Divide and Conquer Select, Multiplication

CS240 Spring 2022

Rui Fan

Selection

- Given list A of n numbers, and $i \leq n$, Select(A, i) finds i'th smallest number in A.
  - Ex $A = [3,1,6,7,2]$, Select(A,4)=6.
  - Assume numbers all distinct, for simplicity.
- Select generalizes median.
  - Median of A is Select$(A, n/2)$.
- We can solve select by first sorting the numbers in $O(n \log n)$ time, then choosing the i'th largest one.
- We show how to solve Select, and hence median, in $O(n)$ time.

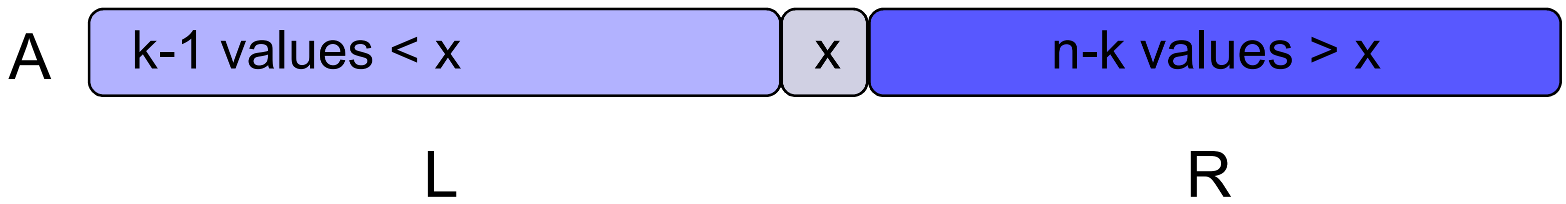
Select algorithm overview

- Break A into many small groups.
- Find the median of each small group directly.
- Recursively find the median of this set of medians (using Select).
- Use the median-of-medians to partition A.
- Keep looking for target in one of the two partitions.

Select algorithm 1

- Say A has n items, and we want i'th largest.
  - Assume for simplicity n divides 5.
- Divide A into $n/5$ groups of 5 elements each.
  - First 5 elements of A in first group, next 5 in second group, etc.
- For each group of 5 elements, find its median.
  - E.g. sort the 5 numbers, take 3^{rd} value.
  - Let B be the set of $n/5$ medians.
- Recursively find the median in B.
  - I.e. do Select($B, n/10$).
  - Say the median is x.

Select algorithm 2



- Partition A using x as pivot.
  - Move all values $< x$ to the left of x, all other values to the right.
  - If A has n values, this takes $O(n)$ time.
- Say there are k values $\leq x$ in A, and $n - k$ values $> x$.
- If $i = k$, return x.
  - We want the i'th largest item in A, which is x.
- If $i < k$, return Select(L, i).
  - Ex If $i = 5$ and $k = 10$, then fifth largest item in A is fifth largest item in L.
- Else $i > k$, return Select$(R, i - k)$.
  - Ex if $i = 15$ and $k = 10$, then fifteenth largest item in A is fifth largest item in R.

