20373067-张凯歌-数学建模作业2

习题1：用多种方法求解差分方程

$$\begin{aligned} 2x_{n+2} - x_{n+1} - 2x_n = 0 \\ x_0 = -2 \\ x_1 = 0 \end{aligned} \tag{1}$$

方法1：差分方程的特征根法（该方法来自PPT第四讲）

差分式的特征方程为：

$$2\lambda^2 - \lambda - 2 = 0 \tag{2}$$

解的特征根为：

$$\lambda_1 = \frac{1+\sqrt{17}}{4}, \lambda_2 = \frac{1-\sqrt{17}}{4} \tag{3}$$

所以通解为:

$$x = c_1(\frac{1+\sqrt{17}}{4})^n + c_2(\frac{1-\sqrt{17}}{4})^n \tag{4}$$

代入$x_0 = -2, x_1 = 0$，有:

$$\begin{aligned} c_1 + c_2 = -2 \\ c_1\frac{1+\sqrt{17}}{4} + c_2\frac{1-\sqrt{17}}{4} = 0 \end{aligned} \tag{5}$$

解得c1c2为：

$$c_1 = \frac{\sqrt{17}-17}{17}, c_2 = -\frac{17+\sqrt{17}}{17} \tag{6}$$

所以结果为：

$$x_n = \frac{\sqrt{17}-17}{17} \times (\frac{1+\sqrt{17}}{4})^n - \frac{17+\sqrt{17}}{17} \times (\frac{1-\sqrt{17}}{4})^n \tag{7}$$

方法2：运用特征值和特征向量求通项（该方法来自PPT第三讲第一部分）

首先将二阶差分方程化为一阶差分方程组。

$$\begin{aligned} x_{n+1} = x_{n+1} \\ x_{n+2} = \frac{1}{2}x_{n+1} + x_n \end{aligned} \tag{8}$$

写成矩阵形式：

$$\alpha_{n+1} = A\alpha_n, n = 0, 1, 2, \ldots, \tag{9}$$

其中，

$$A = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}, \alpha_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}, \alpha_0 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \tag{10}$$

递推可得：

$$\alpha_n = A^n\alpha_0, n = 1, 2, 3, \ldots \tag{11}$$

于是求x_n的问题归结为求α_n，即求A^n的问题。由

$$|\lambda E - A| = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda - \frac{1}{2} \end{bmatrix} = \lambda^2 - \lambda - \frac{1}{2} \tag{12}$$

得A的特征值为$\lambda_1=\frac{1-\sqrt{17}}{4},\lambda_2=\frac{1+\sqrt{17}}{4}$

对应λ_1，λ_2的特征向量为

$$\xi_0=\begin{bmatrix}-\frac{1+\sqrt{17}}{4}\\1\end{bmatrix},\xi_2=\begin{bmatrix}-\frac{1-\sqrt{17}}{4}\\1\end{bmatrix} \tag{13}$$

令

$$P=\begin{bmatrix}-\frac{1+\sqrt{17}}{4} & -\frac{1-\sqrt{17}}{4}\\1 & 1\end{bmatrix} \tag{14}$$

于是有

$$A=P\begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}P^{-1},A^n=P\begin{bmatrix}\lambda_1^n & 0\\0 & \lambda_2^n\end{bmatrix}P^{-1} \tag{15}$$

所以

$$\alpha_n=A^n\alpha_0=A^n\begin{bmatrix}-2\\0\end{bmatrix}=\text{见程序解} \tag{16}$$

```
程序解：
2**(1 - 4*k)*(-2**(2*k + 3)*(1 + sqrt(17))**k - 9*(4 - 4*sqrt(17))**k - sqrt(17)*(4 - 4*sqrt(17))**k)/(sqrt(17) + 17)
```

