4 Electromagnetic waves in material

> Let's let everything move!

4.1 Displacement current

Conservation of charge

$$\nabla \cdot \mathbf{J_s} = -\frac{\partial \rho_f}{\partial t}$$

However, this is incompatible with

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$

Lets take the divergence for both sides, we get

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}\underbrace{\nabla \cdot \mathbf{D}}_{\rho_\mathbf{f}}$$

$\Rightarrow$ add an additional term to the current density $\mathbf{J_f} = \nabla \times \mathbf{H}$

$$\Rightarrow \nabla \times \mathbf{H} = \mathbf{J_f} + \underbrace{\frac{\partial \mathbf{D}}{\partial t}}_{\text{displacement current}}$$

LHS: $\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \frac{1}{\mu_0}\nabla \times \mathbf{B} - \underbrace{\nabla \times \mathbf{M}}_{\mathbf{J_b}}$

Where $\mathbf{J_b}$ is the bound current density.

RHS $= \mathbf{J_s} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J_s} + \frac{\partial \mathbf{P}}{\partial t} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

$\Rightarrow$ we could write $\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J_f} + \mathbf{J_b} + \mathbf{J_p}\right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

where J_p is the polarization current density. which equals $\frac{\partial \mathbf{P}}{\partial t}$

Note that $\nabla \cdot \mathbf{J_p} = -\frac{\partial \rho_p}{\partial t}$ from conservation of charge.

Thus the polarization current responds to changes to bound charge, and hence in $\mathbf{P}$

4.2 Maxwell's equations in insulating linear dielectrics

Since it is insulating linear dielectrics, we have $\mathbf{J}_f = 0$ and $\mathbf{J}_b = 0$

Hence, we could get Maxwell's equation

$$\nabla \cdot \mathbf{D} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

remember, $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E} + \mathbf{P}$ and $\mathbf{B} = \mu_0\mu_r\mathbf{H}$

which gives

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0\mu_r\epsilon_0\epsilon_r\frac{\partial \mathbf{E}}{\partial t}$$

Consider

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\nabla \times \mathbf{B}$$

$$\Rightarrow \nabla^2\mathbf{E} = \underbrace{\mu_0\mu_r\epsilon_0\epsilon_r}_{\frac{1}{v^2}}\frac{\partial^2\mathbf{E}}{\partial t^2}$$

which is wave equation.

$\Rightarrow v = \frac{c}{n}$ where c = $\frac{1}{\sqrt{\epsilon_0\mu_0}}$ and n = $\sqrt{\epsilon_r\mu_r}$ where n is also called *refractive index*

Plane waves solutions

Lets choose propagation parallel to z, and hence
$\frac{\partial \mathbf{\Psi}}{\partial \mathbf{x}} = \frac{\partial \mathbf{\Psi}}{\partial \mathbf{y}} = 0$

remember that $\nabla \cdot \mathbf{E} = 0 \Rightarrow \frac{\partial E_z}{\partial z} = 0$
similarly, $\nabla \cdot \mathbf{B} = 0 \Rightarrow \frac{\partial B_z}{\partial z} = 0$

we also have $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \frac{\partial B_z}{\partial t} = 0$

And $\nabla \times \mathbf{B} = \mu_0\mu_r\epsilon_0\epsilon_r\frac{\partial \mathbf{E}}{\partial t} \Rightarrow \frac{\partial E_z}{\partial t} = 0$

Hence, E_z and B_z are constant in z and t, they are not part of wave motion

now analyze the x,y components of curl:

$$-\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}, -\frac{\partial B_y}{\partial z} = \frac{1}{v^2}\frac{\partial E_x}{\partial t}$$

$\Rightarrow E_x, B_y$ are solutions

Lets then take

$$\frac{\partial^2 E_x}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2}$$

$$\Rightarrow E_x(z,t) = E_{x0}e^{i(\pm kz - \omega t)}\hat{\mathbf{x}}$$

Then, we could have

$$\Rightarrow B(z,t) = B_0 e^{i(\pm kz - \omega t)}\hat{\mathbf{y}}$$

And then we could get the wave travelling in $\pm\mathbf{z}$ direction

$$\begin{aligned} &\Rightarrow \mp kE_0 = -\omega B_0 \\ \text{and } &\pm kB_0 = \frac{\omega}{v^2} E_0 \\ &\Rightarrow \frac{E_0}{B_0} = \pm\frac{\omega}{k} = \pm v \end{aligned}$$

Define Impedance Z as

$$Z = \left|\frac{E_0}{H_0}\right| = \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}}$$

remember that $H_0 = \frac{B_0}{\mu_0\mu_r}$

The motivation of doing so is that $v = -\int \mathbf{E}\cdot d\mathbf{l}$ and $I = \oint \mathbf{H}\cdot d\mathbf{l}$

