Computing k-Centers of Uncertain Points on a Real Line

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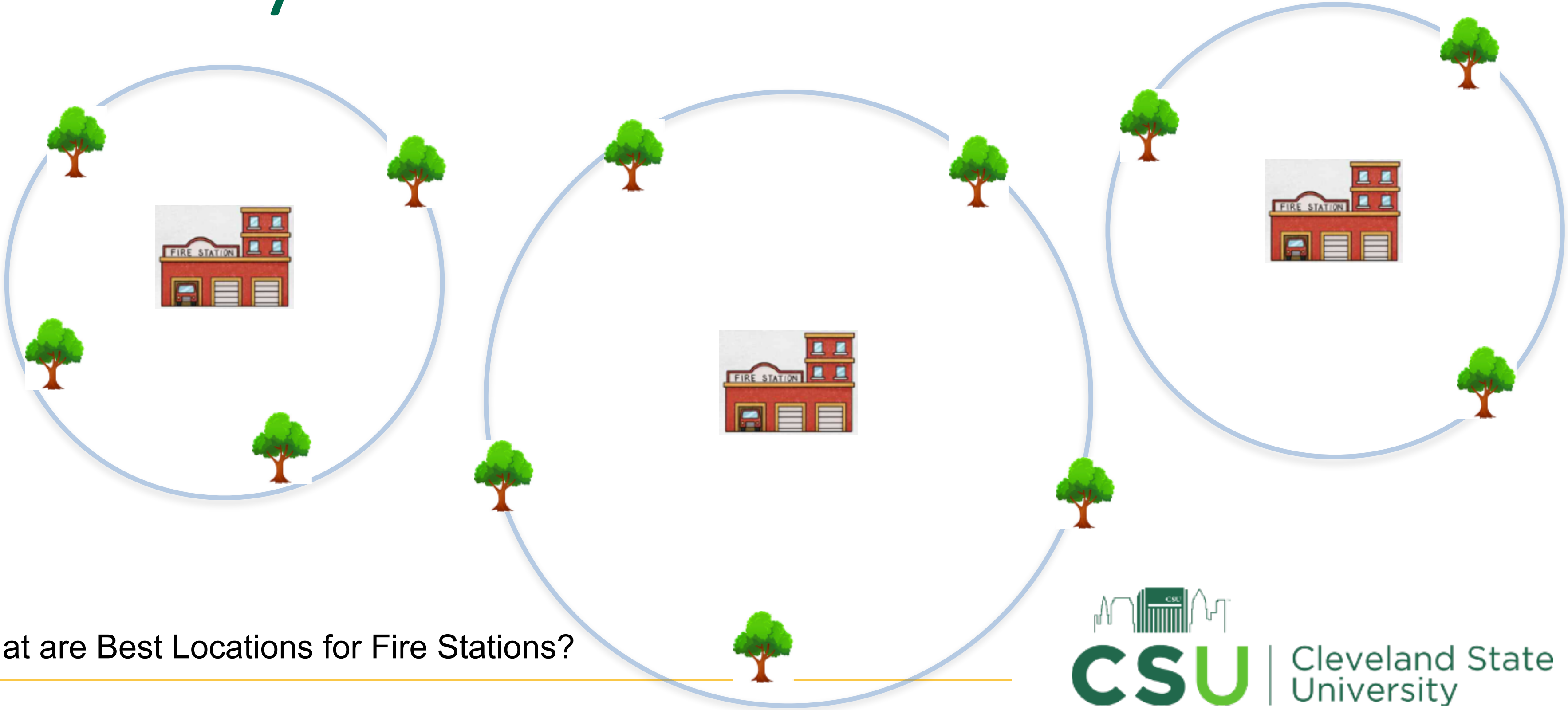
Facility Locations



What are Best Locations for Fire Stations?

