Divide and conquer 3 Linear time sorting

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Polynomial multiplication

- Let $A(x) = \sum_{j=0}^{n-1} a_j x^j$ and $B(x) = \sum_{j=0}^{n-1} b_j x^j$ be two polynomials.
- Compute the product $C(x) = A(x)B(x)$.

$$
\begin{array}{rrrrrrrrr}
 & & & & 6x^3 & + & 7x^2 & - 10x & + & 9 & A(x) \\
 & & & & - & 2x^3 & & + & 4x & - & 5 & B(x) \\
\hline
 & & & & - 30x^3 & - & 35x^2 & + 50x & - & 45 & \\
 & & & 24x^4 & + 28x^3 & - & 40x^2 & + 36x & & & \\
 & - 12x^6 & - & 14x^5 & + 20x^4 & - & 18x^3 & & & & \\
\hline
 & - 12x^6 & - & 14x^5 & + 44x^4 & - & 20x^3 & - 75x^2 & + & 86x - 45 &
\end{array}
$$

- $C(x)$'s degree is at most 2n-2.
- $C(x) = \sum_{j=0}^{2n-2} c_j x^j$, where $c_j = \sum_{k=0}^{j} a_k b_{j-k}$.
- The naive method takes $O(n^2)$ time.
- Using FFT and divide & conquer, we'll do it in $O(n \log n)$ time.

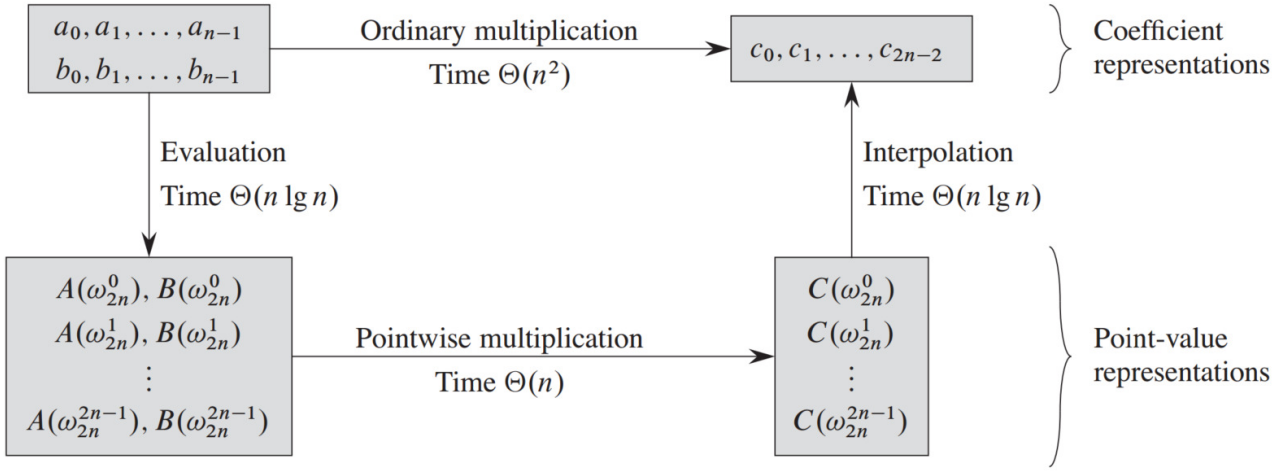
Polynomial representations

- The coefficient representation of a polynomial is $A(x) = \sum_{j=0}^{n-1} a_j x^j$.
- The point-value representation is $\{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$, where $y_k = A(x_k)$.
- If we have $A(x), B(x)$ in point-value form, computing $C(x) = A(x)B(x)$ can be done in $O(n)$ time.
  - Pick $2n$ points $x_0, \ldots, x_{2n-1}$. Then $C(x_k) = A(x_k)B(x_k)$ for $k = 0, \ldots, 2n-1$.
  - In coefficient form, naive multiplication takes $O(n^2)$ time.
- Given $A(x)$ represented in point-value form, we can reconstruct the coefficient representation.

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

- Matrix V is the Vandermonde matrix. It's nonsingular and invertible.
  - So $a = V^{-1}y$.
  - Can compute a more efficiently than general matrix inversion and multiplication.

