CS271 Computer Graphics II

Lecture 4

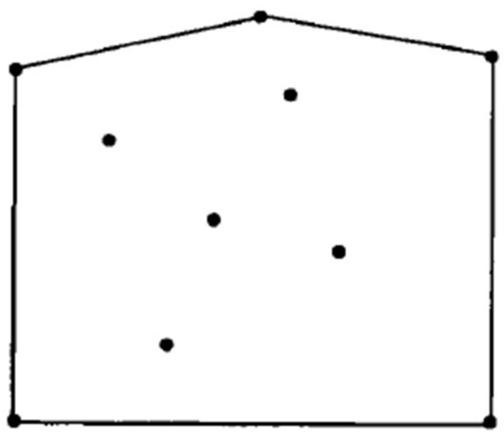
Computational Geometry – Delaunay

Overview

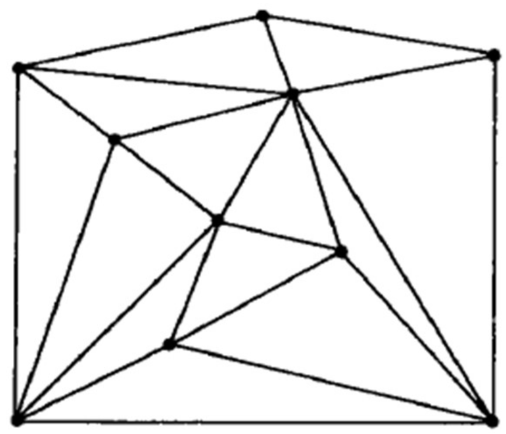
- Triangulation
- Delaunay Triangulation
- Constrained Delaunay Triangulation
- 3D Delaunay Triangulation

Triangulation

Triangulation for a point set $P = \{p_1, p_2, \ldots, p_n\}$ is a division for the area ω, which contains P. ω can be any polygon area. If not given, the convex hull of P is ω.



ω is the convex hull of P



A triangulation of P

Triangulation

Triangulation for a point set $P = \{p_1, p_2, \dots, p_n\}$ is a division for the area ω, which contains P. ω can be any polygon area. If not given, the convex hull of P is ω.

Requirements

- Divide ω into triangles.
- All the vertices of triangles belong to P and the boundary of ω, and all the points of P and vertices of the boundary of ω are the vertices of triangles.
- All the triangles only share edges and vertices.
- Each point in the triangle must belong to ω and each point in ω must belong to a triangle.

Properties of Triangulation

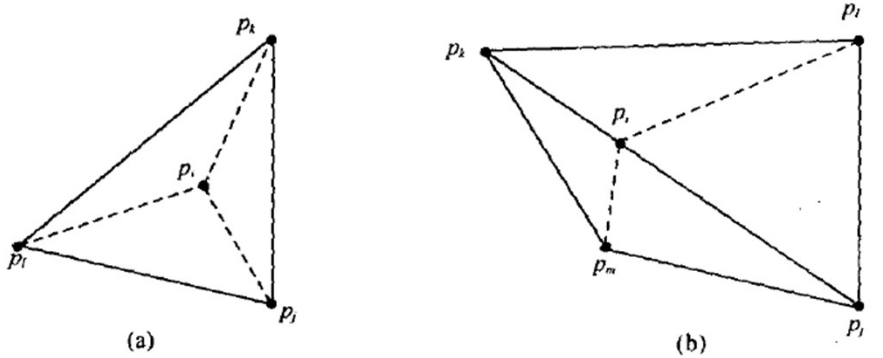
For the triangulation of the point set $P = \{p_1, p_2, \ldots, p_n\}$, the number of triangles t, vertices v, and edges e have the following relationship.

- $t = e - v + 1$
- $e \leq 3v - 6$
- $t \leq 2v - 5$

According to the ***Euler's formula*** of the planar graph

Construction of Triangulation

- ***Input:*** point set $P = \{p_1, p_2, \ldots, p_n\}$
- ***Output:*** triangulation of P: $T(P)$