So dimension would work

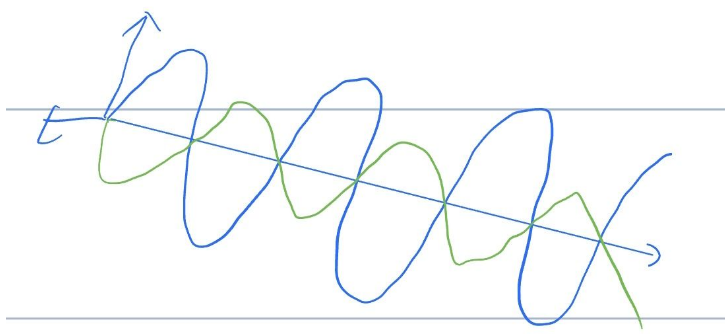
For free space, then, $\epsilon_r = \mu_r = 1$ and $Z = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\Omega$

Remember that $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$, and use E, B $\propto e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, we could get

$$\begin{aligned} i\mathbf{k}\times\mathbf{E} &= -(-i\omega)\mathbf{B} \\ &= i\omega\mu_0\mu_r\mathbf{H} \\ \Rightarrow z = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| &= \sqrt{\frac{\mu_0\mu_r\omega}{k}} \end{aligned}$$

Which gives the same answer because $v = \frac{c}{n} = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0\epsilon_r\mu_0\mu_r}}$

which is this wave



4.3 conductors

> Remember that
> $\nabla\cdot\mathbf{J_f} = -\frac{\partial\rho_f}{\partial t}$
> $\nabla\cdot\mathbf{J_p} = -\frac{\partial\rho_p}{\partial t}$
> $\nabla\cdot\mathbf{J_b} = 0$
> The last one is 0 because $\nabla\cdot(\nabla\times\mathbf{M}) = 0$

For conductors, we have

$\rho_f = 0$ since there are no free charges in equilibrium
$\mathbf{J}_f = \sigma \mathbf{E}$ from Ohm's law where σ is the conductivity
$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$ and $\mathbf{B} = \mu_0 \mu_r \mathbf{H}$ from linearity

Then we could get Maxwell's equation in conductors

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \underbrace{\mu_0 \mu_r \sigma \mathbf{E}}_{\text{conduction } \mathbf{J}} + \underbrace{\mu_0 \mu_r \sigma_0 \sigma_r \frac{\partial \mathbf{E}}{\partial t}}_{\text{dispalcement } \mathbf{J}}$$

Free charge will decay to zero in a short time τ, and it is easy to prove (said Blundell)

$\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t}$

Where $\nabla \cdot \mathbf{J}$ is equal to $\sigma \cdot \nabla \cdot \mathbf{E}$ from Ohm's law

and $\nabla \cdot \mathbf{E}$ is equal to $\frac{\rho}{\epsilon_0 \epsilon_r}$ from Gauss's law

$\Rightarrow \rho(f) = \rho(0) e^{-\frac{t}{\tau}}$

Where $\tau = \frac{\epsilon_0 \epsilon_r}{\sigma}$

If the metal has great conductivity, then τ is very small, and hence $\rho(f)$ is very small.

Let's consider the electromagnetic wave having frequency ω, so we would like to compare $\frac{1}{\omega}$ with τ:

Condition	**Conductor Type**	**Charge Response**	**Conductivity**
$\omega\tau \ll 1$	Good conductor	Charges respond very quickly	$\sigma \gg \epsilon_0 \epsilon_r \omega$ Conduction current dominates
$\omega\tau \gg 1$	Bad conductor	Charges respond very slowly	$\sigma \ll \epsilon_0 \epsilon_r \omega$ Displacement current dominates

Take real life examples

	$\sigma(\Omega m)$	ϵ_r	$\frac{\sigma}{\epsilon_0 \epsilon_r}(S^{-1})$
metal	10^7	1	10^{19}
Silicon	$4 \cdot 10^{-4}$	11.7	10^7
Glass	10^{-10}	5	10

Note that visible light would have frequency ~ $5 \cdot 10^{14}$ Hz

Let's now do some electromagnetism

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{E}) &= \underbrace{\nabla(\nabla\cdot\mathbf{E})}_{0}-\nabla^2\mathbf{E}\\ &= -\frac{\partial}{\partial t}\nabla\times\mathbf{B}\\ \nabla^2\mathbf{E} &= \mu_0\mu_r\sigma\frac{\partial\mathbf{E}}{\partial t}+\mu_0\mu_r\sigma_0\sigma_r\frac{\partial^2\mathbf{E}}{\partial t^2}\end{aligned}$$

Again, this would yield transverse plane waves with $\mathbf{E},\mathbf{B}\perp$ to each others

$\Rightarrow \mathbf{E}=\mathbf{E}_0\hat{\mathbf{x}}e^{i(\widetilde{\mathbf{k}}\cdot\mathbf{z}-\omega t)}$

with $\widetilde{\mathbf{k}}=\mathbf{k}+i\kappa$

$\Rightarrow \widetilde{k}^2=i\underbrace{\mu_0\mu_r\sigma\omega}_{\mu}+\underbrace{\mu_0\mu_r}_{\mu}\underbrace{\epsilon_0\epsilon_r}_{\epsilon}\omega^2$

$$\begin{aligned}(k+i\kappa)^2 &= k^2-\kappa^2+2ik\kappa\\ k^2-\kappa^2 &= \mu_0\epsilon_0\omega^2\\ 2k\kappa &= \mu\sigma\omega\Rightarrow k=\frac{\mu\sigma\omega}{2\kappa}\\ 0 &= \left(\frac{\mu\sigma\omega}{2}\right)^2\frac{1}{k^2}-k^2-\mu\epsilon_0\omega^2\\ 0 &= (k^2)^2+\mu\epsilon\omega^2(k^2)-\left(\frac{\mu\sigma\omega}{2}\right)^2\\ \kappa^2 &= -\frac{\mu\epsilon\omega^2}{2}\pm\sqrt{\left(\frac{\mu\epsilon\omega^2}{2}\right)^2+\left(\frac{\mu\sigma\omega}{2}\right)^2}\\ \kappa^2 &= \frac{\mu\epsilon\omega^2}{2}\left[\pm\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]\end{aligned}$$

