Question 1

Find the eigenvalues of the X operator. Show that the X, Y, Z operators (the Pauli matrices) are traceless.

Since $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\det(X - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$$

Thus $\lambda = \pm 1$

$$\operatorname{tr} X = 0 + 0 = 0$$
$$\operatorname{tr} Y = 0 + 0 = 0$$
$$\operatorname{tr} Z = 1 + (-1) = 0$$

Indeed, all their traces are zero (traceless)

Question 2

Find the eigenvalues of $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$

$$\det(B - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 4 \\ 1 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)(2-\lambda) - 2(3-\lambda) = -\lambda(\lambda - 3)^2 = 0$$

So, $\lambda = 0, 3$

Question 3

Using the matrix representation of the Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

write down the matrix $H \otimes H$ and find $(H \otimes H)(|0\rangle \otimes |1\rangle)$. Show that this is equivalent to

$$|\phi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$H \otimes H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and thus

$$(H \otimes H)(|0\rangle \otimes |1\rangle) = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

which is equal to

$$|\phi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{1}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle) = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Question 4

The beam splitter gate has a matrix representation given by

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

Show that B generates superposition states out of the computational basis states $|0\rangle$ and $|1\rangle$. In particular, show that

$$B \otimes B |00\rangle = \left(\frac{i|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{i|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

Show that two applications of the beam splitter gate on the same state, namely that $B(B|\psi\rangle)$ act analogously to the NOT gate, giving the same probabilities of finding $|0\rangle$ and $|1\rangle$

B indeed generates superposition states out of $|0\rangle$ and $|1\rangle$

$$B|0\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$$

$$B|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

and

$$B \otimes B = \frac{1}{2}\begin{pmatrix} -1 & i & i & 1 \\ i & -1 & 1 & i \\ i & 1 & -1 & i \\ 1 & i & i & -1 \end{pmatrix}$$

$$(B \otimes B)|00\rangle = \frac{1}{2}(-|00\rangle + i|01\rangle + i|10\rangle + |11\rangle) = \left(\frac{i|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{i|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

and

$$\begin{aligned}
B(B|\phi\rangle) &= B(B(\alpha|0\rangle + \beta|1\rangle)) \\
&= B(\frac{1}{\sqrt{2}}((\alpha i + \beta)|0\rangle + (\alpha + \beta i)|1\rangle)) \\
&= \frac{1}{2}((\alpha i + \beta)(i|0\rangle + |1\rangle) + (\alpha + \beta i)(|0\rangle + i|1\rangle)) \\
&= \frac{1}{2}(-\alpha|0\rangle + \alpha i|1\rangle + \beta i|0\rangle + \beta|1\rangle + \alpha|0\rangle + \beta i|0\rangle + \alpha i|1\rangle - \beta|1\rangle) \\
&= \frac{1}{2}(2\beta i|0\rangle + 2\alpha i|1\rangle) \\
&= \beta i|0\rangle + \alpha i|1\rangle
\end{aligned}$$

That is, B^2 acting quite similar to X, flip the $|0\rangle$ and $|1\rangle$. Except that it add a global phase of $i = e^{i\pi/4}$

Question 5

(a)

> Show that the matrix representation of $HP(\theta)HP(\phi)$ is given by
>
> $$HP(\theta)HP(\phi) = e^{i\theta/2} \begin{pmatrix} \cos\frac{\theta}{2} & -ie^{i\phi}\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$
\begin{aligned}
HP(\theta)HP(\phi) &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta} \\ 1 & -e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 & e^{i\phi} \\ 1 & -e^{i\phi} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+e^{i\theta} & e^{i\phi}-e^{i\theta}e^{i\phi} \\ 1-e^{i\theta} & e^{i\phi}+e^{i\theta}e^{i\phi} \end{pmatrix} \\
&= e^{i\theta/2}\begin{pmatrix} \frac{e^{-i\theta/2}+e^{i\theta/2}}{2} & -ie^{i\phi}\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i} \\ -i\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i} & e^{i\phi}\frac{e^{-i\theta/2}+e^{i\theta/2}}{2} \end{pmatrix} \\
&= e^{i\theta/2}\begin{pmatrix} \cos\frac{\theta}{2} & -ie^{i\phi}\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}
\end{aligned}
$$

(b)

> Write down the Hadamard transform of three Hadamard gates acting on the product state $|1\rangle|1\rangle|0\rangle$

That is basically

$$(H|1\rangle)(H|1\rangle)(H|0\rangle) = \frac{1}{\sqrt[3]{2}}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt[3]{2}}(|000\rangle + |001\rangle - |010\rangle - |011\rangle - |100\rangle - |101\rangle + |110\rangle + |111\rangle)$$