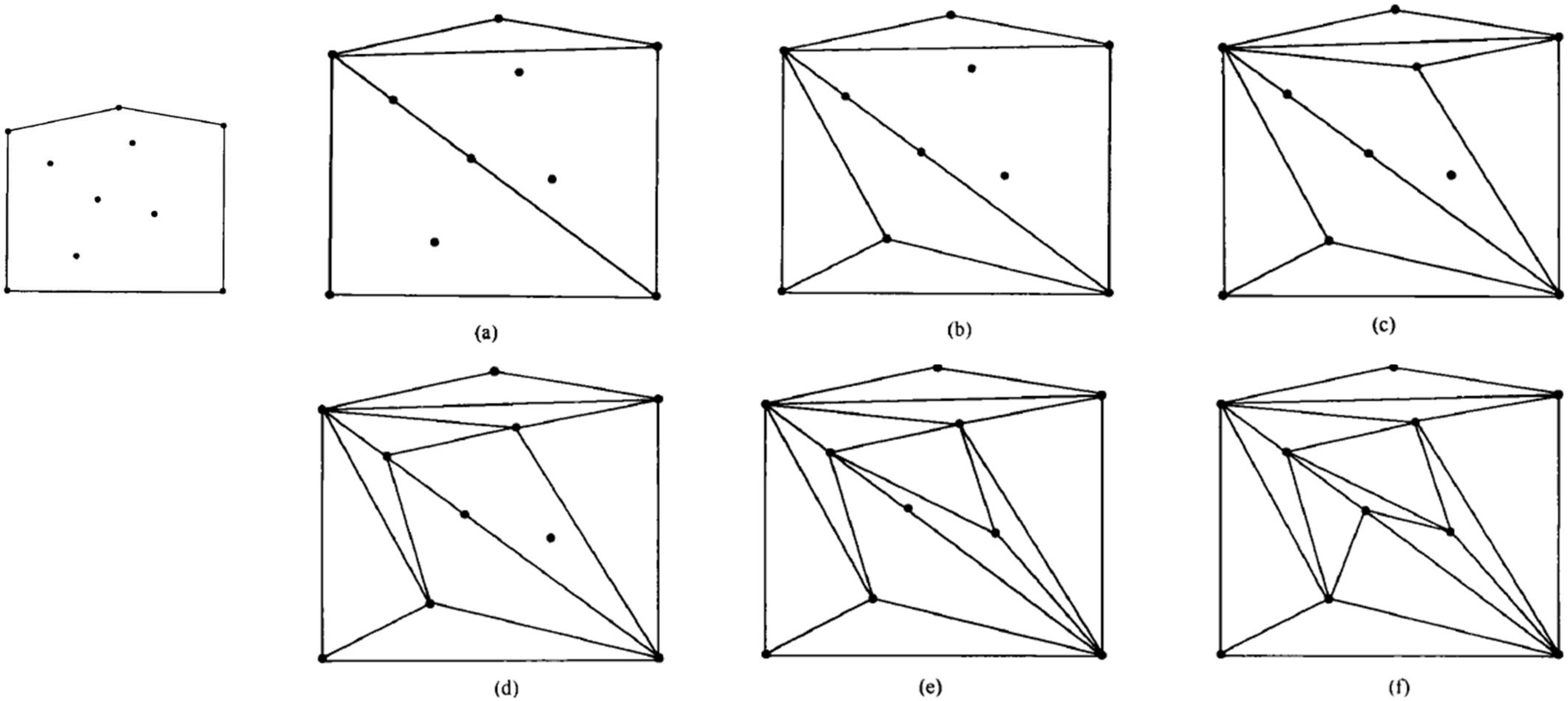


1. **compute the convex hull** $CH(P)$;
2. **compute a triangulation of** $CH(P)$;
3. **for (point** $p_i \in P \cap p_i \notin CH(P)$) {
4. **find the located triangle** $\Delta p_l p_j p_k \in T$;
5. **if** p_i **is inside** $\Delta p_l p_j p_k$, **link** p_i **to** p_l, p_j, p_k **and get new** T;
6. **if** p_i **is on the edge** $p_l p_j$ **of** $\Delta p_l p_j p_k$, **link** p_i **to** p_k **and link** p_i **to** p_m **if** $\Delta p_l p_j p_m$ **exists, and get new** T;
7. }

Triangulation

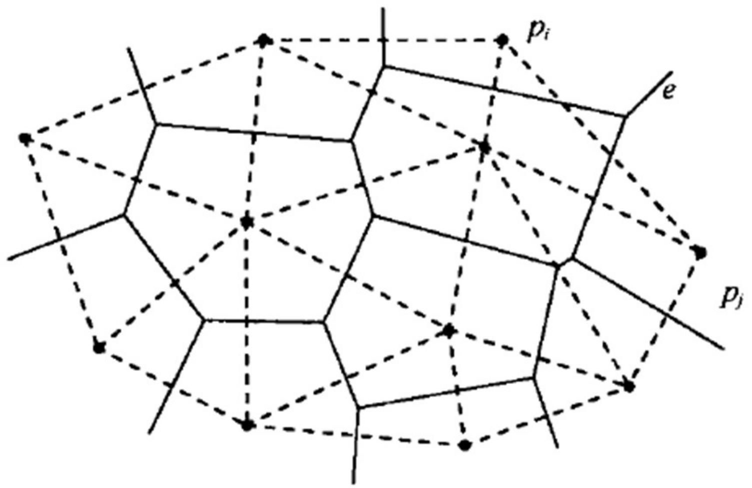
The result depends on the triangulation methods and the sequence of adding points.

