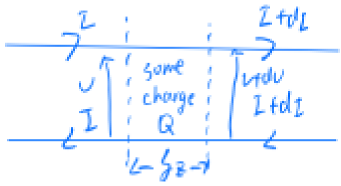
5. Confined EM waves

5.1 Transmission lines

> An example of guided wave

Lets think in a super long circuit

If the time for signal to transverse the circuit is not $\ll \frac{1}{\omega}$, we need to consider the wave behavior of the signal.



We could have

> Remember: we are defining capacitance as C per unit length , and inductance L per unit length

$$
\begin{aligned}
Q &= (C\delta z)v \\
\frac{dQ}{dt} &= C\delta z \frac{\partial v}{\partial t} \\
&= I(z,t) - I(z+dz,t) \\
&= \delta z \left(\frac{-\partial I}{\partial z} \right) \\
\Rightarrow \frac{\partial I}{\partial z} &= -C\frac{\partial v}{\partial t} \qquad \text{equation 1} \\
\Phi &= (L\delta z)I \\
\frac{d\Phi}{dt} &= L\delta z \frac{\partial I}{\partial t} \\
&= V(z,t) - V(z+\delta z,t) \\
&= \delta z \left(-\frac{\partial V}{\partial z} \right) \\
\Rightarrow \frac{\partial V}{\partial z} &= -L\frac{\partial I}{\partial t} \qquad \text{equation 2}
\end{aligned}
$$

If we plug equation 1 into equation 2, we get

$\frac{\partial^2 V}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V}{\partial z^2}$

For wavelike situations, we have

$V(z,t) = f(z-vt) + g(z+vt)$

where f and g are arbitrary functions, and v is the wave velocity which is $\frac{1}{\sqrt{LC}}$.

> Note: the wave velocity is not the speed of light, but the speed of the wave in the circuit.

From equation 1, wee could have
$\frac{dV}{dI} = \frac{1}{C\frac{dz}{dt}} = \frac{1}{C\frac{1}{\sqrt{LC}}} = \frac{1}{\sqrt{\frac{L}{C}}}$

The impedance would be, then $Z = \pm\sqrt{\frac{L}{C}}$ where thr $\pm$ is the direction of the wave.

Instantaneous power

$$= V\underbrace{(\frac{1}{2}LI^2 + \frac{1}{2}CV^2)}_{\text{energy stored/length}}$$

$$= \frac{1}{2}I^2z + \frac{1}{2}\frac{V^2}{z} = I^2z = \frac{LC}{z} = IV$$

<u>Example</u>

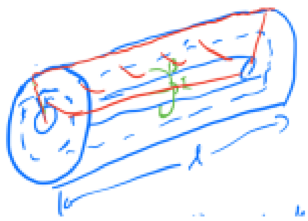
We have coaxial transmission line, which has

$C = \frac{Q}{V}$

> Remember: C is per unit length

We would have

$E \cdot 2\pi rl = \frac{Q}{\epsilon_r\epsilon_0}$



Hence, we have

$$V = -\int_b^a \frac{Q}{2\pi\epsilon_0\epsilon_r}dr = \frac{Q}{2\pi\epsilon_0\epsilon_r}\ln\frac{b}{a}$$
$$\Rightarrow C = \frac{2\pi\epsilon_r\epsilon_0}{\ln\frac{b}{a}}$$

$B \cdot 2\pi r = \mu_r\mu_0 I$

$$\Phi = frac\Phi I = \frac{\mu_r\mu_0}{2\pi}\ln\frac{b}{a}$$

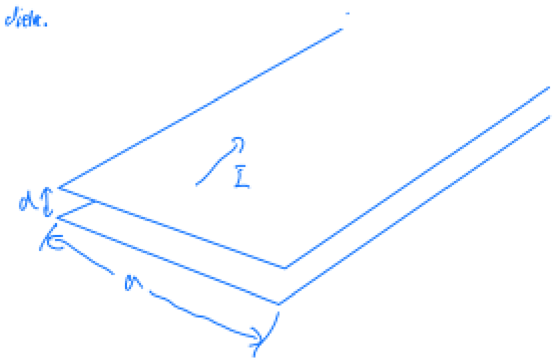
Where, remember, Φ is the flux per unit length

$$\Rightarrow \sigma = \frac{1}{\sqrt{LC}} = \underbrace{\frac{1}{\sqrt{\mu_r\mu_0\epsilon_r\epsilon_0}}}_{\text{Speed of light in dielectric}}$$

$$Z = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu_r\mu_0}{\epsilon_0\epsilon_r}\frac{\ln\frac{b}{a}}{2\pi}}$$

