Problem 1

> Suppose that Alice and Bob share the entangled states
>
> $$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|0_A\rangle\,|0_B\rangle + |1_A\rangle\,|1_B\rangle}{\sqrt{2}}$$

(a)

> Write down the density operator for this state

The density operator ρ is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

(b)

> Compute the density matrix. Verify that $\mathrm{Tr}(\rho) = 1$, and determine if this is a pure state.

Writing density operator in matrix form

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

We could see that the $\mathrm{Tr}(\rho) = 1$, and that

$$\mathrm{Tr}\left(\rho^2\right) = \frac{1}{4}\mathrm{Tr}\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} = 1$$

which means that it's a pure state.

(c)

> Find the density matrix that represents the reduced operator as seen by Alice

$$\rho_A = \mathrm{Tr}_B(\rho) = \langle 0_B|\rho\,|0_B\rangle + \langle 1_B|\rho\,|1_B\rangle$$

$$\begin{aligned}
\langle 0_A|\rho\,|0_A\rangle =& \frac{1}{2}(|0_A\rangle\langle 0_B|0_B\rangle\langle 0_B|0_B\rangle\,\langle 0_A| \\
&+ |1_A\rangle\langle 0_B|1_B\rangle\langle 0_B|0_B\rangle\,\langle 0_A| \\
&+ |0_A\rangle\langle 0_B|0_B\rangle\langle 1_B|0_B\rangle\,\langle 1_A| \\
&+ |1_A\rangle\langle 0_B|1_B\rangle\langle 1_B|0_B\rangle\,\langle 1_A| \\
=& \frac{1}{2}|0_A\rangle\langle 0_A|
\end{aligned}$$

$$\begin{aligned}\langle 1_B|\rho|1_B\rangle =&\frac{1}{2}(|0_A\rangle\langle 1_B|0_B\rangle\langle 0_B|1_B\rangle\,\langle 0_A| \\ &+|1_A\rangle\langle 1_B|1_B\rangle\langle 0_B|1_B\rangle\,\langle 0_A| \\ &+|0_A\rangle\langle 1_B|0_B\rangle\langle 1_B|1_B\rangle\,\langle 1_A| \\ &+|1_A\rangle\langle 1_B|1_B\rangle\langle 1_B|1_B\rangle\,\langle 1_A| \\ =&\frac{1}{2}|1_A\rangle\langle 1_A|\end{aligned}$$

$$\rho_A=\frac{1}{2}(|0\rangle\langle 0|+|1\rangle\langle 1|)$$

(d)

> Show that the reduced density operator as seen by Alice is a completely mixed state

we see that $\rho=\frac{1}{2}I$, and $\mathrm{Tr}(\rho^2)=\frac{1}{4}\mathrm{Tr}(I)=\frac{1}{2}=\frac{1}{n}$, which means the reduced density operator for Alice is a completely mixed state.

Problem 2

> Consider the following matrix
>
> $$\rho=\begin{pmatrix}2/5 & -i/8\\ i/8 & 3/5\end{pmatrix}$$

(a)

> Show that this matrix is Hermitian

$$\rho^\dagger=\begin{pmatrix}2/5 & -i/8\\ -(-i/8) & 3/5\end{pmatrix}=\rho$$

(b)

> Verify that the eigenvalues are $\lambda_{1,2}=(20\pm\sqrt{41})/40$

$$\begin{aligned}\det(\rho-\lambda I)=&\begin{pmatrix}2/5-(20\mp\sqrt{41})/40 & -i/8\\ i/8 & 3/5-(20\mp\sqrt{41})/40\end{pmatrix}\\ =&(2/5-(20\mp\sqrt{41})/40)(3/5-(20\mp\sqrt{41})/40)-\frac{1}{64}\\ =&0\end{aligned}$$

with some help from Mathematica

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Simplify[(2/5 - (20 + Sqrt[41]) / 40) * (3/5 - (20 + Sqrt[41]) / 40) - 1/64]
0
```

```
Simplify[(2/5 - (20 - Sqrt[41]) / 40) * (3/5 - (20 - Sqrt[41]) / 40) - 1/64]
0
```

They are indeed eigenvalues.

(c)

> Does this matrix represent a valid density matrix?

we could find that $\mathrm{Tr}(\rho)=1$, and ρ is Hermitian stated in (a), and ρ is positive operator since its eigenvalue $\lambda_{1,2}=(20\pm\sqrt{41})/40$. Thus, ρ is a valid density matrix.

(d)

> Show that the probability of finding the system in the $|0\rangle$ state is 0.4

$$\mathrm{Pr}_0 = \langle 0|\rho\,|0\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}2/5 & -i/8 \\ i/8 & 3/5\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} = 2/5 = 0.4$$

(e)

> Compute the components of the Bloch vector, and show that this is a mixed state

$$\begin{aligned}
\langle X\rangle = \mathrm{Tr}(\rho X) = \mathrm{Tr}\begin{pmatrix}-i/8 & 2/5 \\ 3/5 & i/8\end{pmatrix} = 0 \\
\langle Y\rangle = \mathrm{Tr}(\rho Y) = \mathrm{Tr}\begin{pmatrix}1/8 & 3i/5 \\ -2i/5 & 1/8\end{pmatrix} = 2/8 \\
\langle Z\rangle = \mathrm{Tr}(\rho Z) = \mathrm{Tr}\begin{pmatrix}2/5 & i/8 \\ i/8 & -3/5\end{pmatrix} = -1/5
\end{aligned}$$

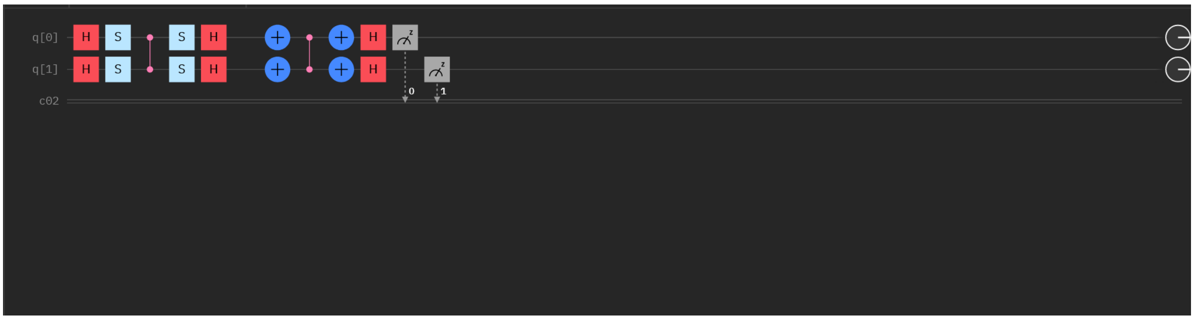
the Bloch vector is

$$\vec{S} = 0\hat{x} + 2/8\hat{y} - 1/5\hat{z}$$

$$|\vec{S}| = \frac{1}{16} + \frac{1}{25} < 1$$

thus this is a mixed states

Problem 3



This is the grover algorithm for finding $|00\rangle$