Taking the positive root

$\Rightarrow \kappa=\sqrt{\frac{\mu\epsilon}{2}}\omega\sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1}$

Sub into origional equation

$k=\frac{\mu\sigma\omega}{2\kappa}=\sqrt{\frac{\mu\epsilon}{2}}\omega\sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)}+1}$
$\Rightarrow \mathbf{E}=\mathbf{E}_0\hat{\mathbf{x}}\underbrace{e^{-\kappa z}}_{e^{-\frac{z}{\delta}}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Where $\delta=\frac{1}{\kappa}$ is the *skin depth*

Reminder: Good conductors have $\sigma\gg\epsilon\omega$

$k=\kappa=\sqrt{\frac{\mu\epsilon}{2}}\omega\sqrt{\frac{\sigma}{\epsilon\omega}}=\sqrt{\frac{\mu\omega\sigma}{2}}$

We could therefore have

$\nabla^2\mathbf{E}=\mu\sigma\frac{\partial\mathbf{E}}{\partial t}+\mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2}$

We could also neglect the last term, since $\frac{\partial^2\mathbf{E}}{\partial t^2}\ll\frac{\partial\mathbf{E}}{\partial t}$

$$\Rightarrow \widetilde{k}^2 = i\mu\sigma\omega$$
$$\Rightarrow \widetilde{k} = \frac{1+i}{\sqrt{2}} sqrt\mu\sigma\omega$$
$$\Rightarrow \widetilde{k} = \kappa = \sqrt{\frac{\mu\sigma\omega}{2}}$$

Hence, we could have

$\delta = \frac{1}{k} = \sqrt{\frac{2}{\mu\sigma\omega}}$

For a typical metal, δ is

$$\begin{cases} \text{few nm - visible light} \\ \text{few}\mu m \text{ - microwave} \\ \text{few mm - radio waves} \end{cases}$$

Lets go to poor conductors

Poor conductors has $\sigma \ll \epsilon\omega$, hence

$k \approx \sqrt{\frac{\mu\epsilon}{2}}\omega\sqrt{2} = \sqrt{\mu\epsilon}\omega$

$\kappa = \sqrt{\frac{\mu\epsilon}{2}}\omega\left(1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 + \cdots - 1\right)^{\frac{1}{2}}$

which equals to $\frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$ which is independent to ω

In an insulating dielectric, $\sigma = 0$, hence $\kappa = 0$, and hence $k = \frac{\omega}{v}$ as expected.

Lets return to previous equation

Lets consider the curl equation in the conductor

$$\frac{\partial \mathbf{E}_x}{\partial z} = \frac{\partial B_y}{\partial t}$$
$$i(k+k)\mathbf{E}_0 = i\omega\mathbf{B}_0$$
$$z = \frac{\mu E_0}{B_0} = \frac{\mu\omega}{k+i\kappa}$$
$$\widetilde{k} = \frac{\mu\omega}{\sqrt{k^2+\kappa^2}} e^{i\phi}$$

If we expand, ϕ would be

$\phi = \tan^{-1}\left(\frac{\sqrt{1+(Q/\epsilon\omega)^2}-1}{\sqrt{1+(Q/\epsilon\omega)^2}+1}\right)^{\frac{1}{2}}$

For a good conductor, $\sigma \gg \epsilon\omega$, hence $\phi \to \tan^{-1} 1 = \frac{\pi}{4}$

So this means that B lags behind E in a metal

Poyting vectors

Work done on charge

$\delta q = \rho = delta\tau$

$$\delta F = \delta q(\mathbf{E} + \mathbf{V} \times \mathbf{B})$$
$$\delta \mathbf{F} \cdot \mathbf{dl} = \delta q(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot \mathbf{v}\delta t$$
$$= \mathbf{E} \cdot \mathbf{J_f}\delta\tau\delta t$$

where $\mathbf{J}_f$ eaquals to $\rho\mathbf{v}$

Rate of work on charges

$$\begin{aligned}\mathbf{F} &= \frac{dw}{dt} \\ &= \mathbf{E}\cdot\mathbf{J_f}d\tau \\ &= \frac{d}{dt}\int \underbrace{u_{mech}}_{\text{Energy density}} d\tau\end{aligned}$$

From Maxwell's equation:
$\mathbf{J}_f = \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}$
We have

$\mathbf{E}\cdot\mathbf{J_f} = \mathbf{E}\cdot\nabla\times\mathbf{H} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial\mathbf{t}}$