Then, a common type of transmission line:

<u>Example</u>: Strip transmission line



$a \gg d$ so we could ignore edge effect

We have

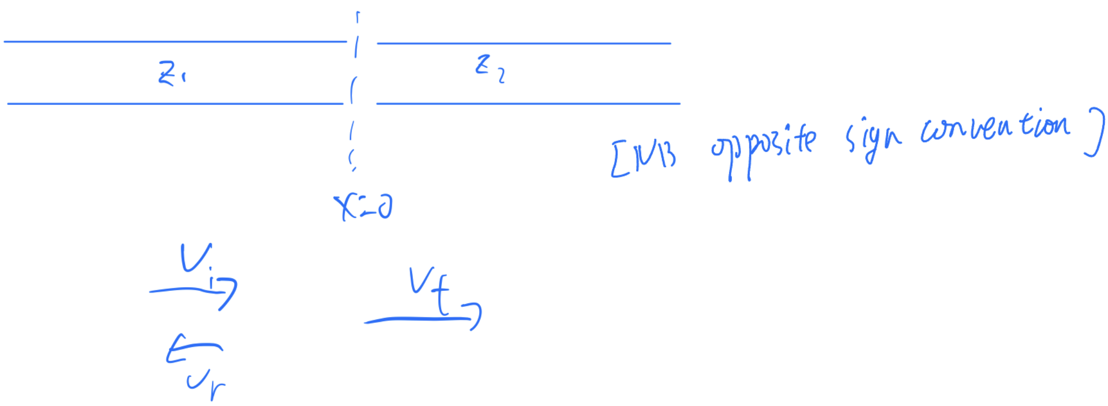
$$C = \frac{\epsilon_0 \epsilon_r a}{d}$$

$$L = \frac{\mu_0 \mu_r d}{a}$$

$$\sigma = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_0 \mu_r \epsilon_0 \epsilon_r}} \quad \text{as before}$$

$$z = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r} \frac{d}{a}}$$

Boundrary between two transmission lines



For x<0, we have

$$V_1 = V_i e^{i(\omega t - kx)} + V_r e^{i(\omega t + kx)}$$

For x>0, we have

$$V_2 = V_t e^{i(\omega t - k_2 x)}$$

At x = 0, we need to match voltages

$$\Rightarrow V_i + V_r = V_t \quad (1)$$

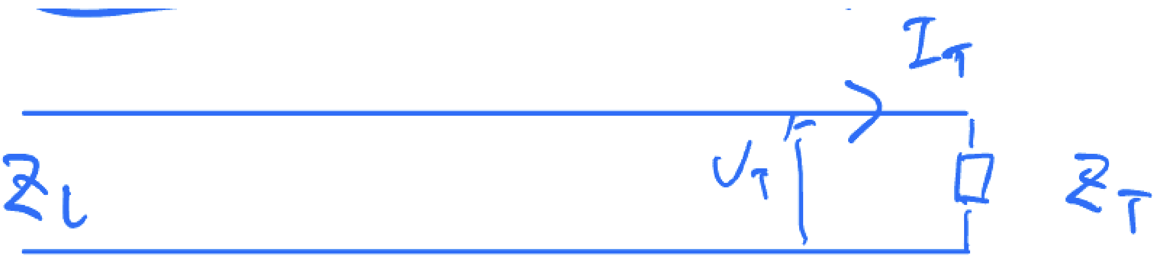
Match currents, we have

$$\frac{V_i - V_r}{Z_1} = \frac{V_t}{Z_2} \quad (2)$$

From (1) and (2), we have $r = \frac{Z_2 - Z_1}{Z_2 + Z_1}, t = \frac{2Z_2}{Z_2 + Z_1}$

As for power, we would have $\frac{V_i^2}{Z_1} = \frac{V_t^2}{Z_2} + \frac{V_r^2}{Z_1}$ as expected.

Termination of load



$\frac{V_T}{I_T} = Z_T$ is the boundary condition

$r = \frac{Z_T - Z_c}{Z_T + Z_c} \quad t = \frac{2Z_T}{Z_T + Z_c}$

We would also like to consider special cases

	Z_T	r	t	V_T	I_T
Short circuit	0	-1	0	0	$2I_i$
Open circuit	∞	1	2	$2V_i$	0
Matched	Z_c	0	1	V_i	I_i

When $Z_T = Z_c$, we would get maximum power transfer

proof

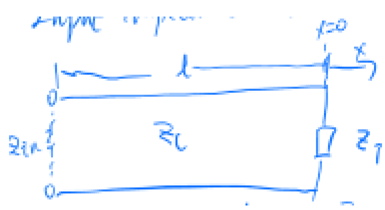
Incident power $= \frac{\langle V_i^2 \rangle}{Z_c} = \frac{1}{2}\frac{V_i^2}{Z_c}$