Select analysis 1

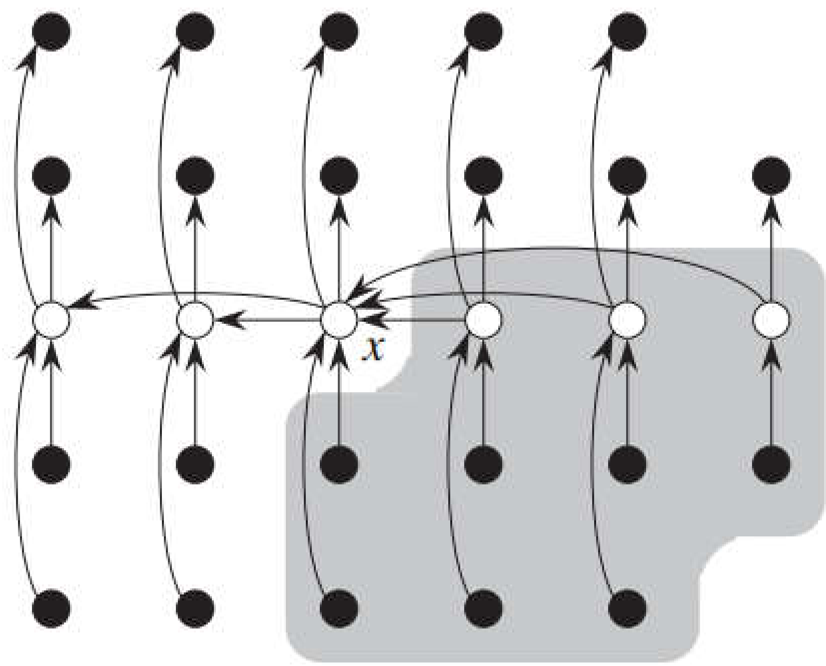
- Let $S(n)$ be an upper bound on select's running time given a list of size n.
- $S(n) = S(n/5) + S(u) + O(n)$
  - u is the size of whichever partition we recurse on.
  - $O(n)$ term comes from two parts.
    - First, for each of the $n/5$ groups of 5 numbers, find their median in constant time $\Rightarrow O(n)$ time overall.
    - Next, partition array in $O(n)$ time.
  - $S(n/5)$ to find the median of the $n/5$ group medians.
  - The key to ensuring select runs fast is ensuring u is small.
  - We will show $u \leq 7n/10$.

Select analysis 2

A	L = {values < x}	x	R = {values > x}

- Let x be the median of medians.
- We show that $|L| \geq 3n/10$, and $|R| \geq 3n/10$.
  - I.e. there are at least 3n/10 values less than x, and 3n/10 values larger than x.
- Then, also get $|L| \leq 7n/10$.
  - $|L| = n - 1 - |R|$.
  - Also, $|R| \leq 7n/10$.
- Thus, $S(n) \leq S(n/5) + S(7n/10) + O(n)$.

Select analysis 3



- Groups of 5 elements are shown in columns.
- Medians are shown in white.
- Medians less than x are shown to the right.

Source: *Introduction to Algorithms* Cormen, Leiserson, Rivest, Stein

- There are $n/5 * 1/2 = n/10$ medians less than x.
- For each such median, there are two more values from the group less than the median.
  - There are 3 values from each group < x.
- So there are at least $3n/10$ values < x.
- I.e. $|L| \geq 3n/10$.
- Similarly, $|R| \geq 3n/10$.

Select analysis 4

- We have $S(n) \leq S(n/5) + S(7n/10) + O(n)$.
  - Since we want to upper bound $S(n)$, we let $S(n) = S(n/5) + S(7n/10) + O(n)$.
  - For sufficiently large n, the $O(n)$ term is bounded by bn, for some constant b.
- We guess $S(n) = O(n)$, and show this satisfies equation above, i.e. the guess is valid.
- Since $S(n) = O(n), S(n) \leq cn$ for sufficiently large n, for some constant c.

Select analysis 5

- From $S(n) = S(n/5) + S(7n/10) + bn$ and $S(n) = cn$, we get

  $cn = c(n/5) + c * 7n/10 + bn$.
- So $cn = 9cn/10 + bn$, and $c = 10b$.
- Thus, $S(n) \leq 10bn = O(n)$ for sufficiently large n, for some constant b.

Multiplying complex numbers

- Compute $(a + bi)(c + di) = x + yi$.
  - i is the imaginary number, i.e. $i^2 = -1$.
- $x = ac - bd, y = ad + bc$
  - 4 multiplications, 2 additions.
- Can we do better?
- Yes! From Gauss, we have $x = ac - bd, y = (a + b)(c + d) - ac - bd$.
  - 3 multiplications: $ac, bd, (a + b)(c + d)$.
  - 5 additions.
- Why does Gauss's method matter?
  - It's useful when addition faster than multiplication.
  - It can be used recursively to speed up integer multiplication.