The k-Center Problem

Input: n points $p_1, p_2, \cdots, p_n$

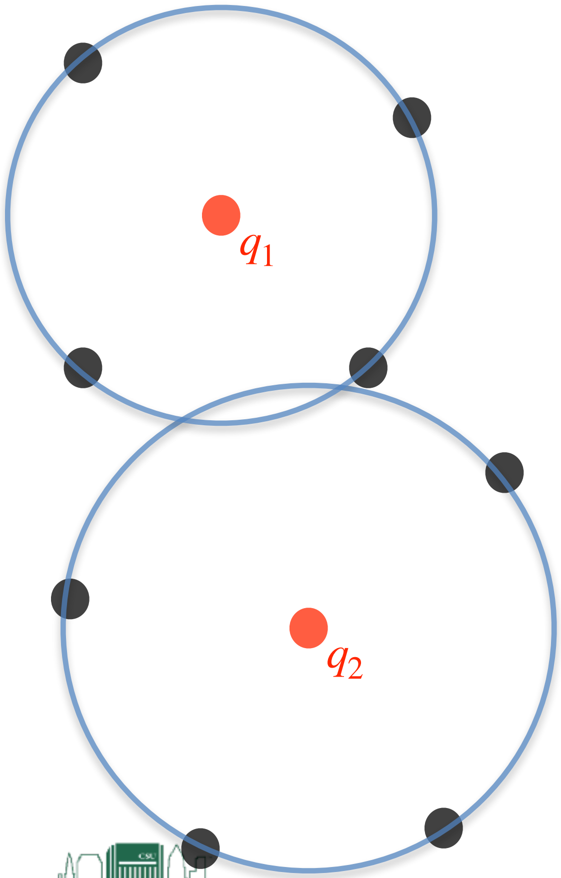
Output: k points (centers) $Q = \{q_1, \cdots, q_k\}$ to

minimize

$$\max_{1 \le i \le n} w_i \cdot d(p_i, Q)$$

where

$$d(p_i, Q) = \min_{q \in Q} d(p_i, q)$$



Uncertain Data

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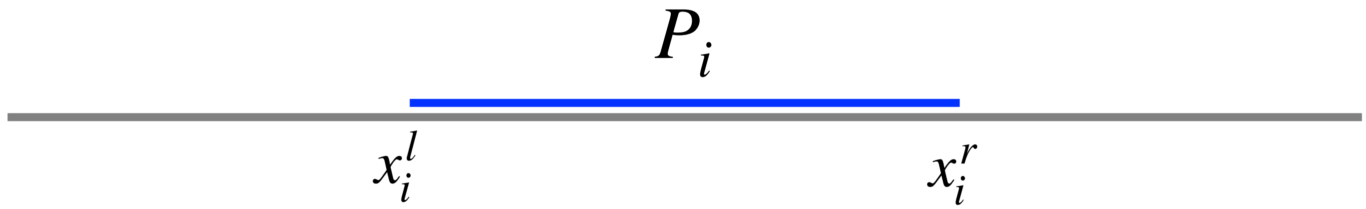
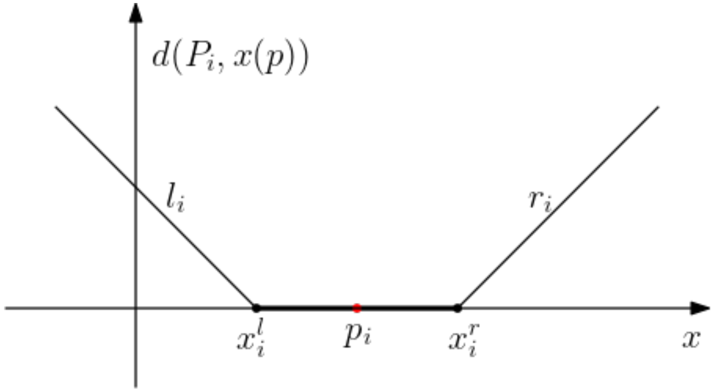
An Uncertain Point on a Line

An uncertain point P_i is associated with a closed segment so that it could appear at any point of this segment without any specified probability density function.