Example

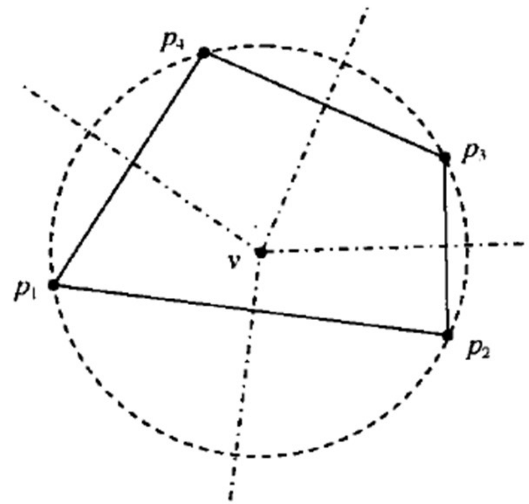


Delaunay Triangulation

For a point set $P = \{p_1, p_2, \ldots, p_n\}$, construct the $VD(P)$. If we link each two seeds, which have shared Voronoi edge, we can get the dual diagram – **Delaunay diagram**.



Voronoi and Delaunay of P



Four seeds are co-circular

If there is no degenerate case, we get Delaunay Triangulation $DT(P)$.

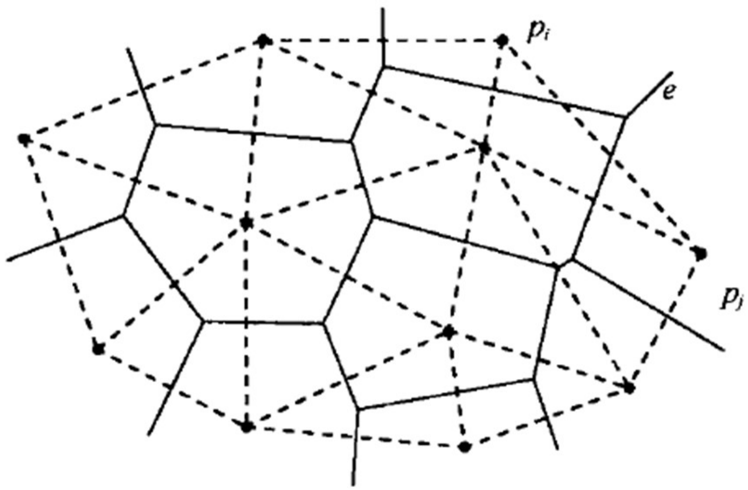
Voronoi and Delaunay Triangulation

The vertices of $DT(P)$ ➡ The facets of $VD(P)$

The triangles of $DT(P)$ ➡ The vertices of $VD(P)$

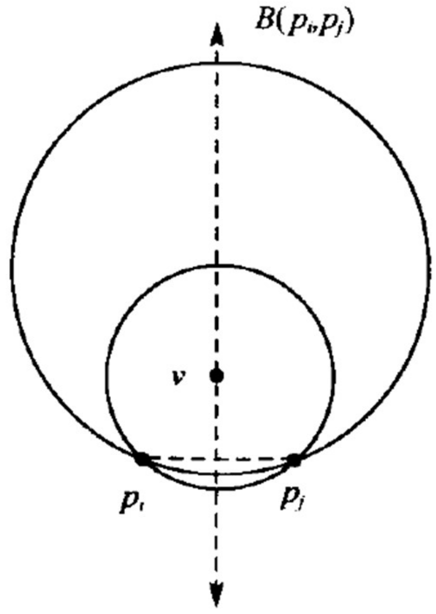
The edges of $DT(P)$ ➡ The edges of $VD(P)$

The boundary of $DT(P)$ ➡ The convex hull of P ➡ The seeds of $VD(P)$ with open area



Properties of Delaunay Triangulation

1. $DT(P)$ has $\leq 3n - 6$ edges and $\leq 2n - 5$ Delaunay triangles.
2. For two points p_i and p_j of P, $p_i p_j$ is one edge of $DT(P)$ if and only if there exists an empty circle only passing p_i and p_j.



The trajectory of the center of the empty circle only passing p_i and p_j form the Voronoi edge.

