(a)

(i)

Consider the fooling set

$$F = \{0^n \mid n \geq 1\}$$

Without loss of generality, We pick the element 0^n and 0^m where $n < m$. We set the suffix to be 2^{n-1}. The 0^n2^{n-1} is in the language because in this case $i + j = n + 0 = n$ and $k + 1 = n - 1 + 1 = n$. Therefore $i + j = k + 1$. However, 0^m2^{n-1} is not in the language since $i + j = m + 0 = m$, and $k + 1 = n - 1 + 1 = n$, and $m \neq n$ obviously from our assumption. Therefore, this fooling set includes infinite elements that are mutually distinguishable.

(ii)

Consider the fooling set

$$F = \{0^i1 \mid i \geq 1\}$$

Without loss of generality, we pick the element 0^i1 and 0^j1 from the fooling set where $i < j$. We set the suffix to be 0^i. The 0^i10^i is thus in the language because there is two empty blocks (namely, 0^i since $i > 0$) that are equal length (length i). However, the 0^i10^j is not in the language because its two non-empty blocks 0^i and 0^j (non-empty since $i > 0$ and $j > 0$) has different length (because $i \neq j$ from our assumption). Therefore, this fooling set includes infinite elements that are mutually distinguishable.

(iii)

Consider the fooling set with recursive definition:

- Base: $F[0] = 0^{k \cdot 2^k}$ where $k \geq 2$ (you could choose any $k \geq 2$ as you want)
- Induction: Write the previous element in the form of $F[n-1] = 0^{i \cdot 2^i}$, let $m = 2^{i+1}$ and

$$F[n] = 0^{m \cdot 2^m}$$

It's obvious that all elements in F is in the language. To show this, we pick an element $0^{i \cdot 2^i}$. We see that

$$0^{i \cdot 2^i} = 0^{\log_2 2^i \cdot 2^i} = 0^{\lceil \log_2 2^i \cdot 2^i \rceil}$$

which match the definition of the language. It's also trivial that if we write the element in the form of $0^{i \cdot 2^i}$, it's always true that $i \geq 2$. It's also trivial to see that if two elements $F[a] = 0^{i \cdot 2^i}$ and $F[b] = 0^{j \cdot 2^j}$, then if $a > b$ then $i > j$.

We are gonna prove that for any two elements $F[a] = 0^{i \cdot 2^i}$ and $F[b] = 0^{j \cdot 2^j}$ in the fooling set where $a > b$ ($i < j$) (without loss of generality), the suffix s that makes $0^{i \cdot 2^i}s = 0^{\lceil (2^i+1) \cdot \log_2(2^i+1) \rceil}$ (this is just the next immediate element of the $0^{i \cdot 2^i}$ in the L) has **smaller** length than the suffix s' that make $0^{j \cdot 2^j}s' = 0^{\lceil (2^j+1) \cdot \log_2(2^j+1) \rceil}$ (the next immediate element of $0^{j \cdot 2^j}$ in L). To show this, we use induction on the size of F:

- Base case, there is the case when there is only $F[0] = 0^{k \cdot 2^k}$ and $F[1] = 0^{m \cdot 2^m}$ in F. In this case, we see that the **length** of the suffix s should be (s itself obviously should be a run of 0 s).

$$
\begin{aligned}
|s| &= \lceil (2^k+1)\cdot \log_2(2^k+1) \rceil - k \cdot 2^k \\
&\le \lceil (2^k+1)\cdot \log_2(2^{k+1}) \rceil - k \cdot 2^k \\
&= (2^k+1)\cdot (k+1) - k \cdot 2^k \\
&= 2^k + k + 1
\end{aligned}
$$

Notice, we use that fact that $2^{k+1} \ge 2^k + 1$ for $k \ge 2$

we have find a upper bound for the length of the suffix. However, we could also calculate the lower bound of the suffix s' that could make $0^{m\cdot 2^m} s' = 0^{\lceil (2^m+1)\cdot \log_2(2^m+1)\rceil}$:

$$
\begin{aligned}
|s'| &= \lceil (2^m+1)\cdot \log_2(2^m+1) \rceil - m \cdot 2^m \\
&\ge \lceil (2^m+1)\cdot \log_2(2^m) \rceil - m \cdot 2^m \\
&= (2^m+1)\cdot m - m \cdot 2^m \\
&= m = 2^{k+1}
\end{aligned}
$$

and we see that $|s'| = 2^{k+1} = 2^k + 2^k > 2^k + k + 1 = |s|$ since $k \ge 2$, which proves the hypothesis in base case.