程序解

```python
#hw2-1-feature.py
import sympy as sp
k = sp.var('k',positive=True, integer=True)
a = sp.Matrix([[0, 1], [1, sp.Rational(1, 2)]])
val = a.eigenvals()                    #求特征值
print(val)
vec = a.eigenvects()                   #求特征向量
P, D = a.diagonalize()                 #把a相似对角化
ak = P @ (D ** k) @ (P.inv())
F = ak @ sp.Matrix([1, 1])
s = sp.simplify(F[0])
print(s)
```

$$x_n=\frac{2^{1-4n}\left(-2^{2n+3}\left(1+\sqrt{17}\right)^n-9\left(4-4\sqrt{17}\right)^n-\sqrt{17}\left(4-4\sqrt{17}\right)^n\right)}{\sqrt{17}+17} \tag{17}$$

结果应该和方法1和3相同。

方法3：生成函数法（由于我上学期选了离散三，所以这是来自离散三的方法）

设数列$x_0,x_1,x_2,\ldots,x_n,\ldots$的生成函数是：

$$g(t)=x_0+x_1t+x_2t^2+\ldots+x_nt^n+\ldots \tag{18}$$

用$-t,-2t^2$乘以生成函数得：

$$\begin{aligned}2g(t)&=2x_0+2x_1t+2x_2t^2+\ldots+2x_nt^n+..\\-tg(t)&=\qquad -x_0t-x_1t^2-\ldots-x_{n-1}t^n+\ldots\\-2t^2g(t)&=\qquad\qquad -2x_0t^2-2x_1t^3-2x_2t^4-\ldots-2x_{n-2}t^n+\ldots\end{aligned} \tag{19}$$

将三式相加得：

$$2g(t)-tg(t)-2t^2g(t)=(2-t-2t^2)g(t)=2x_0+2x_1t-x_0t=-4+2t \tag{20}$$

化简得：

$$
\begin{aligned}
g(t) &= \frac{-4+2t}{2-t-2t^2} \\
&= \frac{-4+2t}{-2(t-\frac{1+\sqrt{17}}{-4})(t-\frac{1-\sqrt{17}}{-4})} \\
&= -\frac{1}{2}(\frac{c_1}{t-\frac{1+\sqrt{17}}{-4}}+\frac{c_2}{t-\frac{1-\sqrt{17}}{-4}})
\end{aligned} \tag{21}
$$

所以有：

$$
\begin{aligned}
c_1+c_2=2 \\
c_1\frac{1-\sqrt{17}}{-4}+c_2\frac{1+\sqrt{17}}{-4}=4
\end{aligned} \tag{22}
$$

解得：

$$
\begin{aligned}
c_1 &= \frac{17+9\sqrt{17}}{17} \\
c_2 &= \frac{17-9\sqrt{17}}{17}
\end{aligned} \tag{23}
$$

所以带入$g(t)$表达式有：

$$
g(t) = -\frac{1}{2}(\frac{\frac{17+9\sqrt{17}}{17}}{t-\frac{1+\sqrt{17}}{-4}}+\frac{\frac{17-9\sqrt{17}}{17}}{t-\frac{1-\sqrt{17}}{-4}}) \tag{24}
$$

因为$\frac{1}{1-ax}=1+ax+a^2x^2+\ldots+a^nx^n+\ldots$，所以对$g(t)$进行变换：

$$
\text{令}a_1=\frac{-4}{1+\sqrt{17}}, a_2=\frac{-4}{1-\sqrt{17}}
$$

$$
\text{则}
$$

$$
\begin{aligned}
g(t) &= -\frac{1}{2}(\frac{\frac{17+9\sqrt{17}}{17}}{t-\frac{1+\sqrt{17}}{-4}}+\frac{\frac{17-9\sqrt{17}}{17}}{t-\frac{1-\sqrt{17}}{-4}}) \\
&= -\frac{1}{2}(\frac{c_1}{t-1/a_1}+\frac{c_2}{t-1/a_2}) \\
&= -\frac{1}{2}(\frac{c_1a_1}{a_1t-1}+\frac{c_2a_2}{a_2t-1}) \\
&= -\frac{1}{2}(-\frac{c_1a_1}{1-a_1t}-\frac{c_2a_2}{1-a_2t}) \\
&= -\frac{1}{2}[(-c_1a_1)\times(\ldots+a_1^n\times t^n+\ldots)+(-c_2a_2)\times(\ldots+a_2^n\times t^n+\ldots)]
\end{aligned} \tag{25}
$$