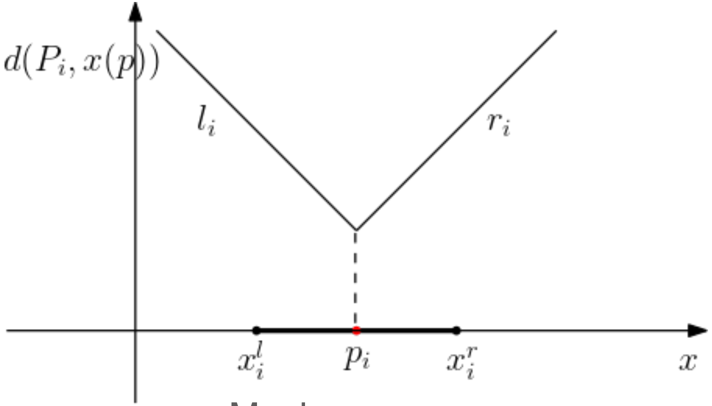


Two Measures for the Distance

The segment appearance of $P_i \in P$ defines a minimum (luckiest) measure and a maximum (unluckiest) measure for distance $d(P_i, x(p))$ of P_i to any point p.



Minimum measure



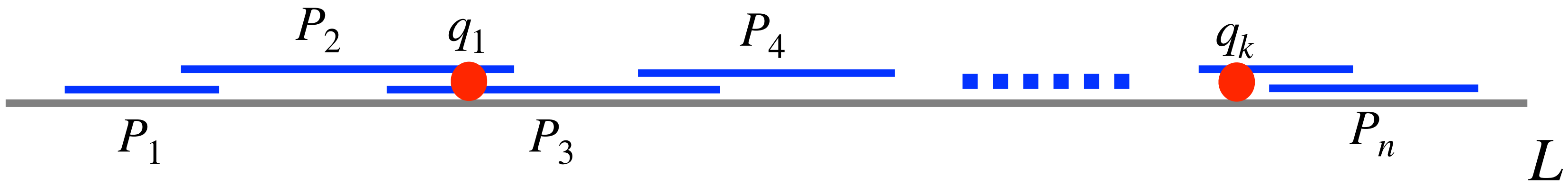
Maximum measure

Our k-Center Problem

Input: a set P of n uncertain points $\{P_1, P_2, \cdots, P_n\}$on a line L

Output: A set $Q = \{q_1, \cdots, q_k\}$ to minimize $\max_{1 \le i \le n} w_i \cdot d(p_i, Q)$ where

$d(p_i, Q) = \min_{q \in Q} d(p_i, q)$ under either measure.



Related Work

The deterministic one-dimensional k-center problem

Time: $O(n \log n)$—Chen et al, 2015

No previous work for our problems

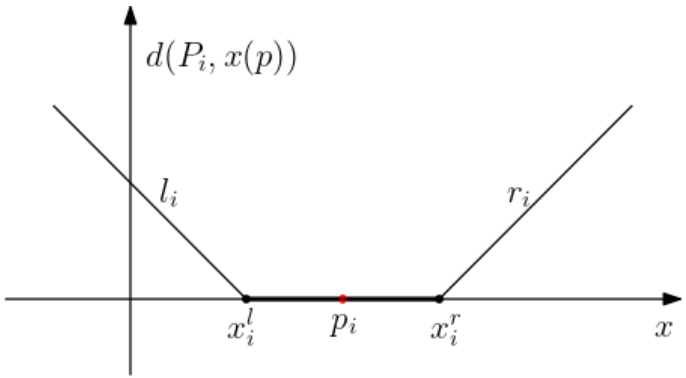
Our Results

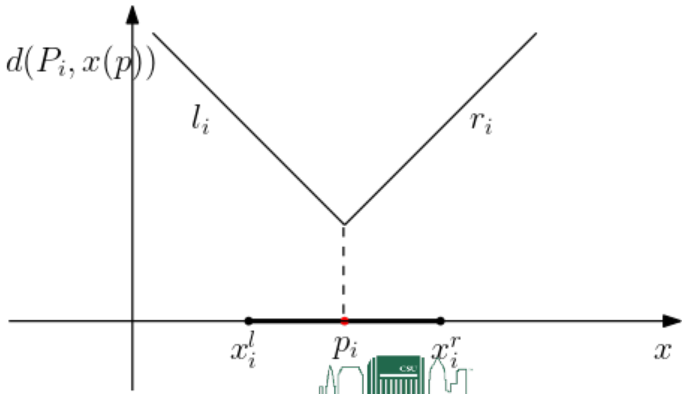
- The weighted k-center problem:
- Our result: $O(n \log n)$
- The unweighted k-center problem:
- Our result: $O(n)$

Ran Hu and Jingru Zhang. Computing k-Centers of Uncertain Points on a Real Line. Operations Research Letters, vol. 50, pages 310-314, 2022.