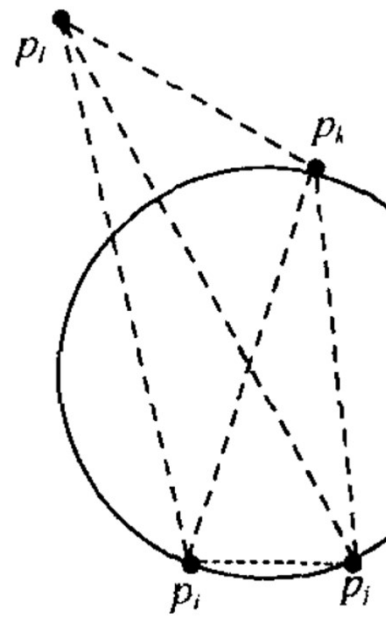
Properties of Delaunay Triangulation

1. $DT(P)$ has $\leq 3n - 6$ edges and $\leq 2n - 5$ Delaunay triangles.
2. For two points p_i and p_j of P, $p_i p_j$ is one edge of $DT(P)$ if and only if there exists an empty circle only passing p_i and p_j.
3. For the point p_i, there must be a $DT(P)$ edge between p_i and its nearest point in P.
4. The point p_i, p_j and p_k is the vertices of a Delaunay triangle if and only if there exists an empty circle only passing p_i, p_j and p_k.
5. There is no other points inside a Delaunay triangle.
6. For any four points, compared with other triangulation methods, the two Delaunay triangles have the property – the minimum angle is the maximum.

Properties of Delaunay Triangulation

For any four points, compared with other triangulation methods, the two Delaunay triangles have the property – the minimum angle is the maximum.

Prove by the relation between the angle of circumference and the arc.



Theorem of Delaunay Triangulation

If no four points in P are co-circular, the Delaunay diagram of P is the Delaunay Triangulation $DT(P)$.

Edge flipping algorithm

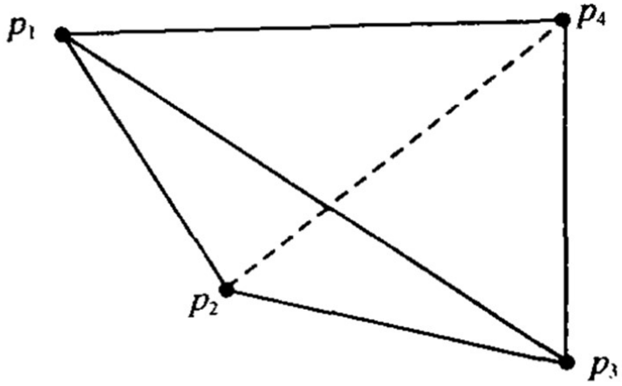
Legal edge: the diagonal $p_i p_k$ of a convex tetragon meets the minimum angle is the maximum.

Legal triangulation: all the edges of the triangulation are legal edges.

Theorem

$\boldsymbol{T(P)}$ is a legal triangulation if and only if it is a Delaunay Triangulation.

Edge flipping algorithm



- ***Input:*** point set $P = \{p_1, p_2, \dots, p_n\}$
- ***Output:*** $DT(P)$

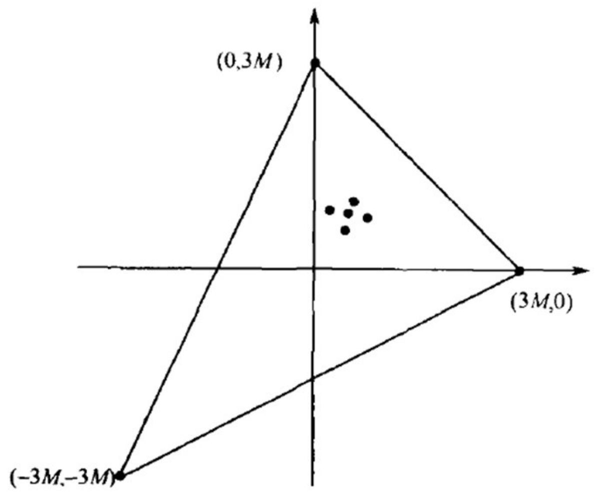
1. compute any a triangulation $T(P)$;
2. While (there exists illegal edge $p_i p_j$) {
3. assume there are two neighbor triangles $\Delta p_i p_j p_k$ and $\Delta p_i p_j p_l$,
4. replace $p_i p_j$ by $p_k p_l$
5. }
6. return T

It is bound to converge because every flip operation will increase the lower bound of six angles.

Incremental algorithm

- ***Input:*** point set $P = \{p_1, p_2, \dots, p_n\}$
- ***Output:*** $DT(P)$

1. compute an initial triangle α which is large enough to cover P.
2. For (each point in P){
3. find the located triangle $\Delta p_l p_j p_k \in T$;
4. compute new T;
5. flip illegal edge to make T become DT.
6. }
7. delete related edges of α;
8. return T



