NP-Completeness Reductions

CS240 Spring 2023

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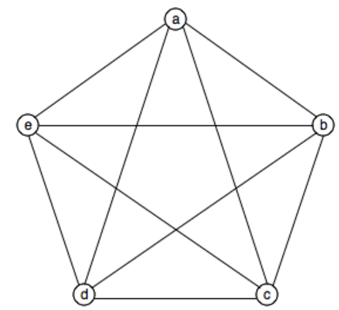
CLIQUE is NP-complete



- CLIQUE Given a graph with n nodes, is there a clique with n/3 nodes?
  - I.e., are there n/3 nodes s.t. they're all connected to each other?
  - Actually, the CLIQUE problem asks if there's a k-clique, for an arbitrary k. But we consider k=n/3 for simplicity.
- CLIQUE $\in$ NP.
  - The witness is a purported n/3-clique.
  - The verifier just checks there are n/3 nodes, and they're are all connected.

CLIQUE is NP-complete

- Show some NP-complete problem reduces to CLIQUE.
  - The problem you reduce from has to be NP-complete, not just in NP.
  - Note, you're reducing from the NPC problem to your problem, not the other way around.
  - You can choose any NP-complete problem to reduce from.
    - Decide on the right problem can make the task a lot easier. But this takes careful thought.

CLIQUE is NP-complete

- 3-CNF-SAT
  - Given a Boolean formula that's an AND of ORs, where each OR has 3 literals, is it satisfiable?
  - $(A \vee B \vee \neg C) \wedge (A \vee \neg B \vee C) \wedge (\neg A \vee \neg B \vee \neg D) \in 3-CNF-SAT.$
    - Set A=B=C=true, D=false.
  - Each OR unit is called a clause. Each literal is either a variable or its negation.
  - A special kind of SAT. SAT allows other formula types, besides ANDs of ORs, and allows any number of variables per clause.
- Assume we've already proven 3-CNF-SAT is NP-complete.
- We show 3-CNF-SAT $\leq_P$ CLIQUE.
- The reduction says that given a 3-CNF-SAT formula ϕ, we can create in polytime a graph G, such that ϕ is satisfiable if and only if G has an n/3-clique.
  - This is actually quite remarkable. Why should a graph be related to a formula?
  - But we'll see how to construct a special graph that captures the satisfiability of a 3-CNF formula.

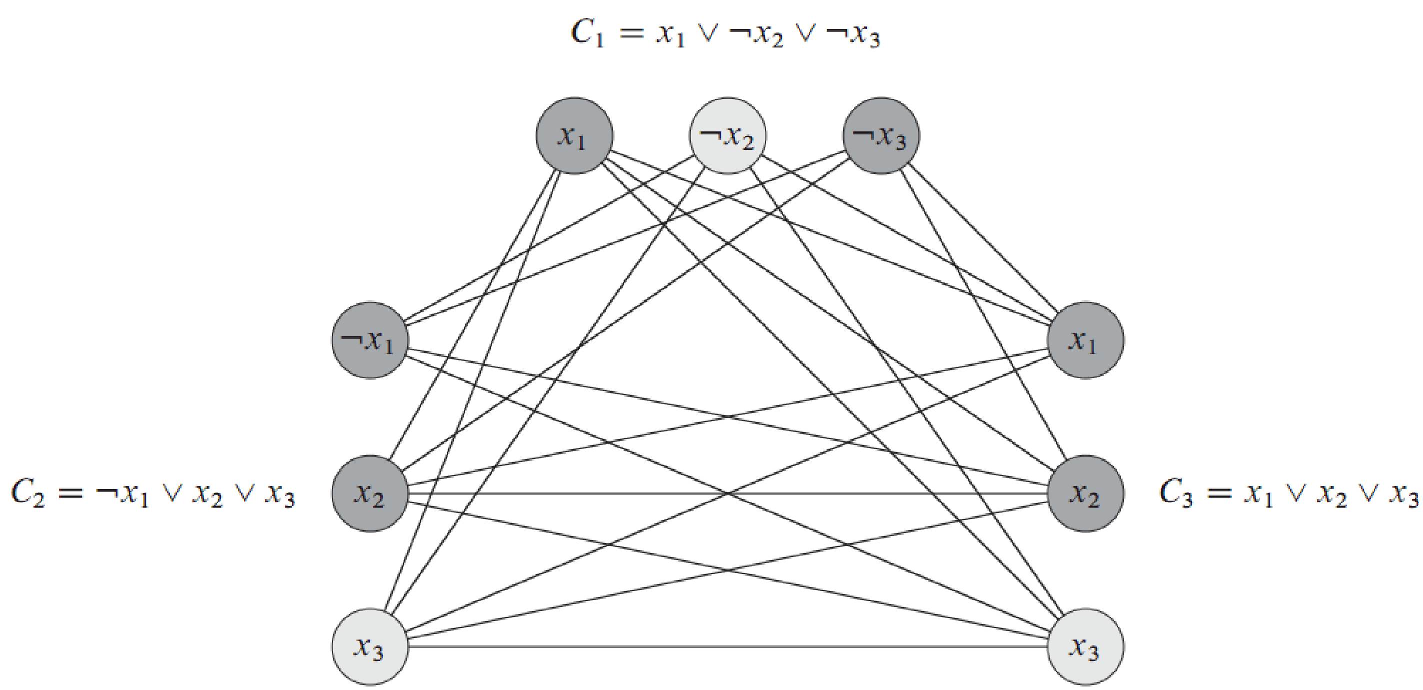
Reducing 3-CNF-SAT to CLIQUE

- Let ϕ be a 3-CNF formula with m clauses.
- Let C be a clause in ϕ. Then C has 3 literals.
  - Make 3 vertices in G corresponding to the literals.
    - So G has 3m vertices total.
    - Let n be the number of nodes in G. Then m = n/3.
- Now, add in an edge between two vertices u, v if both conditions below hold.
  - u, v correspond to literals from different clauses of ϕ.
  - The literals corresponding to u and v are not negations of each other.
    - We say u and v are consistent.