Complexity of multiplication

- Long multiplication of two integers
  - Multiply each digit of one number by all digits of other number.
  - Shift, carry and sum as needed.
  - If each number has n digits, takes $O(n^2)$ time.
- Can we multiply faster than $O(n^2)$ time?
- Yes! Using Karatsuba's algorithm (1962).

```
    5127
  x 4265
  ------
   25635
  307620
 1025400
20508000
--------
21866655
```

Divide and conquer multiplication

$$a = \underbrace{1000}_{a_1}\underbrace{1101}_{a_0} \qquad b = \underbrace{1110}_{b_1}\underbrace{0001}_{b_0}$$

$$\begin{aligned} a &= 2^{n/2} \cdot a_1 + a_0 \\ b &= 2^{n/2} \cdot b_1 + b_0 \\ ab &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot (a_1 b_0 + a_0 b_1) + a_0 b_0 \end{aligned}$$

- To multiply binary numbers a and b, split their digits in half and cross multiply.
  - Ex 10_2 x 11_2 = 1*1*2^2 + (1*1+0*1)*2^1 + 0*1*2^0 = 6
- Divide and conquer multiplication does 4 multiplications of $n/2$ bit numbers.
- It turns out this is not faster than long multiplication.

Karatsuba multiplication

$a = \underbrace{1000}_{a_1}\underbrace{1101}_{a_0}$ $\qquad$ $b = \underbrace{1110}_{b_1}\underbrace{0001}_{b_0}$

$$(a+bi)(c+di) = x + yi$$
$$x = ac - bd$$
$$y = (a+b)(c+d) - ac - bd$$

$$
\begin{aligned}
a &= 2^{n/2} \cdot a_1 + a_0 \\
b &= 2^{n/2} \cdot b_1 + b_0 \\
ab &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0 \\
&= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0\right) + a_0 b_0
\end{aligned}
$$

- Karatsuba multiplication rearranges the terms in divide and conquer multiplication.
- It does 3 multiplications of $n/2$ digit numbers instead of 4.
- Applied recursively, this makes all the difference!
- Notice the similarity between Karatsuba's and Gauss's method.