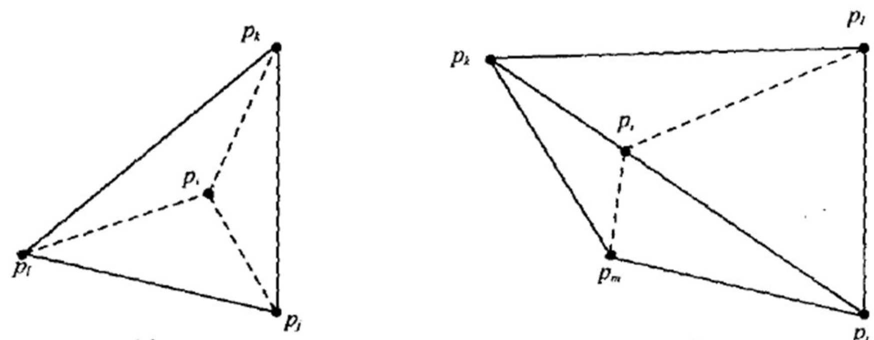
Incremental algorithm

$O(nlogn)$

For the new added point p_i

- All the new triangles has the vertex p_i.
- All the temporary illegal edges must exist in the opposite edges of p_i of new added triangles.
- Recursively check and flip.

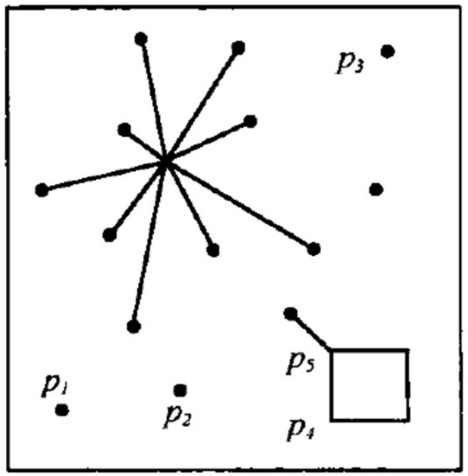
Use a directed acyclic graph to store the triangle division for fast location query.

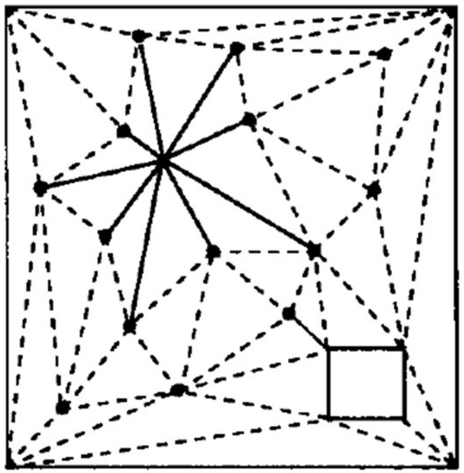


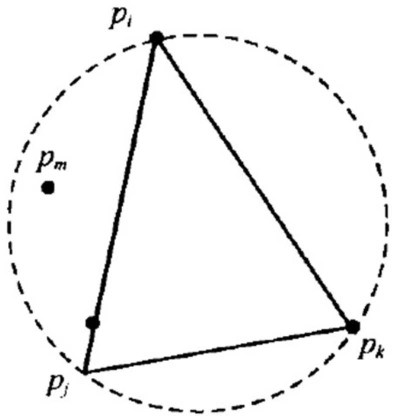
(a) (b)

Constrained Delaunay Triangulation (CDT)

For a point set P and line segment set $LSS(P_1, L)$, where L is line segments and P_1 is vertices, P and P_1 should become vertices and L should become edges in CDT.



