

20373067 张凯歌

结果

```
1 # numpy 数值解
2 [1.33333333 1.33333333 1.33333333]
```

```
1 # sympy 解析解
2 (-56*100**k/3 + 4*100**k*sqrt(7)/3 - 4*(60 - 10*sqrt(7))**k - sqrt(7)*(60 -
10*sqrt(7))**k - 16*(10*sqrt(7) + 60)**k/3 + 5*sqrt(7)*(10*sqrt(7) +
60)**k/3)/(100**k*(-14 + sqrt(7)))
3 (-112*100**k + 8*100**k*sqrt(7) - 3*sqrt(7)*(60 - 10*sqrt(7))**k + 15*(60 -
10*sqrt(7))**k + sqrt(7)*(10*sqrt(7) + 60)**k + 13*(10*sqrt(7) +
60)**k)/(6*100**k*(-14 + sqrt(7)))
4 (56*100**k*(14 - sqrt(7)) - 63*(1 + sqrt(7))*(60 - 10*sqrt(7))**k + (-7 +
5*sqrt(7))*(14 - sqrt(7))*(10*sqrt(7) + 60)**k)/(42*100**k*(14 - sqrt(7)))
```

求解思路

解析解

$$L = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.1 & 0.9 & 0 \\ 0.3 & 0 & 0.7 \end{bmatrix} \quad (1)$$

欲求 $x_k = L^k x_0$ ，将问题归结为求 L^k 。由

$$|\lambda E - L| = \begin{vmatrix} \lambda - 0.6 & -0.1 & -0.3 \\ -0.1 & \lambda - 0.9 & 0 \\ -0.3 & 0 & \lambda - 0.7 \end{vmatrix} = k^3 - 2.2 \times k^2 + 1.49 \times k - 0.29 \quad (2)$$

得 L 的特征值为 $\lambda_1 = 0.6 + \frac{\sqrt{7}}{10}$ ， $\lambda_2 = 0.6 - \frac{\sqrt{7}}{10}$ 和 $\lambda_3 = 1$

根据特征向量求解公式

公式

分别带入三个特征值，解的特征向量为：

```
1 [(1, 1, [Matrix([
2 [1],
3 [1],
4 [1]])]),
5 (3/5 - sqrt(7)/10, 1, [Matrix([
6 [-sqrt(7)/3 - 1/3],
7 [-2/3 + sqrt(7)/3],
8 [1]])]),
9 (sqrt(7)/10 + 3/5, 1, [Matrix([
10 [-1/3 + sqrt(7)/3],
11 [-sqrt(7)/3 - 2/3],
12 [1]])])]
```

$$\begin{aligned}\xi_1 &= [\frac{-1+\sqrt{7}}{3}, -\frac{\sqrt{7}+2}{3}, 1]^T \\ \xi_2 &= [-\frac{\sqrt{7}+1}{3}, \frac{-2+\sqrt{7}}{3}, 1]^T \\ \xi_3 &= [1, 1, 1]^T\end{aligned}\tag{3}$$

令 $P = [\xi_1, \xi_2, \xi_3]$ 于是有

$$\begin{aligned}L &= P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P^{-1} \\ L^k &= P \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix} P^{-1}\end{aligned}\tag{4}$$

从而可以求得解析解。

数值解

由上式，有

$$\begin{aligned}\lambda_3 > 0 \wedge \lambda_3 > \lambda_1 > |\lambda_2| \\ \therefore x_0 = \lambda_3^k P \begin{bmatrix} (\lambda_1/\lambda_3)^k & 0 & 0 \\ 0 & (\lambda_2/\lambda_3)^k & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1} x_0\end{aligned}\tag{5}$$

$$\begin{aligned}\because |\frac{\lambda_1}{\lambda_3}| < 1, |\frac{\lambda_2}{\lambda_3}| < 1, \\ \therefore \lim_{k \rightarrow \infty} x_k = P \text{diag}(0, 0, 1) P^{-1} x_0\end{aligned}\tag{6}$$

记列向量 $P^{-1}x_0$ 的第三个元素为 c （常数），则上式可化为

$$\lim_{k \rightarrow \infty} \frac{1}{\lambda_1^k} x_k = [\xi_1, \xi_2, \xi_3][0, 0, c]^T = c\xi_3\tag{7}$$

于是，当 k 充分大时，近似地成立

$$x_k = c\lambda_3^k \xi_3 = c[1, 1, 1]\tag{8}$$

其中 $c = 4/3$

所以近似解为 $x_k = [4/3, 4/3, 4/3]$ 。

求解程序

Sympy求解析解

```
1 import sympy as sp
2 sp.var('k', positive=True, integer=True)
3 L = sp.Matrix([
4     [sp.Rational(6, 10), sp.Rational(1, 10), sp.Rational(3, 10)],
5     [sp.Rational(1, 10), sp.Rational(9, 10), 0],
6     [sp.Rational(3, 10), 0, sp.Rational(7, 10)]
7 ])
8 val = L.eigenvals() #求特征值
```

```

9  vec = L.eigenvects() #求特征向量
10 P, D = L.diagonalize() #把L相似对角化
11 Lk = P @ (D ** k) @ (P.inv())
12 F = Lk @ sp.Matrix([2, 1, 1])
13 x = []
14 x.append(sp.simplify(F[0]))
15 x.append(sp.simplify(F[1]))
16 x.append(sp.simplify(F[2]))
17 print(x[0])
18 print(x[1])
19 print(x[2])

```

Numpy求数值解

```

1  import numpy as np
2  import sympy as sp
3  x0 = np.array([2, 1, 1])
4  L = np.array([[0, 4, 3], [0.5, 0, 0], [0, 0.25, 0]])
5  x1 = L @ x0; x2 = L @ x1 #@表示矩阵乘法
6  x3 = L @ x2
7  Ls = sp.Matrix([
8      [sp.Rational(6, 10), sp.Rational(1, 10), sp.Rational(3, 10)],
9      [sp.Rational(1, 10), sp.Rational(9, 10), 0],
10     [sp.Rational(3, 10), 0, sp.Rational(7, 10)]
11 ]) #符号矩阵
12 sp.var('lamda') #定义符号变量
13 p = Ls.charpoly(lamda) #计算特征多项式
14 w1 = sp.roots(p) #计算特征值
15 w2 = Ls.eigenvals() #直接计算特征值
16 v = Ls.eigenvects() #直接计算特征向量
17 print("特征值为: ",w2)
18 print("特征向量为: \n",v)
19 P, D = Ls.diagonalize() #相似对角化
20 Pinv = P.inv() #求逆阵
21 Pinv = sp.simplify(Pinv)
22 cc = Pinv @ x0
23 print('P=\n', P)
24 print('c=', cc[0])

```