Reducing 3-CNF-SAT to CLIQUE

- 3 vertices for each clause.
- For vertices u,v, add edge (u,v) if u,v are from different clauses, and are consistent (not negations of each other).

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$



Source: Introduction to Algorithms, Cormen et al

Proving the reduction works

- We first need to show the reduction runs in polytime.
  - Yes. If there are n clauses, the reduction takes $O(n^2)$ time.
- Recall the graph has $n = 3m$ nodes, so $m = n/3$.
- Show $\phi \in$ 3-CNF-SAT $\Leftrightarrow$ $G \in$ m-CLIQUE.
  - ($\Rightarrow$) If ϕ has a satisfying assignment, then G has an m clique.
  - ($\Leftarrow$) If G has an m clique, then ϕ is satisfiable.

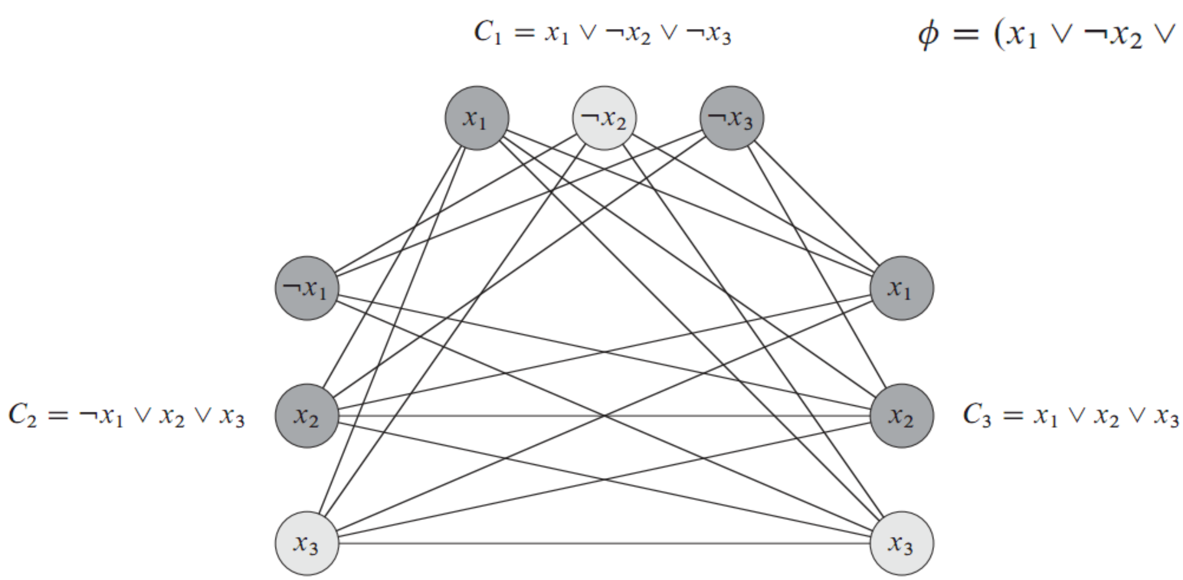
$\exists$ sat. assignment $\Rightarrow \exists$ m clique

- In the satisfying assignment, every clause has to be true, since we AND them.
- In each clause, at least one literal has to be true, since we OR them.
- So for each clause, pick a true literal.
  - We pick m = n/3 literals.
- The true literal corresponds to a vertex in the graph.
  - Pick m vertices corresponding to the m literals we picked.
- Claim The selected vertices form an m-clique.
- Proof Consider any 2 vertices u, v we selected.
  - u,v come from different triples.
    - Because they come from literals from different clauses.

$\exists$ sat. assignment $\Rightarrow$ $\exists$ m clique

- Proof ctd u,v are consistent. I.e. they don't correspond to a literal in one clause, and its negation in another clause.
  - Because we only picked true literals.
  - So there's an edge (u,v), by construction.
  - So any 2 of the m selected vertices are connected. So the vertices are an m-clique.
- Ex ϕ has a satisfying assignment $x_1 = x_2 = x_3 = T$.
  - The corresponding nodes form a 3-clique.

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$



$\exists$ m clique $\Rightarrow$ $\exists$ sat. assignment

- Consider the m vertices in the clique.
- None of the vertices come from literals in the same clause in ϕ.
  - For any pair of vertices, they're connected.
  - There are no edges between vertices from the same clause.
- None of the vertices correspond to a literal and its negation in ϕ.
  - We don't add edges between such vertices.
- For the literals corresponding to the clique vertices, set all of them to be true in the formula.
  - This is a valid assignment, since we never set a literal and its negation both to true.
  - We have one true literal per clause.
  - So every clause is true.
  - So the formula is true.

CLIQUE is NP-complete

- We've shown CLIQUE $\in$ NP.
- We've shown 3-CNF-SAT $\leq_P$ CLIQUE.
  - We found a polytime reduction, constructing a graph G s.t. for every 3-CNF-SAT formula ϕ
    - If ϕ is satisfiable, G has an n/3-clique.
    - If G has an n/3-clique, then ϕ is satisfiable.
- So CLIQUE is NP-complete.

SUBSET-SUM is NP-complete

- Recall that in SUBSET-SUM, we are given a set of numbers $S = \{s_1, \ldots, s_n\}$ and a target value t, and we want to find a subset $S' \subseteq S$ summing to t, i.e. $\sum_{s \in S'} s = t$.
- SUBSET-SUM ∈ NP.
  - The witness is a subset S' of S.
  - The verifier simply checks that S' sums up to t.
- To show SUBSET-SUM is NP-complete, we show 3-CNF-SAT $\leq_P$ SUBSET-SUM.
  - 3-CNF-SAT is a flexible problem used in many reductions.
- Given a 3-CNF formula ϕ, we construct in polytime a set S and target t s.t.
  - ϕ is satisfiable implies there's a subset of S summing to t.
  - If there's a subset of S summing to t, then ϕ is satisfiable.
  - This construction is the polytime reduction $\leq_P$.