By dotting everything, and then

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}$$
$$\mathbf{E}\cdot\mathbf{J_f} = \mathbf{H}\cdot\underbrace{\nabla\times\mathbf{E}}_{-\frac{\partial\mathbf{B}}{\partial\mathbf{t}}} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$$

Where we call $\mathbf{H}\cdot\underbrace{\nabla\times\mathbf{E}}_{-\frac{\partial\mathbf{B}}{\partial\mathbf{t}}} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$ "$\frac{\partial}{\partial t}u_{EM}$" (remember that $u_{EM} = \frac{1}{2}(\mathbf{B}\cdot\mathbf{H}+\mathbf{E}\cdot\mathbf{D})$ whcih equals to Energy stored in EM field per unit volume) and $\nabla\cdot(\mathbf{E}\times\mathbf{H})$ as "$\mathbf{S}=\mathbf{E}\times\mathbf{H}$" or poyting vector.

> How is this working?

Assume that we are using a linear media:

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} = \frac{1}{2}\mathbf{E}\cdot\frac{\partial(\mathbf{E}\cdot\mathbf{D})}{\partial t}$$
$$\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\mathbf{H}\cdot\frac{\partial(\mathbf{B}\cdot\mathbf{H})}{\partial t}$$

Remember that

$$\left.\begin{aligned}\frac{1}{2}\epsilon_0E^2 &= \frac{1}{2}DE \\ \frac{B^2}{2\mu_0} &= \frac{1}{2}BH\end{aligned}\right\}\text{In free space}$$

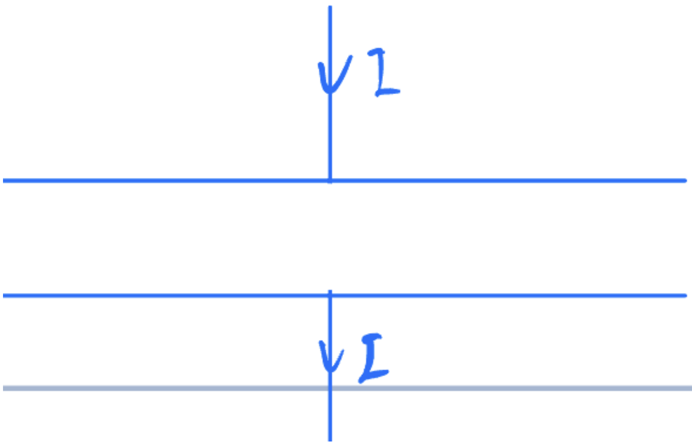
Brining everything together, we could get
$\Rightarrow \frac{d}{dt}(u_{\text{mech}}+u_{\text{EM}}) + \nabla\cdot\mathbf{S} = 0$

Where $\mathbf{S}$ is the poyting vector, or equiviantly,

$\frac{d}{dt}\int(u_{\text{mech}}+u_{\text{EM}})d\tau + \oint\mathbf{S}\cdot d\mathbf{a} = 0$

We could say that, therefore $\mathbf{S}$ is the energy flux density, or the rate of flow of energty per unit area in the direction of S.

Example: a capacitor



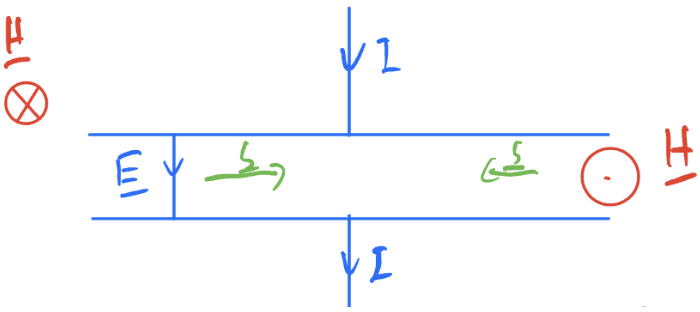
$$C = \frac{\varepsilon_0 A}{d}$$

$$I = \frac{dQ}{dt}$$

The stored energy increase at rate

$\dot{U} = \frac{Q}{C}\frac{dQ}{dt} \quad U = \frac{Q^2}{2C}$

also:



$$H \cdot 2\pi r = I$$

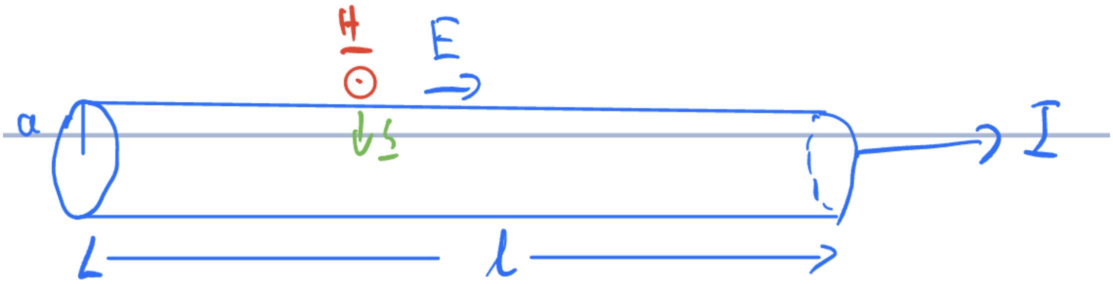
$$\underbrace{Ed}_{V} = \frac{Q}{C}$$

Hence, we have

$\dot{U} = S \cdot 2\pi r d = \frac{Q}{c}\frac{dQ}{dt}$

where $S = EH = \frac{Q}{dc}\frac{\frac{dQ}{dt}}{2\pi r}$

There is another example



$$H \cdot S\pi a = I$$
$$E = \frac{V}{l}$$
$$\Rightarrow S = \frac{V}{l}\frac{I}{2\pi a}$$

