

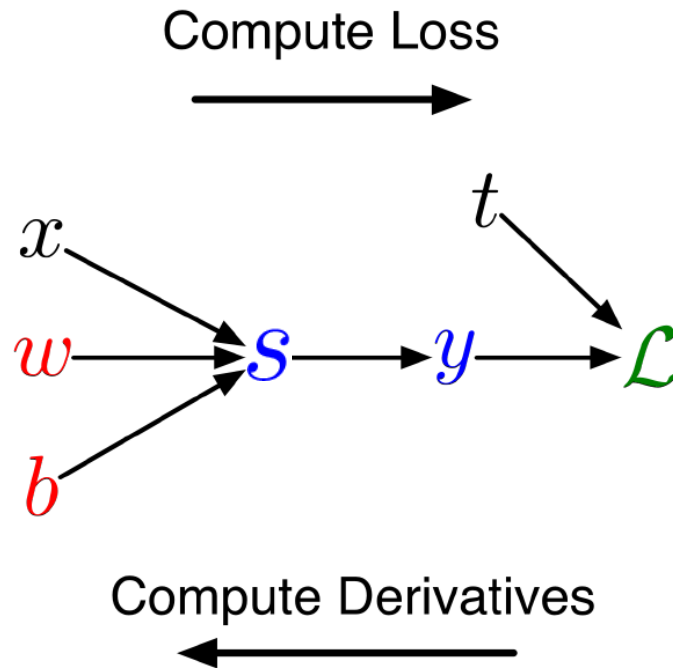


Lecture 4: CNNs I - Architecture & Equivariance

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Computation graph

- Represent the computations using a **computation graph**
 - Nodes: inputs & computed quantities
 - Edges: which nodes are computed directly as function of which other nodes



General Backpropagation

- Given a computation graph

Let v_1, \dots, v_N be a **topological ordering** of the computation graph (i.e. parents come before children.)

v_N denotes the variable we're trying to compute derivatives of (e.g. loss)

forward pass

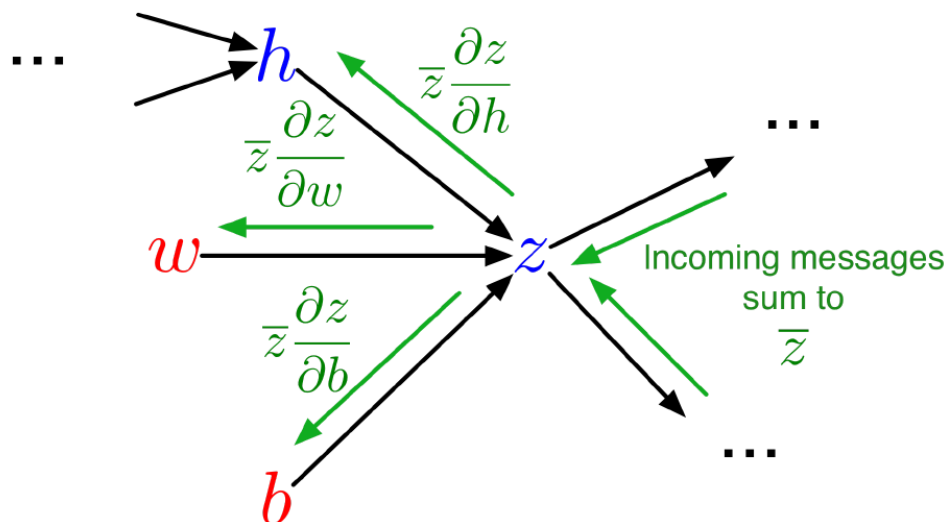
For $i = 1, \dots, N$
Compute v_i as a function of $\text{Pa}(v_i)$

backward pass

$\delta_{v_N} = 1$
For $i = N - 1, \dots, 1$
$$\delta_{v_i} = \sum_{j \in \text{Ch}(v_i)} \delta_{v_j} \frac{\partial v_j}{\partial v_i}$$

General Backpropagation

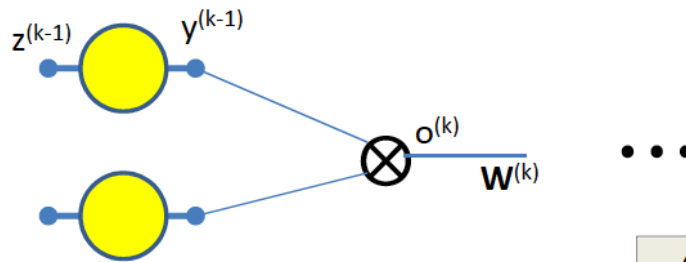
- Backprop as message passing:



- Each node receives a set of messages from its children, which are aggregated into its error signal, then it passes messages to its parents
- **Modularity:** each node only has to know how to compute derivatives w.r.t. its arguments – **local computation in the graph**

Patterns in backward flow

- Multiplicative node

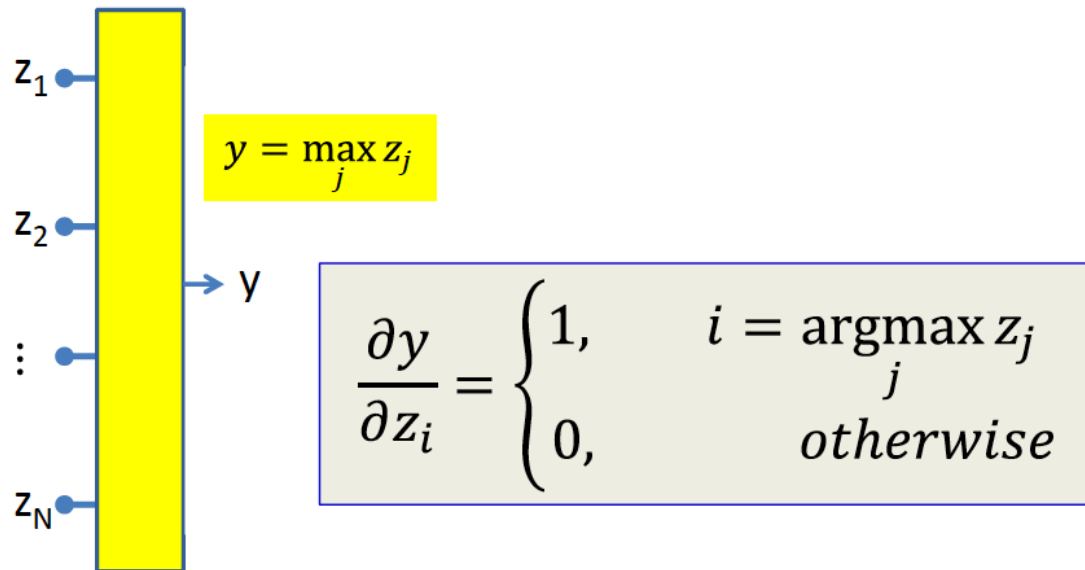


Forward:
$$o_i^{(k)} = y_j^{(k-1)} y_l^{(k-1)}$$

$$\frac{\partial L}{\partial y_j^{(k-1)}} = \frac{\partial L}{\partial o_i^{(k)}} \frac{\partial o_i^{(k)}}{\partial y_j^{(k-1)}} = y_l^{(k-1)} \frac{\partial L}{\partial o_i^{(k)}}$$

Patterns in backward flow

■ Max node



- Vector equivalent of subgradient
 - 1 w.r.t. the largest incoming input
 - Incremental changes in this input will change the output
 - 0 for the rest
 - Incremental changes to these inputs will not change the output

Vector form of BackProp

- Review: Jacobian of vector functions

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

- Vectorized chain rule

$$\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$$

$$g : \mathbb{R}^m \rightarrow \mathbb{R}^n, \mathbf{y} = g(\mathbf{x})$$

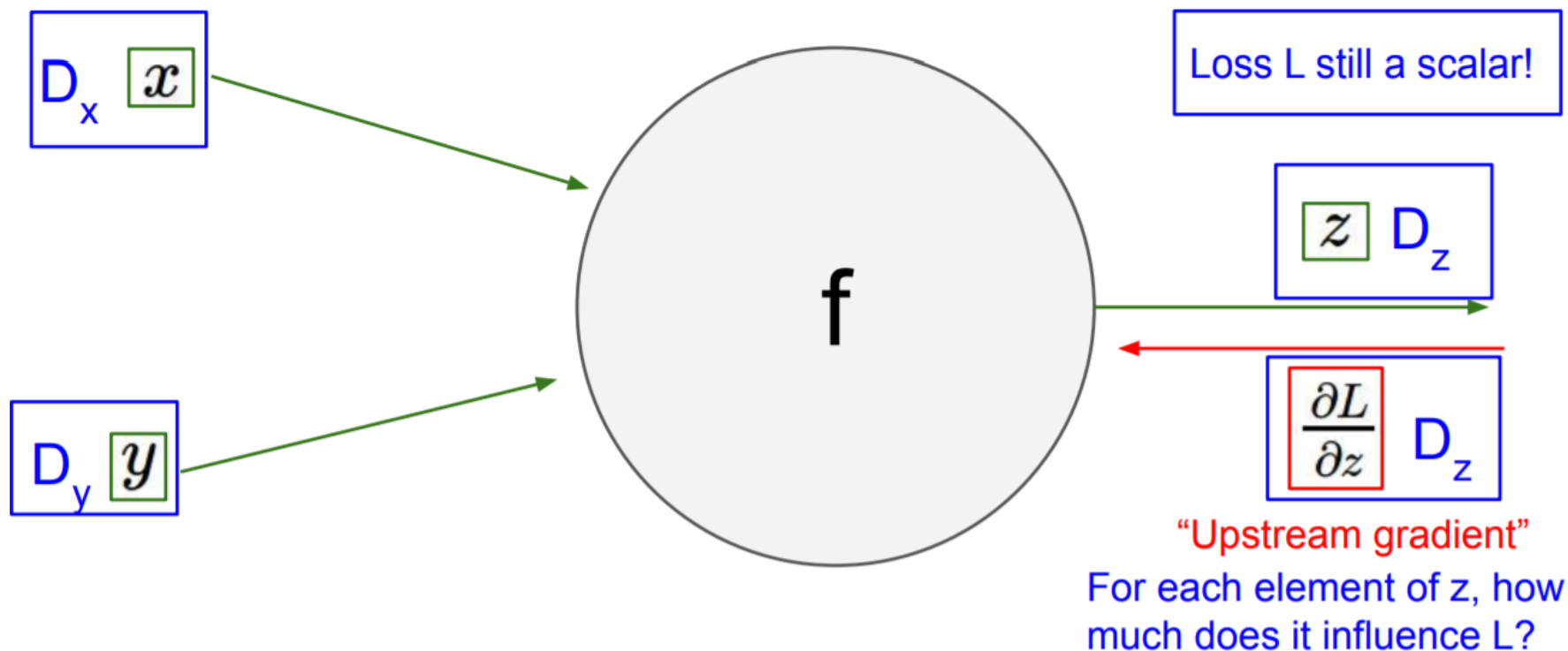
$$\frac{\partial z}{\partial \mathbf{x}_i} = \sum_{j=1}^n \frac{\partial z}{\partial \mathbf{y}_j} \frac{\partial \mathbf{y}_j}{\partial \mathbf{x}_i}$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, z = f(\mathbf{y})$$

$$\nabla_{\mathbf{x}} z = \left[\frac{\partial \mathbf{y}_j}{\partial \mathbf{x}_i} \right] \nabla_{\mathbf{y}} z = \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right)^T \nabla_{\mathbf{y}} z$$

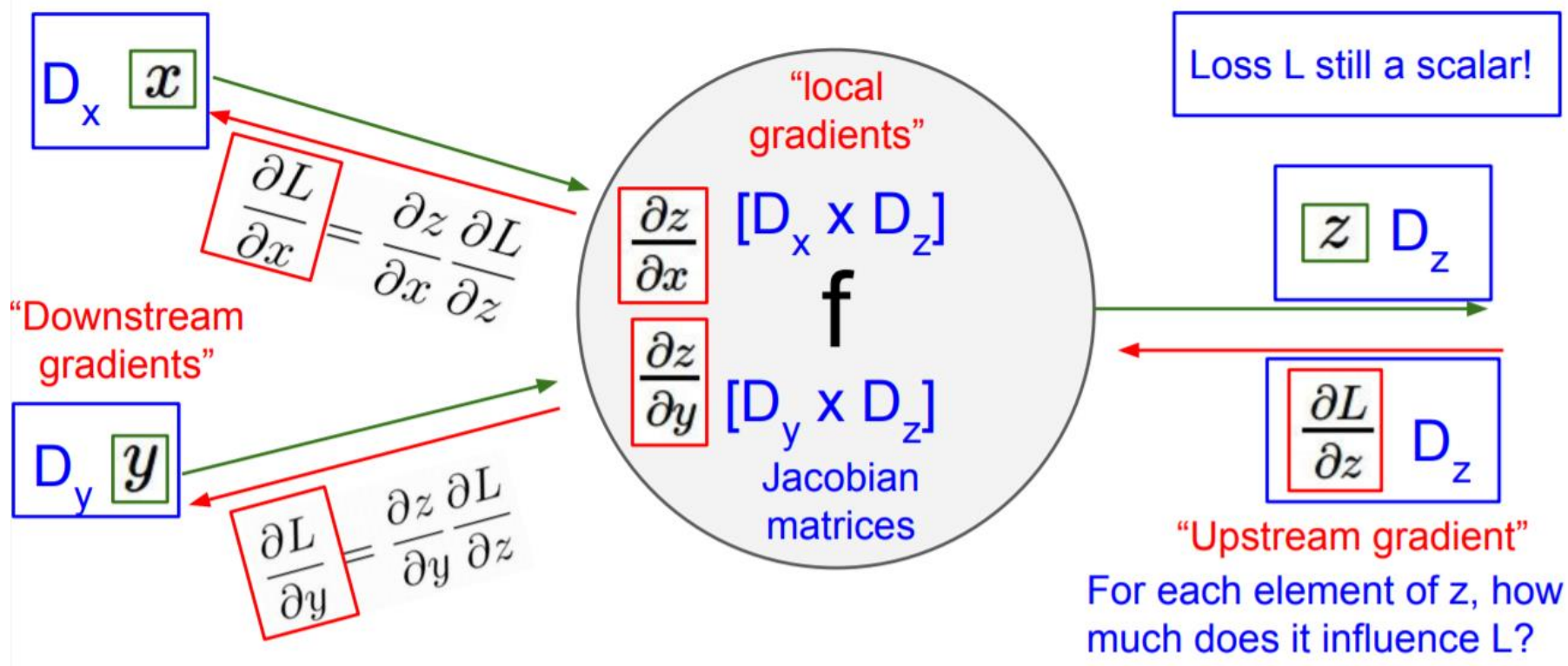
Vector form of BackProp

- Forward pass with vectors



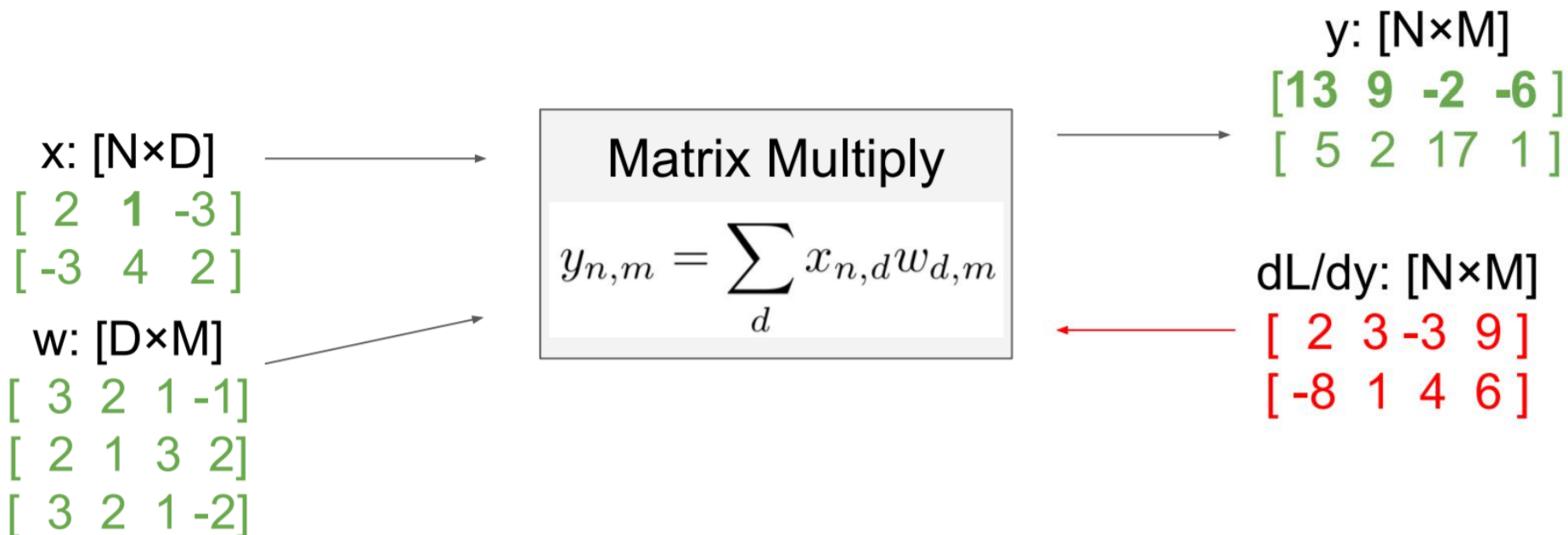
Vector form of BackProp

- Note: here the Jacobian matrices are actually the transpose of the standard version.