An Observation

For any uncertain point $P_i \in P$, distance $d(P_i, x(p))$ under both the minimum and maximum measures reaches the minimum at the midpoint p_i of P_i's appearance segment.





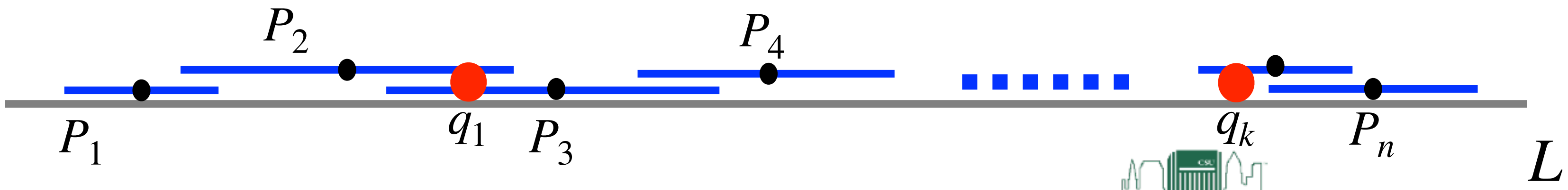
Our k-Center Problem

Input: a set P of n uncertain points $\{P_1, P_2, \cdots, P_n\}$on a line L

Output: A set $Q = \{q_1, \cdots, q_k\}$ to minimize $\max_{1 \leq i \leq n} w_i \cdot d(p_i, Q)$ where $d(p_i, Q) = \min_{q \in Q} d(p_i, q)$

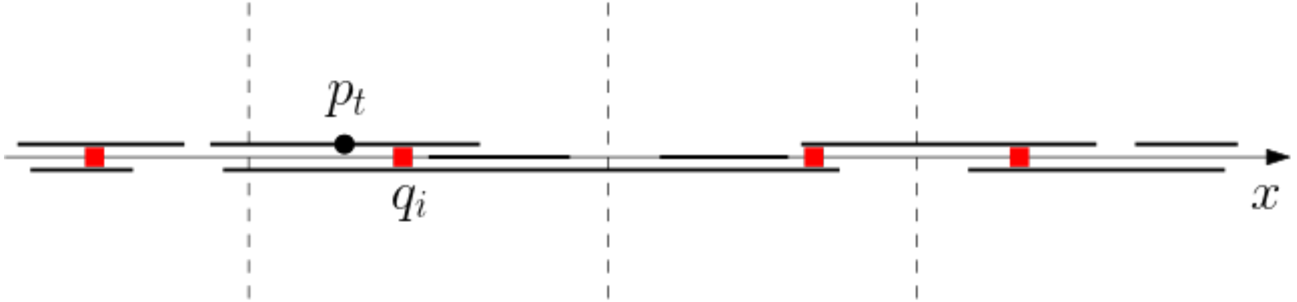
under either measure.

$P_1, P_2, \cdots, P_n$ are given sorted by the midpoints of their appearance segments.



A Key Lemma

For the k-center problem under each distance measure, there must exist an optimal solution in which the uncertain points of P served by the same facility are consecutive in their index order.

