行列式——测验卷

1. $\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \\ n & 0 & 0 & \cdots & 0 \end{vmatrix} = \underline{(-1)^{n+1} n!}$

2. $\begin{vmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 2 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ n & \cdots & 0 & 0 \end{vmatrix} = \underline{(-1)^{\frac{n(n-1)}{2}} n!}$

3. 行列式$\begin{vmatrix} 1 & -1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$中第一行元素的代数余子式之和为$\underline{\quad 0 \quad}$.

4. 计算下列行列式(D_n 为 n 阶行列式)：

(1) $D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{vmatrix}$；

$$= \begin{vmatrix} 1 & -1 & 1 & \cdots & 1 \\ 1 & 0 & 2 & \cdots & 2 \\ 1 & 0 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 3 & \cdots & n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & \cdots & 2 \\ 1 & 3 & \cdots & 3 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 3 & \cdots & n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & \cdots & 2 \\ 1 & 0 & \cdots & 3 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & n \end{vmatrix} = \begin{vmatrix} 1 & 3 & \cdots & 3 \\ 1 & 4 & \cdots & 4 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 4 & \cdots & n \end{vmatrix} = \begin{vmatrix} 1 & n-2 & n-2 \\ 1 & n-1 & n-1 \\ 1 & n-1 & n \end{vmatrix} = \begin{vmatrix} 1 & -1 & n-2 \\ 1 & 0 & n-1 \\ 1 & 0 & n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & n-1 \\ 1 & n \end{vmatrix} = n-(n-1)$$

$$= 1$$

(4) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$;

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & b^2(b^2-a^2) & c^2(c^2-a^2) & d^2(d^2-a^2) \end{vmatrix}$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2(b+a) & c^2(c+a) & d^2(d+a) \end{vmatrix}$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ 0 & c-b & d-b \\ 0 & c^2(c+a)-b^2(b+a) & d^2(d+a)-b^2(b+a) \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b) \begin{vmatrix} 1 & 1 \\ c(c+b)+ac & d(d+b)+da \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(a+b+c+d)(c-d)$$

(5) $D_{2n} = \begin{vmatrix} a_n & & & & & b_n \\ & \ddots & & & \iddots & \\ & & a_1 & b_1 & & \\ & & c_1 & d_1 & & \\ & \iddots & & & \ddots & \\ c_n & & & & & d_n \end{vmatrix}$，未写出的元素都是 0.

将第 2n 行与上一行调换.至第 2 行

再将第 2n 列与前一列调换至第 2 列

则 $D_n = (-1)^{2(2n-2)} \begin{vmatrix} a & b & 0 & \cdots & & 0 \\ c & d & 0 & \cdots & & 0 \\ 0 & 0 & a & & & b \\ \vdots & \vdots & & a & b & \\ & & & c & d & \\ 0 & 0 & c & & & d \end{vmatrix}$

$$D_{2n} = D_2 D_{2(n-1)} = (ad-bc) D_{2(n-1)}$$
$$= (ad-bc)^2 D_{2(n-2)} = \cdots = (ad-bc)^{n-1} D_2 = (ad-bc)^n$$

(2) $D_n=\begin{vmatrix} a_1+b_1 & a_2 & \cdots & a_n \\ a_1 & a_2+b_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n+b_n \end{vmatrix}\ (b_i\neq 0)$;

$$= \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & a_1+b_1 & a_2 & \cdots & a_n \\ 0 & a_1 & a_2+b_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_1 & a_2 & \cdots & a_n+b_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & b_1 & 0 & \cdots & 0 \\ -1 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & b_n \end{vmatrix}$$