Karatsuba complexity

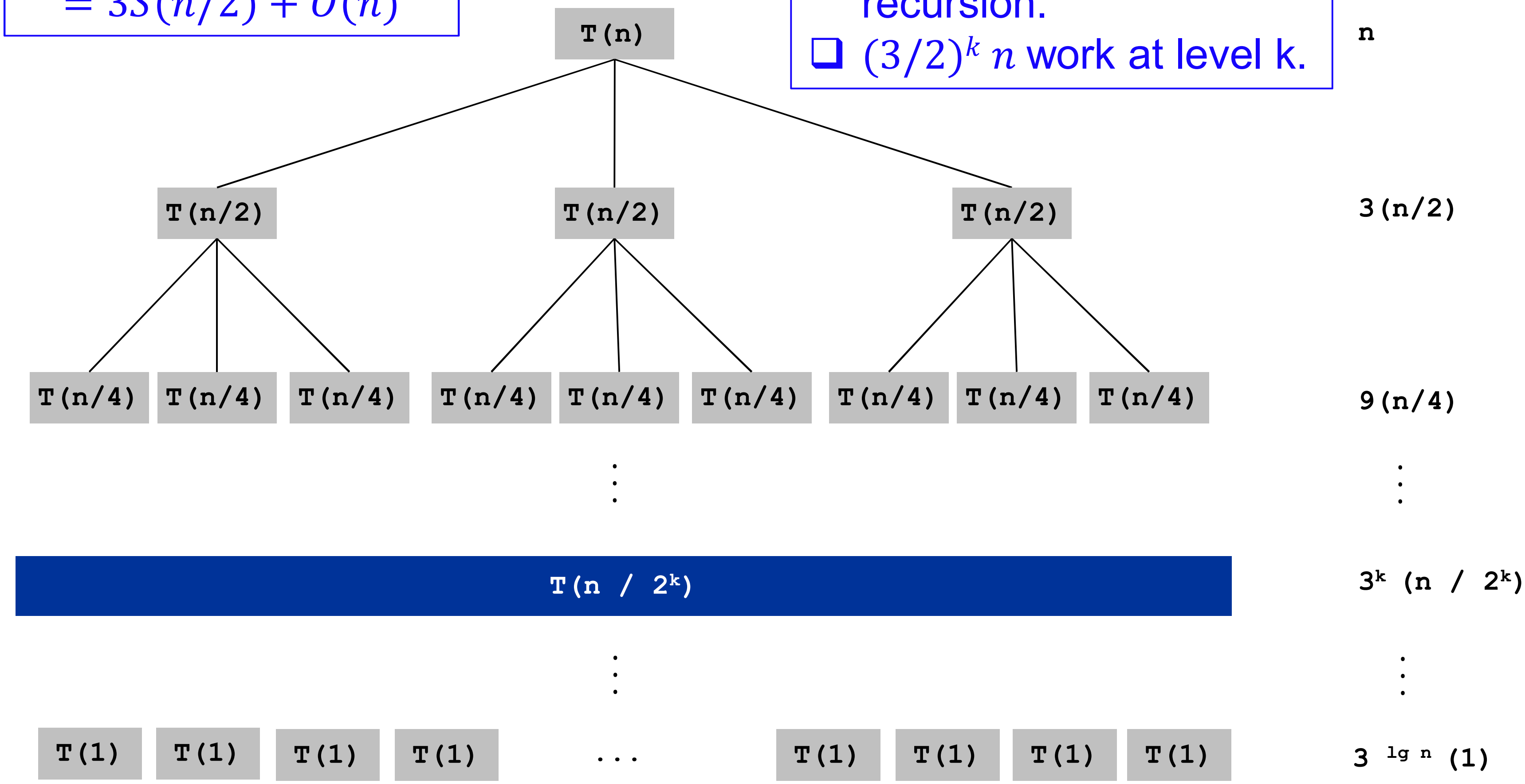
$$
\begin{aligned}
a &= 2^{n/2} \cdot a_1 + a_0 \\
b &= 2^{n/2} \cdot b_1 + b_0 \\
ab &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0 \\
&= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0\right) + a_0 b_0
\end{aligned}
$$

- S(n) = time to multiply two n digit numbers using Karatsuba.
- 3 multiplications of $n/2$ digit numbers.
  - $a_0 b_0, a_1 b_1, (a_1 + a_0)(b_1 + b_0)$.
  - Takes $3S(n/2)$ time.
- Multiply by 2^n and $2^{n/2}$.
  - Done by shifting in $O(n)$ time.
  - Ex $2^4 * 1011_2 = 10110000_2$.
- 5 additions / subtractions of $2n$ digit numbers.
  - $O(n)$ time.
- $S(n) = 3S(n/2) + O(n)$.
- $S(1) = 1$.

Karatsuba recursion tree

$S(n)$
$= 3S(n/2) + O(n)$

- $\lg n = \log_2 n$ levels of recursion.
- $(3/2)^k\, n$ work at level k.



Source: http://www.cs.bu.edu/fac/byers/courses/330/S13/handouts/05multiply.ppt

Karatsuba complexity

- $S(n) = n + \left(\frac{3}{2}\right) n + \left(\frac{3}{2}\right)^2 n + \cdots + \left(\frac{3}{2}\right)^{\lg n} n$
- $S(n) = n \left(\frac{\left(\frac{3}{2}\right)^{1+\lg n} - 1}{\frac{3}{2} - 1} \right) = 2n \left(\frac{3n^{\lg 3}}{2 \cdot 2^{\lg n}} - 1 \right) = 3n^{\lg 3} - 2n$
  - $\lg 3 \approx 1.59.$
- So Karatsuba's method runs in $O\left(n^{1.59}\right)$ instead of $O(n^2)$ time!
  - Practical for moderate n > 100.
- Useful for e.g. RSA cryptography.
- Even faster methods exist.
  - Schonhage-Strassen's Fast Fourier Transform based algorithm runs in $O(n \log n \log \log n)$ time.
  - Only practical for large n (> 10,000).

