Question 1

> In the Example 3.17 we showed that $[\sigma_1, \sigma_2] = 2i\sigma_3$. Following the same procedure, show that $[\sigma_2, \sigma_3] = 2i\sigma_1$ and $[\sigma_3, \sigma_1] = 2i\sigma_2$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 2i\sigma_1$$

$$[\sigma_3, \sigma_1] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2i\sigma_2$$

> and write down the uncertainty principle corresponding to simultaneous measurements of σ_1 and σ_2. Is the uncertainty principle for σ_2 and σ_3 different?

$$\Delta\sigma_1\Delta\sigma_2 \geq \frac{1}{2}|\langle 2i\sigma_3\rangle|$$

The general uncertainty principle works for all operators, so it should follow the same equation that

$$\Delta A\Delta B \geq \frac{1}{2}|\langle [A, B]\rangle|$$

which is

$$\Delta\sigma_2\Delta\sigma_3 \geq \frac{1}{2}|\langle 2i\sigma_1\rangle|$$

Question 2

> Write Z in terms of its projection operators (in the $|0\rangle$ and $|1\rangle$ basis). Using this form of Z, calculate the probability of obtaining the result $|1\rangle$ if you measure Z for the (unnormalized) state $2|0\rangle + 3i|1\rangle$

Z is equal to

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

and

$$|\psi\rangle = Z(2|0\rangle + 3i|1\rangle) = 2|0\rangle - 3i|1\rangle$$

the probability is

$$\text{Pr} = \frac{(3i)(-3i)}{4 + (3i)(-3i)} = \frac{9}{4+9} = \frac{9}{13}$$

Question 3

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i$$

we could find out that

$$\sigma_X\sigma_Y = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$$
$$\sigma_Y\sigma_X = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}-i & 0\\0 & i\end{pmatrix}$$
$$\sigma_X\sigma_Z = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$$
$$\sigma_Z\sigma_X = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$
$$\sigma_Y\sigma_Z = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}0 & i\\i & 0\end{pmatrix}$$
$$\sigma_Z\sigma_Y = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \begin{pmatrix}0 & -i\\-i & 0\end{pmatrix}$$
$$\sigma_X\sigma_X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$
$$\sigma_Y\sigma_Y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$
$$\sigma_Z\sigma_Z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

and thus

$$\begin{aligned}
\{\sigma_X, \sigma_X\} &= \sigma_X\sigma_X + \sigma_X\sigma_X = 2I \\
\{\sigma_Y, \sigma_Y\} &= \sigma_Y\sigma_Y + \sigma_Y\sigma_Y = 2I \\
\{\sigma_Z, \sigma_Z\} &= \sigma_Z\sigma_Z + \sigma_Z\sigma_Z = 2I \\
\{\sigma_X, \sigma_Y\} &= \sigma_X\sigma_Y + \sigma_Y\sigma_X = 0 \\
\{\sigma_Y, \sigma_X\} &= \sigma_Y\sigma_X + \sigma_X\sigma_Y = 0 \\
\{\sigma_X, \sigma_Z\} &= \sigma_X\sigma_Z + \sigma_Z\sigma_X = 0 \\
\{\sigma_Z, \sigma_X\} &= \sigma_Z\sigma_X + \sigma_X\sigma_Z = 0 \\
\{\sigma_Y, \sigma_Z\} &= \sigma_Y\sigma_Z + \sigma_Z\sigma_Y = 0 \\
\{\sigma_Z, \sigma_Y\} &= \sigma_Z\sigma_Y + \sigma_Y\sigma_Z = 0
\end{aligned}$$

and thus it's proved.

Question 4

> Given that $\langle a|b\rangle = 1/2$ and $\langle c|d\rangle = 3/4$, calculate $\langle\psi|\phi\rangle$, where $|\psi\rangle = |a\rangle \otimes |c\rangle$ and $|\phi\rangle = |b\rangle \otimes |d\rangle$

$$\langle\psi|\phi\rangle = (\langle a| \otimes \langle c|)(|b\rangle \otimes |d\rangle) = \langle a|b\rangle\langle c|d\rangle = 3/8$$

Question 5

> Calculate the tensor product of
>
> $$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |\phi\rangle = \frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$$

$$|\psi\rangle \otimes |\phi\rangle = \frac{1}{2\sqrt{2}} = \begin{pmatrix} 1 \\ \sqrt{3} \\ 1 \\ \sqrt{3} \end{pmatrix}$$

Question 6

> Can $|\psi\rangle = \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle)$ be written as a product state?

Yes,

$$|\psi\rangle = |-\rangle \otimes |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |\psi\rangle = \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle$$

Question 7

> Can
>
> $$|\psi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{2}$$
>
> be written as a product state?

No, write the state as vector

$$|\psi\rangle = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}$$

Suppose it could be written as a product of two states $|x\rangle$ and $|y\rangle$

$$|x\rangle \otimes |y\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}$$

which means

$$\begin{aligned} ac \cdot bd = 1/4 \quad &\text{and} \quad ad \cdot bc = 0 \\ abcd = 1/4 \quad &\text{and} \quad abcd = 0 \end{aligned}$$

and that is contradiction. Thus, there are no such states $|x\rangle$ and $|y\rangle$ that $|x\rangle \otimes |y\rangle = |\psi\rangle$. and thus $|\psi\rangle$ could not be written as product of state.