Fast polynomial multiplication

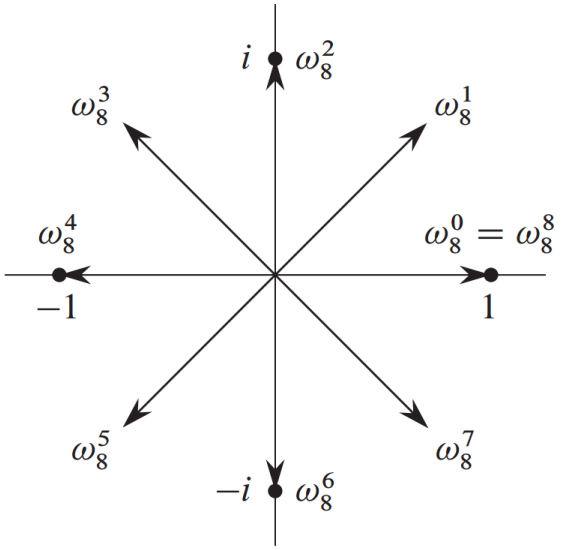


Source: Introduction to Algorithms, Cormen et al

- We want to multiply polynomials A and B quickly.
- Evaluate A and B at $2n$ points to get point-value representations.
  - Evaluation can be at any points. But we'll do it at the $2n$'th roots of unity.
  - Evaluating A and B at $2n$'th roots of unity by FFT takes $O(n \log n)$ time.
- Pointwise multiply A and B to get point-value representation for C.
  - This takes $O(n)$ time.
- Transform point-value representation of C to coefficient form.
  - This is done using inverse FFT in $O(n \log n)$ time.

Roots of unity

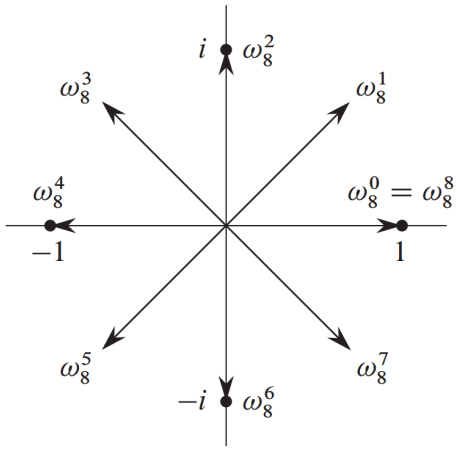
- An n'th root of unity is a complex number ω s.t. $\omega^n = 1$.
- There are n n'th roots of unity, and they have the form $e^{\frac{2\pi ik}{n}}$, for $0 \le k \le n-1$.
  - Write $\omega_n = e^{\frac{2\pi i}{n}}$, so that the n'th roots of unity are $\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}$.
- Below, assume n is a power of 2.
- Fact 1 $\left(\omega_n^k\right)^n = 1$ for any $0 \le k \le n-1$.
- Fact 2 $\omega_n^{k+\frac{n}{2}} = -\omega_n^k$.
- Fact 3 $\left(\omega_n^k\right)^2 = \omega_n^{2k} = \omega_{n/2}^k$.
- Fact 4 For any $0 \le k \le n-1$, $\sum_{j=0}^{n-1}\left(\omega_n^k\right)^j = 0$.



Discrete Fourier Transform

- Given a degree $n-1$ polynomial $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$, DFT computes $A(\omega_n^0), A(\omega_n^1), \ldots, A(\omega_n^{n-1})$.
- FFT is a fast algorithm for DFT that runs in $O(n \log n)$ time using divide and conquer.
  - Assume n is a power of 2.
- FFT has a vast number of applications in CS and EE.
- One of IEEE's top 10 most important algorithms of the 20th century.
  - Popularized by Cooley and Tukey in 1965.
  - But variants known to Gauss in 1805!