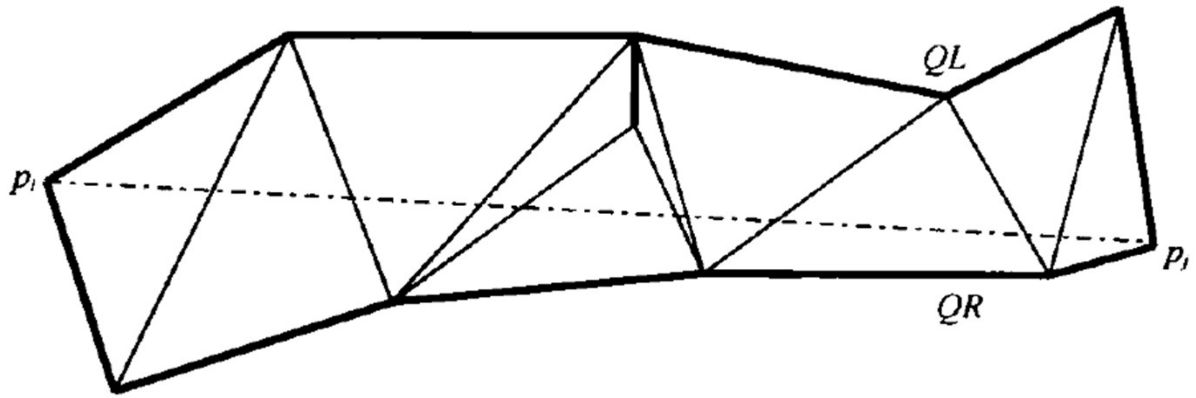




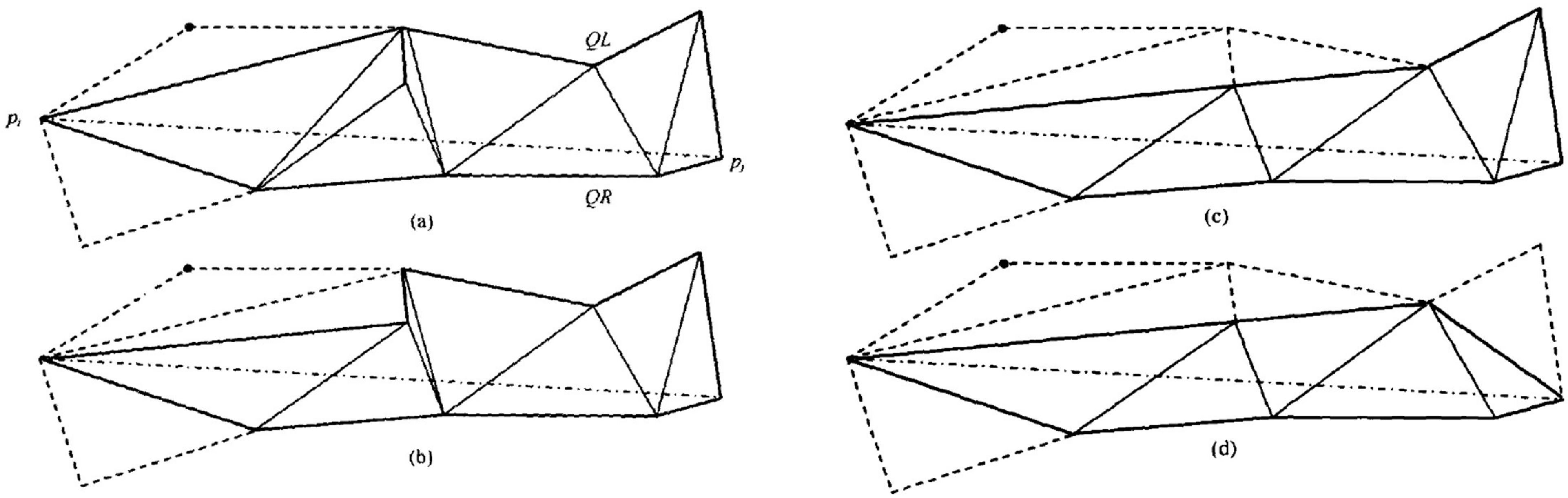
$\Delta p_i p_j p_k$ is legal if $p_i p_j$ is constrained edge.

Constrained Delaunay Triangulation (CDT)

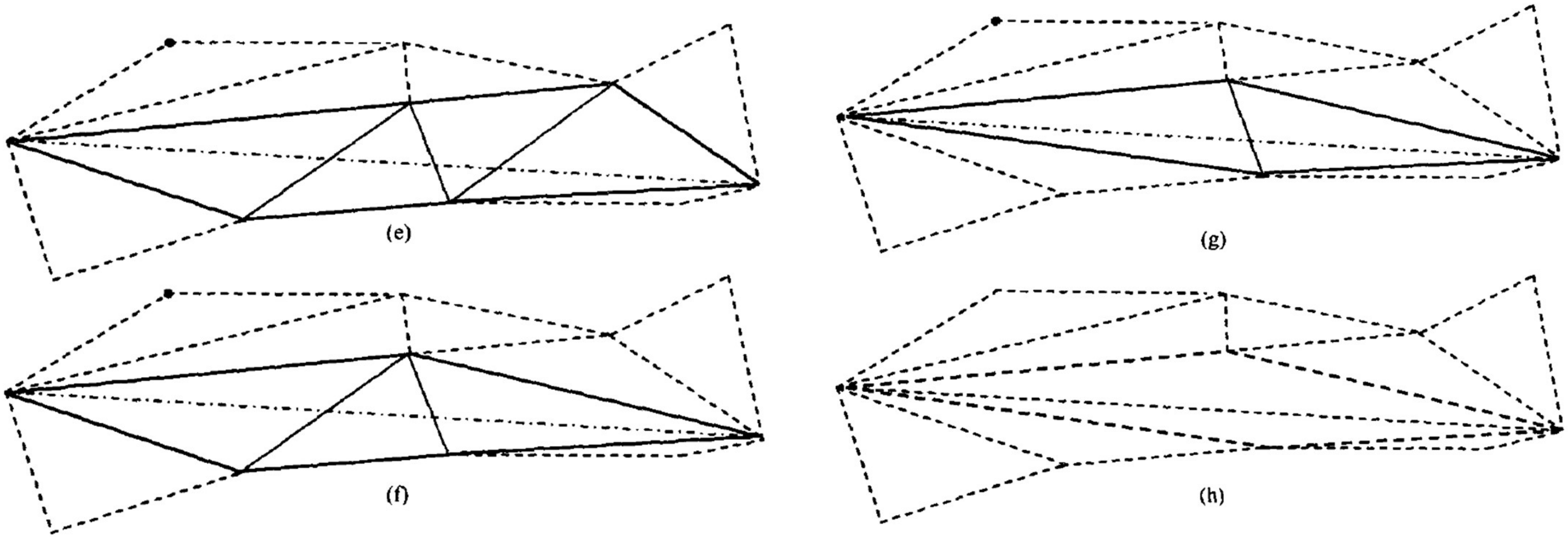
- Compute DT for $P \cap P_1$
- Insert L
- Flip edges for DT



