行列式——行列式的性质

知识点巩固练习

1. D 与 D^{T} 的值 相等 .
2. 对换行列式的两行(列),则行列式 互为相反数
3. $\begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & & \vdots \\ ka_{i1} & \cdots & ka_{in} \\ \vdots & & \vdots \\ ka_{n1} & \cdots & ka_{nn} \end{vmatrix} = \underline{\ k^{n}\ } \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$.
4. $\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1}+a'_{i1} & \cdots & a_{in}+a'_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a'_{i1} & \cdots & a'_{i1} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$.
5. $\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{j1}+ka_{i1} & \cdots & a_{jn}+ka_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \underline{\ 1\ } \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$.
6. 若行列式中有两行(列)成比例,则行列式为 0 .

练习题

1. 计算下列各行列式:

(1) $\begin{vmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 0 & 1 & 1 & 7 \end{vmatrix}$;

$$= \begin{vmatrix} -6 & -4 & 0 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 3 & 0 & -14 \\ 0 & 1 & 1 & 7 \end{vmatrix}$$

$$= 2\begin{vmatrix} -3 & -2 & 2 \\ 1 & 2 & 2 \\ 10 & 3 & -14 \end{vmatrix}$$

$$= 2\begin{vmatrix} -3 & -4 & 8 \\ 1 & 0 & 0 \\ 10 & 17 & -34 \end{vmatrix} = 17\times 8\times \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} = 0$$

(2) $\begin{vmatrix} -ab & ac & ae \\ bd & -cd & de \\ bf & cf & -ef \end{vmatrix}$；

$= bce\begin{vmatrix} -a & a & a \\ d & -d & d \\ f & f & -f \end{vmatrix}$

$= bce\begin{vmatrix} -a & 0 & 0 \\ d & 0 & 2d \\ f & 2f & 0 \end{vmatrix}$

$= -abce\begin{vmatrix} 0 & 2d \\ 2f & 0 \end{vmatrix}$

$= 4abcdef$

(3) 已知 $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}=1$，求 $\begin{vmatrix} 2a_{11} & 2a_{12} & 2a_{13} & 2a_{14} \\ 2a_{21} & 2a_{22} & 2a_{23} & 2a_{24} \\ 2a_{31} & 2a_{32} & 2a_{33} & 2a_{34} \\ 2a_{41} & 2a_{42} & 2a_{43} & 2a_{44} \end{vmatrix}$；$\begin{vmatrix} a_{11} & a_{12}-3a_{14} & a_{13} & -a_{14} \\ a_{21} & a_{22}-3a_{24} & a_{23} & -a_{24} \\ a_{31} & a_{32}-3a_{34} & a_{33} & -a_{34} \\ a_{41} & a_{42}-3a_{44} & a_{43} & -a_{44} \end{vmatrix}$.

$$\begin{vmatrix} 2a_{11} & 2a_{12} & 2a_{13} & 2a_{14} \\ 2a_{21} & 2a_{22} & 2a_{23} & 2a_{24} \\ 2a_{31} & 2a_{32} & 2a_{33} & 2a_{34} \\ 2a_{41} & 2a_{42} & 2a_{43} & 2a_{44} \end{vmatrix} = 2^4\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 16$$

$$\begin{vmatrix} a_{11} & a_{12}-3a_{14} & a_{13} & -a_{14} \\ a_{21} & a_{22}-3a_{24} & a_{23} & -a_{24} \\ a_{31} & a_{32}-3a_{34} & a_{33} & -a_{34} \\ a_{41} & a_{42}-3a_{44} & a_{43} & -a_{44} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = -1$$

2. 已知 $\begin{vmatrix} a & b \\ c & d \end{vmatrix}=m$，则 $\begin{vmatrix} 2a & 5a+3b \\ 2c & 5c+3d \end{vmatrix}=$ <u>6m</u>.

思考题

若 n 阶行列式 $D=\det(a_{ij})$，记 $D_1=\begin{vmatrix} a_{n1} & \cdots & a_{nn} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{1n} \end{vmatrix}$，$D_2=\begin{vmatrix} a_{1n} & \cdots & a_{nn} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{n1} \end{vmatrix}$，$D_3=\begin{vmatrix} a_{nn} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{11} \end{vmatrix}$.