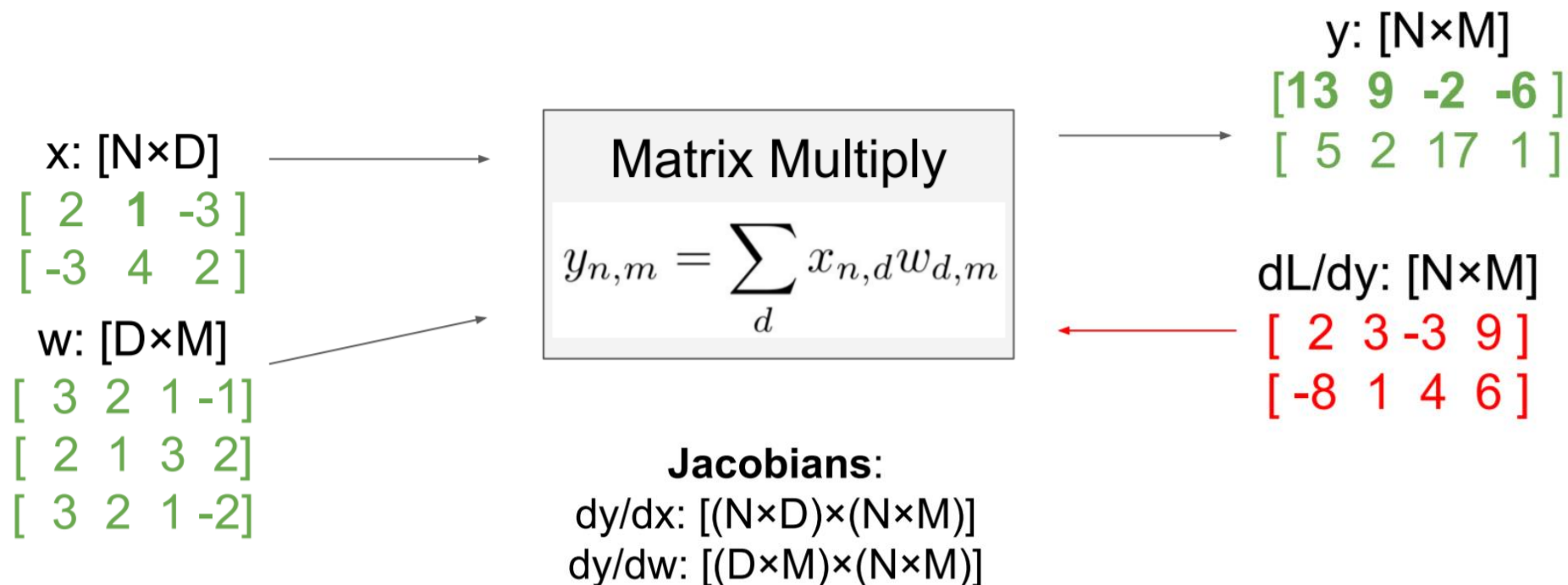
Matrix form of BackProp

- Often used in mini-batches
 - N is the batch size, for instance.



Matrix form of BackProp

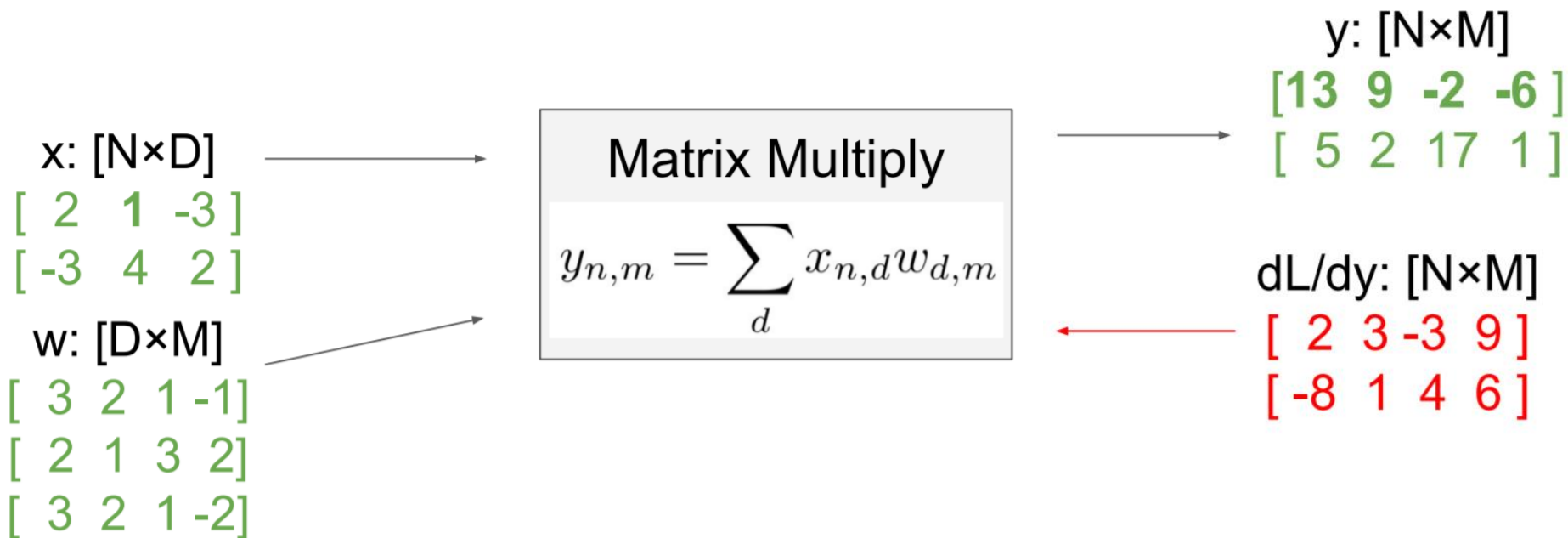
- Often used in mini-batches
 - N is the batch size, for instance.



For a neural net we may have
 $N=64, D=M=4096$
Each Jacobian takes 256 GB of memory!
Must work with them implicitly!

Matrix form of BackProp

- Often used in mini-batches
 - N is the batch size, for instance.



Backward Pass:

Gradient dL/dy : $[N \times M]$

$$\begin{bmatrix} 2 & 3 & -3 & 9 \\ -8 & 1 & 4 & 6 \end{bmatrix}$$

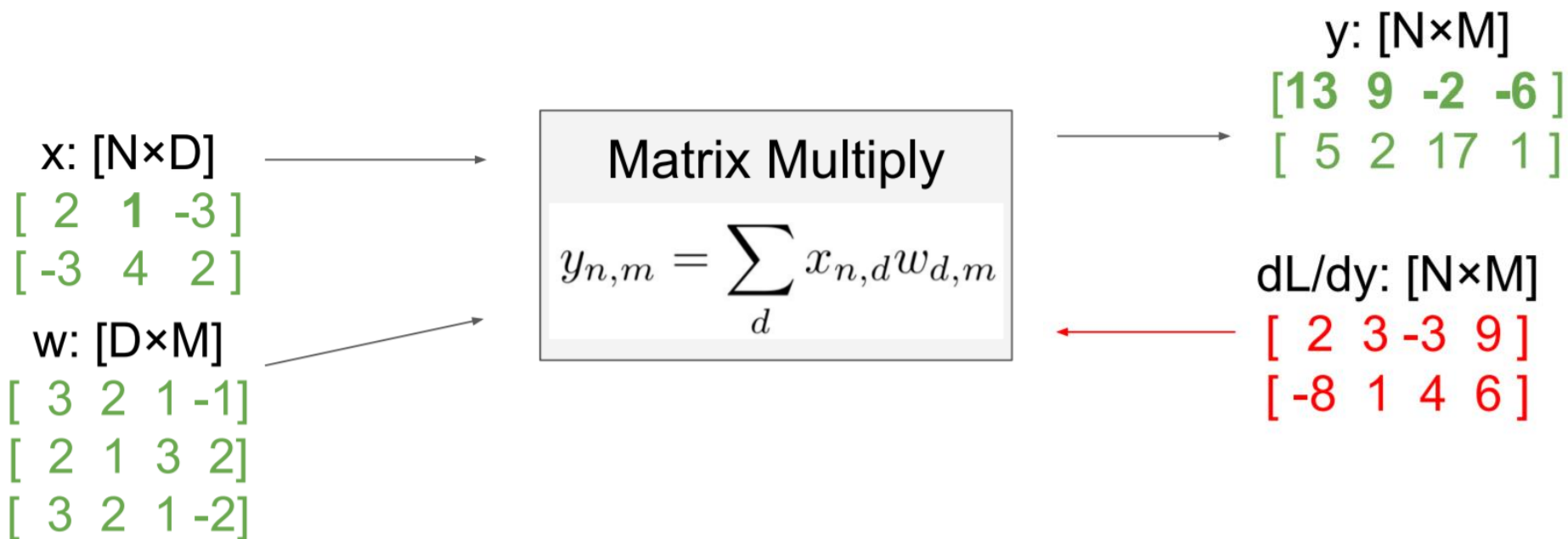
Dimensions: $[N \times D] \quad [N \times M] \quad [M \times D]$

$$\frac{\partial L}{\partial x} = \left(\frac{\partial L}{\partial y} \right) w^T$$

$$\frac{\partial L}{\partial x_{n,d}} = \sum_m \frac{\partial L}{\partial y_{n,m}} \frac{\partial y_{n,m}}{\partial x_{n,d}} = \sum_m \frac{\partial L}{\partial y_{n,m}} w_{d,m}$$

Matrix form of BackProp

- Often used in mini-batches
 - N is the batch size, for instance.



$[N \times D]$ $[N \times M]$ $[M \times D]$

$$\frac{\partial L}{\partial x} = \left(\frac{\partial L}{\partial y} \right) w^T$$

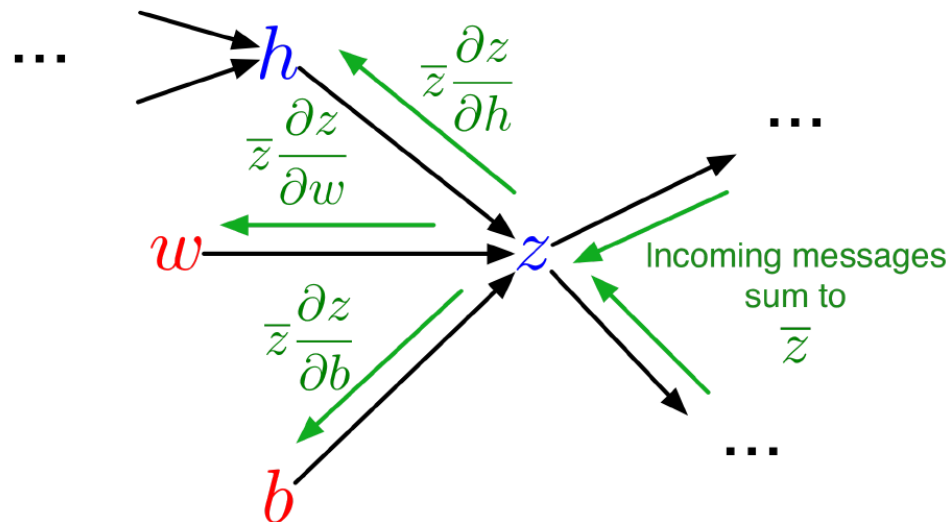
$[D \times M]$ $[D \times N]$ $[N \times M]$

$$\frac{\partial L}{\partial w} = x^T \left(\frac{\partial L}{\partial y} \right)$$

These formulas are easy to remember: they are the only way to make shapes match up!

Computation cost

- Forward pass: one add-multiply operation per weight
- Backward pass: two add-multiply operations per weight



- For a multilayer network, the cost is linear in the number of layers, quadratic in the number of units per layer

Backpropagation

- Backprop is used to train the majority of neural nets
 - Even generative network learning, or advanced optimization algorithms (second-order) use backprop to compute the update of weights
- However, backprop seems biologically implausible
 - No evidence for biological signals analogous to error derivatives
 - All the existing biologically plausible alternatives learn much more slowly on computers.
 - So how on earth does the brain learn???

Outline

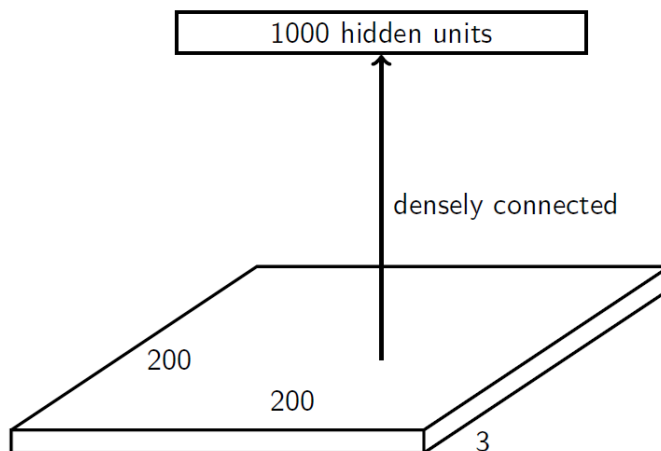
- Why Convolutional Neural Network (CNN)?
 - Motivation and overview
- What is the CNN?
 - Convolution layers & model complexity
 - Closer look at activation functions
 - Pooling layers & model complexity
 - Math properties
- Examples of CNNs

Acknowledgement: Roger Grosse@UofT & Feifei Li's cs231n notes

Motivation

■ Visual recognition

- Suppose we aim to train a network that takes a 200x200 RGB image as input



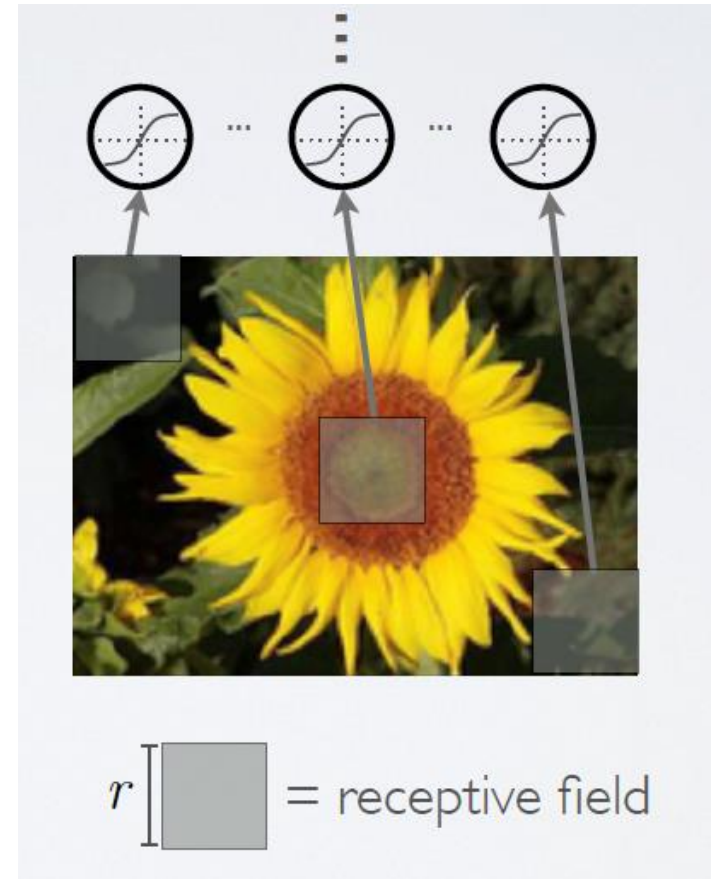
- What is the problem with have full connections in the first layer?
 - Too many parameters! $200 \times 200 \times 3 \times 1000 = 120$ million
 - What happens if the object in the image shifts a little?