Complexity of matrix multiplication

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Source: http://www.cs.bu.edu/fac/byers/courses/330/S13/handouts/05multiply.ppt

```
for (i=0; i<n; i++) {
  for (j=0; j<n; j++) {
    C[i,j] = 0;
    for (k=0; k<n; k++)
      C[i,j] = C[i,j] + A[i,k]*B[k,j]
```

- Multiplying two $n \times n$ matrices the naive way takes $\Theta(n^3)$ time.
- Can we do better?
- Yes! Use Strassen's algorithm (1969).

Block matrix multiplication

$$
\underbrace{\begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}}_{C_{11}} \ldots
$$

$$
\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}
$$

$$
C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}
$$

- Break A, B and C into blocks. Multiply them as in normal matrix multiplication.
- Each block can be broken into subblocks and multiplied same way.
- Leads to recursive matrix multiplication method.

Block matrix multiplication

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

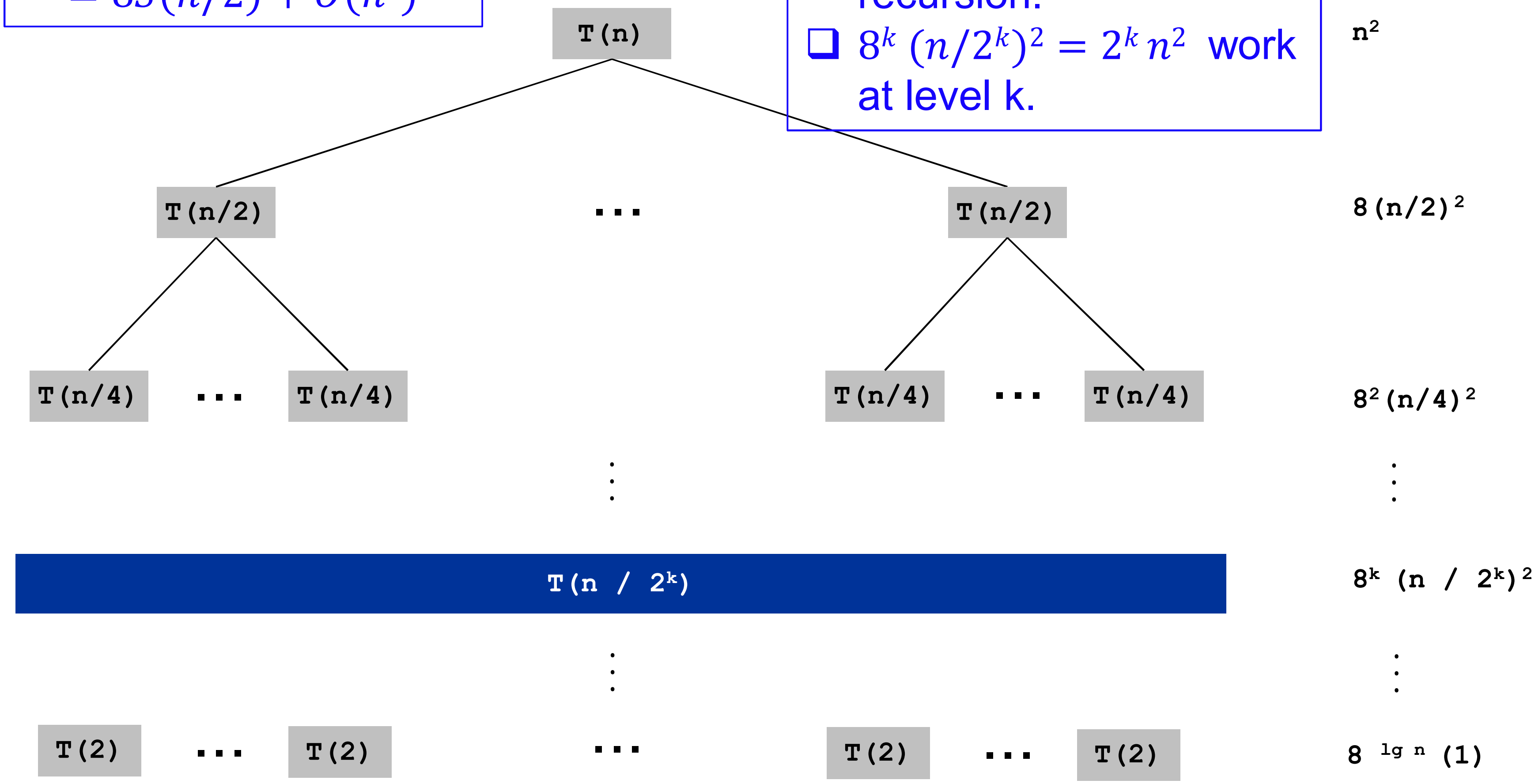
$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