The reduction

- Suppose ϕ contains n variables $x_1, \ldots, x_n$ and k clauses $C_1, \ldots, C_k$.
  - Assume WLOG that no clause contains a variable and its negation, since those clauses are automatically satisfied.
- The reduction creates a set S with 2n+2k numbers, two for each variable and clause.
  - Each number has n+k digits, with one digit corresponding to each variable and each clause.
  - The numbers are in base 10.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

- SUBSET-SUM instance for $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$.
- Each row except last represents a base-10 number in S. Last row is target t.

The reduction

- For each variable x_i, S contains two numbers v_i and v_i'.
  - v_i and v_i' both have a 1 in x_i's digit, and 0's in all the other variable digits.
  - If x_i appears in clause C_j, then the j'th clause digit in v_i is 1.
  - If $\neg x_i$ appears in clause C_j, then the j'th clause digit in v_i' is 1.
  - All other clause digits in v_i and v_i' are 0.
- For each clause C_j, S contains two numbers s_j and s_j'.
  - s_j has a 1 in the C_j digit, and s_j' has a 2 in this digit.
  - s_j and s_j' are 0's elsewhere.
- Target t is 1 in all the variable digits and 4 in all the clause digits.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

- SUBSET-SUM instance for $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$.
- Each row except last represents a base-10 number in S. Last row is target t.

$\phi \in 3\text{–}CNF\text{–}SAT \Rightarrow (S, t) \in SUBSET\text{–}SUM$

- Suppose there's a satisfying assignment ρ to ϕ.
- We form a subset S' of S summing to t based on ρ.
  - If $x_i = T$ in ρ, include v_i in S'.
  - If $x_i = F$ in ρ, include v_i' in S'.
- Claim 1 Any variable digit x_i sums to 1.
  - Either v_i or v_i' is in S', but not both.
  - Both v_i and v_i' cause digit x_i to be 1.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

- SUBSET-SUM instance for $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$.
- Each row except last represents a base-10 number in S. Last row is target t.
- Lightly shaded rows sum to t, and correspond to a satisfying assignment $x_1 = F, x_2 = F, x_3 = T$.

$\phi \in 3\text{–}CNF\text{–}SAT \Rightarrow (S, t) \in SUBSET\text{–}SUM$

- Claim 2 Any clause digit C_j sums to ≥ 1.
  - Since ρ is a satisfying assignment, C_j must have one true literal in ρ.
  - If the literal is x_i, then $x_i = T$ in ρ, and $v_i \in S'$.
    - v_i has a 1 in clause digit C_j, by construction.
  - If the literal is $\neg x_i$, then $x_i = F$ in ρ, and $v_i' \in S'$.
    - v_i' has a 1 in clause digit C_j, by construction.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

- SUBSET-SUM instance for $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$.
- Each row except last represents a base-10 number in S. Last row is target t.
- Lightly shaded rows sum to t, and correspond to a satisfying assignment $x_1 = F, x_2 = F, x_3 = T$.

$\phi \in 3\text{–}CNF\text{–}SAT \Rightarrow (S,t) \in SUBSET\text{–}SUM$

- Claim 3 Any clause digit C_j sums to $\leq$ 3.
  - C_j includes 3 literals.
  - The v or v' corresponding to each literal is either in S' or not.
  - If it's in S', it contributes 1 to C_j.
- Each clause digit C_j sums to between 1 to 3 using the current elements of S'.
  - Add s_j to S' if the sum is 3.
  - Add s_j' to S' if the sum is 2.
  - Add $\{s_j, s_j'\}$ to S' if the sum is 1.
- Now digit C_j sums to 4.
- Since all the variable digits sum to 1 by Claim 1, we have that S' sums to t.
- Thus, ϕ is satisfiable $\Rightarrow$ there's a subset of S summing to t.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

- SUBSET-SUM instance for $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$.
- Each row except last represents a base-10 number in S. Last row is target t.
- Lightly shaded rows sum to t, and correspond to a satisfying assignment $x_1 = F, x_2 = F, x_3 = T$.

$(S, t) \in SUBSET\text{–}SUM \Rightarrow \phi \in 3\text{–}CNF\text{–}SAT$

- Assume there's a subset S' summing to t.
  - We use S' to form a satisfying assignment ρ for ϕ.
- Notice the largest sum in any digit is 6.
  - Each variable digit sums to ≤ 2.
  - Each clause digit has three 1's among the v_i, v_i' values, since the clause contains 3 literals.
  - The s_j, s_j' values also sum to ≤ 3.
- Thus, there are no "carries" when we add values from S, i.e. the sum in each column comes only from values in that column.
- So since S' sums to 1 in the variable digits, it contains either v_i or v_i', but not both.
- If $v_i \in S'$, set $x_i = T$. If $v_i' \in S'$, set $x_i = F$.
  - Call this assignment ρ. Note ρ is valid, i.e. either $x_i = T$ or $x_i = F$, but not both.
  - We show ρ satisfies ϕ.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

- SUBSET-SUM instance for $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$.
- Each row except last represents a base-10 number in S. Last row is target t.
- Lightly shaded rows sum to t, and correspond to a satisfying assignment $x_1 = F, x_2 = F, x_3 = T$.

$(S, t) \in SUBSET{-}SUM \Rightarrow \phi \in 3{-}CNF{-}SAT$

- Consider a clause C_j.
  - Since C_j's clause digit in t is 4, and s_j and s_j' sum to ≤ 3 in this digit, S' must contain either a v_i or v_i' that has a 1 in clause digit C_j.
  - If $v_i \in S'$ and v_i has 1 in digit C_j, then x_i occurs in clause C_j.
    - Since we set $x_i = T$ in ρ, clause C_j is satisfied.
  - If $v_i' \in S'$ and v_i' has 1 in digit C_j, then $\neg x_i$ occurs in clause C_j.
    - Since we set $x_i = F$ in ρ, clause C_j is satisfied.
- In this way, all clauses satisfied.
  - So $\phi \in 3{-}CNF{-}SAT$.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