Our goal

- Visual Recognition: Design a neural network that
 - Much deal with very **high-dimensional inputs**
 - Can exploit the **2D topology** of pixels in images
 - Can build in **invariance/equivariance to certain variations** we can expect
 - Translation, small deformations, illumination, etc.
- Convolution networks leverage these ideas
 - Local connectivity
 - Parameter sharing
 - Pooling/subsampling hidden units

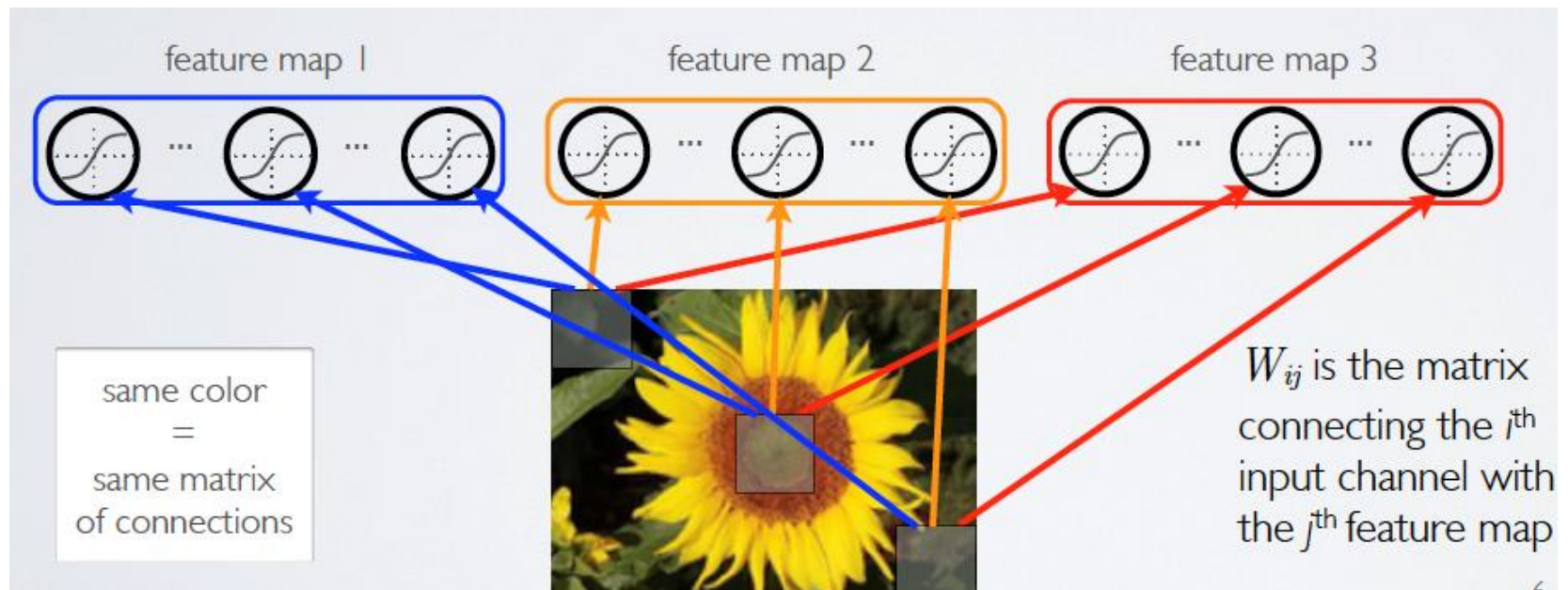
Overview of CNNs

- First idea: Use a local connectivity of hidden units
 - Each hidden unit is connected only to a subregion (patch) of the input image
 - Usually it is connected to all channels
 - Each neuron has a local receptive field



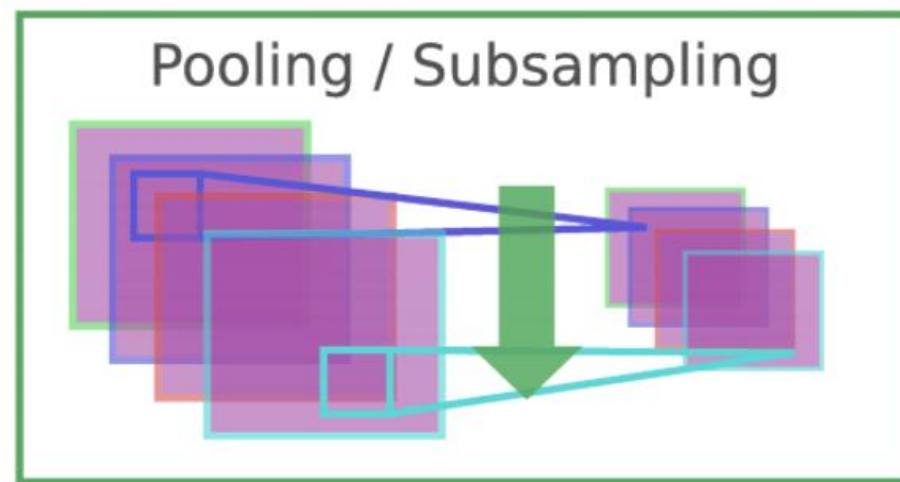
Overview of CNNs

- Second idea: share weights across certain units
 - Units organized into the same “feature map” share weight parameters
 - Hidden units within a feature map cover different positions in the image



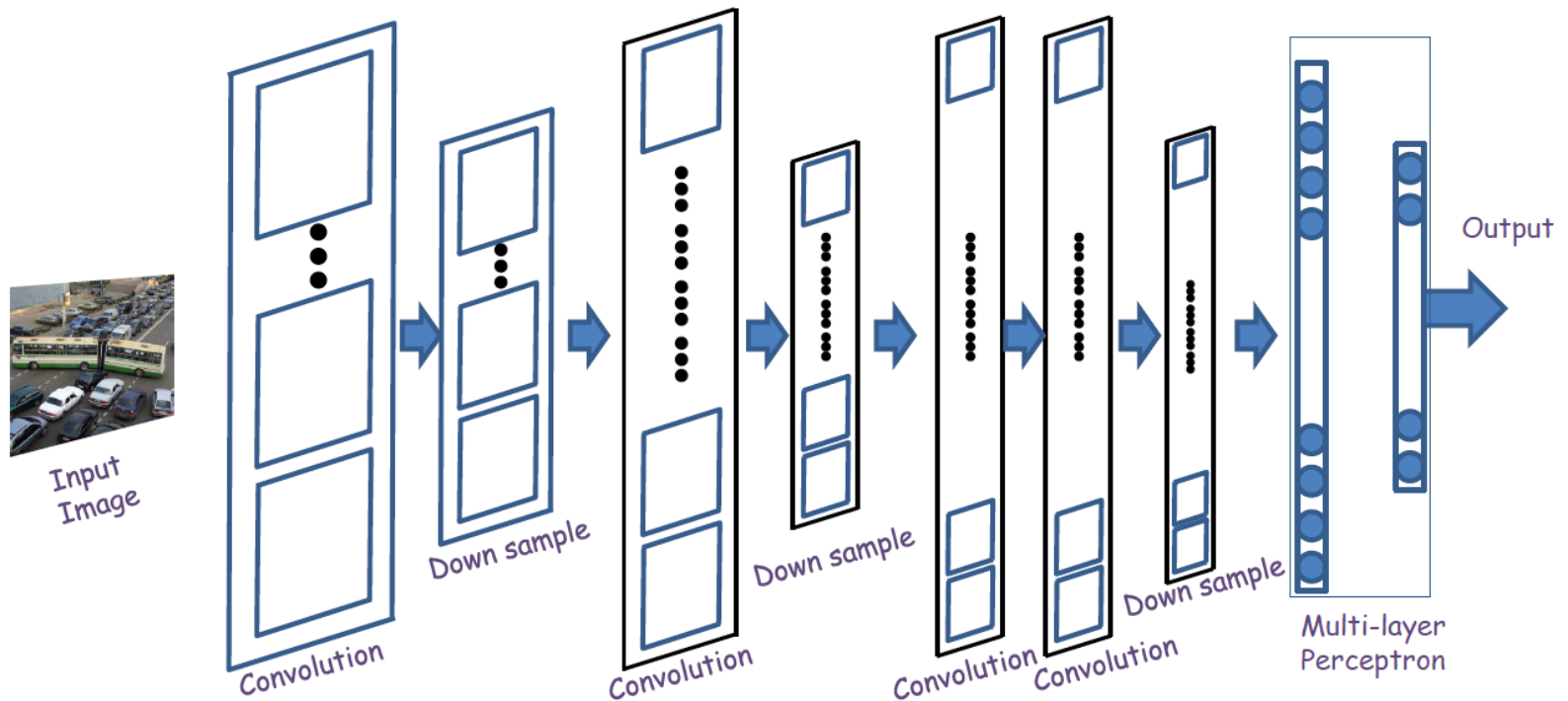
Overview of CNNs

- Third idea: pool hidden units in the same neighborhood
 - Averaging or Discarding location information in a small region
 - Robust toward small deformations in object shapes by ignoring details.



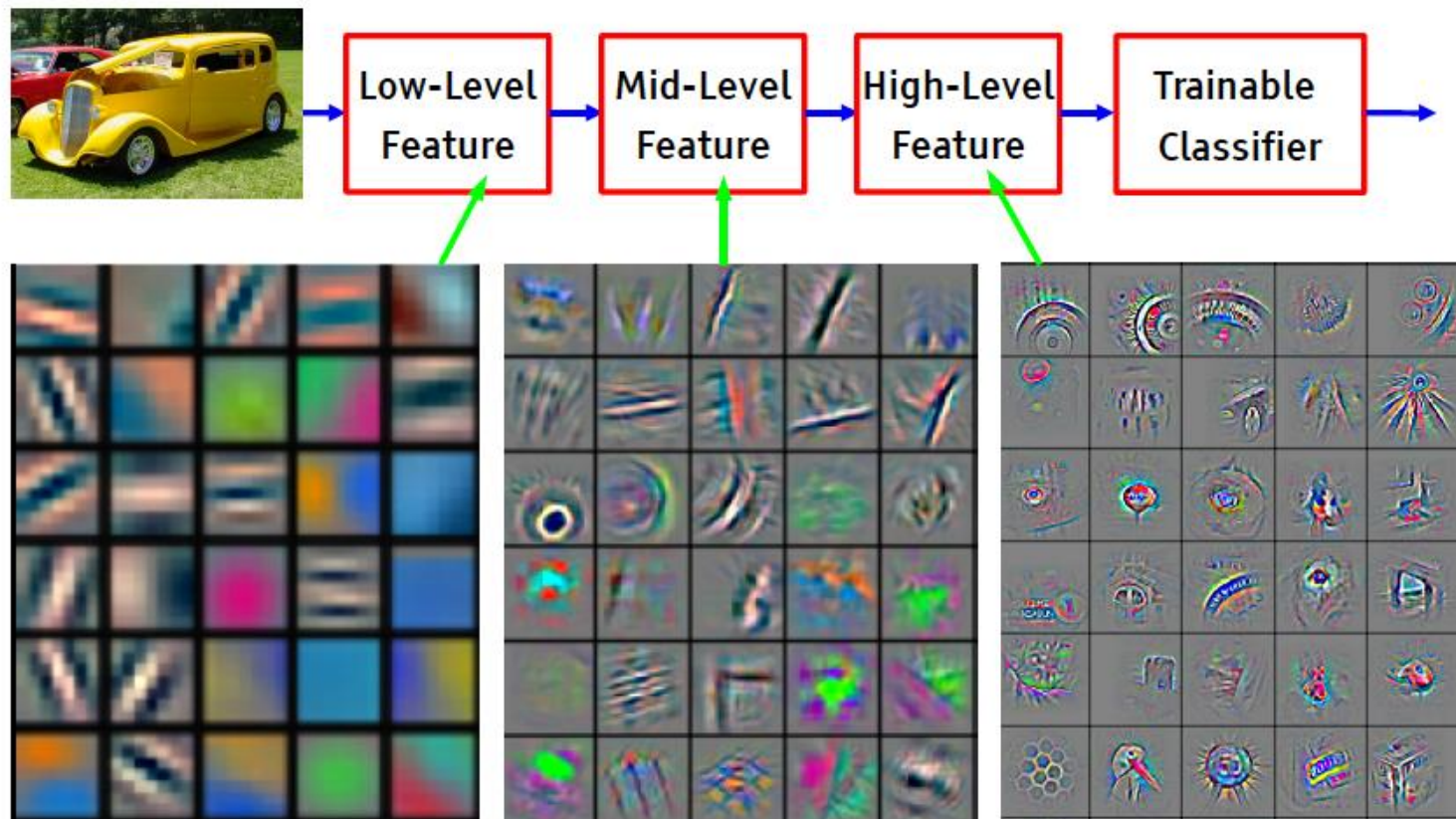
Overview of CNNs

- Fourth idea: Interleaving feature extraction and pooling operations
 - Extracting abstract, compositional features for representing semantic object classes



Overview of CNNs

- Artificial visual pathway: from images to semantic concepts (Representation learning)



Feature visualization of convolutional net trained on ImageNet from [Zeiler & Fergus 2013]

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 - Pooling layers & model complexity
 - Math properties
- Examples of CNNs