- S(n) = time for $n \times n$ block matrix multiplication.
- C_{11} requires two $(n/2) \times (n/2)$ block matrix multiplications $\Rightarrow 2S(n/2)$ time.
- C_{12}, C_{21} and C_{22} also require $2S(n/2)$ time each.
- We also add four pairs of $(n/2) \times (n/2)$ matrices.
  - Takes $O(n^2)$ time.
- So $S(n) = 8S(n/2) + O(n^2)$.
  - $S(2) = O(1)$.

Block MM recursion tree

$S(n)$
$= 8S(n/2) + O(n^2)$

- $\lg n = \log_2 n$ levels of recursion.
- $8^k\,(n/2^k)^2 = 2^k\,n^2$ work at level k.



Block MM complexity

- $S(n) = n^2 + 2n^2 + 4n^2 + \ \ldots + 2^{\lg n} n2 =$ $n^2\,(1 + 2 + \cdots + 2^{\lg n}) = n^2(2n - 1) = \Theta(n^3)$.
- So simple block matrix multiplication takes same time as naive matrix multiplication.
- Problem is each recursion does 8 $(n/2) \times (n/2)$ MMs.
- Strassen's algorithm (1969) does 7 $(n/2) \times (n/2)$ MMs in each recursion.
  - Complexity becomes $O(n^{\lg_2 7}) = O\,(n^{2.81})$.

