\\
&= e^{i\theta/2}\begin{pmatrix}\cos\frac{\theta}{2} & -ie^{i\phi}\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix}
\end{aligned}$$

(b)

> Write down the Hadamard transform of three Hadamard gates acting on the product state $|1\rangle|1\rangle|0\rangle$

That is basically

$$(H|1\rangle)(H|1\rangle)(H|0\rangle) = \frac{1}{\sqrt[3]{2}}(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)(|0\rangle+|1\rangle)$$

$$= \frac{1}{\sqrt[3]{2}}(|000\rangle+|001\rangle-|010\rangle-|011\rangle-|100\rangle-|101\rangle+|110\rangle+|111\rangle)$$