Power on the load $= \frac{\langle V_T^2 \rangle}{Z_T} = \frac{1}{2}\frac{V_T^2}{Z_T} = \frac{1}{2}\frac{4Z_T V_T^2}{(Z_T+Z_c)^2} \quad (3)$

Hence,

$$\frac{\text{Power transmitted}}{\text{incident power}} = \frac{(3)}{(4)}$$
$$= \frac{4Z_c Z_T}{(Z_T + Z_c)^2}$$
$$\frac{d}{dZ_T}\left[\frac{Z_T}{(Z_T - Z_c)^2}\right] = 0$$
$$\frac{(Z_T + Z_c)^2 \cdot 1 - 2(Z_T + Z_c)Z_T}{(Z_T + Z_c)^2} = 0$$
$$\Rightarrow (Z_T + Z_c) \cdot (Z_T + Z_c - 2Z_T) = 0$$
$$\Rightarrow Z_T = Z_c$$

Input impedance of short sections



Input impedance

$$
\begin{aligned}
Z_{in} &= \frac{V(-l)}{I(-l)} \\
&= \frac{V_i e^{ikl} + V_r e^{-ikl}}{\frac{V_i}{Z_T} e^{ikl} - \frac{V_r}{Z_c} e^{-ikl}}
\end{aligned}
$$

$$
\text{and } V_r = rV_i \quad \text{where} \quad r = \frac{Z_T - Z_c}{Z_T + Z_c}
$$

$$
\Rightarrow Z_{in} = \left(\frac{Z_T \cos(kl) + iZ_c \sin(kl)}{Z_c \cos(kl) + iZ_T \cos(kl)} \right) Z_c
$$

$$
\begin{aligned}
&\text{For a } \frac{\lambda}{2} \text{ line } (kl = \pi) \\
&\qquad \Rightarrow Z_{in} = Z_T \\
&\text{For a } \frac{\lambda}{4} \text{ line } (kl = \frac{\pi}{2}) \\
&\qquad \Rightarrow Z_{in} = \frac{Z_c}{Z_T}
\end{aligned}
$$

If Z_T = 0 $\Rightarrow Z_{in} = iZ_c \tan(kl)$
If Z_T = $\infty \Rightarrow Z_{in} = -iZ_c \cot(kl)$

For a $\frac{\lambda}{4}$ line, we have $Z_{in} = \frac{Z_c^2}{Z_T}$

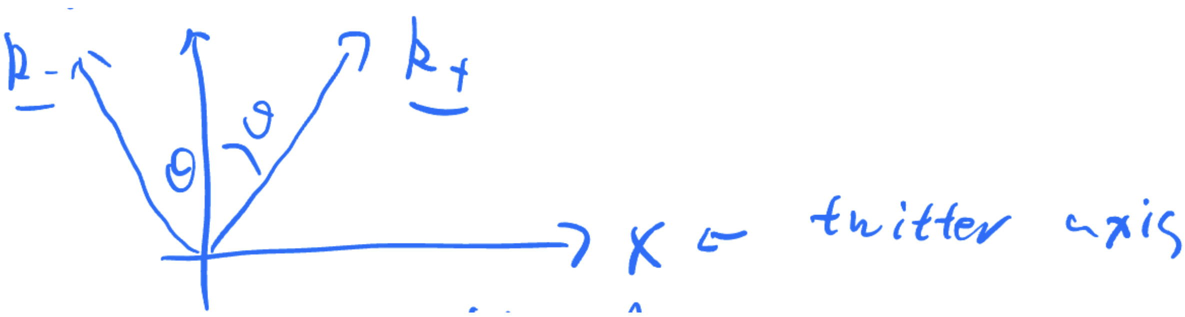
Choose $\frac{Z_c'^2}{Z_T} = Z_c$
i.e. $Z_c' = \sqrt{Z_c Z_t}$
The transmission line has been perfectly terninated and there is no reflected wave

Waveguides

<u>Example of interfering EM waves</u>

Consider 2 EM waves with $E \parallel y$, travelling along $\mathbf{k}_\pm = \mathbf{k}_0(\pm \sin\theta, 0 \cos\theta)$