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2D Convolution

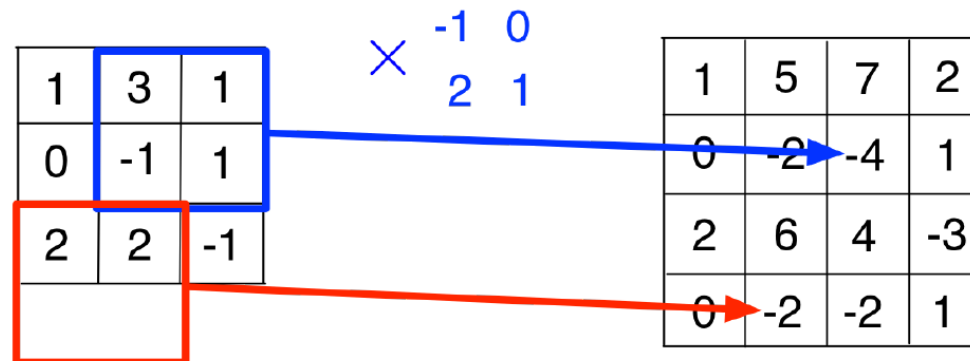
If A and B are two 2-D arrays, then:

$$(A * B)_{ij} = \sum_s \sum_t A_{st} B_{i-s, j-t}.$$

1	3	1
0	-1	1
2	2	-1

 $*$

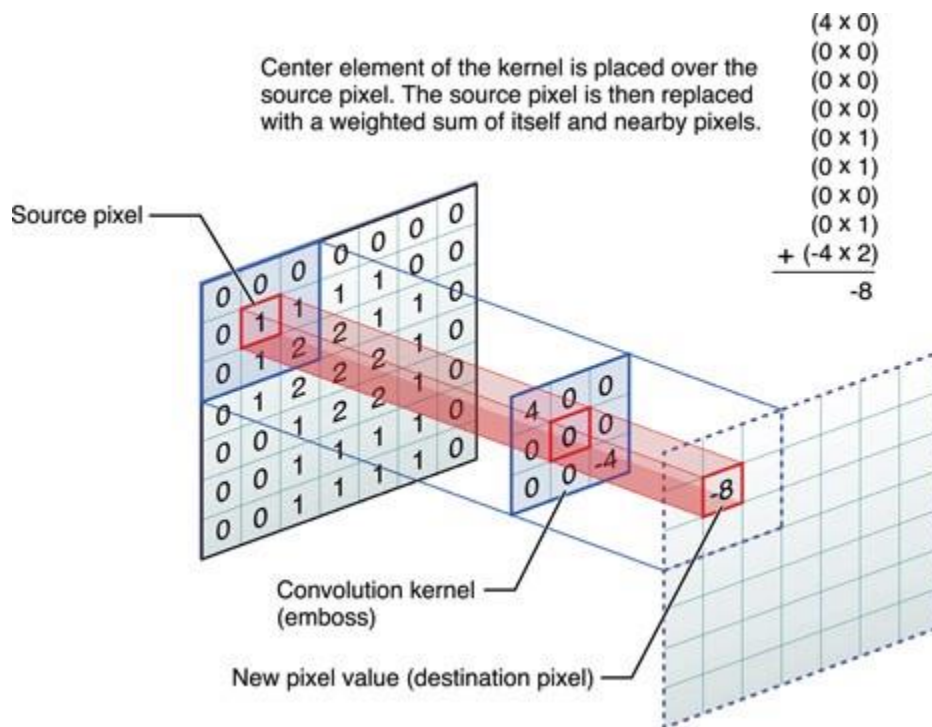
1	2
0	-1



2D Convolution

If A and B are two 2-D arrays, then:

$$(A * B)_{ij} = \sum_s \sum_t A_{st} B_{i-s, j-t}.$$



1 _{x1}	1 _{x0}	1 _{x1}	0	0
0 _{x0}	1 _{x1}	1 _{x0}	1	0
0 _{x1}	0 _{x0}	1 _{x1}	1	1
0	0	1	1	0
0	1	1	0	0

Image

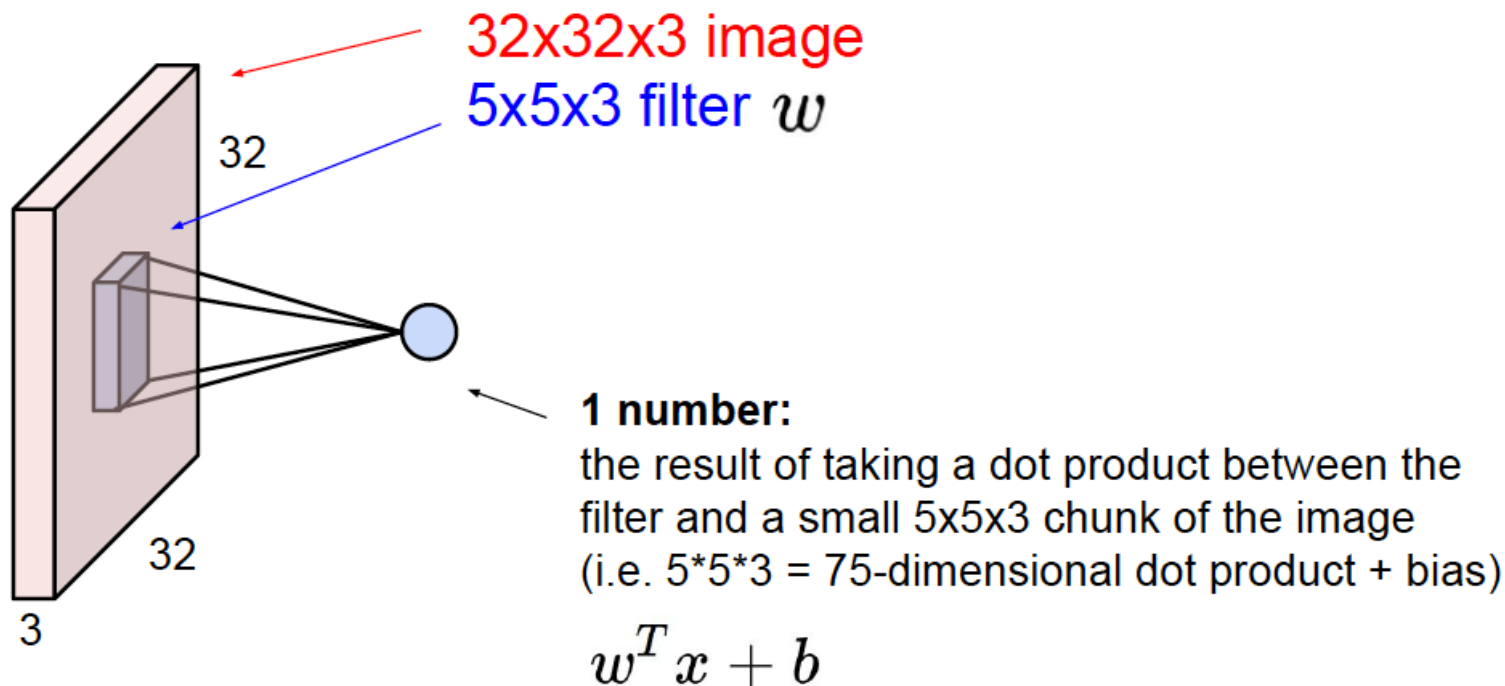
4		

Convolved Feature

Picture Courtesy: developer.apple.com

Convolution Layers

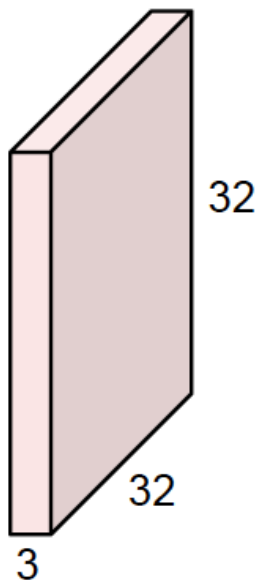
- Formal definition



Convolution Layers

- Define a neuron corresponding to a 5x5 filter

32x32x3 image



5x5x3 filter

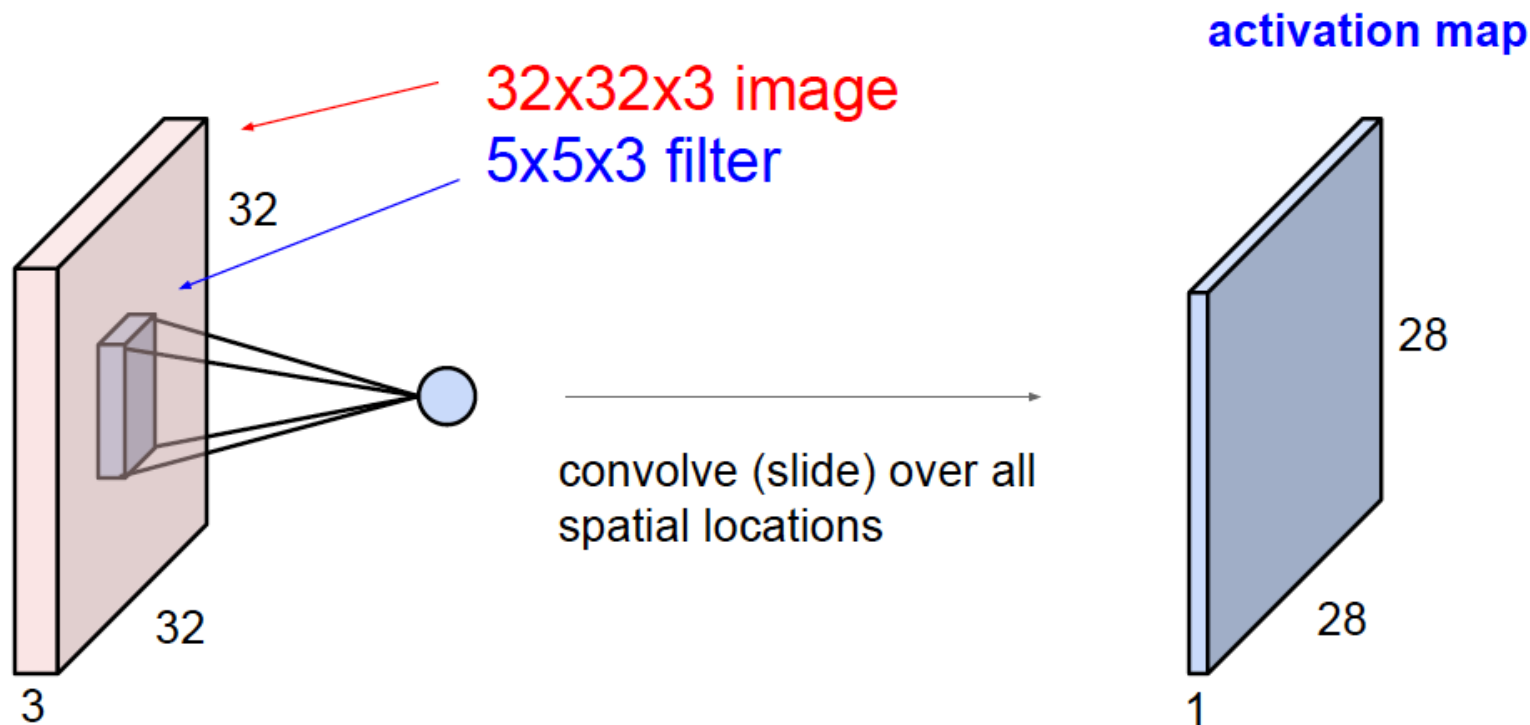


Convolve the filter with the image
i.e. “slide over the image spatially,
computing dot products”

Convolution Layers

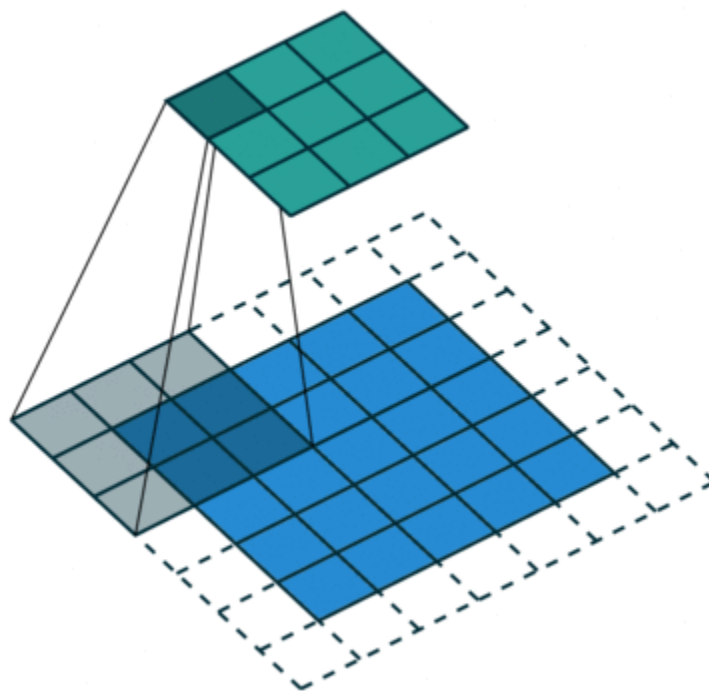
■ Convolution operation

- Parameter sharing
- Spatial information



Convolution Layers

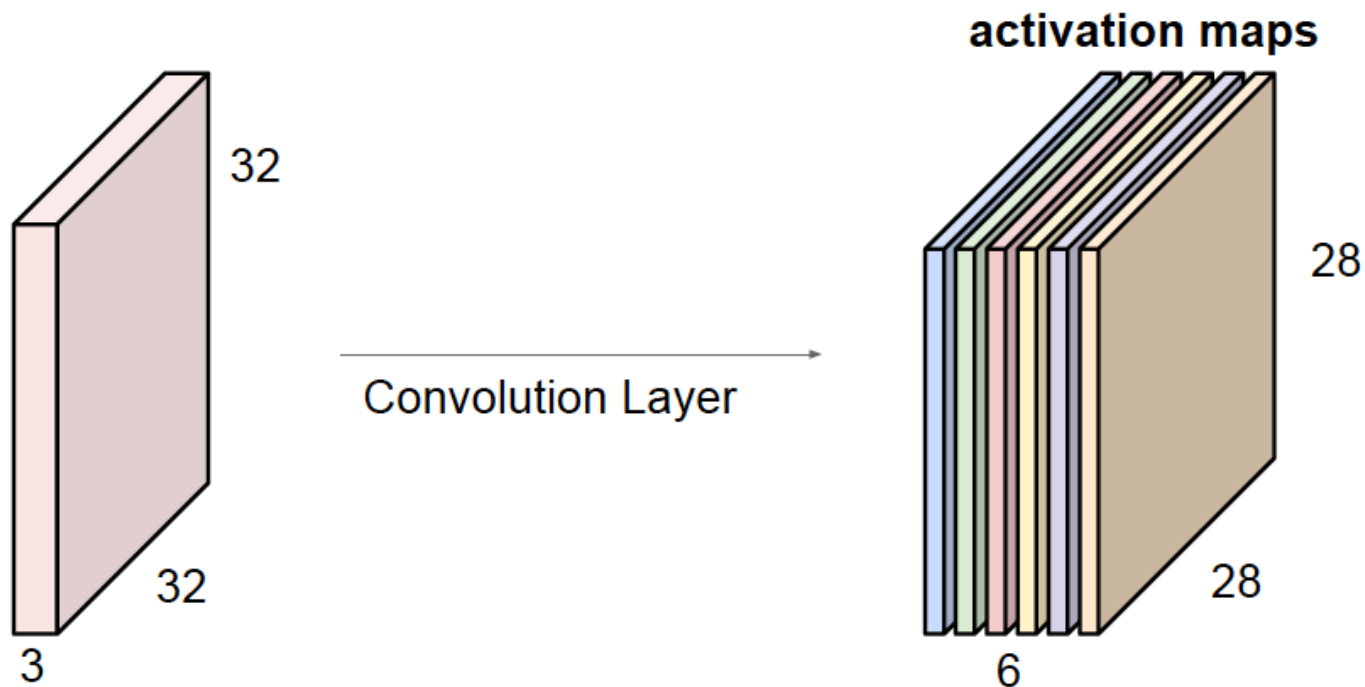
- Convolution operation
 - Parameter sharing
 - Spatial information