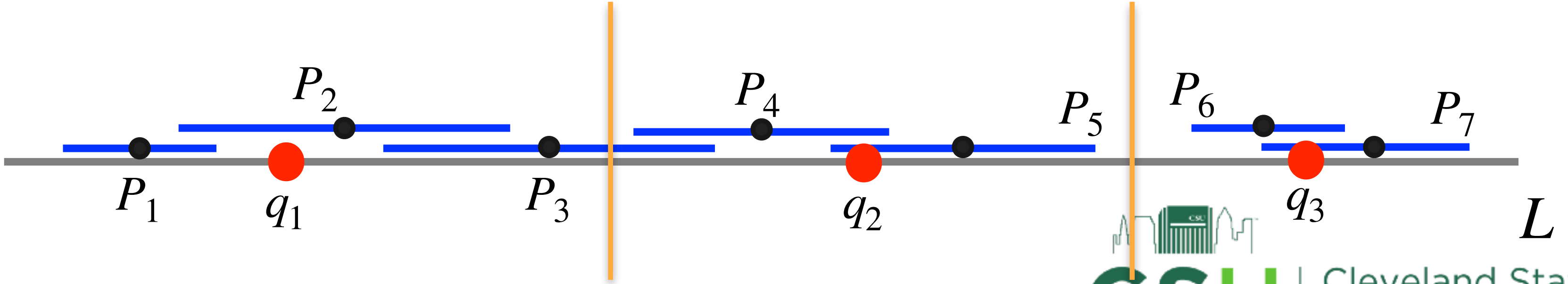


Reduction into a Min-Max Partition Problem

Input: a set P of n uncertain points $P_1, P_2, \cdots, P_n$ on a line L

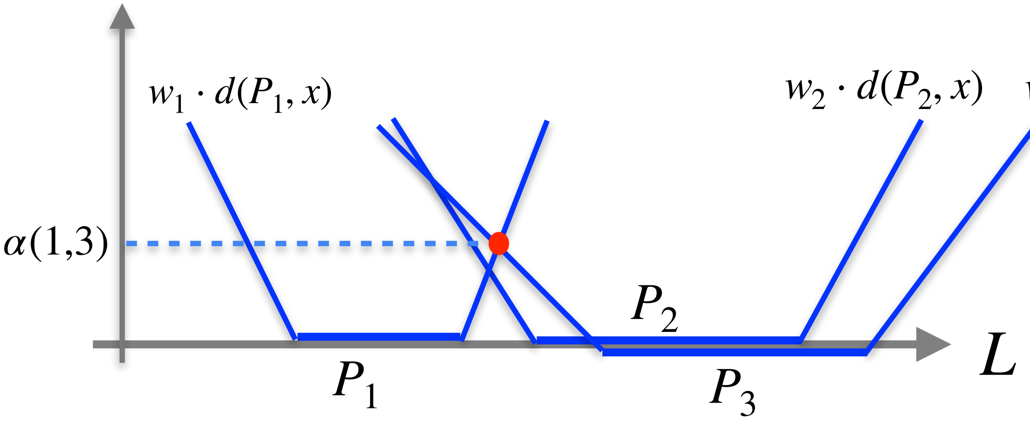
Output: Computing k+1 integers $i_0 = 1,\ i_1, \cdots,\ i_{k-1},\ i_k = n$ in order to minimize $\max\limits_{0 \le t \le k-1} \alpha(i_t, i_{t+1})$

where $\alpha(i_t, i_{t+1}) = \min\limits_{x \in L} \max\limits_{i_t \le j \le i_{t+1}} w_j \cdot d(P_j, x)$



The Candidate Set for ϵ^*

Observation: ϵ^* is decided by the **y-coordinate** of an intersection of distance functions $w_i \cdot d(P_i, x)$ for all $1 \leq i \leq n$.