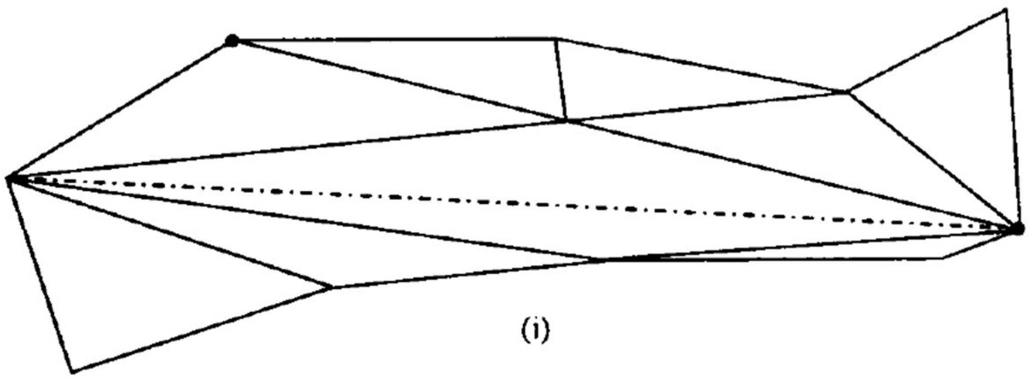
Example of the process



Example of the process



Example of the process

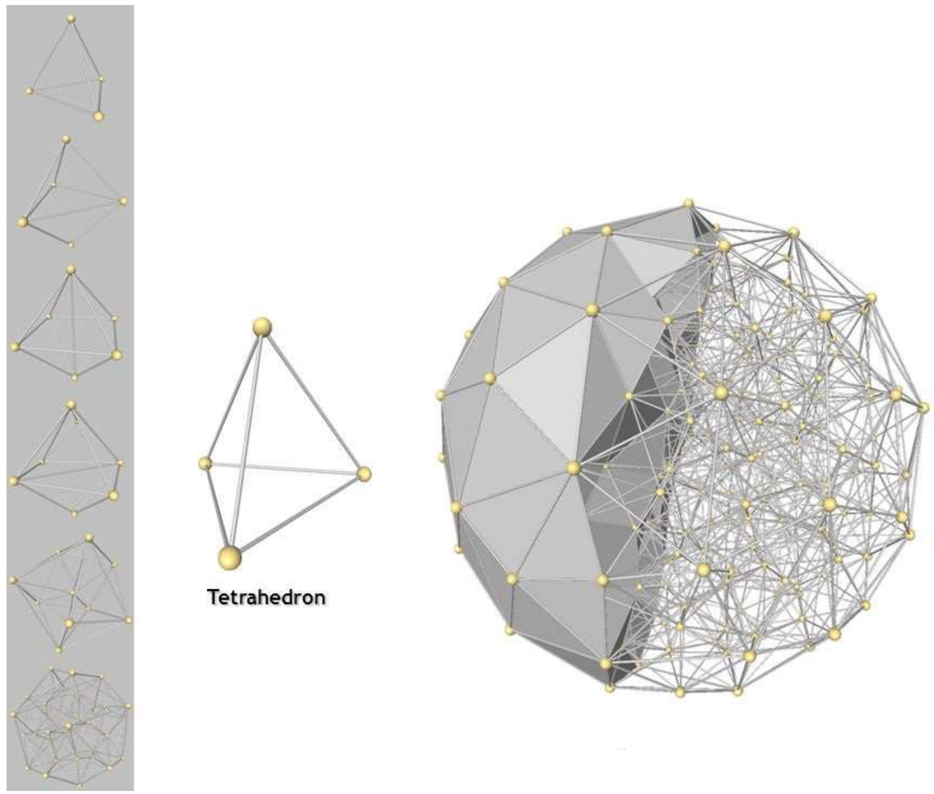


(i)