Fast Fourier Transform



- Let $A^{[0]} = a_0 + a_2x^1 + a_4x^2 + \cdots + a_{n-2}x^{\frac{n}{2}-1}$

  $A^{[1]} = a_1 + a_3x^1 + a_5x^2 + \cdots + a_{n-1}x^{\frac{n}{2}-1}$.
- Then $A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$.
- So can compute $A(\omega_n^0), A(\omega_n^1), \ldots, A(\omega_n^{n-1})$ by computing $A^{[0]}$ and $A^{[1]}$ at $(\omega_n^0)^2, (\omega_n^1)^2, \ldots, (\omega_n^{n-1})^2$, and multiplying some terms by $\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}$.
- But $\left(\omega_n^{k+n/2}\right)^2 = \omega_n^{2k+n} = \omega_n^{2k} = \left(\omega_n^k\right)^2$ by Fact 1.
  - So $\{(\omega_n^0)^2, (\omega_n^1)^2, \ldots, (\omega_n^{n-1})^2\} = \left\{(\omega_n^0)^2, (\omega_n^1)^2, \ldots, \left(\omega_n^{n/2-1}\right)^2\right\}$, i.e. only need to evaluate $A^{[0]}$ and $A^{[1]}$ at $n/2$ points instead of n!
- Also, $\left(\omega_n^k\right)^2 = \omega_n^{2k} = \omega_{n/2}^k$.
  - So $\left\{(\omega_n^0)^2, (\omega_n^1)^2, \ldots, \left(\omega_n^{n/2-1}\right)^2\right\} = \left\{\omega_{n/2}^0, \omega_{n/2}^1, \ldots, \omega_{n/2}^{n/2-1}\right\}$, i.e. only need to evaluate $A^{[0]}$ and $A^{[1]}$ at the $(n/2)$'th roots of unity.

Fast Fourier Transform

- Thus, computing $A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$ for $x \in \{\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}\}$ requires
  - Computing for $A^{[0]}(x)$ and $A^{[1]}(x)$ for $x \in \left\{\omega_{n/2}^0, \omega_{n/2}^1, \ldots, \omega_{n/2}^{n/2-1}\right\}$.
  - These are also DFT's, so can be done recursively using two $n/2$-point FFT's.
- For $0 \leq k \leq \frac{n}{2} - 1$
  - $A\left(\omega_n^k\right) = A^{[0]}\left(\omega_{n/2}^k\right) + \omega_n^k \, A^{[1]}\left(\omega_{n/2}^k\right)$
  - $A\left(\omega_n^{k+n/2}\right) = A^{[0]}\left(\omega_{n/2}^{k+n/2}\right) + \omega_n^{k+n/2} \, A^{[1]}\left(\omega_{n/2}^{k+n/2}\right)$
    $= A^{[0]}\left(\omega_{n/2}^k\right) - \omega_n^k \, A^{[1]}\left(\omega_{n/2}^k\right)$

FFT algorithm

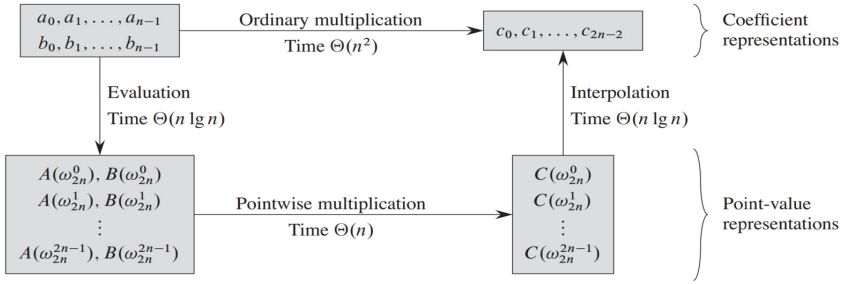
```
FFT(n, a_0, a_1, …, a_{n-1}) {
    if (n == 1) return a_0

    (e_0, e_1, …, e_{n/2-1}) ← FFT(n/2, a_0, a_2, a_4, …, a_{n-2})
    (d_0, d_1, …, d_{n/2-1}) ← FFT(n/2, a_1, a_3, a_5, …, a_{n-1})

    for k = 0 to n/2 - 1 {
        ω^k ← e^{2πik/n}
        y_k ← e_k + ω^k d_k
        y_{k+n/2} ← e_k - ω^k d_k
    }
    return (y_0, y_1, …, y_{n-1})
}
```

- Let $T(n)$ be the time to compute a size n FFT. Then $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$, so $T(n) = \Theta(n \log n)$.