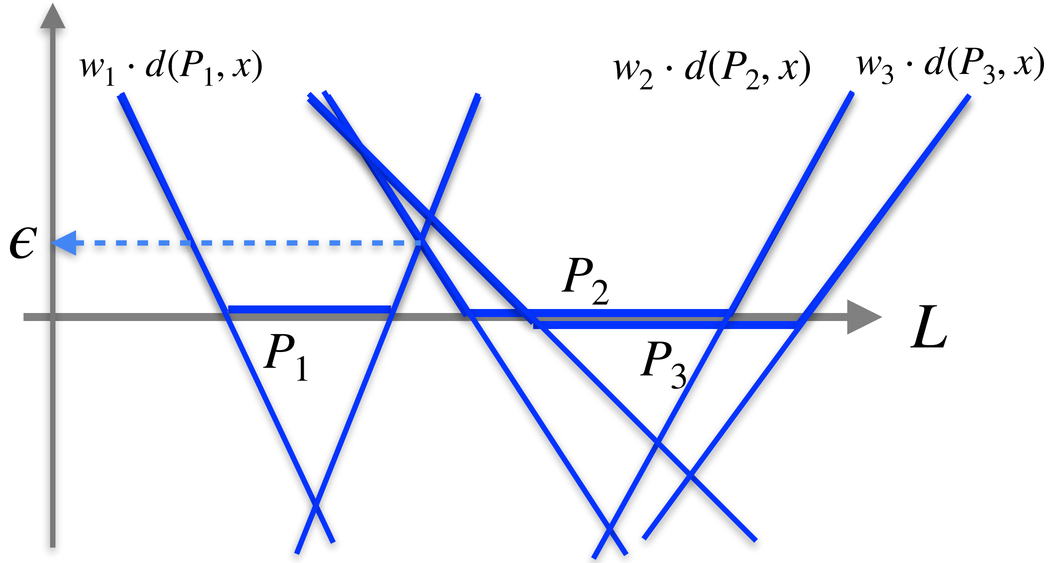
Our $O(n \log n)$ Algorithm for Computing ϵ^*

Determining $w_i \cdot d(P_i, x)$ for all $1 \leq i \leq n$

Extending half-lines on all $w_i d(P_i, x)$ into lines

Forming a line arrangement A of lines

Applying the line arrangement search technique to search ϵ^* among vertices of A with the assistance of our decision algorithm



Our decision algorithm decides in $O(n)$ time whether a given ϵ is feasible or not.

The Decision Problem

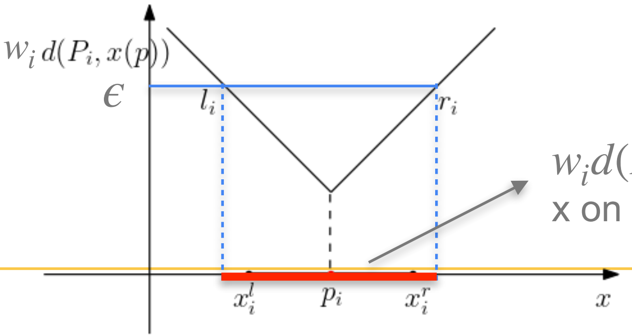
Input: A set P of n uncertain points $P_1, P_2, \cdots, P_n$ on a line L and a value $\epsilon > 0$

Goal: Deciding if at most *k* centers $Q = \{q_1, \cdots, q_k\}$ exist so that $\max_{1 \le i \le n} w_i \cdot d(p_i, Q) \le \epsilon$