- Induction, suppose the above hypothesis works for $F = \{F[0], F[1], \cdots, F[n]\}$. We add a new element $F[n+1]$ according to its definition, and we want to prove the hypothesis still holds. Consider any two elements, $F[l]$ ($0 \le l \le n$) and $F[n+1]$. We write them in the form: $F[l] = 0^{i\cdot 2^i}$ and $F[n] = 0^{j\cdot 2^j}$, and $F[n+1] = 0^{p\cdot 2^p}$. From the inductive hypothesis, we know that the suffix s that make $F[l]s = 0^{\lceil (2^i+1)\cdot \log_2(2^i+1)\rceil}$ and the suffix s' that makes $F[n]s = 0^{\lceil (2^j+1)\cdot \log_2(2^j+1)\rceil}$ satisfy the relation that $|s| \le |s'|$ (notice the condition when they are equal is when $l = n$). We know that

$$
\begin{aligned}
|s'| &= \lceil (2^j+1)\cdot \log_2(2^j+1) \rceil - j \cdot 2^j \\
&\le \lceil (2^j+1)\cdot \log_2(2^{j+1}) \rceil - j \cdot 2^j \\
&= (2^j+1)\cdot (j+1) - j \cdot 2^j \\
&= 2^j + j + 1
\end{aligned}
$$

  and suppose that the suffix s'' make $F[n+1]s'' = 0^{\lceil (2^p+1)\cdot \log_2(2^p+1)\rceil}$. Then

$$
\begin{aligned}
|s''| &= \lceil (2^p+1)\cdot \log_2(2^p+1) \rceil - p \cdot 2^p \\
&\ge \lceil (2^p+1)\cdot \log_2(2^p) \rceil - p \cdot 2^p \\
&= (2^p+1)\cdot p - p \cdot 2^p \\
&= p = 2^{j+1}
\end{aligned}
$$

  using the same trick in base case, we see that $|s''| = 2^{j+1} = 2^j + 2^j > 2^j + j + 1 = |s'|$, since $j \ge 2$. From this, we get $|s''| > |s|$, and we prove the hypothesis.

Once this hypothesis proved, for any two different elements in our set $F[a]$ and $F[b]$ with $a > b$, we could find one suffix s that $F[a]s$ is next immediate element of $F[a]$ in language L. However, since hypothesis said the suffix s' that makes $F[b]s'$ the next immediate element of $F[b]$ in language L is longer than s, that means $F[b]s$ is not in the language. Therefore, this infinite fooling set is valid with its elements mutually distinguishable.

(b)

For L_k, consider the fooling set F

$$F = \{\{0,1\}^k\}$$

this set has size of 2^k. To prove it's working, we first define a size k tuple t associate with each string s in F:

- $t[0] = s[0]$
- $t[i] = t[i-1] + s[i]$, $k > i > 0$

(Note the implicit conversion from string `0` and `1` to number). Informally, each entry $t[k]$ counts the number of 1 in s from $s[0]$ to $s[i]$. Note, two different string u and v in F will have different t_u and t_v. Prove this by contradiction: suppose $t_u = t_v$. Then

- $u[0] = t_u[0] = t_v[0] = v[0]$
- $u[i] = t_u[i] - t_u[i-1] = t_v[i] - t_v[i-1] = v[i]$. for $k > i > 0$.

This means $u = v$, which violates our assumption that u are v are different. So, two different strings u and v in F must have different t_u and t_v.

Therefore, to prove that this fooling set is valid, we pick two different string u and v. Since they are different, we have a different t_u and t_v. Since the tuple are different, they need to have at least one entry such that $t_u[i] \neq t_v[i]$. Use a suffix s that is $2k - (i+1)$ long and contains $k - t_u[i]$ `1` s. Such suffix exists because $2k - (i+1) \geq k$ since $i < k$, and $k - t_u[i] \leq k$ since $t_u[i] \geq 0$. With the suffix s, both us and vs has length $2k - (i+1) + k \geq 2k$, fulfilling one requirement for L. However, the last $2k$ character for us will be $u[i:0]s$ with $t_u[i] + k - t_u[i] = k$ `1` s (and hence k `0` s), this will make `0` and `1` s in last $2k$ equal and us not accepted by the language. vs, on the other hand, have last $2k$ character $v[i:0]s$ with $t_u[i] + k - t_v[i] \neq k$ since $t_u[i] \neq t_v[i]$. This will make the number of `1` s and `0` s in the last $2k$ character unequal and thus v is in the L. Therefore, this size 2^k fooling set F is valid with its element mutually distinguishable.

(c)

First notice that

$$(L \backslash L') \cup (L \cap L') = L$$

We use proof by contradiction. Suppose $L \backslash L'$ is regular, then since L' is finite, the intersection $L \cap L'$ is finite too. This means $L \cap L'$ is also regular. (We could just loop through every string s in $L \cap L'$ and construct a regular expression r that is union of all the strings, this will make $L(r) = L \cap L'$). Now we know that both $L \backslash L'$ and $L \cap L'$, there union is also regular since regular language is closed under union. However, that means L is regular, which contradicts with the assumption. Therefore, $L \backslash L'$ is not regular.

For the example, consider the $L = \{1^n 0^n \mid n \geq 0\}$, and L' represented by 1^*0^*. Obviously, this means that $L \backslash L' = \emptyset$, and it's obviously regular.