$$\int \mathbf{S} \cdot \mathbf{da} = -IV = -I^2R$$

4.5 Radiation pressure

EM waves are made up of photons, and hecne they have momentum

$E = pc$

$\Rightarrow$ Transport of energy is appoinated by transport of momentum

$P_{rad} = \frac{<S>}{c}$

For a perfect absorber, where P_{rad} is the radiation pressure

Example For a plane EM wave in free space, we have

$$\begin{aligned} U &= \frac{1}{2}\epsilon_0 E^2 + frac12\frac{B^2}{\mu_0} \\ \text{but } E &= cB \\ \Rightarrow U &= \epsilon_0 E^2 \\ \mathbf{E} &= \frac{1}{2} E_0 \cos(kz - \omega t)\hat{\mathbf{x}} \\ &= E_0 \cos^2(kz - \omega t) \\ < E^2 > &= \frac{1}{2} E_0^2 \\ \Rightarrow < u > &= \frac{1}{2}\epsilon_0 E_0^2 \\ < S > &= \frac{1}{2}\epsilon_0 E_0^2 c = I \end{aligned}$$

Where I is the intensity of wave

$$\Rightarrow P_rad = \begin{cases} \frac{1}{2}\epsilon_0 E_0^2 & \text{perfect absorber} \\ \epsilon_0 E_0^2 & \text{perfect reflector} \end{cases}$$

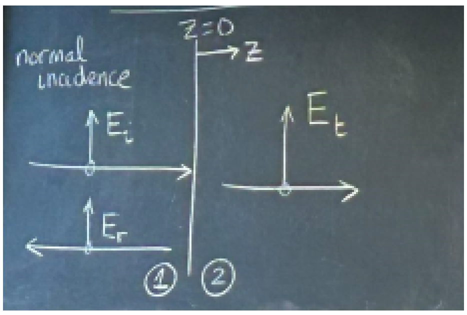
Sunlight: I~$1kWm^{-2}$
$\Rightarrow P_{rad} = 10^{-5} Pa$
FYI, $P_{atm} = 10^5 Pa$

Example Consider a star which is growing by accretion

i.e. matter is falling onto it uniformly in all directions

The star has luminosity L

4.6 EM waves - reflection and refraction



Left:
$E_i e^{i(k_1 x - \omega t)} + E_r e^{i(-k_1 - \omega t)}$

Right:
$E_t e^{i(k_2 x-\omega t)}$

Using electrom boundery conditions, we could get

$E^{\parallel}$ is continuous
$E_i^{\parallel}+E_r^{\parallel}=E_t^{\parallel}$

$H^{\parallel}$ is continuous
$\frac{E_i}{Z^1}-\frac{E_r}{Z_1}=\frac{E_f}{Z_2}$
Putting two equations together

$\frac{E_r}{E_i}=\frac{Z_2-Z_1}{Z_2+Z_1} \quad \frac{E_t}{E_i}=\frac{2Z_2}{Z_2+Z_1}$
Where $Z=\sqrt{\frac{\mu}{\epsilon}}$

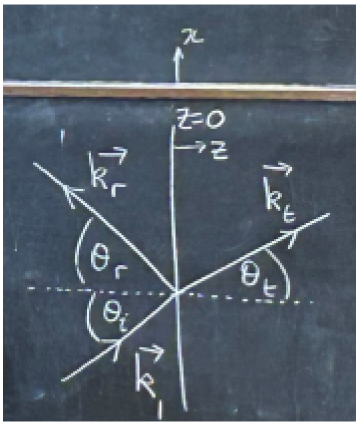
|Poyting vector| $=S=|\mathbf{E}\times\mathbf{H}|=\frac{E^2}{Z}$

We expect $S_{incident}=S_{reflected}+S_{transmitted}$

Where they equal to

$\frac{E_i^2}{Z_1}+\frac{E_r^2}{Z_1}=\frac{E_t^2}{Z_2}$
separately

Lets now have angles
$E_r e^{i(\mathbf{k_r}\cdot\mathbf{r}-\omega t)}$



$\mathbf{E_i}e^{i(\mathbf{k_i}\cdot\mathbf{r}\omega t)}$

$E_t e^{i(\mathbf{k_t}\cdot\mathbf{r}-\omega t)}$

Choose $\mathbf{k}_i$ in x-z plane

At z = 0, $E_{\parallel}$ is cointinuous and this holds for all x y and t
$\Rightarrow \omega$ must be the same