Convolution Layers

- Multiple kernels/filters

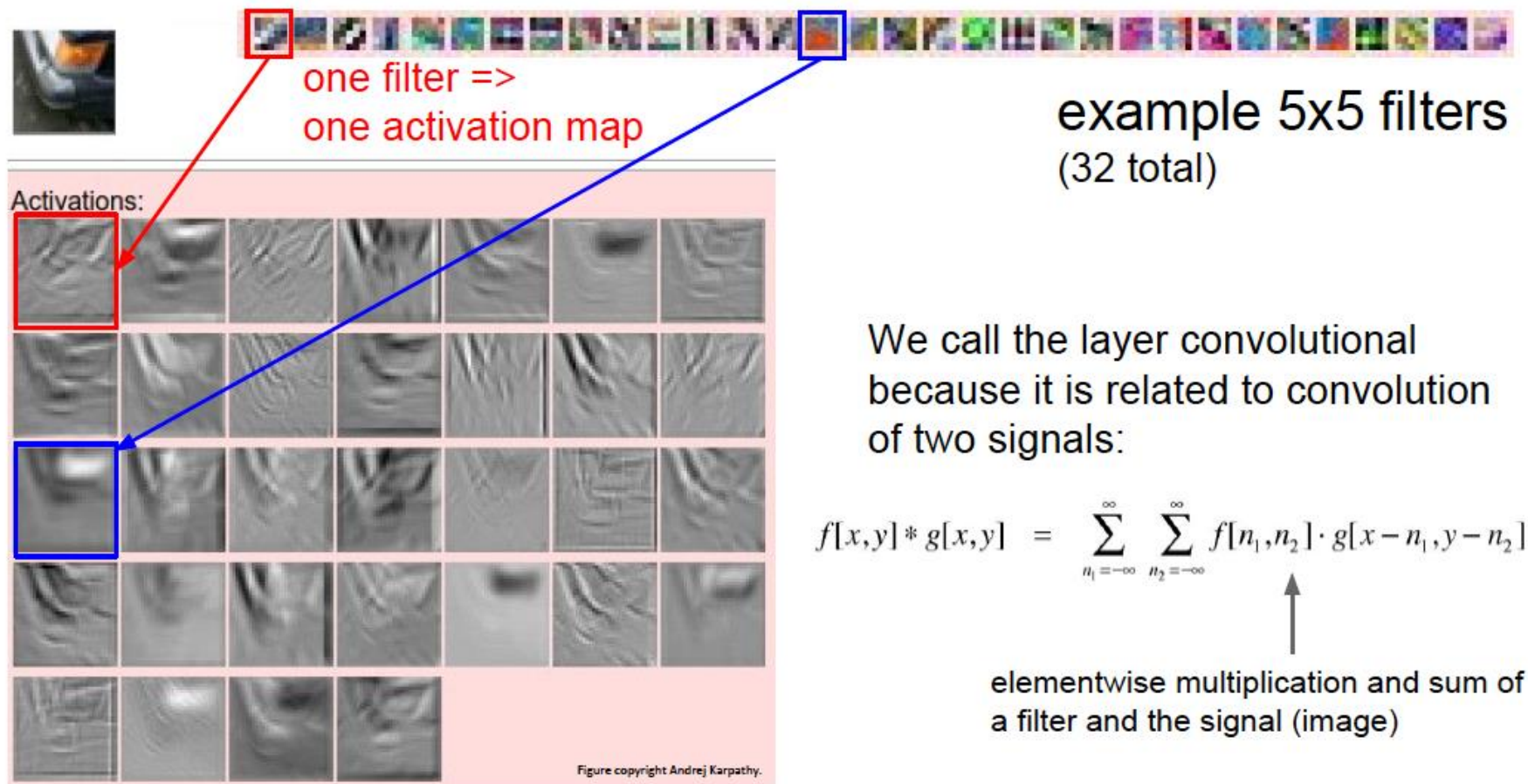
For example, if we had 6 5x5 filters, we'll get 6 separate activation maps:



We stack these up to get a “new image” of size 28x28x6!

Convolution Layers

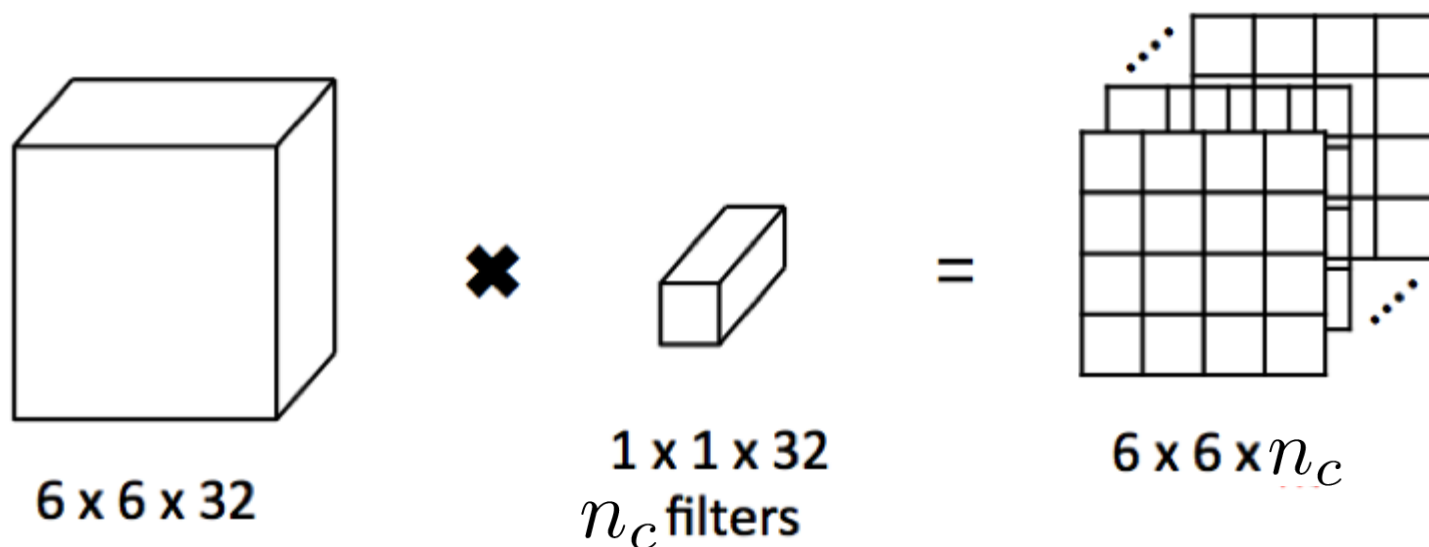
- Visualizing the filters and their outputs



Special Convolutions

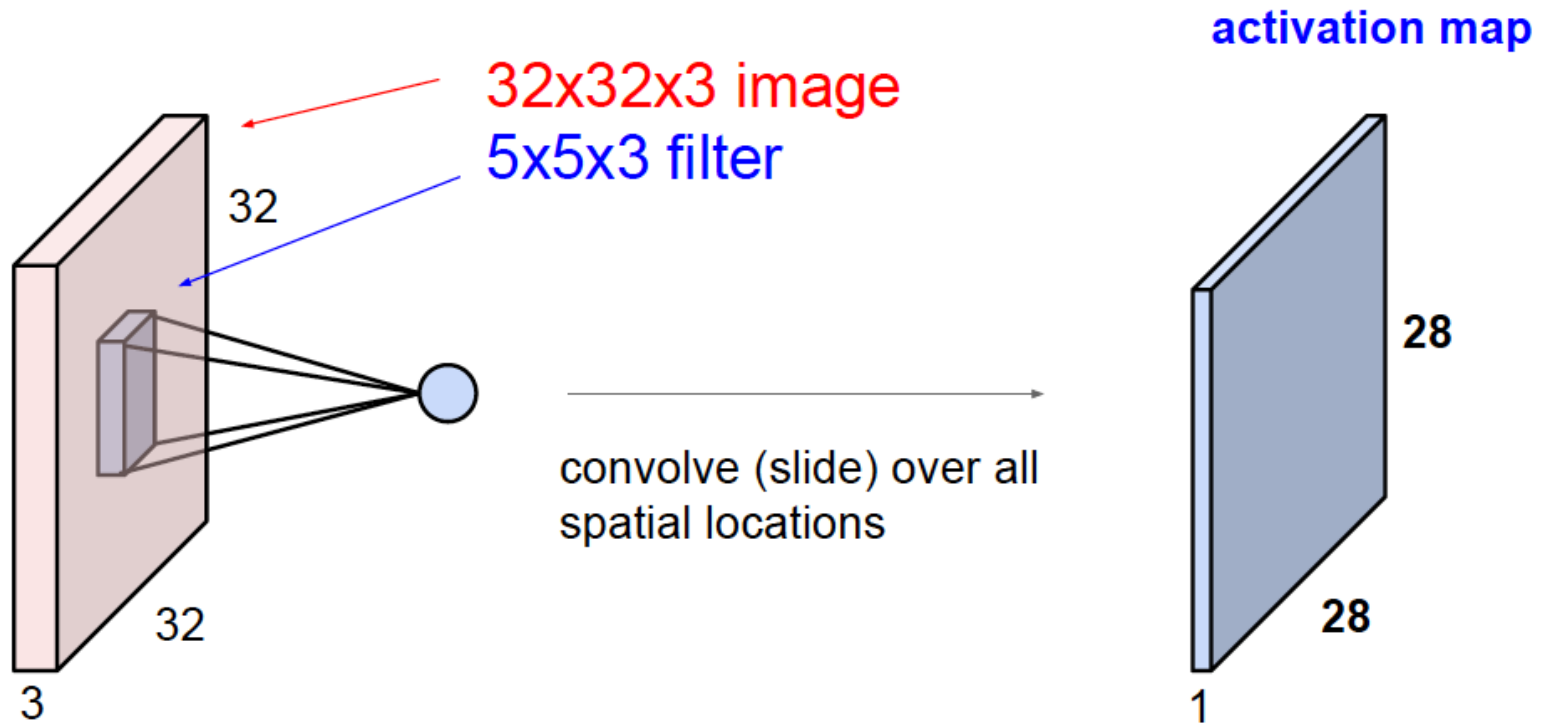
■ 1x1 convolutions

- Used in Network-in-network, GoogLeNet
- Reduce or increase dimensionality
- Can be considered as ‘feature pooling’



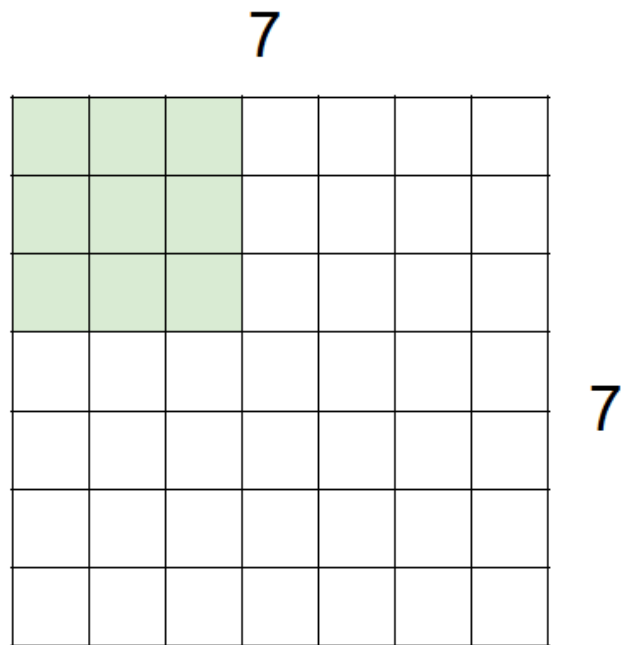
Complexity of Convolution Layers

- Sizes of activation maps and number of parameters



Complexity of Convolution Layers

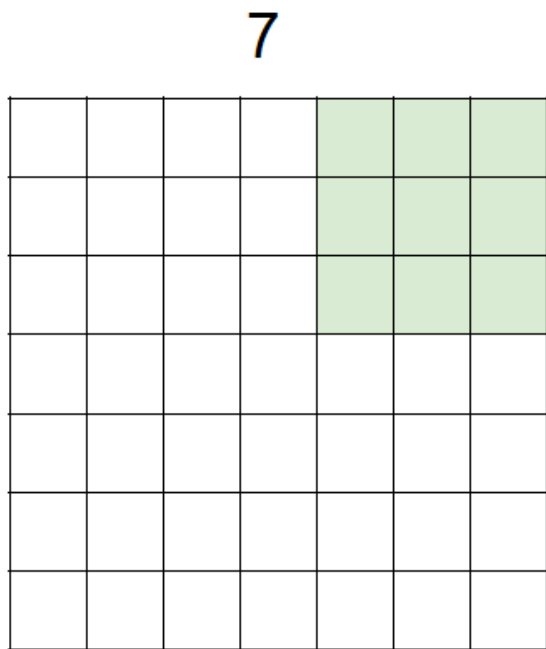
- Size of activation maps



7x7 input (spatially)
assume 3x3 filter

Complexity of Convolution Layers

- Size of activation maps

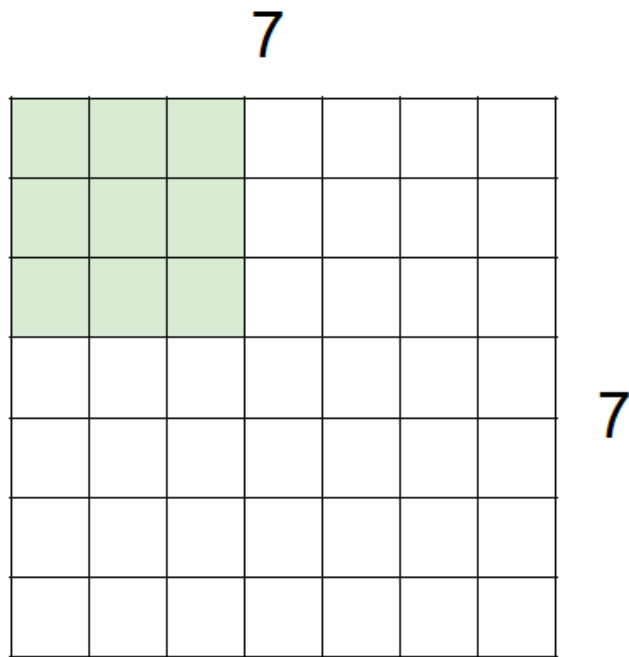


7x7 input (spatially)
assume 3x3 filter

=> 5x5 output

Complexity of Convolution Layers

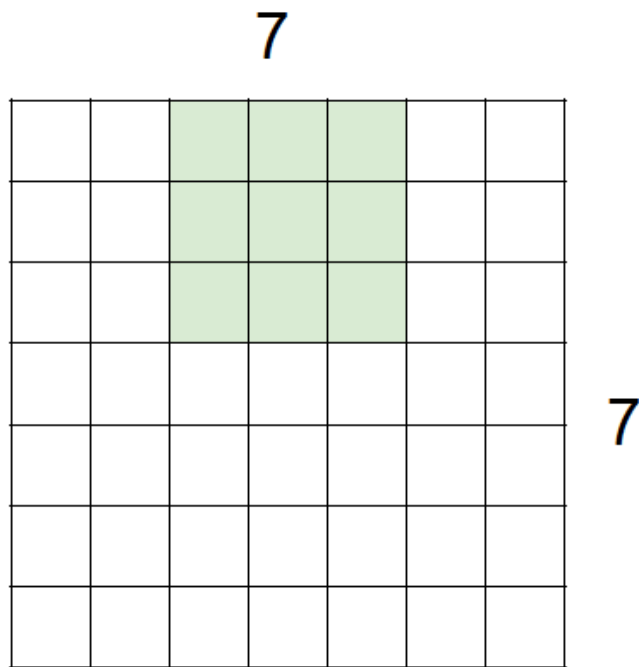
- Case: Stride > 1



7x7 input (spatially)
assume 3x3 filter
applied **with stride 2**

Complexity of Convolution Layers

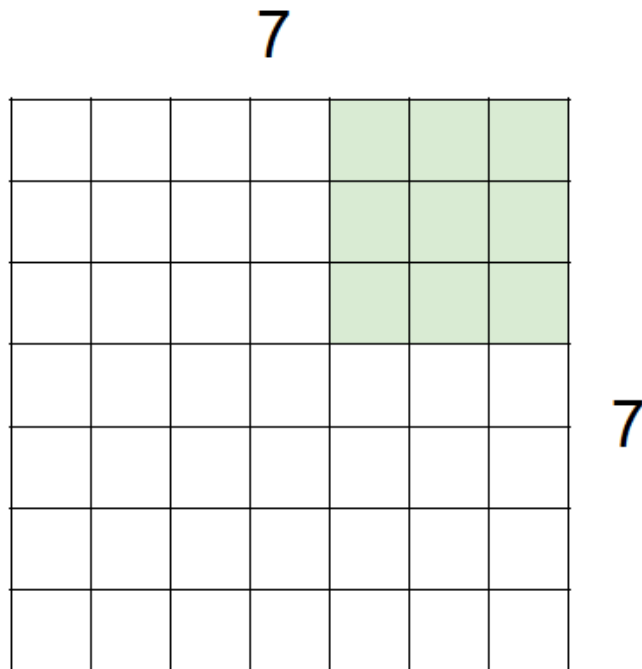
- Case: Stride > 1



7x7 input (spatially)
assume 3x3 filter
applied **with stride 2**

Complexity of Convolution Layers

- Case: Stride > 1



7x7 input (spatially)
assume 3x3 filter
applied **with stride 2**
=> 3x3 output!