Inverse FFT



- So far we have
  - Computed A, B at the $2n$'th roots of unity.
  - Pointwise multiplied A, B to get point-value representation of C.
- To multiply A and B, the last step is to convert C back to coefficient representation using inverse FFT.
- Inverse FFT takes a polynomial represented in point-value form and computes the polynomial's coefficient form.
  - I.e. given $[y_0\ y_1 \ldots y_{n-1}]^T$ below, it computes $[a_0\ a_1\ \ldots\ a_{n-1}]^T$.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

- For the polynomial multiplication problem, the a vector represents C.
  - $a = V_n^{-1} y$.
- Need to know the inverse of Vandermonde matrix V_n.

Inverse FFT

- Thm For $j, k = 0, \ldots, n-1$, the (j,k) entry of V_n^{-1} is ω_n^{-kj}/n.

$$V_n^{-1} = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

- Proof

$$[V_n^{-1}V_n]_{jj'} = \sum_{k=0}^{n-1}\left(\frac{\omega_n^{-kj}}{n}\right)\left(\omega_n^{kj'}\right) = \frac{1}{n}\sum_{k=0}^{n-1}\omega_n^{k(j'-j)}$$

- By Fact 4, (j, j') entry is 1 exactly when $j = j'$, and 0 otherwise. So $V_n^{-1}V_n$ is the identity matrix.

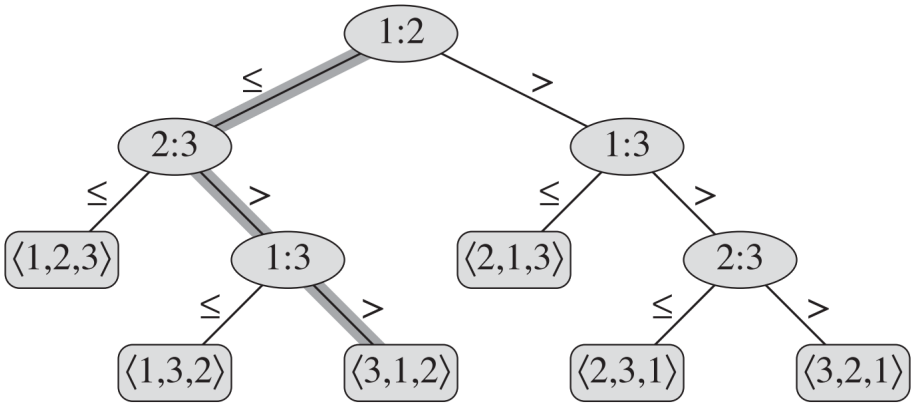
Inverse FFT

$$V_n^{-1} = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

- Since $[a_0\ a_1\ \ldots a_{n-1}]^T = V_n^{-1}[y_0\ y_1\ \ldots y_{n-1}]$, we have $a_j = \frac{1}{n}\sum_{k=0}^{n-1} y_k \omega_n^{-kj}$.
- Consider the degree $n-1$ polynomial $Y(x) = \frac{1}{n}(y_0 + y_1 x + y_2 x^2 + \ldots + y_{n-1}x^{n-1})$.
  - We want to compute Y at $1, \omega_n^{-1}, \omega_n^{-2}, \ldots, \omega_n^{-n}$.
- This is just another DFT, which we can do in $O(n \log n)$ time.
- Thus, we can multiply two degree $n-1$, or more generally, do $n-1$ point convolution in $O(n \log n)$ time.

Comparison sorts

- In insertion sort, mergesort and Quicksort, the output order only depended on comparisons between the input values.
  - Ex In insertion sort, we compare a value with other values to determine its sorted position.
  - These algorithms are comparison sorts.
  - They all take $\Omega(n \log n)$ time to sort n inputs.
- There is a general $\Omega(n \log n)$ lower bound on the time complexity of any comparison sort algorithm.
  - Any algorithm in which the output is only determined by comparisons between input values takes at least $\Omega(n \log n)$ steps.