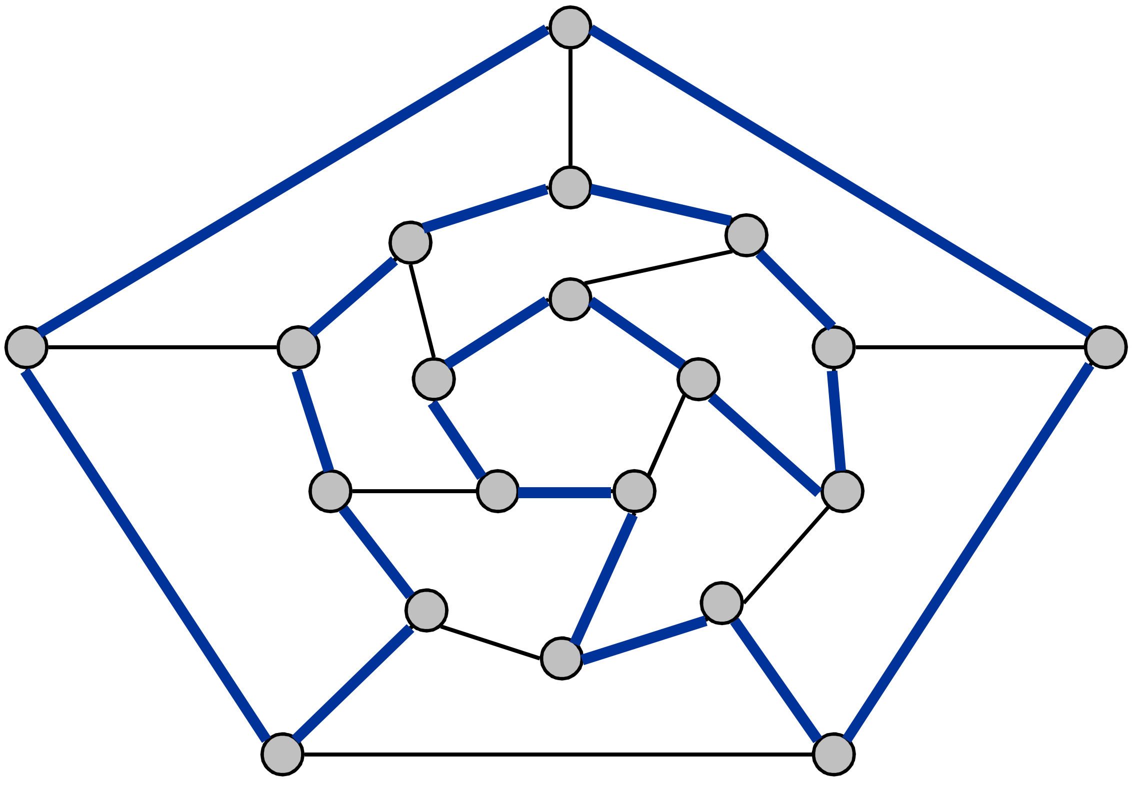
- SUBSET-SUM instance for $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$.
- Each row except last represents a base-10 number in S. Last row is target t.
- Lightly shaded rows sum to t, and correspond to a satisfying assignment $x_1 = F, x_2 = F, x_3 = T$.

SUBSET-SUM is NP-complete

- The reduction runs in polynomial time.
  - It creates 2n+2k+1 numbers, each with n+k digits. Each digit is computed in O(1) time.
- So 3-CNF-SAT $\leq_P$ SUBSET-SUM. Since SUBSET-SUM ∈ NP, then SUBSET-SUM is NP-complete.
- But didn't we show SUBSET-SUM ∈ P by dynamic programming?
  - The dynamic program ran in O(nW) time, where n is the number of elements in S, and W is the sum of the elements.
  - So did we prove P=NP!?
- No ☹, because O(nW) is not polynomial in the input size.
  - The input has n numbers, each with O(log W) bits.
  - So the input size is O(n log W).
  - The running time O(nW) is exponential in the input size.
  - For example, in the previous reduction, we had 2n+2k+1 numbers, but the sum of the numbers is $W \leq (2n + 2k + 1)10^{n+k}$.

Hamiltonian Cycle

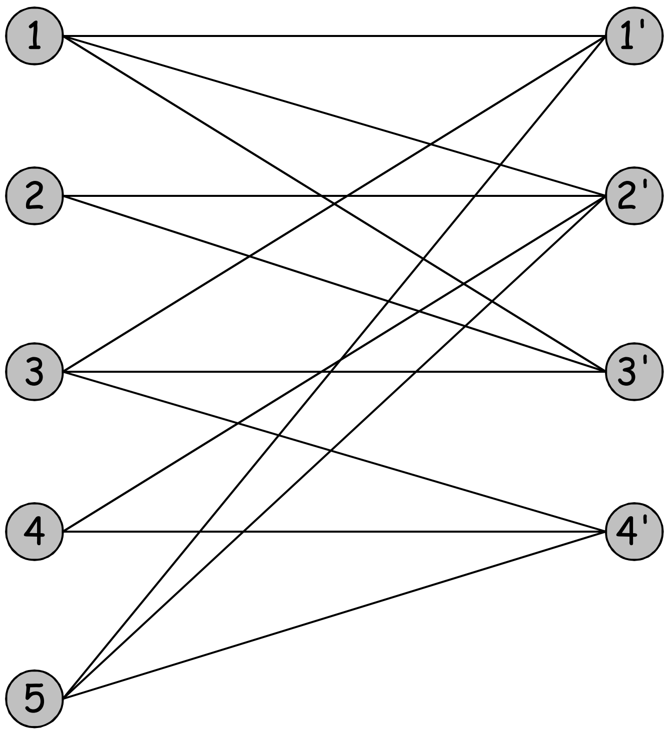
HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle Γ that contains every node in V.



YES: vertices and faces of a dodecahedron.

Hamiltonian Cycle

HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle Γ that contains every node in V.



NO: bipartite graph with odd number of nodes.