Strassen's algorithm

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

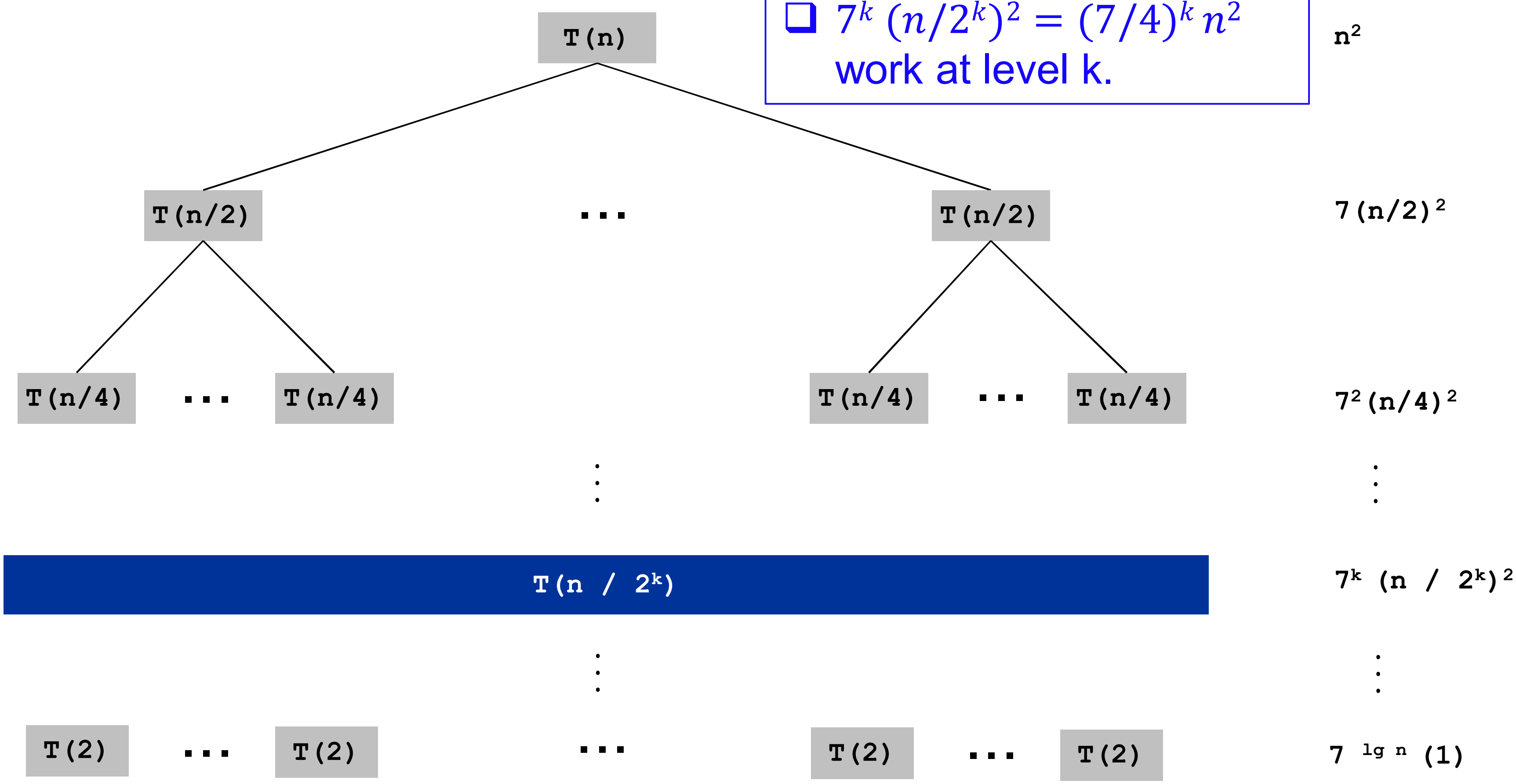
$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

- Strassen's algorithm does 7 multiplications and 18 additions / subtractions of $(n/2) \times (n/2)$ matrices.
- Let S(n) be Strassen's algorithm's complexity.
- $S(n) = 7S(n/2) + O(n^2)$.
  - $O(n^2)$ comes from the matrix addition/subtractions.
- S(2) = O(1).

Strassen recursion tree

$S(n) = 7S(n/2) + O(n^2)$

- $\lg n = \log_2 n$ levels of recursion.
- $7^k\,(n/2^k)^2 = (7/4)^k\,n^2$ work at level k.



Strassen's algorithm complexity

- $S(n) = n^2 + (7/4)n^2 + (7/4)^2 n^2 + \ \ldots + (7/4)^{\lg n} n^2$

  $= n^2 \left(1 + (7/4) + \cdots + (7/4)^{\lg n}\right)$

  $= n^2 \left(\left((7/4)^{\lg n + 1} - 1\right) / (7/4 - 1)\right)$

  $= n^2 \, O(7^{\lg n} / 4^{\lg n})$

  $= n^2 \, O(n^{\lg 7} / n^2)$

  $= \Theta(n^{\lg 7})$

  $= O(n^{2.81})$

More about matrix multiplication

- Strassen's algorithm was a huge surprise. Before Strassen, it was widely believed matrix multiplication required $\Omega(n^3)$ time.
- Not clear how Strassen discovered the algorithm. Maybe inspiration from Karatsuba?
- Strassen's algorithm is practical. Beats naive method for $n > 20 - 1000$, depending on hardware architecture.
- Since Strassen, more sophisticated algorithms (but impractical) algorithms with $O(n^{2.375})$time discovered.
- Some conjecture matrix multiplication can be done in $O(n^{2+\epsilon})$ time, for any $\epsilon > 0$.
  - This is nearly optimal, since even writing out $n \times n$ matrix output requires $O(n^2)$ time.