Beyond comparison sorts

- To sort faster than $\Omega(n \log n)$ time, we need to use other operations besides comparison.
- An algorithm can sort in $O(n)$ linear time by reading the value of inputs, and using these for array indexing, comparing digits, etc.
- $O(n)$ time is asymptotically the best possible, since we need to read all n inputs.
- We'll look at counting and radix sort.

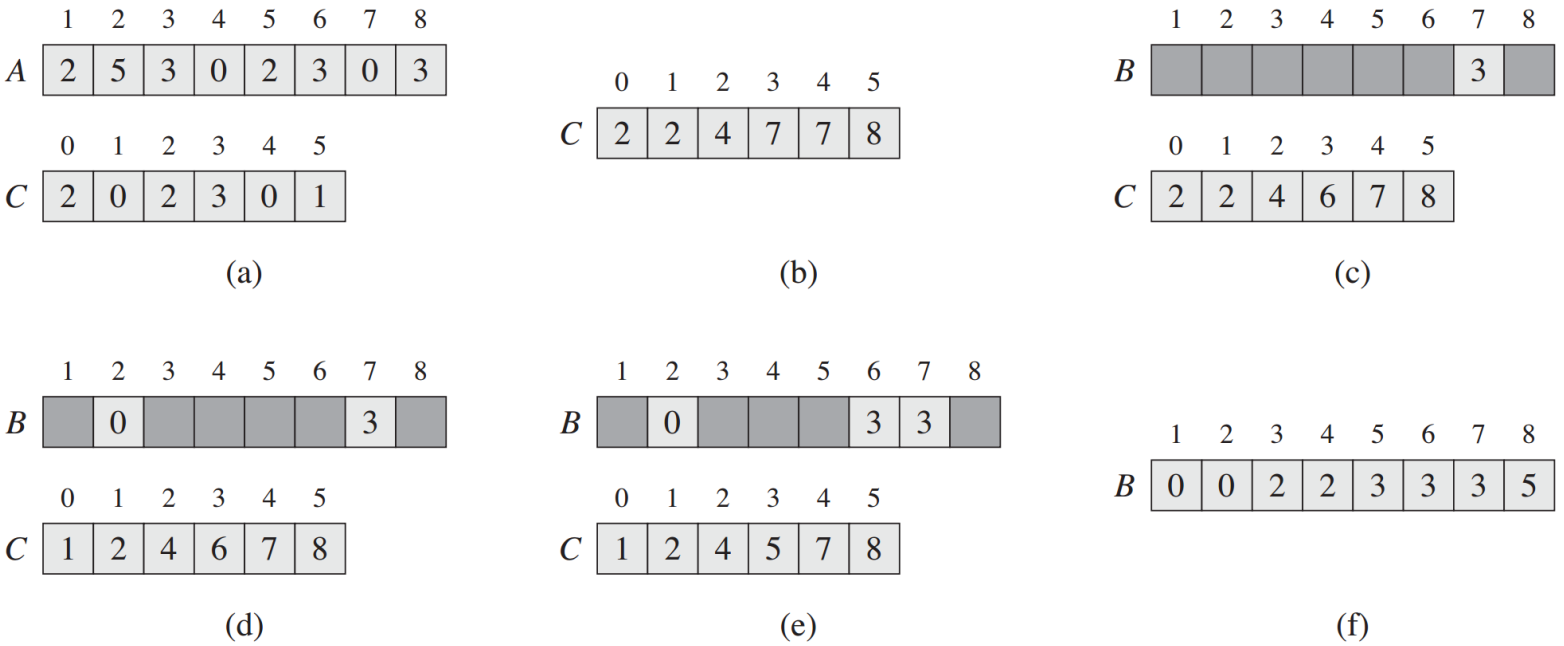
Counting sort

- Counting sort assumes all input values are integers in the range 0 to k, for some k.
- The algorithm runs in $\Theta(n)$ time when k is small.
  - As k gets larger, the algorithm becomes increasingly inefficient.
  - Counting sort is used as a stand-alone algorithm, and also as a subroutine in other algorithms, e.g. radix sort.
  - Radix sort ensures k is small, so counting sort is fast.