3-SAT Reduces to Directed Hamiltonian Cycle

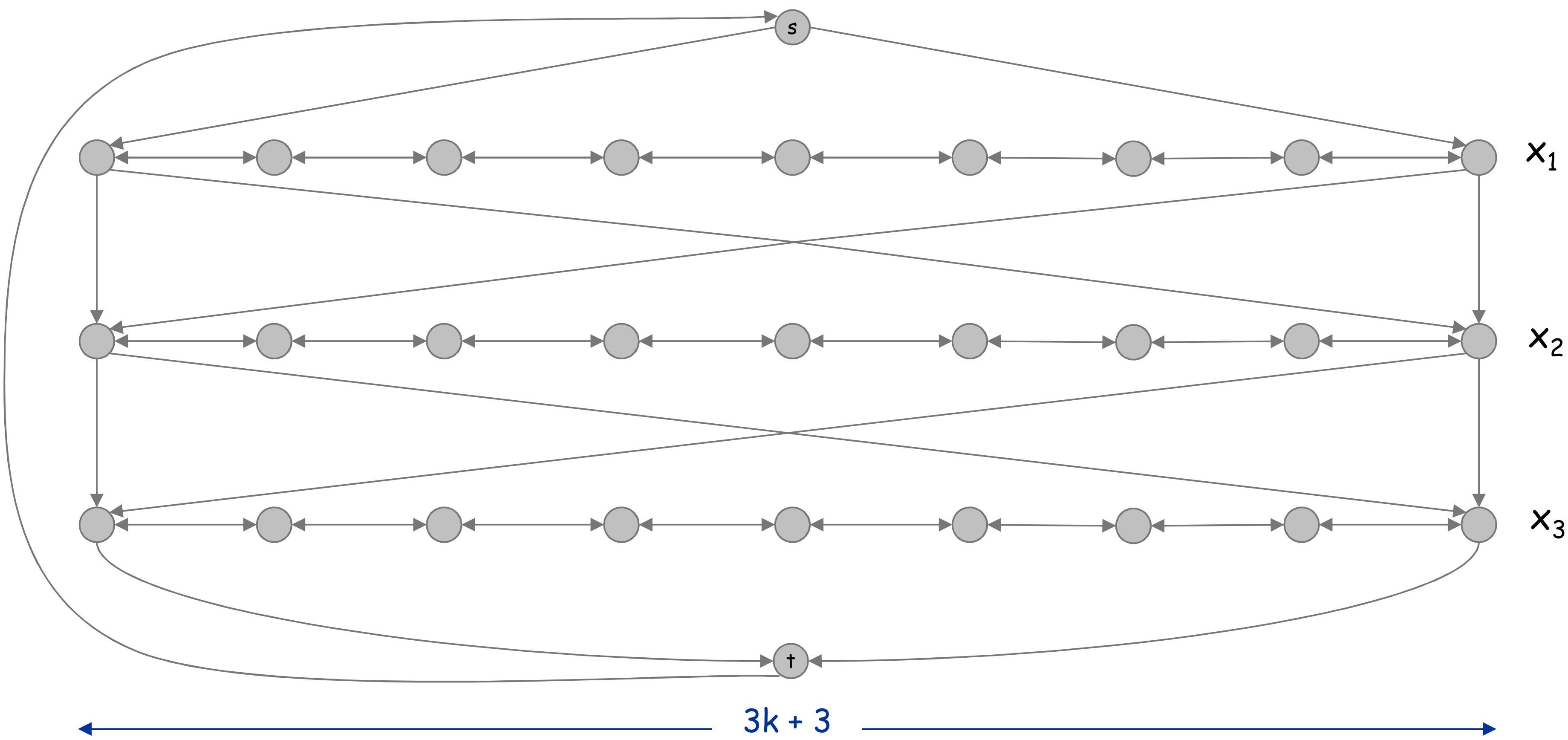
Claim. 3-SAT $\leq_P$ DIR-HAM-CYCLE.

Pf. Given an instance Φ of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff Φ is satisfiable.

3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

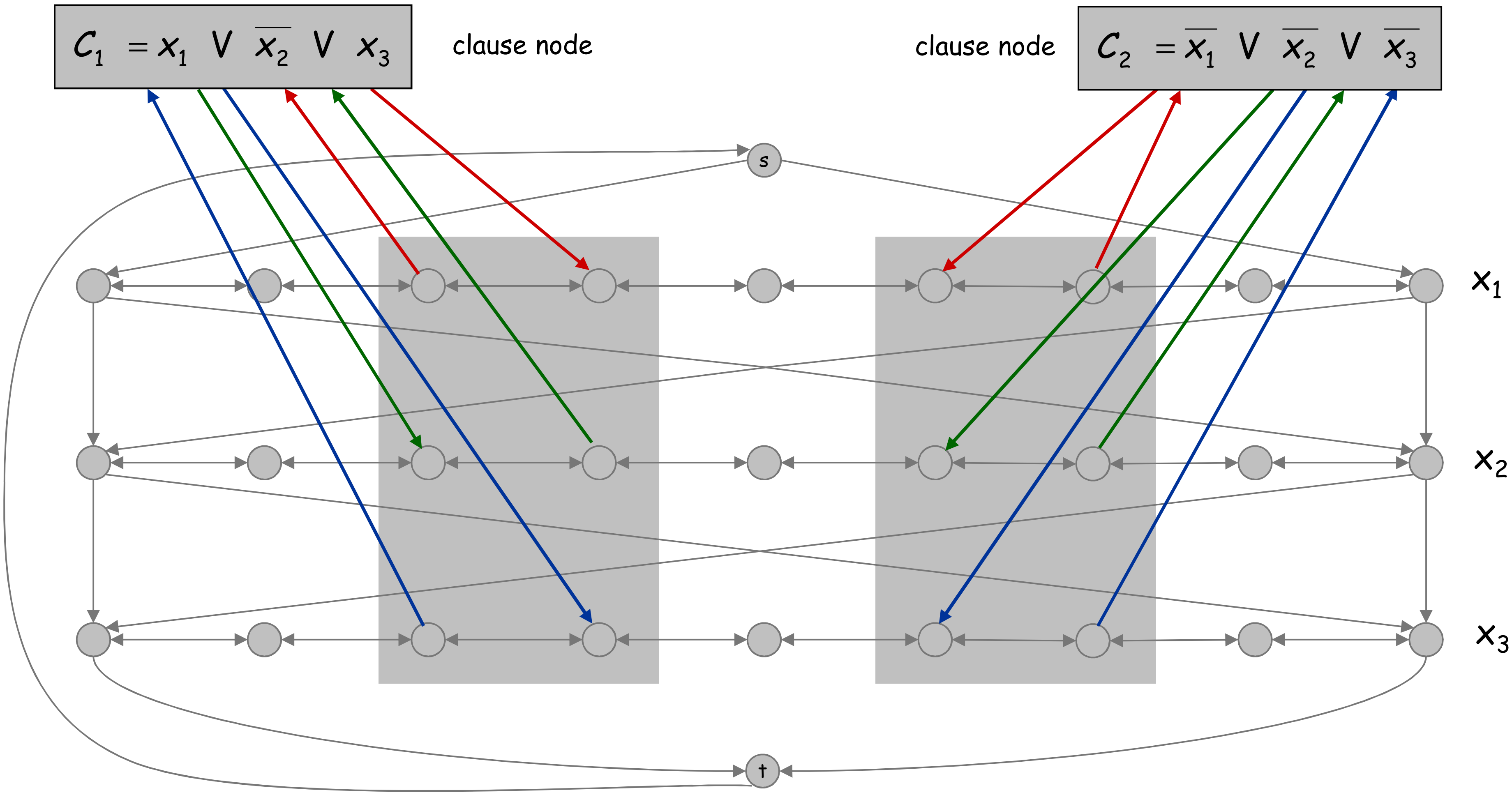
- Construct G to have 2^n Hamiltonian cycles.
- Intuition: traverse path i from left to right $\Leftrightarrow$ set variable $x_i = 1$.



