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结果

```
# numpy 数值解
[1.33333333 1.33333333 1.33333333]
```

```
# sympy 解析解
(-56*100**k/3 + 4*100**k*sqrt(7)/3 - 4*(60 - 10*sqrt(7))**k - sqrt(7)*(60 - 10*sqrt(7))**k - 16*(10*sqrt(7) + 60)**k/3 + 5*sqrt(7)*(10*sqrt(7) + 60)**k/3)/(100**k*(-14 + sqrt(7)))
(-112*100**k + 8*100**k*sqrt(7) - 3*sqrt(7)*(60 - 10*sqrt(7))**k + 15*(60 - 10*sqrt(7))**k + sqrt(7)*(10*sqrt(7) + 60)**k + 13*(10*sqrt(7) + 60)**k)/(6*100**k*(-14 + sqrt(7)))
(56*100**k*(14 - sqrt(7)) - 63*(1 + sqrt(7))*(60 - 10*sqrt(7))**k + (-7 + 5*sqrt(7))*(14 - sqrt(7))*(10*sqrt(7) + 60)**k)/(42*100**k*(14 - sqrt(7)))
```

求解思路

解析解

$$L=\begin{bmatrix}0.6 & 0.1 & 0.3\\ 0.1 & 0.9 & 0\\ 0.3 & 0 & 0.7\end{bmatrix} \tag{1}$$

欲求$x_k=L^kx_0$，将问题归结为求L^k。由

$$|\lambda E-L|=\begin{vmatrix}\lambda-0.6 & -0.1 & -0.3\\ -0.1 & \lambda-0.9 & 0\\ -0.3 & 0 & \lambda-0.7\end{vmatrix}=k^3-2.2\times k^2+1.49\times k-0.29 \tag{2}$$

得L的特征值为$\lambda_1=0.6+\frac{\sqrt{7}}{10}$，$\lambda_2=0.6-\frac{\sqrt{7}}{10}$和$\lambda_3=1$

根据特征向量求解公式

公式

分别带入三个特征值，解的特征向量为：

```
[(1, 1, [Matrix([
[1],
[1],
[1]])]),
 (3/5 - sqrt(7)/10, 1, [Matrix([
[-sqrt(7)/3 - 1/3],
[-2/3 + sqrt(7)/3],
[                1]])]),
(sqrt(7)/10 + 3/5, 1, [Matrix([
[-1/3 + sqrt(7)/3],
[-sqrt(7)/3 - 2/3],
[                1]])])]
```

$$
\begin{aligned}
\xi_1 &= [\frac{-1+\sqrt{7}}{3}, -\frac{\sqrt{7}+2}{3}, 1]^T \\
\xi_2 &= [-\frac{\sqrt{7}+1}{3}, \frac{-2+\sqrt{7}}{3}, 1]^T \\
\xi_3 &= [1,1,1]^T
\end{aligned} \tag{3}
$$

令$P = [\xi_1, \xi_2, \xi_3]$于是有

$$
\begin{aligned}
L &= P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P^{-1} \\
L^k &= P \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix} P^{-1}
\end{aligned} \tag{4}
$$

从而可以求得解析解。

数值解

由上式，有

$$
\begin{aligned}
\lambda_3 > 0 \wedge \lambda_3 > \lambda_1 > |\lambda_2| \\
\therefore x_0 = \lambda_3^k P \begin{bmatrix} (\lambda_1/\lambda_3)^k & 0 & 0 \\ 0 & (\lambda_2/\lambda_3)^k & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1} x_0
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\because |\frac{\lambda_1}{\lambda_3}| < 1, |\frac{\lambda_2}{\lambda_3}| < 1, \\
\therefore \lim_{k \to \infty} x_k = P \mathrm{diag}(0,0,1) P^{-1} x_0
\end{aligned} \tag{6}
$$

$$
\begin{gathered}
\text{记列向量} P^{-1}x_0 \text{的第三个元素为} c \text{（常数），则上式可化为} \\
\lim_{k \to \infty} \frac{1}{\lambda_1^k} x_k = [\xi_1, \xi_2, \xi_3][0,0,c]^T = c\xi_3
\end{gathered} \tag{7}
$$

于是，当k充分大时，近似地成立

$$
x_k = c\lambda_3^k \xi_3 = c[1,1,1] \tag{8}
$$

其中$c = 4/3$

所以近似解为$x_k = [4/3, 4/3, 4/3]$。

求解程序

Sympy求解析解

```python
import sympy as sp
sp.var('k',positive=True, integer=True)
L = sp.Matrix([
    [sp.Rational(6, 10), sp.Rational(1, 10), sp.Rational(3, 10)],
    [sp.Rational(1, 10), sp.Rational(9, 10), 0],
    [sp.Rational(3, 10), 0, sp.Rational(7, 10)]
])
val = L.eigenvals() #求特征值
```

```python
vec = L.eigenvects() #求特征向量
P, D = L.diagonalize() #把L相似对角化
Lk = P @ (D ** k) @ (P.inv())
F = Lk @ sp.Matrix([2, 1, 1])
x = []
x.append(sp.simplify(F[0]))
x.append(sp.simplify(F[1]))
x.append(sp.simplify(F[2]))
print(x[0])
print(x[1])
print(x[2])
```

Numpy求数值解

```python
import numpy as np
import sympy as sp
X0 = np.array([2, 1, 1])
Ls = sp.Matrix([
    [sp.Rational(6, 10), sp.Rational(1, 10), sp.Rational(3, 10)],
    [sp.Rational(1, 10), sp.Rational(9, 10), 0],
    [sp.Rational(3, 10), 0, sp.Rational(7, 10)]
]) #符号矩阵
sp.var('lamda') #定义符号变量
p = Ls.charpoly(lamda) #计算特征多项式
w1 = sp.roots(p) #计算特征值
w2 = Ls.eigenvals() #直接计算特征值
v = Ls.eigenvects() #直接计算特征向量
print("特征值为：",w2)
print("特征向量为：\n",v)
P, D = Ls.diagonalize() #相似对角化
Pinv = P.inv() #求逆阵
Pinv = sp.simplify(Pinv)
cc = Pinv @ X0
print('P=\n', P)
print('c=', cc[0])
```

k取1000时数值解

```python
import numpy as np
L = np.array([[0.6, 0.1, 0.3], [0.1, 0.9, 0], [0.3, 0, 0.7]])
x = np.array([2, 1, 1]).T

k = 1000
Lk = L
for i in range(k):
    Lk = Lk.dot(L)

print(Lk.dot(x))
```