Counting sort

- Since we assume all inputs are in $[0, k]$, we use a size k array C to store how many inputs have each value.
  - $C[i] = c$ if there are c inputs with value i.
- Iterate through input array A.
  - For input value $A[i]$, increment $C[A[i]]$ to record an additional occurrence of value $A[i]$.
- Once we know number of occurrences of each value, we know the sorted output.
  - Ex If $C = [2,1,3,0,0,1]$, then output is $0,0,1,2,2,2,5$.
- Value i occurs $C[i]$ times.
  - It appears after values $0,1, \ldots, i-1$.
  - There are $\sum_{j=0}^{i-1} C[j]$ values $0,1, \ldots, i-1$.
  - So first occurrence of i is at index $\sum_{j=0}^{i-1} C[j] + 1$, and last occurrence is at $\sum_{j=0}^{i} C[j]$.
- After computing count array C, compute prefix sum C' of C.
  - $C'[i] = \sum_{j=0}^{i} C[j]$. Also, set $C'[-1] = 0$.
  - Value i occurs in indices $[C'[i-1] + 1, C'[i]]$ of output.

Example



Source: Introduction to Algorithms, Cormen et al.

- In (a), C contains number of occurrences of each input value in A.
- In (b), compute the prefix sum of C.
- In (c)-(f), iterate through A in reverse order.
  - Put value $A[i]$ in position $C[A[i]]$ of output, then decrement $C[A[i]]$ (because we output one more copy of $A[i]$).

Pseudocode and complexity

```
COUNTING-SORT(A, B, k)
 1  let C[0..k] be a new array
 2  for i = 0 to k
 3      C[i] = 0
 4  for j = 1 to A.length
 5      C[A[j]] = C[A[j]] + 1
 6  // C[i] now contains the number
 7  for i = 1 to k
 8      C[i] = C[i] + C[i − 1]
 9  // C[i] now contains the number
10  for j = A.length downto 1
11      B[C[A[j]]] = A[j]
12      C[A[j]] = C[A[j]] − 1
```

- First for loop resets all counts.
  - $O(k)$ time.
- Second for loop counts occurrences of each value.
  - $O(n)$ time.
- Third loop computes prefix sum.
  - $O(k)$ time.
- Last for loop uses prefix sum to scatter inputs to output positions.
  - $O(n)$ time.
- Overall complexity $\Theta(n + k)$.
- If $k = O(n)$, then complexity is $\Theta(n)$.