3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

- For each clause: add a node and 6 edges.



3-SAT Reduces to Directed Hamiltonian Cycle

Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. $\Rightarrow$

- Suppose 3-SAT instance has satisfying assignment x^*.
- Then, define Hamiltonian cycle in G as follows:
  - if $x^*_i = 1$, traverse row i from left to right
  - if $x^*_i = 0$, traverse row i from right to left
  - for each clause C_j , there will be at least one row i in which we are going in "correct" direction to splice node C_j into tour

3-SAT Reduces to Directed Hamiltonian Cycle

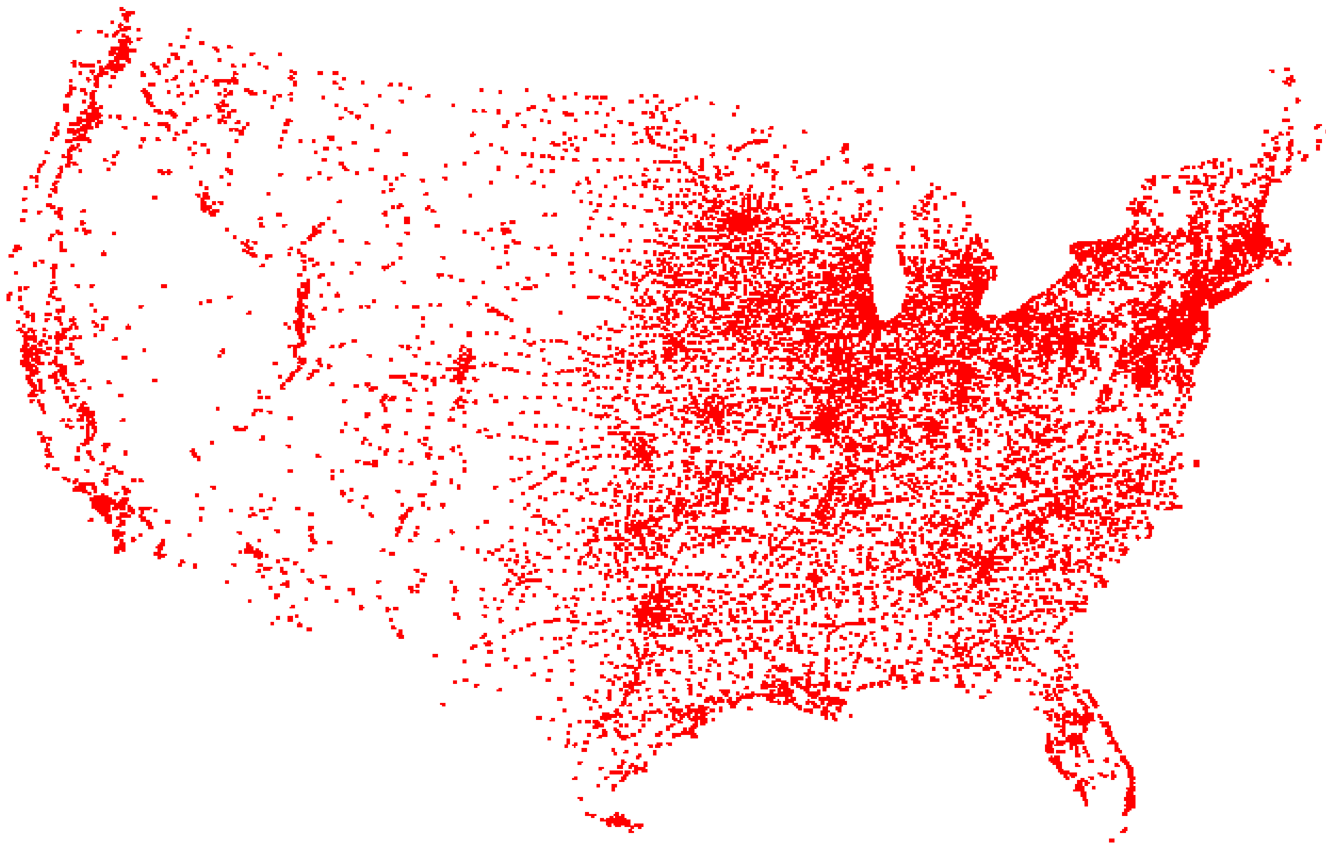
Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. $\Leftarrow$

