

2. Electric polarization

Matter is not free space, it contains atoms

2.1 Bound charges

Electric field can induce a dipole moment in a neutral atom

Example 4.1. A primitive model for an atom consists of a point nucleus ($+q$) surrounded by a uniformly charged spherical cloud ($-q$) of radius a (Fig. 4.1). Calculate the atomic **polarizability** of such an atom.

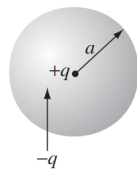


FIGURE 4.1

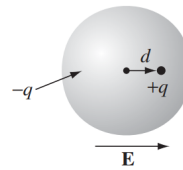


FIGURE 4.2

And the electric moment is defined as $\mathbf{p} = \alpha \mathbf{E}$ where α is the **polarizability** of the atom. A list of polarizability can be seen from the handout.

If, for example, the outside $E \sim 10^6$ V/m, then the displacement would be about 10^{-15} m, which is much less the size of an atom.

Also, there are polar molecules which already have a dipole moment, but they are randomized because of the present of inner energy $K_B T$

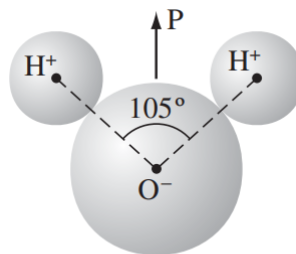


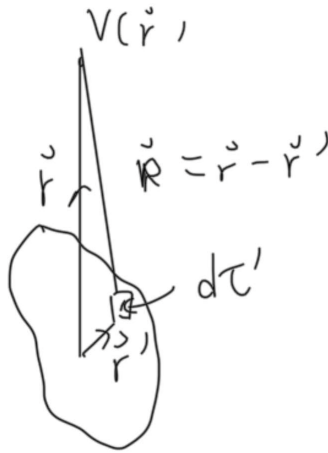
FIGURE 4.4

However, an E field can line them up

For a box of aligned dipoles (per unit volume), define the polarization \mathbf{P} as the dipole moment per unit volume as $\mathbf{P} = n\mathbf{p}$

If they are not aligned, then the polarization is zero.

Question: what is the electric potential produced by a box of dipoles?



$$\begin{aligned}
 V(\mathbf{R}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{P}(\mathbf{r}) \cdot \mathbf{R}}{|\mathbf{R}|^3} d^3\tau' \\
 &= \frac{1}{4\pi\epsilon_0} \int \mathbf{P}(\mathbf{r}) \cdot \nabla \frac{1}{R} d\tau' \\
 \nabla \frac{1}{R} &= -\frac{\mathbf{R}}{R^3} \\
 \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \\
 \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}
 \end{aligned}$$

Note that $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$

$$\begin{aligned}
 \Rightarrow V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int \nabla' \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{R}|} d\tau' - \frac{1}{4\pi\epsilon_0} \int \frac{1}{R} \nabla \cdot \mathbf{P}(\mathbf{r}') d\tau' \\
 &= \frac{1}{4\pi\epsilon_0} \int_{s'} \frac{\sigma_b dS'}{R} + \frac{1}{4\pi\epsilon_0} \int_{\tau'} \frac{\rho_b d\tau'}{R}
 \end{aligned}$$

where $\sigma_b = \mathbf{P} \cdot \mathbf{n}$ is the surface bounded charge density and $\rho_b = -\nabla \cdot \mathbf{P}$ is the volume bound charge density (the minus sign is because \mathbf{P} is defined as the dipole moment per unit volume).

The derivation can be seen from Griffiths' book page 176.

Remember to drop primes!

In bulk, Gauss's Law is $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$, but $\rho = \rho_{free} + \rho_{bound}$ remember that $\rho_{bound} = -\nabla \cdot \mathbf{P}$ and hence $\nabla \cdot \mathbf{E} = \frac{\rho_{free}}{\epsilon_0} - \frac{\nabla \cdot \mathbf{P}}{\epsilon_0}$

We define electric displacement

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\Rightarrow \nabla \cdot \mathbf{D} = \rho_{free}$$

That is much easier, since it only depends on free charges.

