6. Special relativity and electromagnetism

> Just a brief introduction, no worries.

We have spacetime four vector $(ct, \mathbf{r})$, who has dot product with itself as

$$x^\mu x_\mu = c^2t^2 - r^2$$

$$\begin{aligned}\tau^2 &= t^2\left(1-\frac{x^2+y^2+z^2}{c^2t^2}\right)\\ &= t^2\left(1-\frac{v^2}{c^2}\right)\\ \Rightarrow \tau &= \frac{t}{\gamma}\\ \gamma &= \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\\ t &= \gamma\tau \quad \text{time dilation}\end{aligned}$$

We also have momentum four vector $(\frac{E}{c}, \mathbf{p})$, who has dot product with itself as

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = m^2c^2$$

$$\text{since} E^2 = p^2c^2 + m^2c^2 \quad (m = \text{rest mass})$$

We also have differential operator ∂_μ which is written as

$$\partial_\mu = \left(\tfrac{1}{c}\tfrac{\partial}{\partial t}, \nabla\right)$$

We would just introduce this here

Then, we could have current density four vector

$$J_\mu = (c\rho, \mathbf{J})$$

We would consider charge at rest, then

$$J^\mu = (c\rho, 0)$$

This could be "boosted" to a moving frame with speed v

$$J'^\mu = (\rho' c, \mathbf{J}')$$

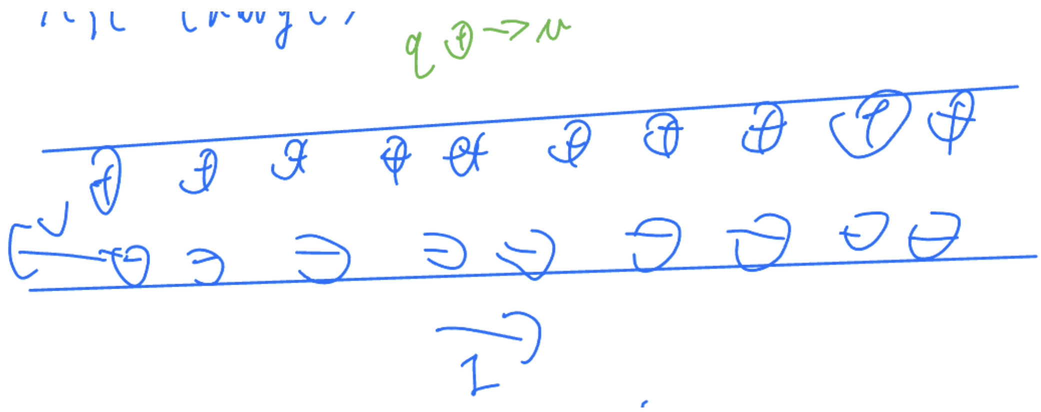
Where $\rho' = \gamma\rho_0$ due to length contraction and $\mathbf{J}' = -\gamma\rho_0\mathbf{v}$

Then, we could have continuity equation

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} &= 0\\ \partial_\mu J^\mu &= 0\end{aligned}$$

In lab frame S, then, we could have

- Line of charge, with density $+\lambda$ which is stationary
- Line of charge, with density $-\lambda$ which is moving with speed $-v$
- Test charge q moving with speed $+u$



remember that the wire itself has no net charge

$\lambda_{\text{tot}} = +\lambda - \lambda = 0$

In the test charge frame S′, we could have

- Test charge is stationary
- + charges move backwards at speed u$\Rightarrow \gamma_+ = \gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$
- - charges move back at $v' = \frac{v+u}{1+\frac{uv}{c^2}} \Rightarrow \gamma_- = \frac{1}{\sqrt{1-\frac{c^2(v+u)^2}{(c^2+uv)^2}}} = \frac{c^2+uv}{\sqrt{(c^2-v^2)(c^2-u^2)}} = \frac{c^2+uv}{\sqrt{(c^2-v^2)(c^2-u^2)}} = \gamma_v \partial_u (1 + \frac{vu}{c})$

Hence $\gamma'_{\text{tot}} = \gamma_+\lambda - \gamma_-\left(-\frac{\lambda}{\gamma_v}\right)$

In the rest frame of negative charges, they have charge density of $\frac{-\lambda}{\gamma_v}$

$$\begin{aligned}
\lambda'_{\text{tot}} &= \gamma_u\lambda - \lambda\gamma_u \cdot \left(1 + \frac{uv}{c^2}\right) \\
&= -\lambda\gamma_u\frac{uv}{c^2} \\
E' &= \frac{\lambda'_{\text{tot}}}{2\pi\epsilon_0 r} = \frac{-\lambda uv\gamma_u}{2\pi\epsilon_0 c^2 r} \\
F' &= qE' = \frac{-\lambda uv\gamma_u q}{2\pi\epsilon_0 c^2 r} \\
F &= \frac{F'}{\gamma_u} = \frac{-(\lambda v)uq}{2\pi\epsilon_0 c^2 r} \quad \text{note that} \quad \lambda v = I \\
&= -qu\left(\frac{\mu_0 I}{2\pi r}\right) \quad \text{note that} \quad \frac{\mu_0 I}{2\pi r} = B \\
&= -quB \quad \text{Which is lorentz force}
\end{aligned}$$

Lets remember Maxwell's equations

$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$E = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$
$$\text{If} \quad V \to V - \frac{\partial \chi}{\partial t}$$
$$\text{and} \quad \mathbf{A} \to \mathbf{A} + \nabla \chi$$

Where χ is a scalar field

$$A^\mu = (\frac{V}{c}, \mathbf{A})$$
$$A_\mu = (\frac{V}{c}, -\mathbf{A})$$
$$A_\mu \to A_\mu - \partial_\mu \chi \quad \text{gauge transformation}$$

We would, therefore, need a new object called the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Where μ, ν are indices

$$F_{\mu\nu} \to F_{\mu\nu} - \partial_\mu \partial_\nu \chi + \partial_\nu \partial_\mu \chi = F_{\mu\nu}$$
$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$
$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_z & 0 \end{pmatrix}$$

And the function $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ gives us the four maxwell's equations. The first half would be simple to deduce, but the second half would be a bit more complicated so we are not going to do it here.

First two:

$$\partial_i F^{i0} = \mu_0 J^0$$
$$\Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\partial_\mu F^{\mu i} = \mu_0 J^i$$
$$\Rightarrow \frac{1}{c^2}\mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$