- Suppose G has a Hamiltonian cycle Γ.
- If Γ enters clause node C_j , it must depart on mate edge.
  - thus, nodes immediately before and after C_j are connected by an edge e in G
  - removing C_j from cycle, and replacing it with edge e yields Hamiltonian cycle on G - { C_j }
- Continuing in this way, we are left with Hamiltonian cycle Γ' in G - { C_1 , C_2 , . . . , C_k }.
- Set $x^*_i = 1$ iff Γ' traverses row i left to right.
- Since Γ visits each clause node C_j , at least one of the paths is traversed in "correct" direction, and each clause is satisfied. ▪

Traveling Salesperson Problem

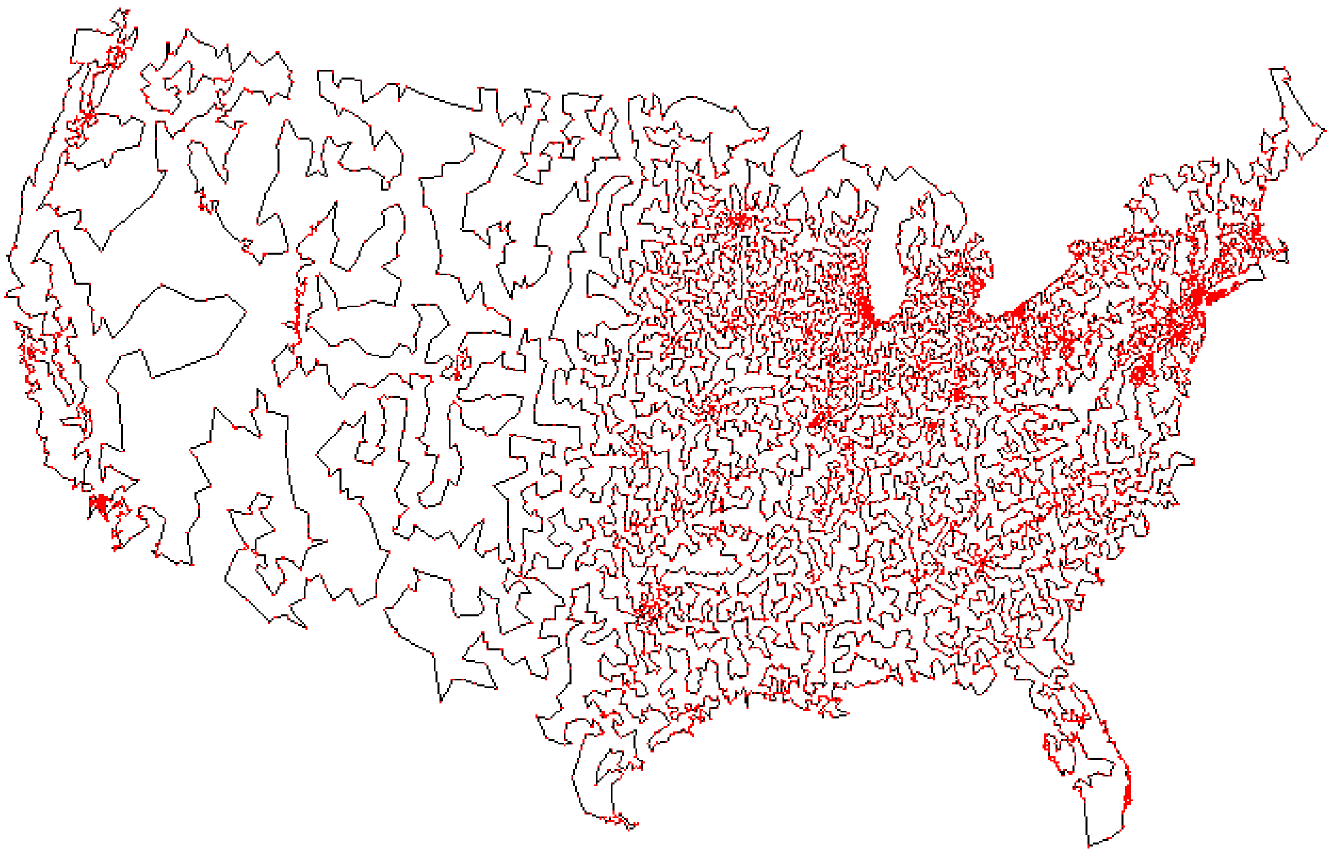
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



All 13,509 cities in US with a population of at least 500
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