对其进行幂级数展开后，t^n前的系数为：

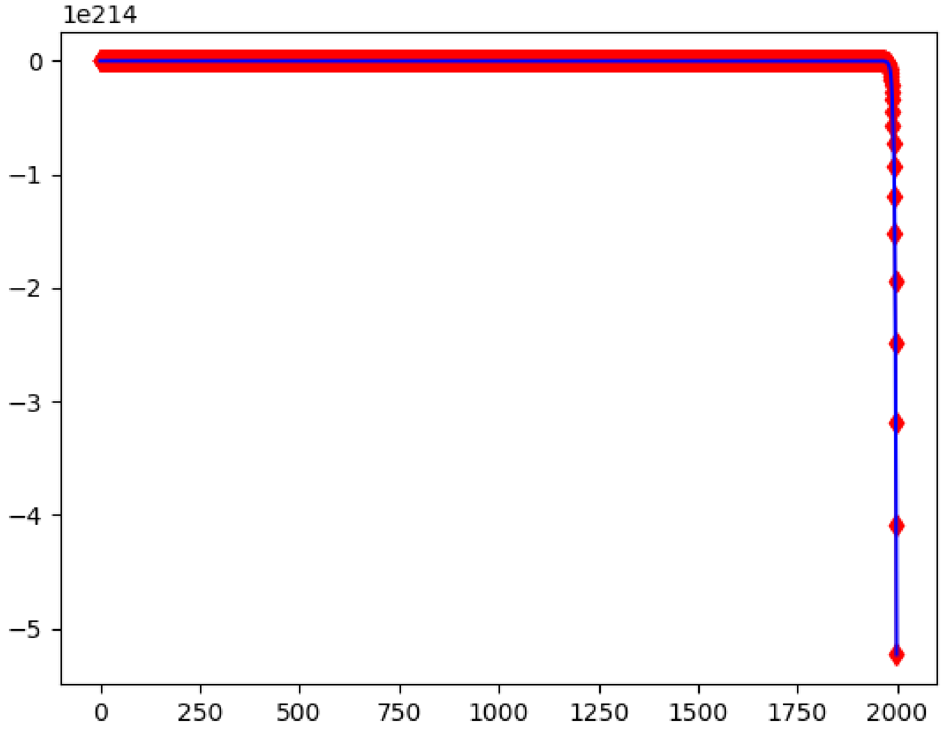
$$
\begin{aligned}
x_n &= -\frac{1}{2}[(-\frac{17+9\sqrt{17}}{17}\times\frac{-4}{1+\sqrt{17}})\times(\frac{-4}{1+\sqrt{17}})^n+(-\frac{17-9\sqrt{17}}{17}\times\frac{-4}{1-\sqrt{17}})\times(\frac{-4}{1-\sqrt{17}})^n] \\
&= -\frac{1}{2}[(-\frac{17+9\sqrt{17}}{17}\times\frac{1-\sqrt{17}}{4})\times(\frac{1-\sqrt{17}}{4})^n+(-\frac{17-9\sqrt{17}}{17}\times\frac{1+\sqrt{17}}{4})\times(\frac{1+\sqrt{17}}{4})^n] \\
&= -\frac{17+\sqrt{17}}{17}\times(\frac{1-\sqrt{17}}{4})^n+\frac{\sqrt{17}-17}{17}\times(\frac{1+\sqrt{17}}{4})^n
\end{aligned} \tag{26}
$$

与方法一所求结果一致。

数值解： 主要采用的方法是带入验证解析解和直接递归差分结果是否一致。

```python
# hw2-1.py
import matplotlib.pyplot as plt
import numpy as np
# 数值差分解
x = [-2, 0]
for i in range(2, 2000):
    x.append(0.5 * x[i - 1] + x[i - 2])
```

```
n = range(0, 2000)
c1 = 'r'
plt.plot(n, x, c1, linewidth=1, marker='d')

# 解析解
x2 = (np.sqrt(17)-17)/17*((1+np.sqrt(17))/4)**n-(17+np.sqrt(17))/17*((1-np.sqrt(17))/4)**n
c2 = 'b'
plt.plot(n, x2, 'b')
plt.show()
```