The integral form is

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{free}$$

Linear dielectrics

For a linear dielectric, $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ where χ_e is the electric susceptibility, which is dimensionless.

$$\Rightarrow \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

$(1 + \chi_e)$ is the relative permittivity ϵ_r and it is dimensionless.

Example:

Field from a point charge in a linear dielectric:

$$\begin{aligned} \int_s \mathbf{D} \cdot d\mathbf{s} &= Q_{free} \\ \Rightarrow \mathbf{D} &= \frac{Q_f}{4\pi r^2} \hat{\mathbf{r}} \\ \Rightarrow \mathbf{E} &= \frac{Q_f}{4\pi \epsilon_0 r^2} \hat{\mathbf{r}} \end{aligned}$$

Recap of lecture 5:

Polarization: $\vec{P} = n\vec{p}$

Bound charges: $\rho_b = -\nabla \cdot \vec{P}$, $\sigma_b = \vec{P} \cdot \hat{n}$

Electric displacement: $\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \epsilon_r \vec{E}$

Maxwell: $\nabla \cdot \vec{D} = \rho_f$

Linear dielectric: $\vec{P} = \epsilon_0 \chi_e \vec{E}$

Where χ_e is the electric susceptibility = $\epsilon_r - 1$

Question Has D solved everything?

-- No. The div equation is easier, but...

Reminder: $\vec{\nabla} \cdot \vec{E} = \frac{\rho_{tot}}{\epsilon_0}$

$\vec{\nabla} \cdot \vec{D} = \rho_f$

And the equation for \vec{D} is $\epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \epsilon_r \vec{E}$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times \vec{D} = \epsilon \vec{\nabla} \times \vec{E} + \vec{\nabla} \times \vec{P} \neq 0$$

Boundary conditions

$E_{//}$ is continuous

$$\vec{\nabla} \times \vec{E} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

But $D_{//}$ is not continuous (if $\nabla \times \vec{P} \neq 0$ at interface)

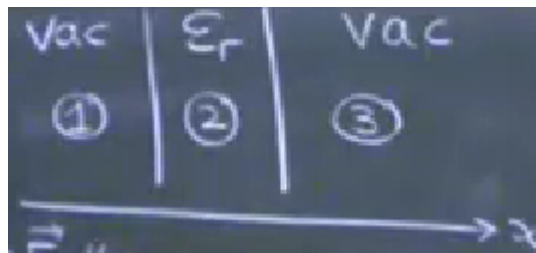
D_{\perp} is continuous

$$\nabla \cdot \vec{D} = \rho_f$$

$$\oint \vec{D} \cdot d\vec{a} = Q_f$$

but E_{\perp} may not be continuous if we have bound charges at the interface

Example



$$\Rightarrow E_0 // x$$

$$E_1 = E_3 = E_0$$

D_{\perp} is continuous

$$\Rightarrow D_1 = D_3 = \epsilon_0 E_0 = D_2$$

$$D_2 = \epsilon_0 \epsilon_r E_2$$

$$\Rightarrow E_2 = \frac{E_0}{\epsilon_r}$$

So there is a smaller E field in the dielectric

$$\begin{aligned} P_2 &= \epsilon_0 (\epsilon_r - 1) E_2 \\ &= \frac{\epsilon_0}{\epsilon_r} (\epsilon_r - 1) E_0 \end{aligned}$$

$$(P_1 = P_3 = 0)$$

Since P_2 is constant, $\nabla \times \vec{P} = 0 \Rightarrow \rho_b = 0$

$$z = 0, \sigma_b = \mathbf{P} \cdot \mathbf{n} = \frac{\epsilon_0}{\epsilon_r} (\epsilon_r - 1) E_0$$

$$z = d, \sigma_b = \mathbf{P} \cdot \mathbf{n} = \frac{\epsilon_0}{\epsilon_r} (\epsilon_r - 1) E_0$$

Example:

There is dielectric sphere, with radius a and relative permittivity ϵ_r . The sphere is in a uniform electric field \mathbf{E}_0 . Find V .

We got to use Laplace's equation

General solution to Laplace's equation in spherical coordinates (with azimuthal symmetry) is

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

where P_l is the Legendre polynomial.