Yes $\Rightarrow$ $\epsilon \ge \epsilon^*$

No $\Rightarrow$ $\epsilon < \epsilon^*$

Interval Piercing Problem

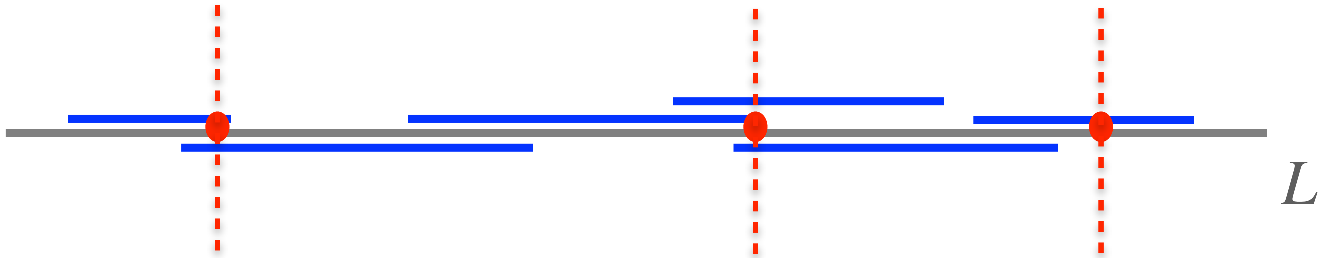


$w_i d(P_i, x) \le \epsilon$ for any x on the red interval

Interval Piercing Problem

Input: A set of n intervals on a line

Output: The minimum points piercing all intervals



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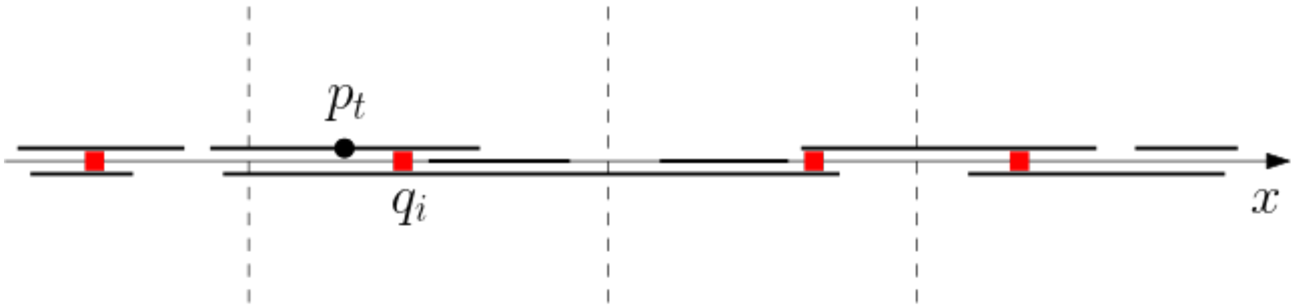
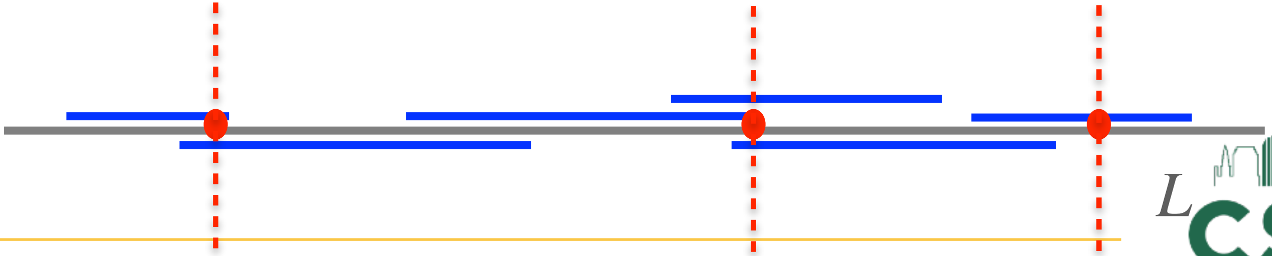


Interval Piercing Problem

Input: A set of n intervals on a line

Output: The minimum points piercing all intervals

Our decision algorithm solves it in $O(n)$ time



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The Unweighted Case

Input: a set P of n uncertain points $\{P_1, P_2, \cdots, P_n\}$on a line L

Output: A set $Q = \{q_1, \cdots, q_k\}$ to minimize $\max_{1 \le i \le n} d(p_i, Q)$ where $d(p_i, Q) = \min_{q \in Q} d(p_i, q)$

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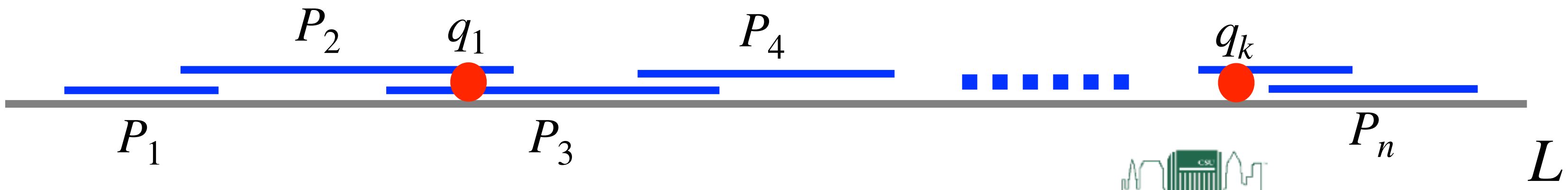