可以看出数值解和解析解结果是十分吻合的。

习题2：求下列微分方程的解析解和数值解，画出曲线图，范围[0,1]

$$\begin{gathered}\frac{\partial x}{\partial t}=x-2y\\ \frac{\partial y}{\partial t}=x+2y\\ x(0)=1,y(0)=0\end{gathered}\tag{27}$$

由(17-1)，可得：

$$y=\frac{1}{2}x-\frac{1}{2}\frac{\partial x}{\partial t}\tag{28}$$

左右对t求偏导得：

$$\frac{\partial y}{\partial t}=\frac{1}{2}\frac{\partial x}{\partial t}-\frac{1}{2}\frac{\partial^2 x}{\partial t^2}\tag{29}$$

将上式带入题目第二个公式得：

$$\frac{1}{2}\frac{\partial x}{\partial t}-\frac{1}{2}\frac{\partial^2 x}{\partial t^2}=2x-\frac{\partial x}{\partial t}\tag{30}$$

化简得：

$$\frac{\partial^2 x}{\partial t^2} - 3\frac{\partial x}{\partial t} + 4x = 0 \tag{31}$$

特征方程为:

$$\lambda^2 - 3\lambda + 4 = 0 \tag{32}$$

解的特征根为:

$$\lambda_{1,2} = \frac{3}{2} \pm \frac{\sqrt{7}}{2} i \tag{33}$$

因此x的通解为:

$$x = (c_1 \sin(\frac{\sqrt{7}}{2}t) + c_2 \cos(\frac{\sqrt{7}}{2}t))e^{3t/2} \tag{34}$$

两边同时对t偏导得:

$$\frac{\partial x}{\partial t} = \frac{3}{2}(c_1 \sin(\frac{\sqrt{7}}{2}t) + c_2 \cos(\frac{\sqrt{7}}{2}t))e^{3t/2} + (\frac{\sqrt{7}}{2}c_1 \cos(\frac{\sqrt{7}}{2}t) - \frac{\sqrt{7}}{2}c_2 \sin(\frac{\sqrt{7}}{2}t))e^{3t/2} \tag{35}$$

带入$y = \frac{1}{2}x - \frac{1}{2}\frac{\partial x}{\partial t}$解得y的通解为:

$$y = -\frac{1}{4}(c_1 \sin(\frac{\sqrt{7}}{2}t) + c_2 \cos(\frac{\sqrt{7}}{2}t))e^{3t/2} - \frac{1}{2}(\frac{\sqrt{7}}{2}c_1 \cos(\frac{\sqrt{7}}{2}t) - \frac{\sqrt{7}}{2}c_2 \sin(\frac{\sqrt{7}}{2}t))e^{3t/2} \tag{36}$$

带入$x(0) = 1, y(0) = 0$有:

$$\begin{aligned} c_2 &= 1 \\ -\frac{1}{4}c_2 - \frac{1}{2}(\frac{\sqrt{7}}{2}c_1) &= 0 \end{aligned} \tag{37}$$

所以:

$$\begin{aligned} c_1 &= -\frac{\sqrt{7}}{7} \\ c_2 &= 1 \end{aligned} \tag{38}$$

带入解得:

$$\begin{aligned} x &= (-\frac{\sqrt{7}}{7}\sin(\frac{\sqrt{7}}{2}t) + \cos(\frac{\sqrt{7}}{2}t))e^{3t/2} \\ y &= \frac{2\sqrt{7}}{7}\sin(\frac{\sqrt{7}}{2}t)e^{3t/2} \end{aligned} \tag{39}$$

使用python求解:

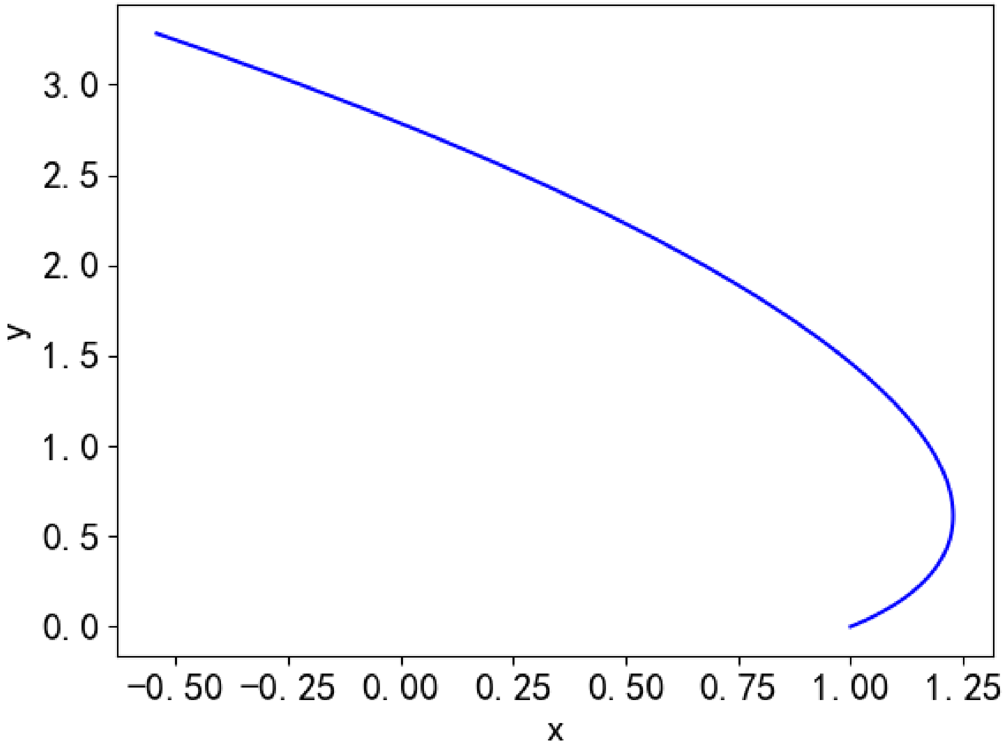
```
# test.py
import sympy as sp
from sympy import Eq, Derivative

x = sp.symbols('x', cls=sp.Function)
y = sp.symbols('y', cls=sp.Function)
t = sp.symbols('t')

eq = (Eq(Derivative(x(t), t, 1), x(t) - 2 * y(t)), Eq(Derivative(y(t), t, 1), x(t) + 2 * y(t)))
result = sp.dsolve(eq, ics={y(0): 0, x(0): 1})
print(result)
```

结果为

```
[
    Eq(x(t), -sqrt(7)*exp(3*t/2)*sin(sqrt(7)*t/2)/7 + exp(3*t/2)*cos(sqrt(7)*t/2)),
    Eq(y(t), 2*sqrt(7)*exp(3*t/2)*sin(sqrt(7)*t/2)/7)
]
```

