Exercise 1.1.1

Suppose for $a_1, a_2 \in A$, $\sigma(a_1) = \sigma(a_2)$. Then since τ is well defined $\tau \circ \sigma(a_1) = \tau \circ \sigma(a_2)$. Then since $\tau \circ \sigma$ is injective, then $a_1 = a_2$.

So, σ is injective.

Exercise 1.1.2

(i)

No, it's not transitive. Consider $x = 1, y = 4, z = 7$. $x \sim y$ because $||x - y|| = 3$. $y \sim z$ since $||y - z|| = 3$. However, $x \nsim z$ since $||x - z|| = 6 > 3$

(ii)

No, it's not reflective, Consider $x = 1$, $x \nsim x$ because $||x - x|| = 0 < 3$

(iii)

Yes, it's reflective, for $x \in \mathbb{N}$, $x \sim x$ since itself have same digit in the 1's place. It's symmetric, for $x, y \in \mathbb{N}$, $x \sim y$ means x and y (i.e. y and x) have same digit in the 1's place, so $y \sim x$. It's also transitive, for $x, y, z \in \mathbb{N}$, if $x \sim y$ and $y \sim z$, that means that x and y have the same digit in the 1's place, and y and z have the same digit in the 1's place. So, x and z have the same digit in the 1's place. So, $x \sim z$

(iv)

No, it's not symmetric, consider $x = 2, y = 1$. $x \sim y$ since $x \geq y$. but $y \nsim x$ since $y < x$.

Exercise 1.2.3

Reflective: $x \sim_f x$ since $f(x) = f(x)$

f(z)=f(x) and thus z∈[x]. So f−1(f(x))⊂[x].

Thus, [x]=f−1(f(x))Symmetrical: if $x \sim_f y$ then $f(x) = f(y)$, then $f(y) = f(x)$, and so that $y \sim_f x$.

Transitive: if $x \sim_f y$ and $y \sim_f z$, then $f(x) = f(y)$ and $f(y) = f(z)$, and then $f(x) = f(z)$ and so $x \sim_f z$

So the $\sim_f$ is an equivalence relation.

We will first show for any $x \in X$ that $[x] \subset f^{-1}(f(x))$

According to definition, the $f^{-1}(f(x)) = f^{-1}(\{f(x)\}) = \{a \in X | f(a) \in \{f(x)\}\}$. It's obvious that $x \in f^{-1}(f(x))$. Also, for any $y \in [x]$, $f(y) = f(x)$ and $f(y) \in \{f(x)\}$ so that $y \in f^{-1}(f(x))$. So $[x] \subset f^{-1}(f(x))$

Conversely, for any $z \in f^{-1}(f(x))$, $f(z) \in \{f(x)\}$ which means $f(z) = f(x)$ and thus $z \in [x]$. So $f^{-1}(f(x)) \subset [x]$.

Thus, $[x] = f^{-1}(f(x))$

Exercise 1.2.3

Because the definition $X/\sim = \{[x] | x \in X\}$, that means for any $S \in X/\sim$, we could find $x \in X$ that $[x] = S$. Thus, since $\pi(x) = [x]$ for all $x \in X$, for any $S \in X/\sim$, we could find $x \in X$ so that $[x] = S$, thus there exist $x \in X$ that $\pi(x) = [x] = S$, thus π is surjective.

If $x, y \in X$, and $x \sim y$. Then for $z \in [x]$, $x \sim z$, since the relation is transitive, $y \sim z$, so $z \in [y]$, that means $[x] \subset [y]$. Converse is also true, and that means $[x] = [y]$. Therefore, $\pi(x) = [x] = [y] = \pi(y)$, and thus $x \sim_\pi y$

Conversely, if $x, y \in X$ and $x \sim_\pi y$, then $\pi(x) = \pi(y)$ and since $\pi(x) = [x]$ and $\pi(y) = [y]$, $[x] = [y]$. That means $y \in [y]$ and thus $y \in [x]$, and thus $x \sim y$.

Thus, for $x, y \in X$, $x \sim y$ if and only if $x \sim_\pi y$, which means the $\sim_\pi$ is exactly $\sim$

Exercise 1.3.1

$\sigma = (1 \quad 4 \quad 2)\,(5 \quad 7 \quad 8)$

$\sigma^{-1} = (2 \quad 4 \quad 1)\,(8 \quad 7 \quad 5)$ (Same as example 1.3.8)

These followed the quick method used in Example 1.3.7

$\sigma^2 = \sigma \circ \sigma = (1 \quad 4 \quad 2)\,(5 \quad 7 \quad 8)\,(1 \quad 4 \quad 2)\,(5 \quad 7 \quad 8) = (1 \quad 2 \quad 4)\,(5 \quad 8 \quad 7)$

$\sigma^3 = \sigma^2 \circ \sigma = (1 \quad 2 \quad 4)\,(5 \quad 8 \quad 7)\,(1 \quad 4 \quad 2)\,(5 \quad 7 \quad 8) = \mathrm{id}$

Exercise 1.3.2

These followed the quick method used in Example 1.3.7

$\sigma \circ \tau = (3 \quad 4 \quad 8)\,(5 \quad 7 \quad 6 \quad 9)\,(1 \quad 9 \quad 3 \quad 5)\,(2 \quad 7 \quad 4) = (1 \quad 5)\,(2 \quad 6 \quad 9 \quad 4)\,(3 \quad 7 \quad 8)$

$\tau \circ \sigma = (1 \quad 9 \quad 3 \quad 5)\,(2 \quad 7 \quad 4)\,(3 \quad 4 \quad 8)\,(5 \quad 7 \quad 6 \quad 9) = (1 \quad 9)\,(2 \quad 7 \quad 6 \quad 3)\,(4 \quad 8 \quad 5)$