Complexity of Convolution Layers

- Zero padding to handle non-integer cases or control the output sizes

0	0	0	0	0	0			
0								
0								
0								
0								

e.g. input 7x7

3x3 filter, applied with **stride 1**

pad with 1 pixel border => what is the output?

7x7 output!

(recall:)

$(N - F) / \text{stride} + 1$

Complexity of Convolution Layers

- Zero padding to handle non-integer cases or control the output sizes

0	0	0	0	0	0			
0								
0								
0								
0								

e.g. input 7x7

3x3 filter, applied with **stride 1**

pad with 1 pixel border => what is the output?

7x7 output!

in general, common to see CONV layers with stride 1, filters of size $F \times F$, and zero-padding with $(F-1)/2$. (will preserve size spatially)

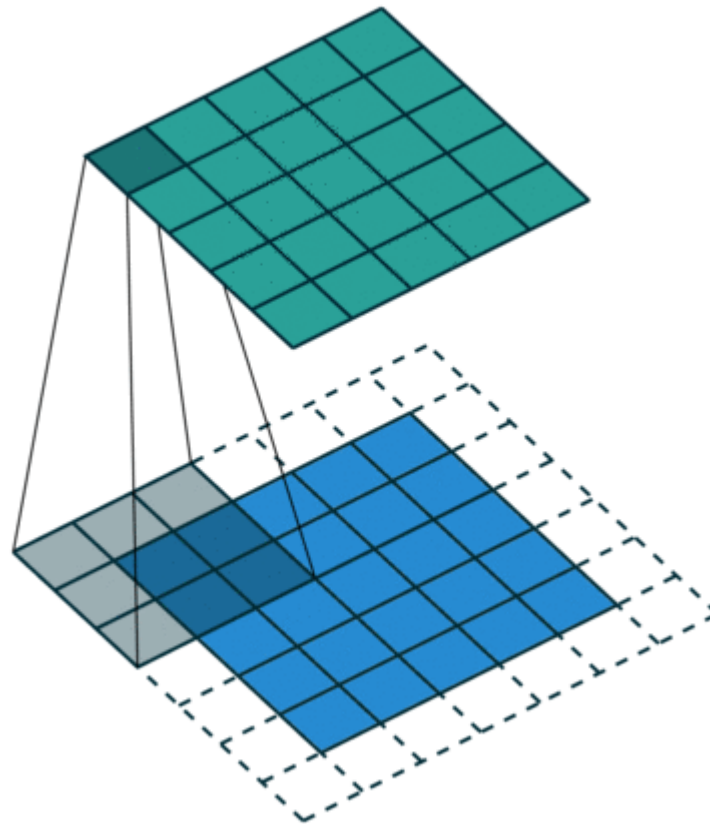
e.g. $F = 3 \Rightarrow$ zero pad with 1

$F = 5 \Rightarrow$ zero pad with 2

$F = 7 \Rightarrow$ zero pad with 3

Complexity of Convolution Layers

- Zero padding to handle non-integer cases or control the output sizes

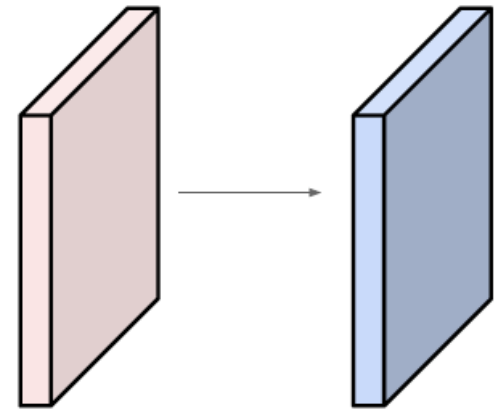


Complexity of Convolution Layers

Examples time:

Input volume: **32x32x3**

10 **5x5** filters with stride **1**, pad **2**



Output volume size:

$(32 + 2 * 2 - 5) / 1 + 1 = 32$ spatially, so

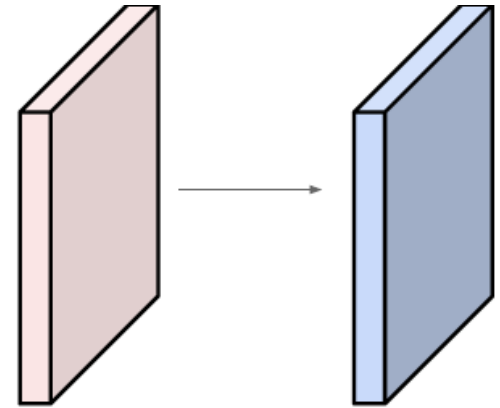
32x32x10

Complexity of Convolution Layers

Examples time:

Input volume: **32x32x3**

10 **5x5** filters with stride 1, pad 2



Number of parameters in this layer?

each filter has $5*5*3 + 1 = 76$ params (+1 for bias)

=> $76*10 = 760$

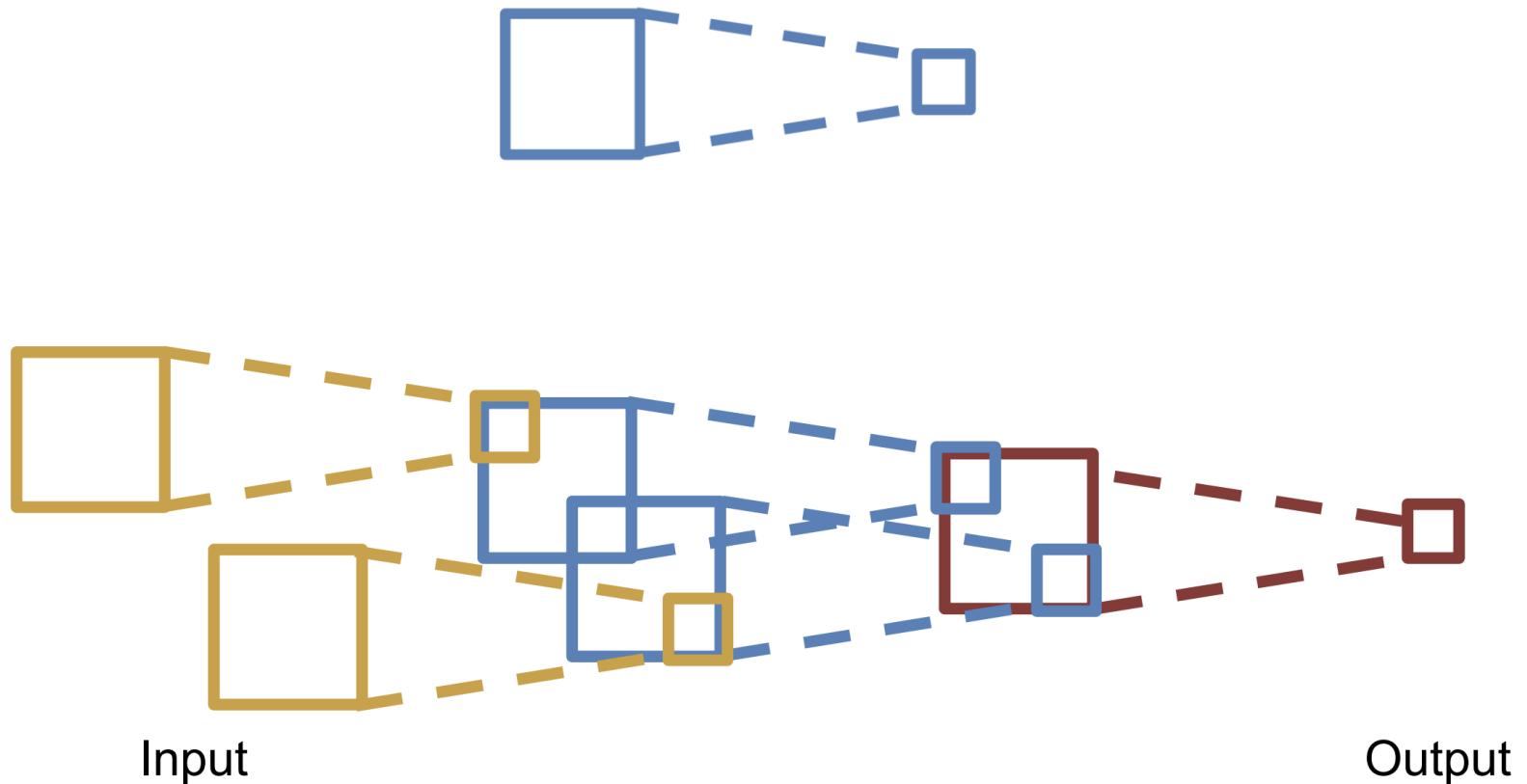
Complexity of Convolution Layers

■ Summary

- Accepts a volume of size $W_1 \times H_1 \times D_1$
- Requires four hyperparameters:
 - Number of filters K ,
 - their spatial extent F ,
 - the stride S ,
 - the amount of zero padding P .
- Produces a volume of size $W_2 \times H_2 \times D_2$ where:
 - $W_2 = (W_1 - F + 2P)/S + 1$
 - $H_2 = (H_1 - F + 2P)/S + 1$ (i.e. width and height are computed equally by symmetry)
 - $D_2 = K$
- With parameter sharing, it introduces $F \cdot F \cdot D_1$ weights per filter, for a total of $(F \cdot F \cdot D_1) \cdot K$ weights and K biases.
- In the output volume, the d -th depth slice (of size $W_2 \times H_2$) is the result of performing a valid convolution of the d -th filter over the input volume with a stride of S , and then offset by d -th bias.

Receptive Fields

- For convolution with kernel size K , each element in the output depends on a $K \times K$ receptive field in the input



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- Examples of CNNs

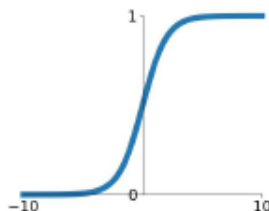
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Review: Activation Function

■ Zoo of Activation functions

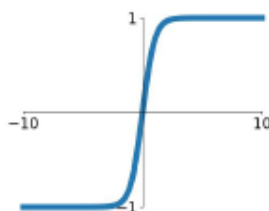
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



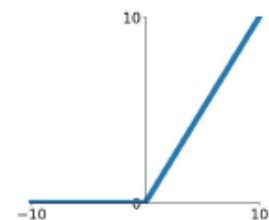
tanh

$$\tanh(x)$$



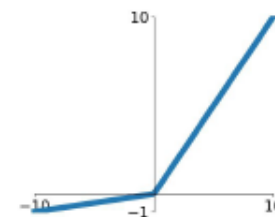
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.1x, x)$$

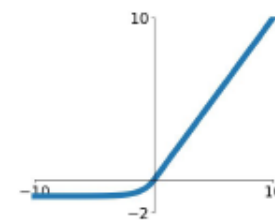


Maxout

$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

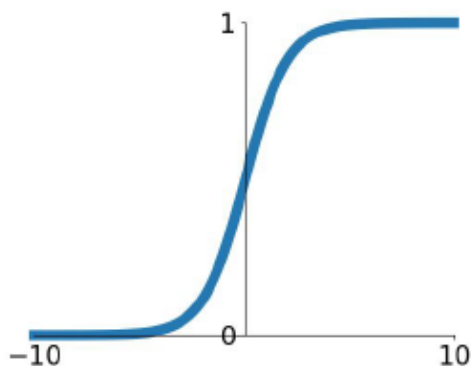
ELU

$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$



Sigmoid function

$$\sigma(x) = 1/(1 + e^{-x})$$



Sigmoid

- Squashes numbers to range [0,1]
- Historically popular since they have nice interpretation as a saturating “firing rate” of a neuron

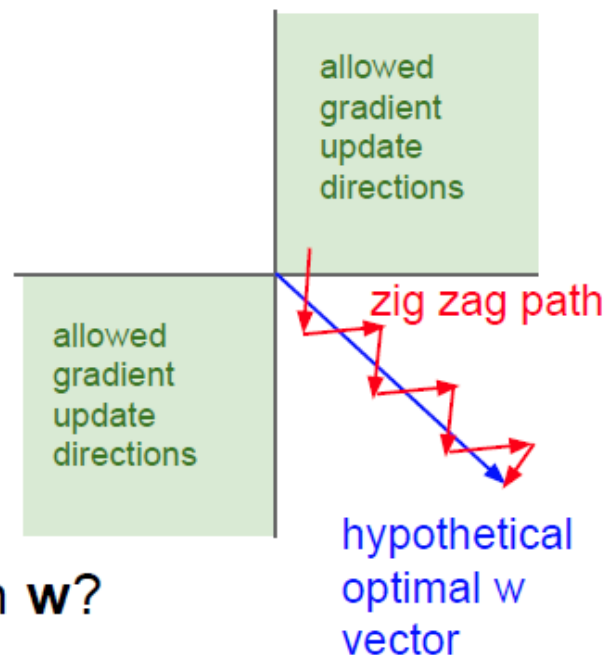
3 problems:

1. Saturated neurons “kill” the gradients
2. Sigmoid outputs are not zero-centered
3. $\exp()$ is a bit compute expensive

Sigmoid function

Consider what happens when the input to a neuron is always positive...

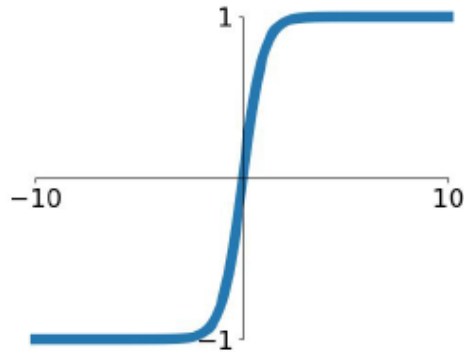
$$f\left(\sum_i w_i x_i + b\right)$$



What can we say about the gradients on \mathbf{w} ?

Always all positive or all negative :(
(this is also why you want zero-mean data!)