$\Rightarrow \mathbf{k_i}\cdot\mathbf{r}=\mathbf{k_r}\cdot\mathbf{r}=\mathbf{k_t}\cdot\mathbf{r}$ for all x, y at z=0

Take $\mathbf{r}=(0,y,0)$

$\Rightarrow \mathbf{k_i},\mathbf{k_r}$ and $\mathbf{k}_t$ all lie iin the xz plane
(*the plane of incidence*)

Take $\mathbf{r}=(x,0,0)$ so $\mathbf{k_i}\cdot\mathbf{r}=k\sin\theta_x$

$$|\mathbf{k_i}| = \mathbf{k_r}| = k_1$$
$$|\mathbf{k_t}| = k_2$$
$$\Rightarrow \underbrace{k_1 \sin\theta_i = k_1 \sin\theta_r}_{\theta_i=\theta_r,\text{ law of reflection}} = k_2 \sin\theta_t$$

Remember that $\frac{\omega}{k} = \frac{c}{n}$, And the last two would lead to

$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_1}{k_2} = \frac{n_2}{n_1}$-law of refraction, or snell's law

Where $n = \sqrt{\epsilon_r \mu_r}$

Fresnel equations

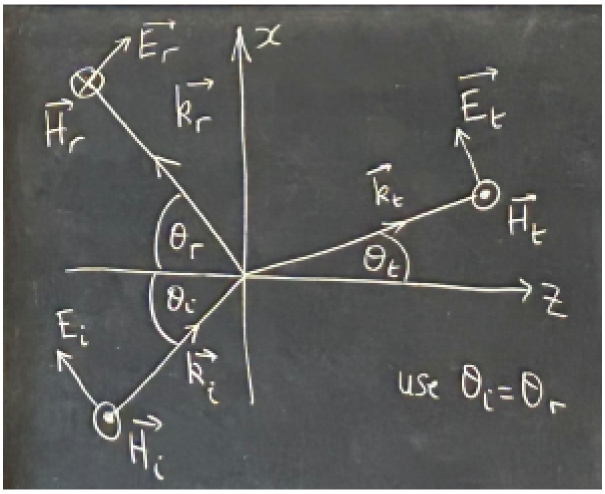
Worring about polarization directions

We work in those steps

1. **$\mathbf{E}$ in the plane of incidence**

> "parallel-like" = parallel

Remid that $\mathbf{E}, \mathbf{H}$ and $\mathbf{k}$ form a right-handed system



	incident	reflected	transmitted
E_x	$E_i \cos\theta_i$	$E_r \cos\theta_r$	$E_t \cos\theta_t$
E_z	$-E_i \sin\theta_i$	$E_r \cos\theta_r$	$-E_t \sin\theta_t$
H_y	$\frac{E_i}{Z_1}$	$-\frac{E_r}{Z_1}$	$\frac{E_t}{Z_2}$

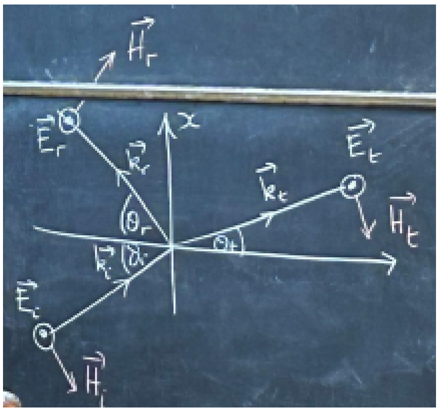
$E_{\|}$ continuous $\Rightarrow$ E_x continuous $\Rightarrow$ $E_i \cos\theta_i + E_r \cos\theta_i = E_t \cos\theta_t$

$$\frac{E_r}{E_i} = \frac{Z_2 \cos_t heta_i - Z_1 \cos\theta_i}{Z_2 \cos\theta_1 + Z_1 \cos\theta_i}$$
$$\frac{E_t}{E_i} = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_i + Z_1 \cos\theta_i}$$

Now look for Fresned equations for p-polarizations

2. **$\mathbf{E}$ perpendicular to the plane of incidence**

> "s-like" s = senkrecht = perpendicular



	incident	reflected	transmitted
E_y	E_i	E_r	E_t
H_x	$-\frac{E_i}{Z_1}\cos\theta_i$	$\frac{E_r}{Z_1}\cos\theta_r$	$-\frac{E_t}{Z_2}\cos\theta_t$
H_z	$\frac{E_i}{Z_1}\sin\theta_i$	$\frac{E_r}{Z_1}\sin\theta_r$	$\frac{E_t}{Z_2}\sin\theta_t$

$E_{||}$ continuous
$\Rightarrow E_y \text{continuous} \quad E_i + E_r = E_t$
$H_{||}$ continuous
$\Rightarrow H_x continuous \quad -\frac{E_i}{Z_1}\cos\theta_i + \frac{E_r}{Z_1}\cos\theta_r = -\frac{E_t}{Z_2}\cos\theta_t$

Remember that

$Z = \sqrt{\frac{\mu_r\mu_0}{\epsilon_r\epsilon_0}} = \frac{Z_0}{n} \quad n = \sqrt{\epsilon_r}\text{while } \mu_r = 1$