Solve separately for $r > a$ and $r < a$

As $r \rightarrow \infty$, $V \rightarrow -E_0 r \cos \theta$

As $r \rightarrow 0$, V should be finite (not blow up)

$\Rightarrow B_l = 0$ for $r < a$

case $r < a$:

$$V = V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

And for $r > a$:

$$V = V_{out} = -E_0 r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

As for $r = a$, $E_{//}$, D_{\perp} , V are continuous

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) = -E_0 a \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos \theta)$$

$$\Rightarrow A_l = \begin{cases} \frac{B_l}{a^{l+1}} & l \neq 1 \\ -E_0 + \frac{B_1}{a^2} & l = 1 \end{cases}$$

$$E_0 = -\frac{1}{r} \frac{\partial V}{\partial \theta} \text{ is continuous}$$

\rightarrow generates same expressions

That is not terribly useful, so we will try another one

$$D_{\perp} = -\epsilon_0 \epsilon_r \frac{\partial V}{\partial r} \text{ is continuous}$$

$$\Rightarrow -\epsilon_0 \epsilon_r \frac{\partial V}{\partial r} \Big|_{r=a} = -\epsilon_0 \frac{\partial V_{out}}{\partial r} \Big|_{r=a}$$

$$\epsilon_r \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos \theta) = -E_0 \cos \theta + \sum_{l=0}^{\infty} -(l+1) \frac{B_l}{a^{l+2}} P_l(\cos \theta)$$

$$A_l = \begin{cases} -\frac{1}{\epsilon_r} \frac{l+1}{l} \frac{B_l}{a^{2l+1}} & l \neq 1 \\ \frac{1}{\epsilon_r} \left(-E_0 - \frac{2B_1}{a^3} \right) & l = 1 \end{cases}$$

Hence, we got two expressions for A in terms of l . They can't both be right unless

$$A_l = B_l = 0 \text{ for } l \neq 1$$

On the other hand, we have

$$A_1 = \underbrace{-E_0 + \frac{B_1}{a^3}}_{\textcircled{1}} = \underbrace{\frac{1}{\epsilon_r} \left(-E_0 - \frac{2B_1}{a^3} \right)}_{\textcircled{2}}$$

$$\epsilon_r (\textcircled{1} - \textcircled{2}) = 0$$

Hence

$$(1 - \epsilon_r)E_0 + (\epsilon_r + 2)\frac{B_1}{a^3} = 0$$

$$\Rightarrow B_1 = a^3 E_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)$$

$$\rightarrow A_1 = -\frac{3E_0}{\epsilon_r + 2}$$

Now, put everything together

$$V = \begin{cases} \underbrace{\frac{3}{\epsilon_r + 2} E_0 r \cos \theta}_{\text{uniform field}} & r < a \\ \underbrace{-E_0 r \cos \theta}_{\text{uniform field}} + \underbrace{\left(\frac{E_r - 1}{E_r + 2} \frac{E_0 a^3 \cos \theta}{r^2} \right)}_{\text{dipole field}} & r > a \end{cases}$$

Interestingly, we all have uniform fields applied inside and outside. And if in vacuum, we will have uniform field outside (dipole field vanishes)

P inside sphere:

$$P = \epsilon_0(\epsilon_r - 1)E_{in}$$

where E_{in} is $\frac{3E_0}{\epsilon_r + 2}$

Dipole moment of sphere

$$p = \frac{4}{3}\pi a^3 P = \frac{4\pi a^3 \epsilon_0 (\epsilon_r - 1) E_0}{\epsilon_r + 2}$$

Polarizability

$$\begin{aligned} \alpha &= \frac{P}{E_0} \\ &= 4\pi a^3 \epsilon_0 \frac{\epsilon_r - 1}{\epsilon_r + 2} \end{aligned}$$

Bound charge on the surface

$$\begin{aligned} \sigma_b &= \mathbf{P} \cdot \hat{\mathbf{n}} \\ &= P \cos \theta \\ &= \frac{3\epsilon_0(\epsilon_r - 1)}{\epsilon_r + 2} E_0 \cos \theta \end{aligned}$$