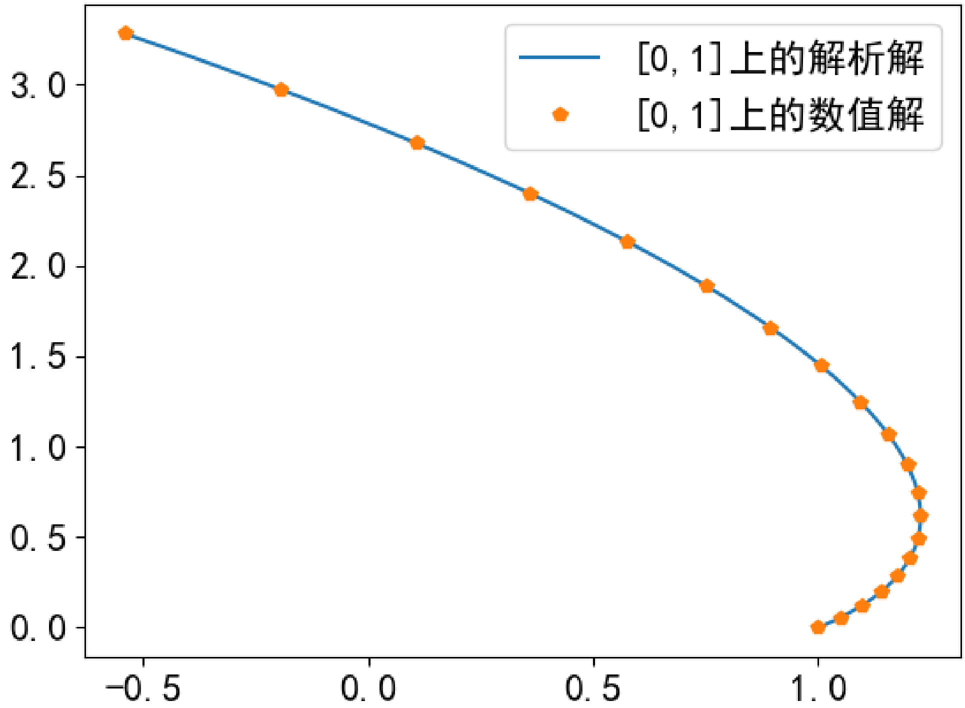
使用python画图

```python
# hw2plot.py
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp
from sympy import Eq, Derivative

x = sp.symbols('x', cls=sp.Function)
y = sp.symbols('y', cls=sp.Function)
t = sp.symbols('t')

plt.rc('font', family='SimHei')
plt.rc('axes', unicode_minus=False)
plt.rc('font', size=16)

eq = (Eq(Derivative(x(t), t, 1), x(t) - 2 * y(t)), Eq(Derivative(y(t), t, 1), x(t) + 2 * y(t)))
result = sp.dsolve(eq, ics={y(0): 0, x(0): 1})
result1 = sp.lambdify(t, result[0].args[1], 'numpy')
result2 = sp.lambdify(t, result[1].args[1], 'numpy')
tt = np.linspace(0, 1, 101)
plt.plot(result1(tt), result2(tt), 'b')
plt.xlabel('x')
plt.ylabel('y') # x, y 轴添加标签

x_major_locator = plt.MultipleLocator(0.25)
ax = plt.gca()
ax.xaxis.set_major_locator(x_major_locator)

# plt.plot(tt, result2(tt), 'r')
plt.show()
```



使用PPT的常微分方程组求数值解：

```
# hw2-2.py
import sympy as sp
import numpy as np
from sympy import Eq, Derivative
import matplotlib.pyplot as plt
from scipy.integrate import odeint

plt.rc('font', family='SimHei')
plt.rc('axes', unicode_minus=False)
plt.rc('font', size=16)

x = sp.symbols('x', cls=sp.Function)
y = sp.symbols('y', cls=sp.Function)
t = sp.symbols('t')

eq = (Eq(Derivative(x(t), t, 1), x(t) - 2 * y(t)), Eq(Derivative(y(t), t, 1), x(t) + 2 *
y(t)))
result = sp.dsolve(eq, ics={y(0): 0, x(0): 1})
result1 = sp.lambdify(t, result[0].args[1], 'numpy')
result2 = sp.lambdify(t, result[1].args[1], 'numpy')
tt = np.linspace(0, 1, 101)
plt.plot(result1(tt), result2(tt))
# plt.plot(tt, result2(tt), )
ttt = np.linspace(0, 1, 20)
df = lambda f, t2: [f[0] - 2 * f[1], f[0] + 2 * f[1]]
s1 = odeint(df, [1, 0], ttt)
print(s1[:, 0])
plt.plot(s1[:, 0], s1[:, 1], 'p')
plt.legend(['[0,1]上的解析解', '[0,1]上的数值解'])
plt.show()

```



$$\frac{2^{1-4k}\left(-2^{2k+3}\left(1+\sqrt{17}\right)^{k}-9\left(4-4\sqrt{17}\right)^{k}-\sqrt{17}\left(4-4\sqrt{17}\right)^{k}\right)}{\sqrt{17}+17} \tag{40}$$