Let's set $\mu_r = 1$
Then, $Z = \sqrt{\frac{\mu_r\mu_0}{\epsilon_r\epsilon_0}} = \frac{Z_0}{n} \quad n = \sqrt{\epsilon_r}$

So we can replace Z_i with $\frac{1}{n_i}$ in expressions involving ratios of Z's.

e.g. Fresnel equations for p-polarization

$r = \frac{E_r}{E_i} = \frac{\frac{1}{n_2}\cos\theta_t - \frac{1}{n_1}\cos\theta_i}{\frac{1}{n_2}\cos\theta_t + \frac{1}{n_1}\cos\theta_i}$
$= \frac{n_1\cos\theta_t - n_2\cos_t heta_i}{n_1\cos\theta_t + n_2\cos\theta_i}$

Use Snell's law

$n_1\sin\theta_i = n_2\sin\theta_t$

$$\Rightarrow r = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i}$$

$$t = \frac{4\sin_t heta_t\cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}$$

For s-polarization, we have

$$r = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$$

$$t = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_t + \theta_i)}$$

We also have

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$
$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$$
$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

Fresnel equations:

/ $n_1 \cos\theta_i$ on top and bottom

$$\alpha = \frac{\cos\theta_t}{\cos\theta_i} = \frac{1}{\cos\theta_i}\sqrt{1-(\frac{n_1}{n_2}\sin\theta_i)^2}$$
$$\beta = \frac{n_2}{n_1}$$

	$E^{\parallel}$**(p)**	$E^{\perp}$**(s)**
r	$\frac{\alpha-\beta}{\alpha+\beta}$	$\frac{1-\alpha\beta}{1+\alpha\beta}$
t	$\frac{2\alpha}{\alpha+\beta}$	$\frac{2}{1+\alpha\beta}$

Remember, EM waves have an energy flux given by

$S = |\mathbf{E}\times\mathbf{H}| = \frac{E^2}{Z}$

Intensify coefficients

$$T = \frac{I_r}{I_i} = |r|^2 = \begin{cases} (\frac{\alpha-\beta}{\alpha+\beta})^2 & (p) \\ (\frac{1-\alpha\beta}{1+\alpha\beta})^2 & (s) \end{cases}$$

$T = \frac{I_t}{I_i} = |t|^2 \frac{n_2}{n_1}\frac{\cos\theta_t}{cos\theta_i}$

where $\frac{n_2}{n_1}$ is due to waves at ifferent speeds. And $\frac{\cos\theta_t}{\cos\theta_i}$ is due to the wavesfronts at different angles.

$$T = |t|^2\alpha\beta = \begin{cases} \alpha\beta(\frac{2}{\alpha+\beta})^2 & (p) \\ \alpha\beta(\frac{2}{1+\alpha\beta})^2 & (s) \end{cases}$$

Lets check in certain cases

- $\theta_i = 0 \Rightarrow \alpha = 1$

$$r_s = r_p = \frac{1-\beta}{1+\beta} = \frac{n_1-n_2}{n_1+n_2}$$
$$t_s = t_p = \frac{2}{1+\beta} = \frac{2n_1}{n_1+n_2}$$
$$R_s = R_p = (\frac{n_1-n_2}{n_1+n_2})^2$$

$$T_s = T_p - \frac{n_2}{n_1}\frac{4n_1^2}{(n_1+n_2)^2} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

Example: air/glass interface

$n_1 = 1 \quad n_2 = 1.5$
$$r_s = r_p = -0.2$$
$$T_s = T_p = 0.8$$
$$R_s = R_p = 0.04$$
$$T_s = T_p = 0.96$$
$\Rightarrow$ 4% of light is reflected, 96% is transmitted

[If n_2, for example, = 1.75, $R_s = R_p = 0.074$, which is sa problem]

Lets take another go at differnet angle

- $\theta_i = 90^\circ$, and set $\beta > 1$

$\sin\theta_i = 1 \quad \cos\theta_i = 0$
$\Rightarrow \alpha \to \infty$
$r_p = 1 \quad r_s = -1$
$t_p = 0 \quad t_s = 0$

- now consider $\beta < 1$, we cna have total interal reflection for $\theta_i > \theta_c$ where θ_c is the critical angle = $\beta = \frac{n_2}{n_1}$

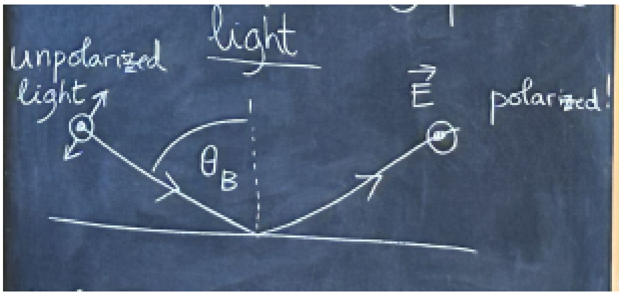
At $\theta_c = \sin^{-1}\beta$, $\alpha = \frac{1}{\cos\theta_c}\sqrt{1 - (\frac{\sin\theta_c}{\beta})^2} = 0$