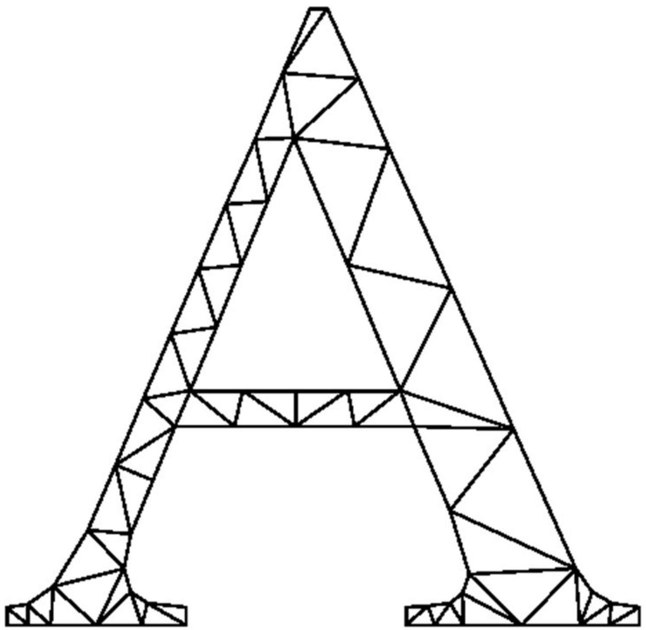
Constrained Delaunay Triangulation

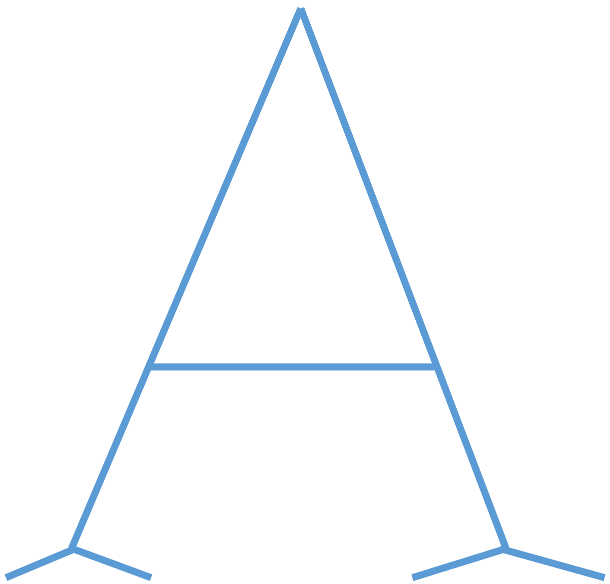
3D Delaunay Triangulation

- The dual of 3D Voronoi
- Consists of tetrahedron



Delaunay Triangulation ➡ MAT



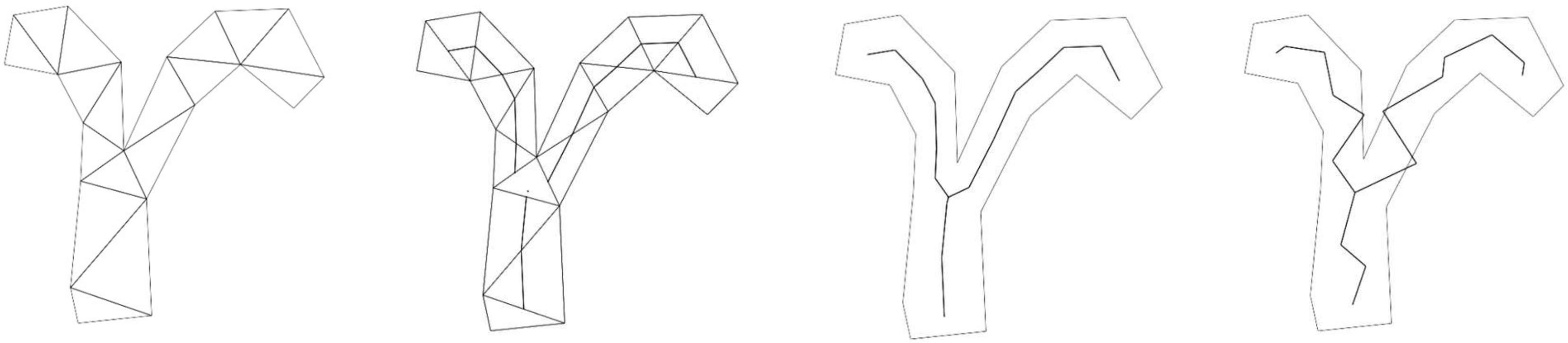


Connect centers of circumcircles of Delaunay Triangles according to the neighborhood. **Dense sampling** has better accuracy.

Chordal Axis Transform (CAT) approximated MAT

Case 1: One edge on the boundary

Case 2: No edge on the boundary



DT

Case 1 process

Case 1&2 process

Compared with the method using centers of circumcircles

Skeleton extraction by CAT

3D CAT