问 D 与 D_1，D_2，D_3 之间的关系如何？

解：以 D_1 为例：将最后一行将 r_{n-1} 调换，以此类推，由 $D_1 \to D$ 需调换 $n-1+n-2+\cdots+1$ 次

即 $\frac{n(n-1)}{2}$ 次

则 $D_1=(-1)^{\frac{n(n-1)}{2}}D$

同理 $D_1=D_2=(-1)^{\frac{n(n-1)}{2}}D$

$D_3=D$

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行列式——行列式的计算

知识点巩固练习

1. n 阶行列式中，(i,j) 元 a_{ij} 的余子式 M_{ij} 是指 把(i,j)元 a_{ij} 所在的第 i 行和第 j 列划去后，留下来的 $n-1$ 阶行列式.
代数余子式 A_{ij} 是指 $(-1)^{i+j}M_{ij}$.
2. $a_{i1}A_{i1}+\cdots+a_{in}A_{in}=$ D_n.
3. $a_{11}A_{21}+\cdots+a_{1n}A_{2n}=$ 0.
4. $D_n=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}=\prod\limits_{n\geq i>j\geq 1}(x_i-x_j)$.

练习题

1. 设 $D=\begin{vmatrix} 3 & 1 & -1 & 2 \\ 1 & -3 & 2 & 0 \\ 2 & 1 & 1 & -1 \\ 0 & 2 & 5 & -3 \end{vmatrix}$，求 $-2A_{31}-4A_{32}+A_{34}$.

$$-2A_{31}-4A_{32}+A_{34}=\begin{vmatrix} 3 & 1 & -1 & 2 \\ 1 & -3 & 2 & 0 \\ -2 & -4 & 0 & 1 \\ 0 & 2 & 5 & -3 \end{vmatrix}$$

$$=\begin{vmatrix} 3 & 1 & -1 & 2 \\ 1 & -3 & 2 & 0 \\ 0 & -10 & 4 & 1 \\ 0 & 2 & 5 & -3 \end{vmatrix}$$

$$=\begin{vmatrix} 0 & 10 & -7 & 2 \\ 1 & -3 & 2 & 0 \\ 0 & -10 & 4 & 1 \\ 0 & 2 & 5 & -3 \end{vmatrix}$$

$$=2\begin{vmatrix} 5 & -7 & 2 \\ -5 & 4 & 1 \\ 1 & 5 & -3 \end{vmatrix}$$

$$=2\begin{vmatrix} 15 & -15 & 2 \\ 0 & 0 & 1 \\ -14 & 17 & -3 \end{vmatrix}=2\begin{vmatrix} 15 & -15 \\ -14 & 17 \end{vmatrix}=90$$

2. 计算下列行列式：

(1) $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix}$；

$$= \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -6 & 1 \\ 0 & -6 & -8 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

$$=4\begin{vmatrix} 1 & 3 & 1 \\ 3 & 2 & 1 \\ -1 & -1 & 3 \end{vmatrix}$$

$$=4\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 0 \\ -4 & -10 & 0 \end{vmatrix}$$

$$=-16\begin{vmatrix} 1 & 1 \\ -1 & 5 \end{vmatrix} = -96$$

(2) $\begin{vmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{vmatrix}$；

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 9 \\ 1 & 4 & 8 & 28 \\ 1 & 6 & 24 & 126 \end{vmatrix}$$

$$=72\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 7 \\ 1 & 4 & 21 \end{vmatrix}$$

$$=72\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 4 \\ 1 & 3 & 18 \end{vmatrix} = 216\begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix} = 432$$

(3) $\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$;

$$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1-ab & b & 1 & 0 \\ a & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

$$= \begin{vmatrix} -1-ab & 1 & 0 \\ a & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$$= \begin{vmatrix} -1-ab & 1 & 0 \\ a & c & 1 \\ -ad & -1-cd & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1-ab & 1 \\ -ad & -1-cd \end{vmatrix} = (ab+1)(cd+1)+ad$$

(4) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$;

$$= \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix} = 0$$

(5) $D_n=\begin{vmatrix} \beta & a & \cdots & a \\ a & \beta & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & \beta \end{vmatrix}$,

$$=[\beta+(n-1)a]\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & \beta & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & \beta \end{vmatrix}=\begin{vmatrix} 1 & 0 & \cdots & 0 \\ a & \beta-a & \cdots & 0 \\ a & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ a & 0 & \cdots & \beta-a \end{vmatrix}[\beta+(n-1)a]$$

$$=[\beta+(n-1)a]\cdot(\beta-a)^{n-1}$$

$$=(\beta-a)^{n-1}[\beta+(n-1)a]$$

(6) $D_n=\begin{vmatrix} a & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & a \end{vmatrix}$，其中对角线上元素都是 a，未写出的元素都是 0.

$$=a\begin{vmatrix} 1 & \cdots & \frac{1}{a} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a-\frac{1}{a} \end{vmatrix}=a\begin{vmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a-\frac{1}{a} \end{vmatrix}=a\cdot a^{n-2}\cdot(a-\frac{1}{a})$$

$$=a^{n-2}(a^2-1)$$