Tanh function



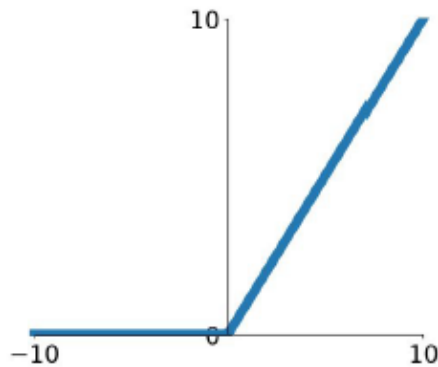
$\tanh(x)$

- Squashes numbers to range $[-1,1]$
- zero centered (nice)
- still kills gradients when saturated :(

[LeCun et al., 1991]

- Recurrent neural networks: LSTM, GRU

Rectified Linear Unit

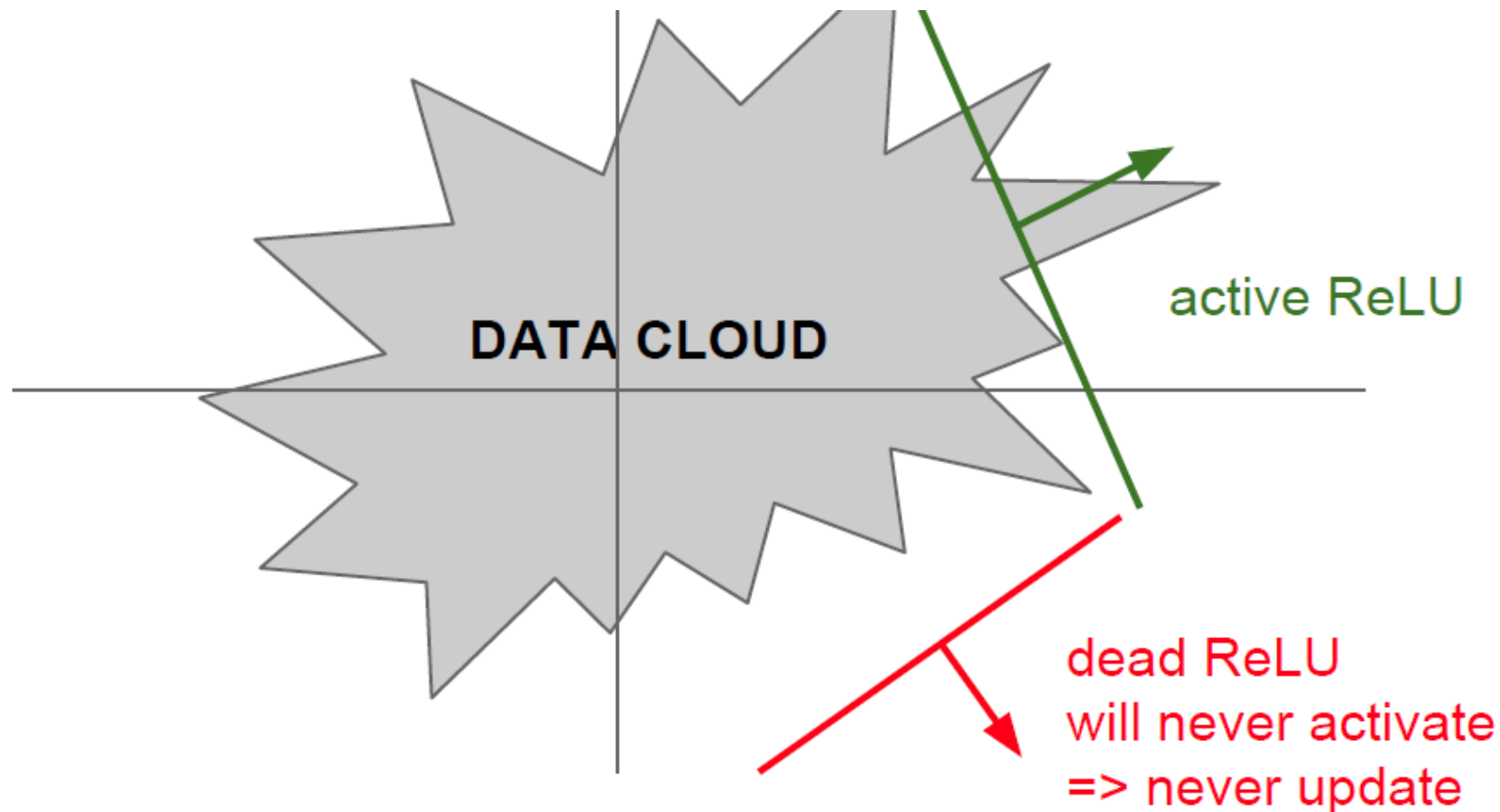


ReLU
(Rectified Linear Unit)

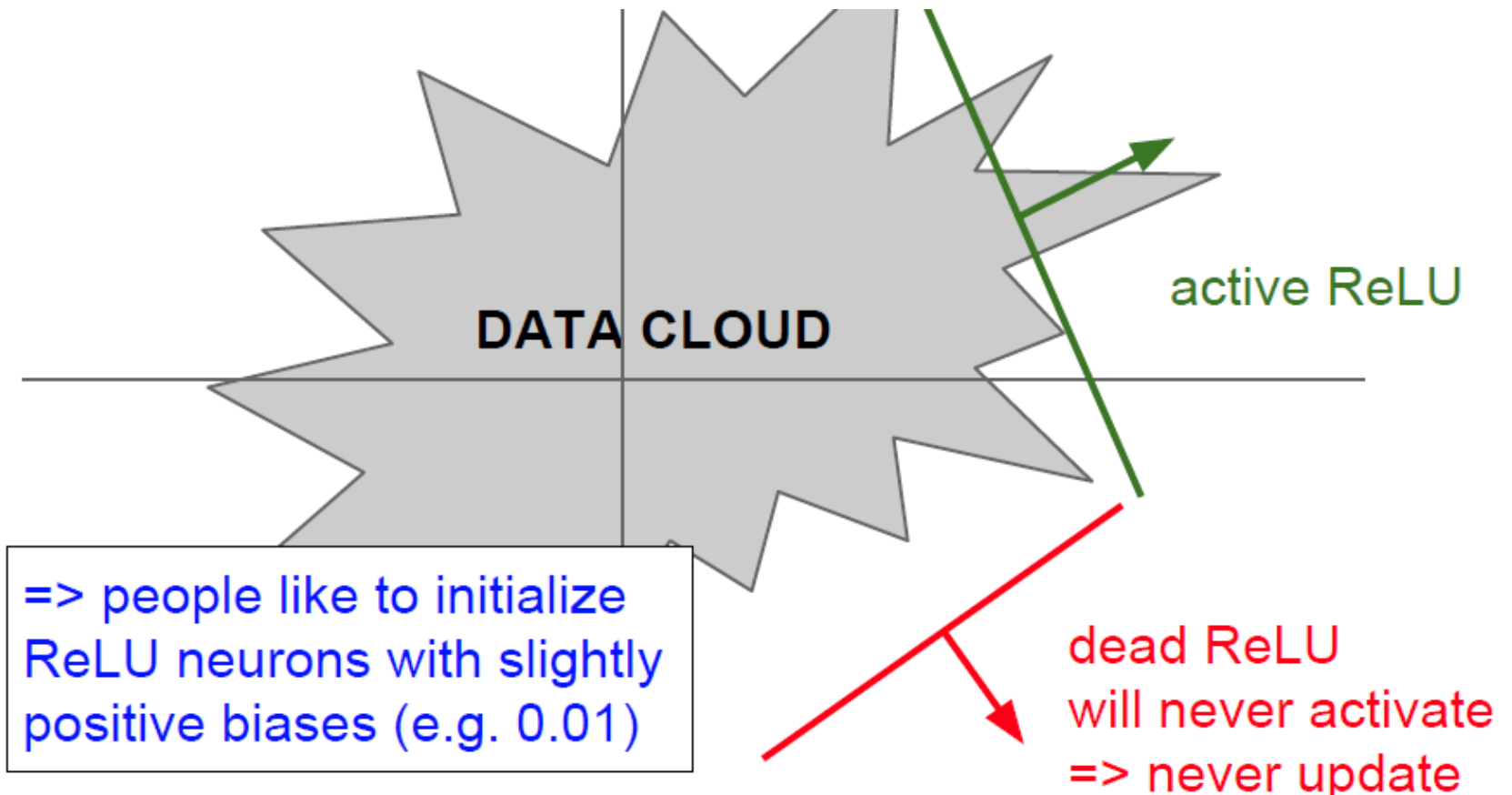
- Computes $f(x) = \max(0, x)$
- Does not saturate (in +region)
- Very computationally efficient
- Converges much faster than sigmoid/tanh in practice (e.g. 6x)
- Actually more biologically plausible than sigmoid
- Not zero-centered output
- An annoyance:

hint: what is the gradient when $x < 0$?

Rectified Linear Unit



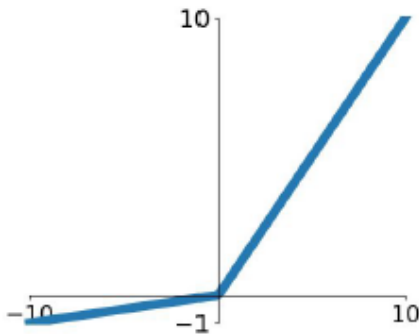
Rectified Linear Unit



Leaky ReLU

[Mass et al., 2013]

[He et al., 2015]



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- Computationally efficient
- Converges much faster than sigmoid/tanh in practice! (e.g. 6x)
- **will not “die”.**

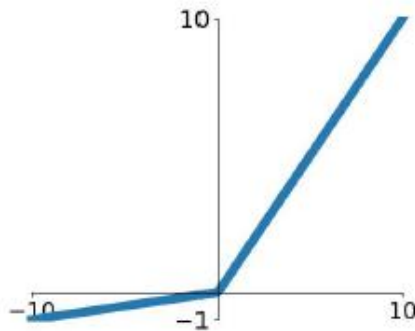
Leaky ReLU

$$f(x) = \max(0.01x, x)$$

Leaky ReLU

[Mass et al., 2013]

[He et al., 2015]



Leaky ReLU

$$f(x) = \max(0.01x, x)$$

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- Computationally efficient
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Parametric Rectifier (PReLU)

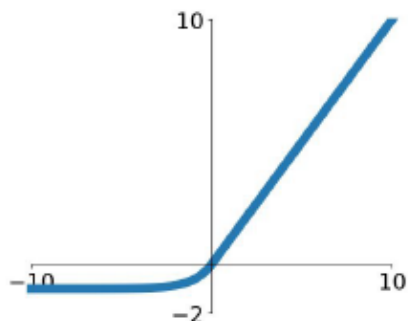
$$f(x) = \max(\alpha x, x)$$

backprop into α
(parameter)

Exponential Linear Units (ELU)

[Clevert et al., 2015]

Exponential Linear Units (ELU)



- All benefits of ReLU
- Closer to zero mean outputs
- Negative saturation regime compared with Leaky ReLU adds some robustness to noise

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ \alpha (\exp(x) - 1) & \text{if } x \leq 0 \end{cases}$$

- Computation requires exp()

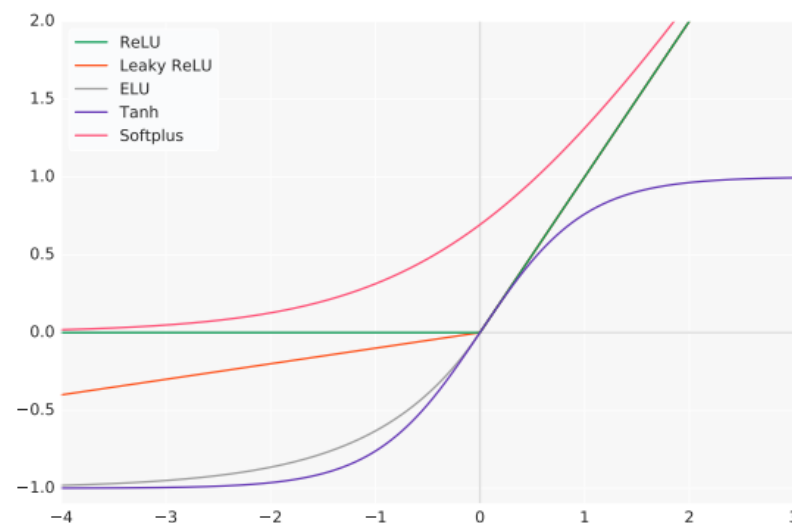
Summary: Activation function

■ For internal layers in CNNs

- Use **ReLU**. Be careful with your learning rates
- Try out **Leaky ReLU** / **Maxout** / **ELU**
- Try out **tanh** but don't expect much
- **Don't use sigmoid**

■ For output layers

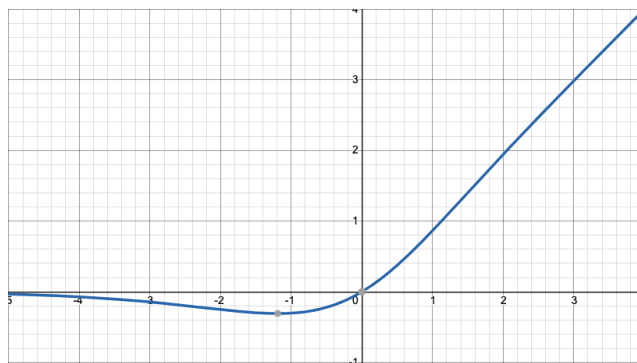
- ☐ Task dependent
- ☐ Related to your loss function



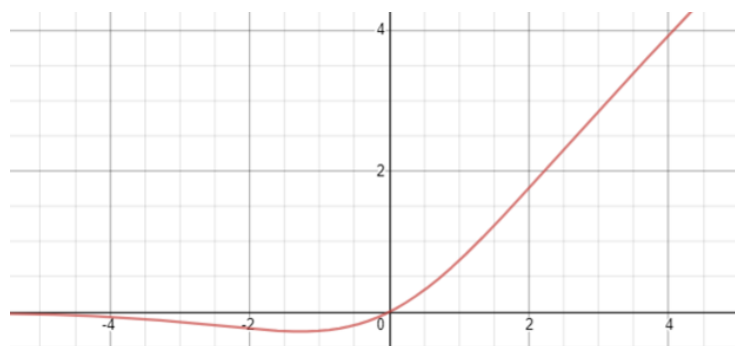
Summary: Activation function

■ Recent progresses

□ Mish $f(x) = x \cdot \tanh(\varsigma(x))$, $\varsigma(x) = \ln(1 + e^x)$,



□ Swish $f(x) = x * (1 + \exp(-x))^{-1}$ <https://arxiv.org/abs/1908.08681>



<https://arxiv.org/abs/1710.05941>

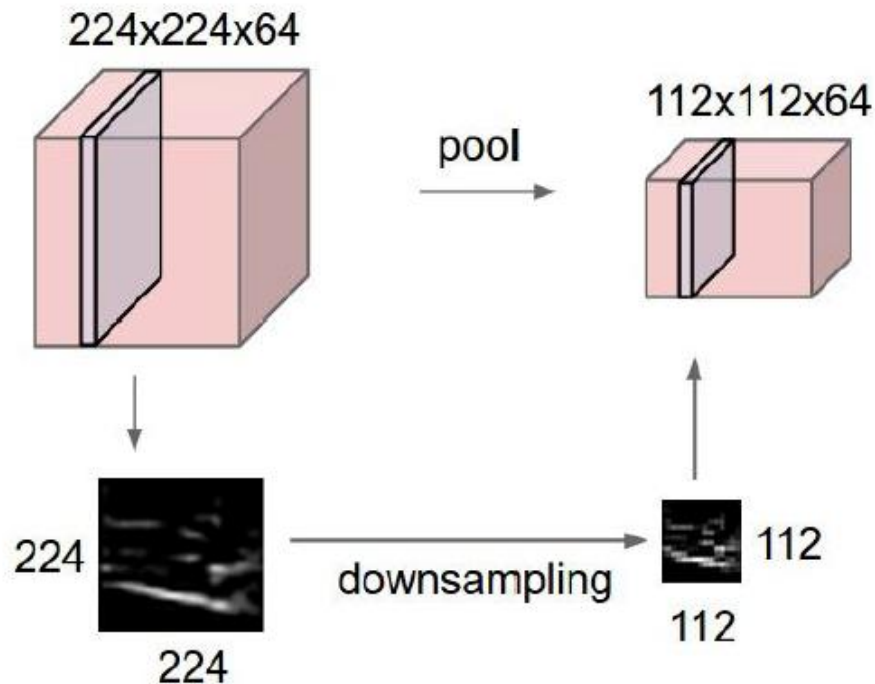
Outline

- Why Convolutional Neural Network (CNN)?
 - Motivation and overview
- What is the CNN?
 - Convolution layers & model complexity
 - Closer look at activation functions
 - Pooling layers & model complexity
 - Math properties
- Examples of CNNs