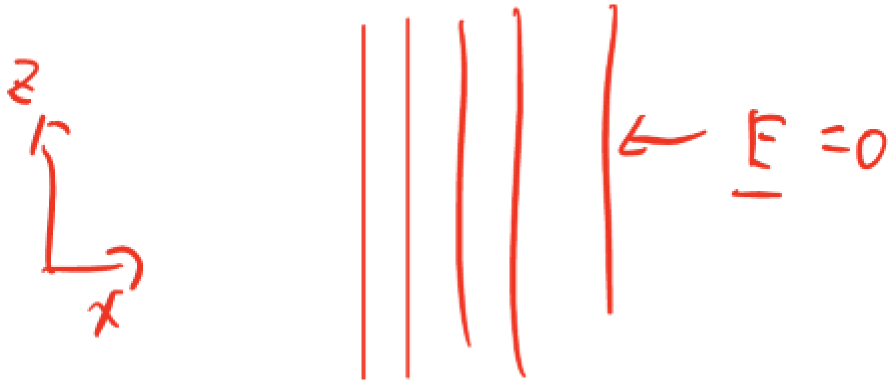
And we add them up



$$
\begin{aligned}
\mathbf{E} &= \frac{\mathbf{E}_0}{2} \left(e^{i(\mathbf{k}_+ \cdot \mathbf{r} - \omega t)} - e^{i(\mathbf{k}_- \cdot \mathbf{r} - \omega t)} \right) \hat{\mathbf{y}} \\
&= \frac{\mathbf{E}_0}{2} e^{i(k_0 z \cos\theta - \omega t)} \left(e^{ik_0 x \sin\theta} - e^{-ik_0 x \sin\theta} \right) \hat{\mathbf{y}} \\
&= \mathbf{E}_0 i \hat{\mathbf{y}} e^{i(k_0 z \cos\theta - \omega t)} \sin(k_0 x \sin\theta)
\end{aligned}
$$

And so it is zero when $k_0 x \sin\theta = n\pi$
i.e. $x = na$ when aa $= \frac{\pi}{k_0 \sin\theta}$

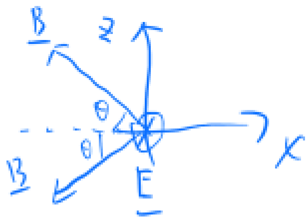
Nodal plane would be like



We could, therefore, insert conducting sheets at places where $\mathbf{E}=0$

This demonstrates (at least in 1 dimension) that guided waves are possible

When confined

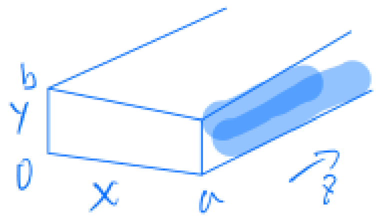


$$\mathbf{B}=\frac{1}{c}\frac{\mathbf{E}_0}{2}e^{i(k_0 z\cos\theta-\omega t)}\left[e^{ik_0\sin\theta}\begin{pmatrix}-\cos\theta\\0\\\sin\theta\end{pmatrix}+e^{-ik_0\sin\theta}\begin{pmatrix}-\cos\theta\\0\\-\sin\theta\end{pmatrix}\right]$$

$$=\frac{\mathbf{E}_0}{2c}e^{i(k_0 z\cos\theta-\omega t)}\begin{pmatrix}-i\cos\theta\sin(k_0 x\sin\theta)\\0\\\sin\theta\cos(k_0 x\sin\theta)\end{pmatrix}$$

$\Rightarrow$ only $\mathbf{E}$ is transverse, $\mathbf{B}$ is not

Rectangular waveguide



Rectangular cross section metal walls

$\mathbf{E},\mathbf{B}\propto e^{-i\omega t}$
$\mathbf{E}=\mathbf{B}=0$ inside conductor walls
$\mathbf{B}_\perp$ and $\mathbf{E}_{||}$ are continuous
$\Rightarrow$ $\mathbf{B}_\perp$ and $\mathbf{E}_{||}$ are zero at the walls

Consider Maxwell's equations inside waveguide
$\nabla\cdot\mathbf{E}=0 \quad \nabla\times\mathbf{E}=i\omega\mathbf{B}$

$$\nabla\cdot\mathbf{B}=0 \quad \nabla\times\mathbf{B}=\frac{i\omega}{c^2}\mathbf{E}$$

Hence, using our old method, we could have
$\nabla\times\nabla\times\mathbf{E}=i\omega\nabla\times\mathbf{B}=\frac{\omega^2}{c^2}\mathbf{E}$
Where the LHS would be $\underbrace{\nabla(\nabla\cdot\mathbf{E})}_{\mathbf{0}}-\nabla^2\mathbf{E}=-\nabla^2\mathbf{E}$, so we can have a second order differential equation