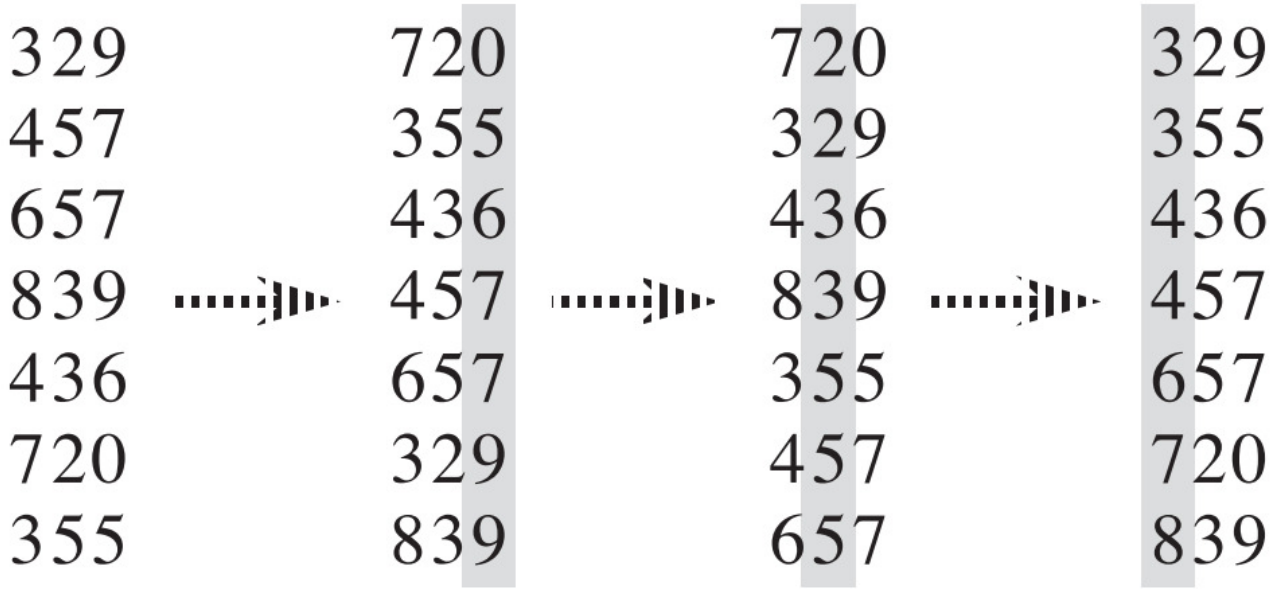
Stability

- A useful property of counting sort is that it's stable.
  - If two inputs are equal, then their order in the output is the same as their order in the input.
  - This is why we iterated through A in reverse order when producing B.
- Ex If input is $4,1_A,5,2_A,1_B,2_B,1_C$, then output is $1_A,1_B,1_C,2_A,2_B,4,5$.
- Stability is necessary when counting sort is used as a subroutine in other sorts, e.g. radix sort.

Radix sort

- Sort digit by digit, from least to most significant digit.
- Take list of input values, sort them on the singles digit.
- Take the new list, sort values on the tens digit.
- Take the new list, sort values on hundreds digit. Etc.
- The sorting algorithm for each digit must be stable.
- We will use counting sort.
  - It's stable.
  - Since we sort a digit at a time, the values being sorted are between 0 and 9.
  - Sorting n inputs takes $O(n)$ time.

Example



Notice due to stability, after sorting by the 10's digit, 436 and 839 (for example) keep the same order they had after sorting by the 1's digit.

Correctness

- Lemma 1 Let x and y be two inputs to radix sort.
  - Let k be the most significant digit on which they differ.
  - Suppose k'th digit of x is less than k'th digit of y.

  After sorting the k'th digit (in nondecreasing order), x will come before y in the remainder of the execution.
- Proof x comes before y right after sorting the k'th digit.
  - x and y are equal on all higher digits.
  - So when sorting on higher digits, x and y are always tied.
  - Since the sort is stable, x stays before y from the k'th sort onwards.

Complexity

- Lemma 2 Suppose we sort n d-digit numbers, where each digit is between 0 to $k-1$. Then radix sort takes $O(d(n+k))$ time.
- Proof Since each digit is between 0 and $k-1$, then counting sort takes $O(n+k)$ time per digit. So the total time is $O(d(n+k))$.
- Ex Sorting n d-digit binary numbers takes $O(dn)$ time.

Complexity

- Lemma 3 Given n b-bit numbers and $r \leq b$. Radix sort takes $O\left(\frac{b}{r}(n + 2^r)\right)$ time.
- Proof Break the b bits into blocks of r digits, having values between 0 and $2^r - 1$.
  - Ex For $b = 6, r = 2$, break the value 100111 into blocks 10, 01 and 11.
  - There are $d = \lceil b/r \rceil$ such blocks.
  - We can think of each b-bit number as a d digit number, where each digit has value between 0 and $2^r - 1$.
  - The lemma follows from Lemma 2.

Complexity

- Lemma 4 Setting $r = \min(\lfloor \log n \rfloor, b)$ minimizes the running time $\Theta\left(\frac{b}{r}(n + 2^r)\right)$.
- Proof If $b < \lfloor \log n \rfloor$, then for any $r \leq b$, we have $n + 2^r = \Theta(n)$. So we set $r = b$ to minimize b/r.
  - If $b \geq \lfloor \log n \rfloor$, setting $r = \lfloor \log n \rfloor$ makes running time $\Theta\left(\frac{bn}{\log n}\right)$.
  - We show $r > \lfloor \log n \rfloor$ and $r < \lfloor \log n \rfloor$ both result in slower running times.
  - If $r > \lfloor \log n \rfloor$, then $2^r > n$, and 2^r in numerator increases faster than r in denominator, so running time increases.
  - If $r < \lfloor \log n \rfloor$, then the b/r term increases, but the $n + 2^r$ remains $\Theta(n)$.
- In other words, radix sort is efficient when there are many short numbers, but not when there are a few long numbers.