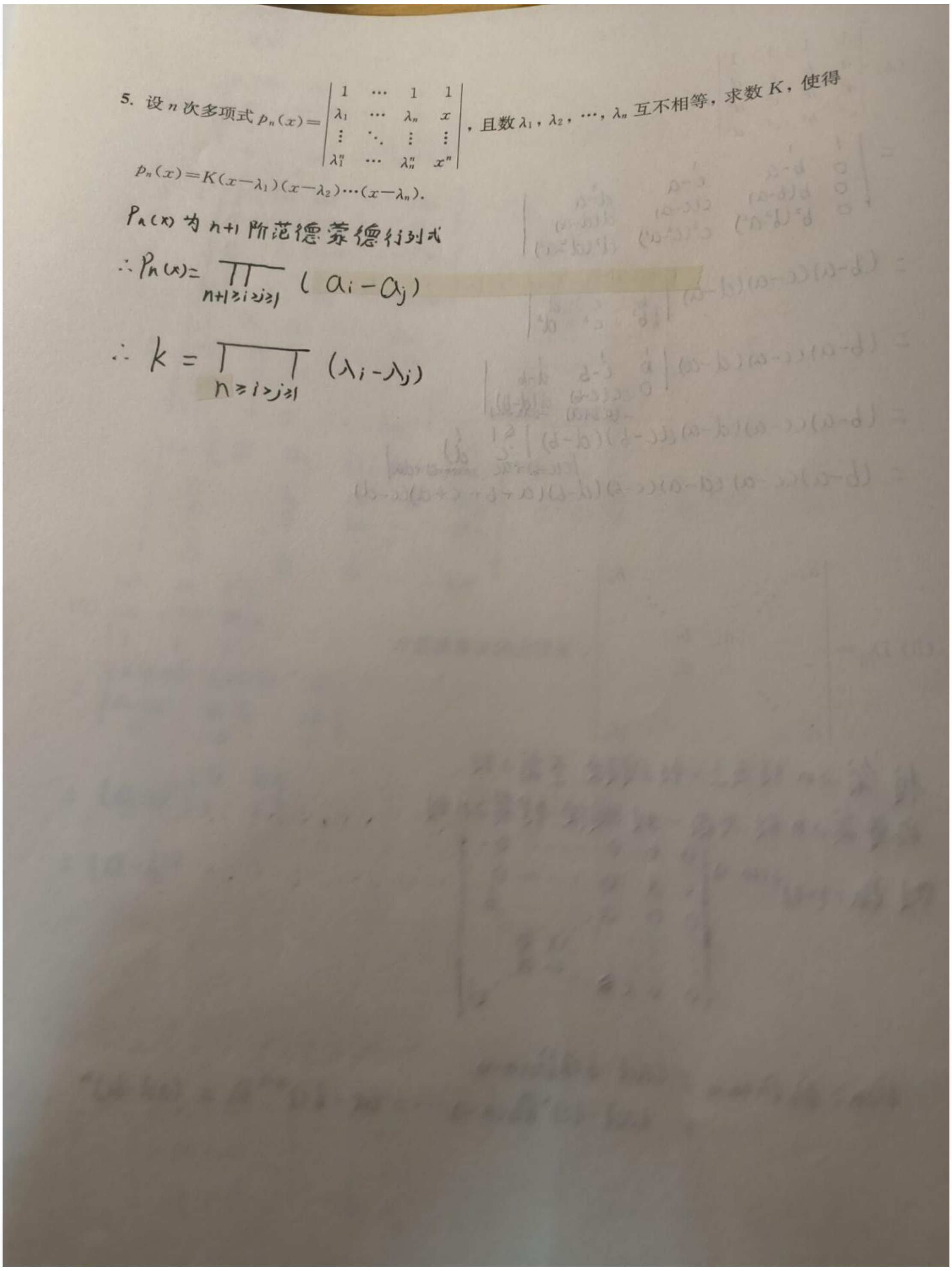
$$= \begin{vmatrix} 1+\sum_{i=1}^{n}\frac{1}{b_i} & a_1 & a_2 & \cdots & a_n \\ 0 & b_1 & 0 & \cdots & a_n \\ 0 & 0 & b_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_n \end{vmatrix} = b_1\cdot b_2\cdots b_n\left(1+\sum_{i=1}^{n}\frac{1}{b_i}\right)$$

(3) $\begin{vmatrix} a^2 & ab & b^2 \\ 2a & a+b & 2b \\ 1 & 1 & 1 \end{vmatrix}$;

$$= \begin{vmatrix} a(a-b) & b(a-b) & b^2 \\ a-b & a-b & 2b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)^2\begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix}$$

$$= (a-b)^3$$



5. 设 n 次多项式 $p_n(x)=\begin{vmatrix} 1 & \cdots & 1 & 1 \\ \lambda_1 & \cdots & \lambda_n & x \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_1^n & \cdots & \lambda_n^n & x^n \end{vmatrix}$，且数 $\lambda_1,\lambda_2,\cdots,\lambda_n$ 互不相等，求数 K，使得 $p_n(x)=K(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$.

$P_n(x)$ 为 $n+1$ 阶范德蒙德行列式

$\therefore P_n(x)=\prod\limits_{n+1\geq i>j\geq 1}(a_i-a_j)$

$\therefore k=\prod\limits_{n\geq i>j\geq 1}(\lambda_i-\lambda_j)$