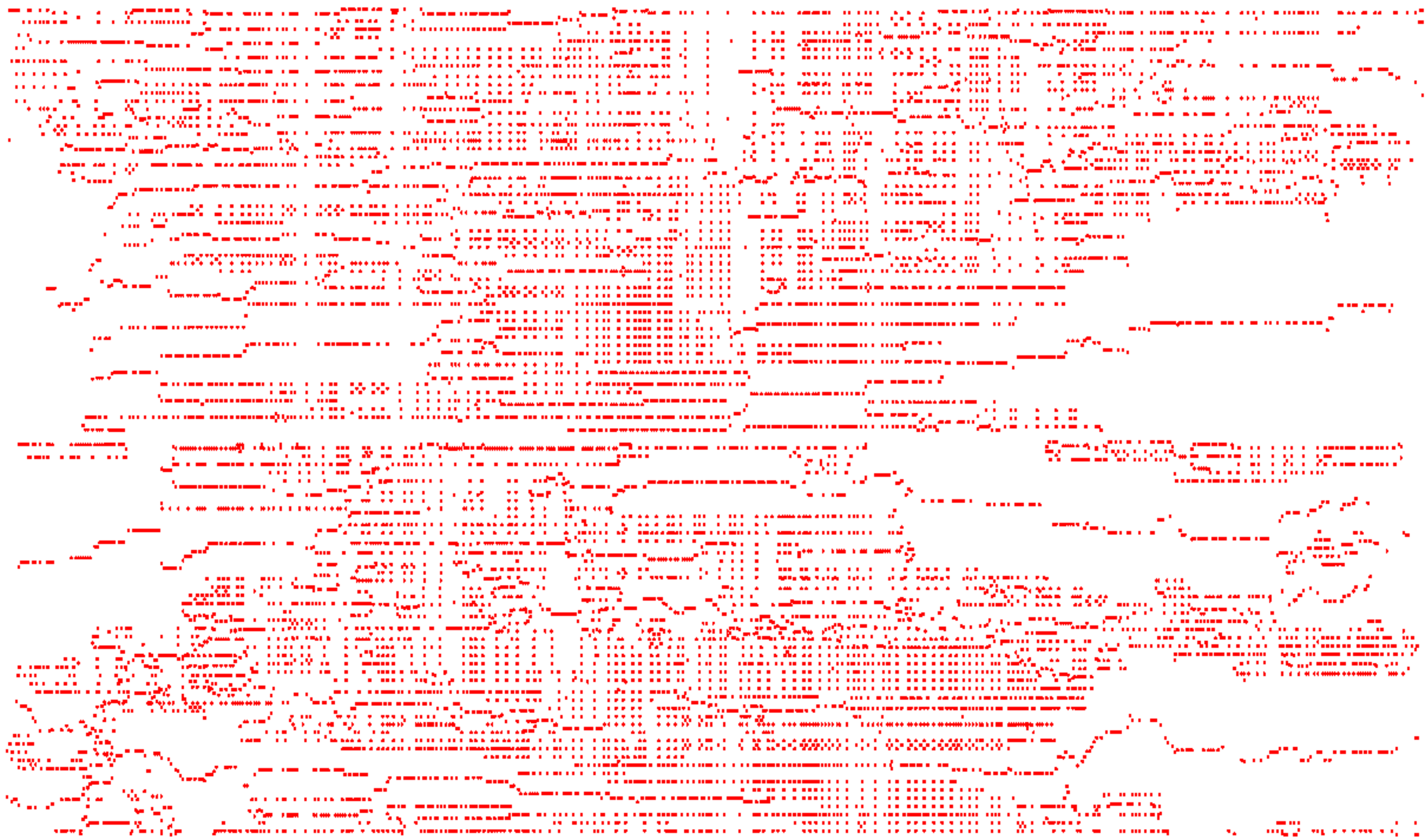
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

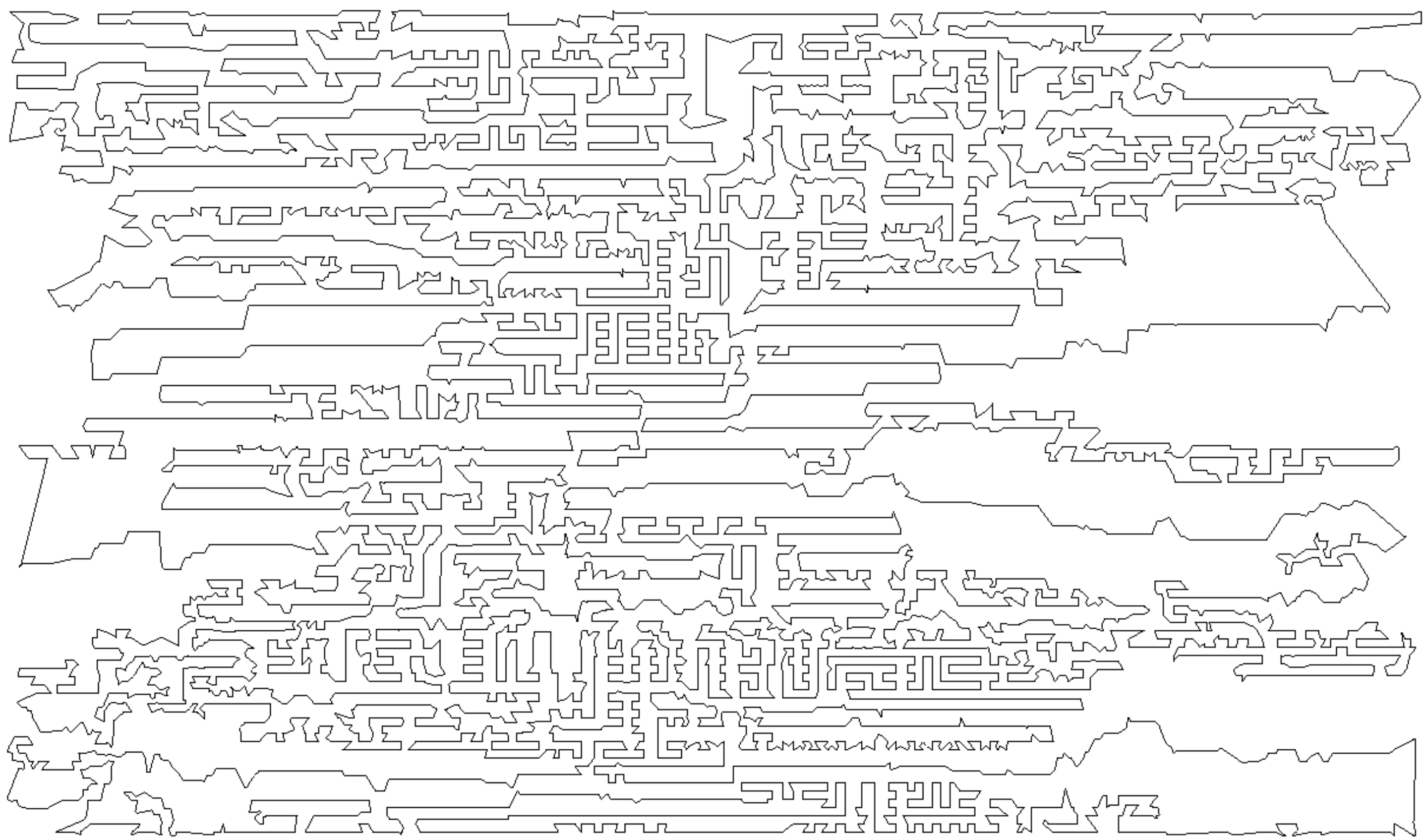
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?



11,849 holes to drill in a programmed logic array
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



Optimal TSP tour
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Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?

HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V?

Claim. HAM-CYCLE $\leq_P$ TSP.

Pf.

- Given instance G = (V, E) of HAM-CYCLE, create n cities with distance function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

- TSP instance has tour of length ≤ n iff G is Hamiltonian. ▪