Acknowledgement: Roger Grosse@UofT & Feifei Li's cs231n notes

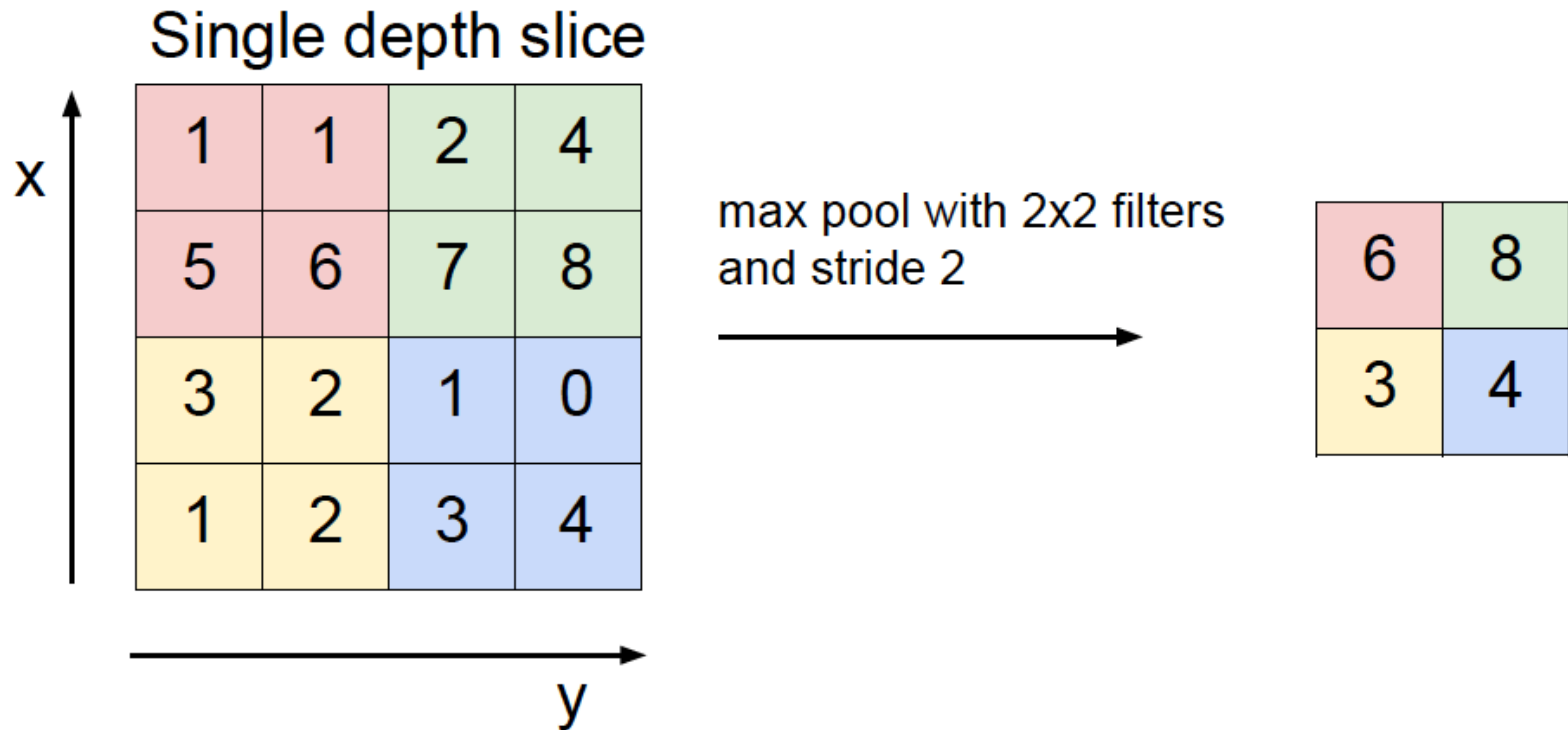
Pooling Layers

- Reducing the spatial size of the feature maps
 - Smaller representations
 - On each activation map independently
 - Low resolution means fewer details



Pooling Layers

- Example: max pooling
- Spatial invariance; no learnable parameters!



Complexity of Pooling Layers

■ Summary

- Accepts a volume of size $W_1 \times H_1 \times D_1$
- Requires three hyperparameters:
 - their spatial extent F ,
 - the stride S ,
- Produces a volume of size $W_2 \times H_2 \times D_2$ where:
 - $W_2 = (W_1 - F)/S + 1$
 - $H_2 = (H_1 - F)/S + 1$
 - $D_2 = D_1$
- Introduces zero parameters since it computes a fixed function of the input
- Note that it is not common to use zero-padding for Pooling layers

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Math Properties of CNNs

- What representations a CNN can capture in general?
- Consider a representation ϕ as an abstract function

$$\phi : \mathbf{x} \rightarrow \phi(\mathbf{x}) \in \mathbb{R}^d$$

- We want to look at how the representation changes upon transformations of input image.
 - Transformations represent the potential variations in the natural images
 - Translation, scale change, rotation, local deformation etc.

Math Properties of CNNs

- Two key properties of representations

- Equivariance

A representation ϕ is equivariant with a transformation g if the transformation can be transferred to the representation output.

\exists a map $M_g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that:

$$\forall \mathbf{x} \in \mathcal{X} : \phi(g\mathbf{x}) \approx M_g\phi(\mathbf{x})$$

- Example: convolution w.r.t. translation



Math Properties of CNNs

- Two key properties of representations

- Invariance

A representation ϕ is invariant with a transformation g if the transformation has no effect on the representation output.

$$\forall \mathbf{x} \in \mathcal{X} : \phi(g\mathbf{x}) \approx \phi(\mathbf{x})$$

- Example: convolution+pooling+FC w.r.t. translation



Math Properties of CNNs

- Recent results on convolution layers
 - Convolutions are equivariant to translation
 - Convolutions are not equivariant to other isometries of the sampling lattice, e.g., rotation

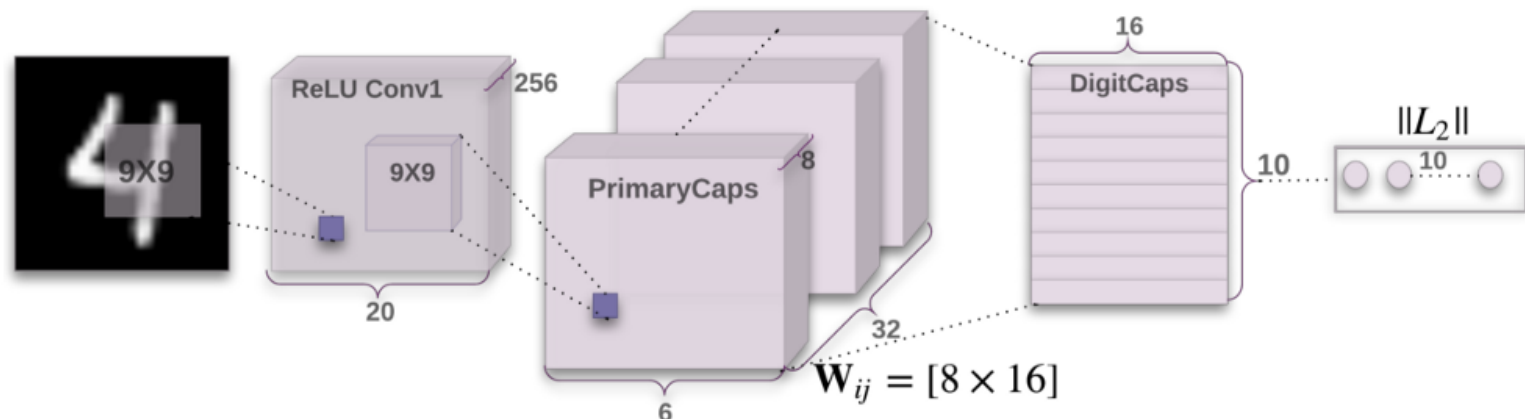


conv2d( , ) = 

- What if a CNN learns rotated copies of the same filter?
 - The stack of feature maps is equivariant to rotation.

Math Properties of CNNs

- Recent results on convolution layers
 - Ordinary CNNs can be generalized to Group Equivariant Networks (Cohen and Welling ICML'16, Kondor and Trivedi ICML'18)
 - Redefining the convolution and pooling operations
 - Equivariant to more general transformation from some group G
 - Replacing pooling by other network designs
 - Capsule network (Sabour et al, 2017)
<https://arxiv.org/abs/1710.09829>



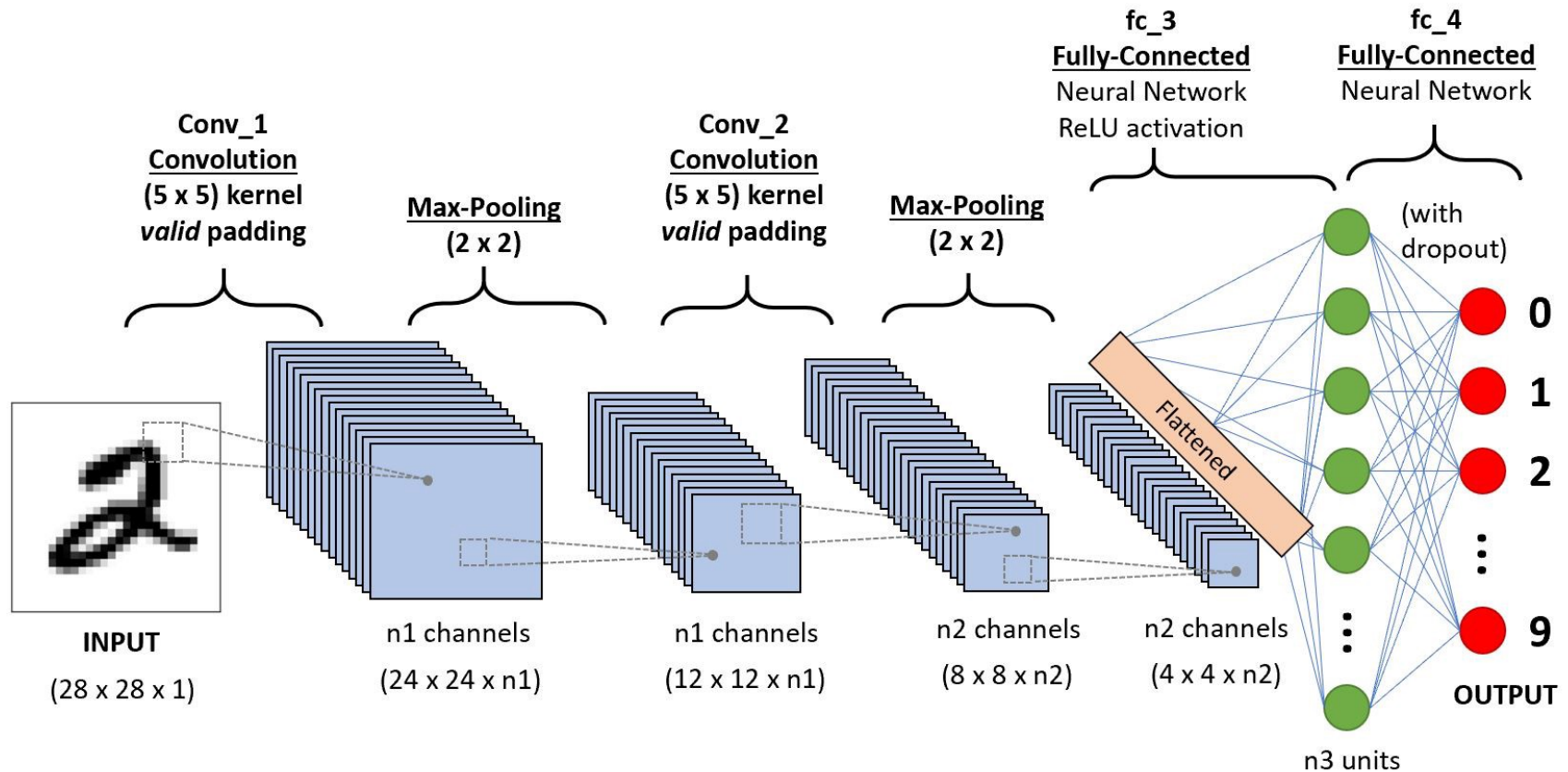
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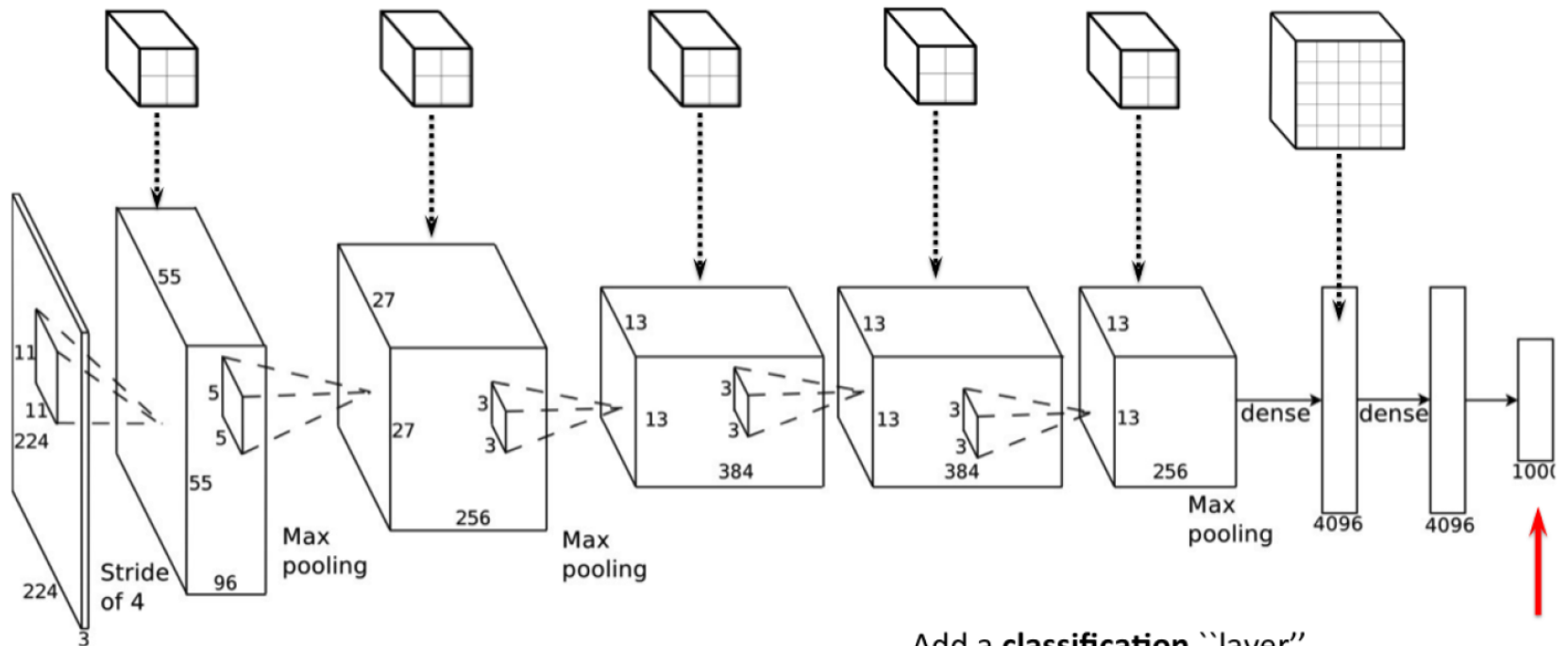
LeNet-5

■ Handwritten digit recognition



AlexNet

■ Deeper network structure



Add a **classification** "layer".

For an input image, the value in a particular dimension of this vector tells you the probability of the corresponding object class.

Summary of CNNs

- CNN properties [Bronstein et al., 2018]
 - Convolutional (Translation invariance)
 - Scale Separation (Compositionality)
 - Filters localized in space (Deformation Stability)
 - $O(1)$ parameters per filter (independent of input image size n)
 - $O(n)$ complexity per layer (filtering done in the spatial domain)
 - $O(\log n)$ layers in classification tasks
- Next time ...
 - Structure design of Modern CNNs
- Reference
 - CS231n course notes <http://cs231n.github.io/convolutional-networks/>
 - D2L Chapter 6 + DLBook Chapter 9