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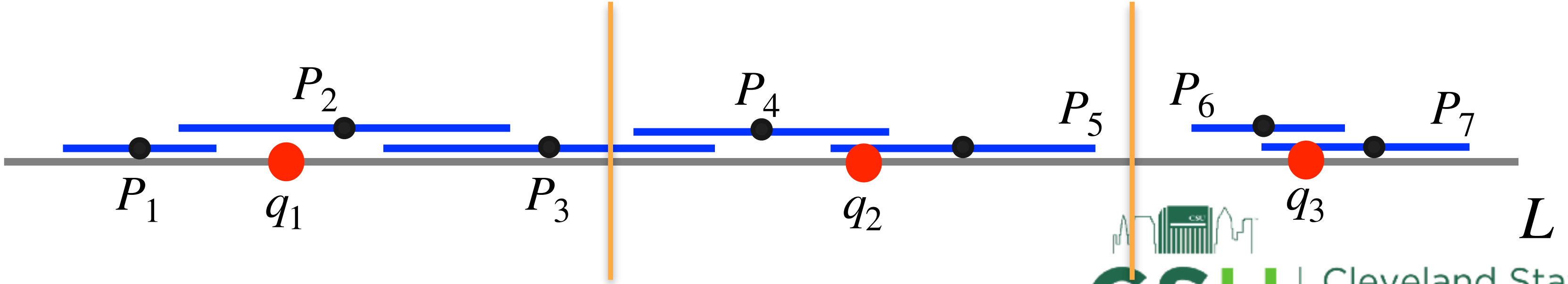

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Solving the Min-Max Partition Problem

Input: a set P of n uncertain points $P_1, P_2, \cdots, P_n$ on a line L

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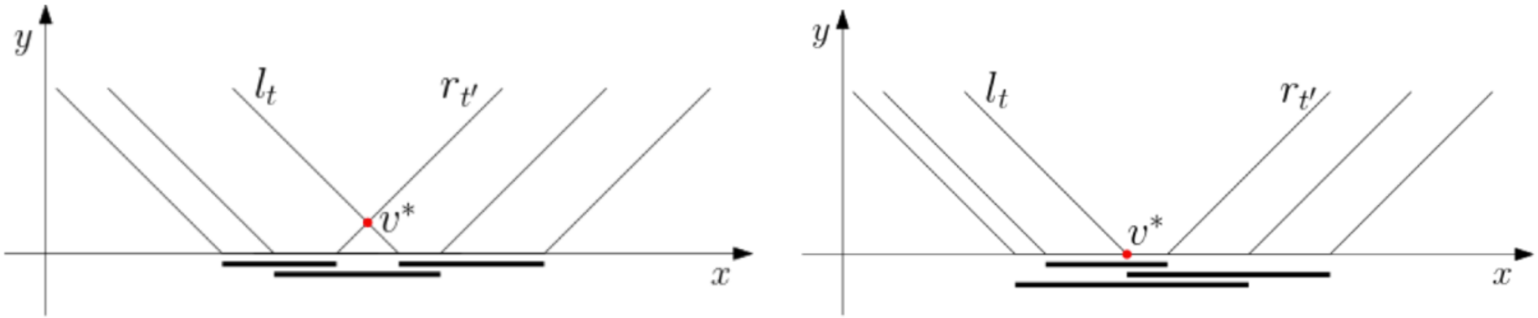
where $\alpha(i_t, i_{t+1}) = \min_{x \in L} \max_{i_t \le j \le i_{t+1}} d(P_j, x)$

The Min-Max Partition problem can be solved in $O(n\tau)$ time where τ is the time of computing $\alpha(i, j)$ for any query $i \le j$.

Our Data Structures

Lemma 4. With $O(n)$ preprocessing work, for any query $i \leq j$, we can compute in constant time $\alpha(i, j)$ under each distance measure.



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Thank You!
Q&A

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