$\nabla^2 \mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} = 0$

Which is called the Helmholtz equation

One set of solution of this equation have $\mathbf{E}$ transverse (called TE modes)

$\Rightarrow E_z = 0$

Since $\mathbf{E}_{\parallel}$ = 0 at walls, we could get

$$\begin{cases} E_x = 0 \quad \text{at} \quad y = 0, b \\ E_y = 0 \quad \text{at} \quad x = 0, a \end{cases}$$

And since $E(\mathbf{r}, t) \propto e^{i(k_g z - \omega t)}$, we would like to put it in Helmholtz equation

$\Rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k_g^2\right)\right]\mathbf{E} = 0$

> Notice that E_z = 0

<u>Boundary conditions</u>

$$E_x \propto \sin\frac{n\pi y}{b}$$

$$E_y \propto \sin\frac{m\pi x}{a}$$

$$E_x = f(x)\sin\frac{n\pi y}{b}e^{i(k_g z - \omega t)}$$

$$E_y = g(y)\sin\frac{m\pi y}{a}e^{i(k_g z - \omega t)}$$

From $\nabla \cdot \mathbf{E} = 0$, we have $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \underbrace{\frac{\partial E_z}{\partial z}}_{0} = 0$, hence we could continue

$$\frac{\partial f}{\partial x}\sin\frac{n\pi y}{b} + \frac{\partial g}{\partial y}\sin\frac{m\pi x}{a} = 0$$

$$\frac{\partial f}{\partial x} = -\sin\frac{m\pi x}{a} \Rightarrow f(x) = -\frac{a}{m\pi}\cos\frac{m\pi x}{a}$$

$$\frac{\partial g}{\partial y} = -\sin\frac{n\pi y}{b} \Rightarrow g(y) = -\frac{b}{n\pi}\cos\frac{n\pi y}{b}$$

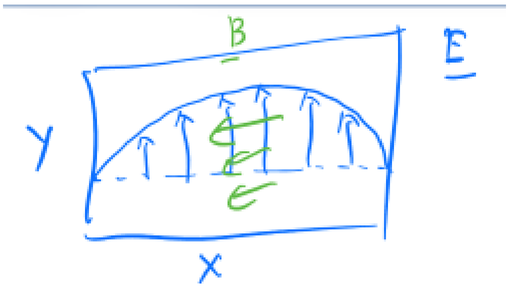
If m = 1, n = 0, we would call it TE10 mode

$$E_x = E_z = 0$$

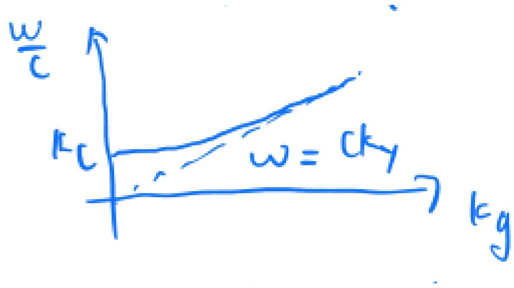
$$E_y = -A\frac{\pi}{a}\sin\frac{\pi x}{a}e^{i(k_g z - \omega t)}$$

$$B_x = A\frac{k_g}{\omega}\sin\frac{\pi x}{a}e^{i(k_g z - \omega t)}$$

$$B_z = i\frac{A}{\omega}\left(\frac{\pi}{a}\right)^2\cos\frac{\pi x}{a}e^{i(k_g z - \omega t)}$$



We could have Helmholtz equation for B_z as well
$\Rightarrow \frac{\omega^2}{c^2} - k_g^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 = 0$
$\Rightarrow \frac{\omega^2}{c^2} = k_g^2 + \underbrace{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}_{k_c^2}$
Where k_c is the cutoff wave vector



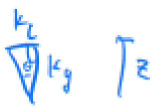
There is a minumym frequency allowed allowed by the waveguide

$$\frac{2\omega d\omega}{c^2} = 2k_g dk_g$$

$$\Rightarrow V_{\text{group}} = \frac{d\omega}{dk_g} = \frac{c^2 k_g}{\omega} = \frac{ck_g}{\sqrt{k_g^2 + k_c^2}}$$

$$v_{\text{phase}} = \frac{\omega}{k_g} = \frac{c\sqrt{k_g^2 + k_c^2}}{k_g}$$

And $v_{\text{group}} v_{\text{phase}} = c^2$
$k_g = k_0 \cos\theta$
$k_c = k_0 \sin\theta$
$\frac{\omega}{c} = k_0$



We can, therefore, consider the TE_{10} mode ad an EM wave bouncing off the walls at angle θ_0 with respect to the walls