Hence, we have
$r_p = -1 \quad r_s = 1$
$t_p = \frac{2}{\beta} \quad t_s = 2$

$\sin\theta_c = \beta$

r_p vanishes at the certain angle called **"Brewster's angle"** θ_B
$r_p = \frac{\alpha-\beta}{\alpha+\beta} = 0$ when $\alpha = \beta = \frac{n_2}{n_1}$
$$\frac{1}{cos\theta_B}\sqrt{1 - (\frac{\sin\theta_B}{\beta})^2} = \beta$$
$$\text{square both sides:}$$
$$\sec^2\theta_B - \frac{1}{\beta^2}\tan^2\theta_B = \beta^2$$
$$(1 - \frac{1}{\beta^2})\tan^2\theta_B = \beta^2 - 1$$
$$\frac{(\beta^2-1)}{\beta^2}\tan^2\theta_B = (\beta^2 - 1)$$
$$\Rightarrow \tan\theta_B = \beta$$

Method of producing polarized light



Reflectted light polarized with $\mathbf{E}$ perpendicular to the plane of incidence

Polarizing sunglasses with transmission axis vertical reduce glare because reflected light is mainly horizontally polarized.

Total internal reflection

$n_1 \sin\theta_i = n_2 \sin\theta_t$

$$\sin\theta_t = \frac{\sin\theta_i}{n_2/n_1}$$

Since $n_2 < n_1$, there will be total internal reflection when $\sin\theta_t > 1$

For transmitted wave, its like

- $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ in medium
  = $e^{i(k_2[x\sin\theta_t + z\cos\theta_t] - \omega t)}$

  since $z\cos\theta t = 1 - \sin^2\theta_t$ = $\underbrace{e^{i(k_2 x \sin\theta_t - \omega_t)}}_{wave}\underbrace{e^{-(\sin^2\theta_t - 1)^{\frac{1}{2}} k_2 z}}_{e^{-z/\delta}}$ where

  $\delta = \frac{\lambda_2}{2\pi}\frac{1}{(\sin^2\theta_t - 1)^{\frac{1}{2}}}$
  This is called an *evanescent wave*

Lets now consider the **Plane travelling wave**

$$E_x = E_0 e^{i(kz-\omega t)}$$

$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} = i\omega B$

$$\begin{aligned}
\mathbf{B} &= \frac{1}{i\omega}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial x & \partial y & \partial z \\ E_x & 0 & 0 \end{vmatrix} \\
&= \frac{1}{i\omega}\begin{pmatrix} 0 \\ \partial_z E_x \\ -\partial_y E_x \end{pmatrix} \\
&= \frac{1}{i\omega}\begin{pmatrix} 0 \\ ikE_x \\ 0 \end{pmatrix} \\
&= \frac{E_x}{\omega/k}\hat{\mathbf{y}} \\
&= \frac{E_x}{c}\hat{\mathbf{y}}
\end{aligned}$$

For s-polarization, $\theta_i > \theta_c$, we have

$$\begin{aligned}
\sin\theta_t &= \frac{\sin\theta_i}{n_2/n_1} > 1 \\
\cos\theta_t &= i\sqrt{\sin^2\theta_t - 1} \\
r_s &= \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} \\
&= \frac{A - iB}{A + iB} \\
\Rightarrow R_s = |r_s|^2 &= \frac{A^2 + B^2}{A^2 + B^2} = 1
\end{aligned}$$

$$t_s = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{2A}{A + iB}$$

So is complex

$\Rightarrow \mathbf{E}_t = \hat{\mathbf{y}} t_s E_i e^{i(k_2[x\sin\theta_t - \omega t])} e^{-z/\delta}$

where $\delta = \frac{\lambda_2}{2\pi}\frac{1}{(\sin^2\theta_t - 1)^{\frac{1}{2}}}$

$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} = i\omega\mathbf{B}$

$$\begin{aligned}
\mathbf{B} &= \frac{1}{i\omega}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_t & 0 \end{vmatrix} \\
&= \frac{1}{i\omega}\begin{pmatrix} -\partial_z E_t \\ 0 \\ \partial_x E_t \end{pmatrix} \\
&= \begin{pmatrix} \frac{iE_t}{\omega\delta} \\ 0 \\ k_2 \sin\theta_t E_t/\omega \end{pmatrix}
\end{aligned}$$

$\Rightarrow B_z$ is in phase with E_y

- $\Rightarrow$ transport of energy along x
- B_x is out of phase with E_y
- $\Rightarrow$ no transport of energy along z
- $<\mathbf{S}> = <\mathbf{E}\times\mathbf{H}^*>$
- $\mathbf{H}^*$ is the complex conjugate of H

Reminder on conductors

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} - \frac{\partial}{\partial t}\nabla\times\mathbf{B}$$

$$\mu_0\mu_r\sigma\mathbf{E} + \mu_0\mu_r\sigma_0\sigma_r\frac{\partial\mathbf{E}}{\partial t} = 0 \quad \text{if good conductor}$$

$$\nabla^2\mathbf{E} = \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t}$$

$$-\tilde{k}^2 = i\omega\mu_0\sigma$$

$$\Rightarrow \tilde{k} = k + i\kappa \quad k = \kappa = \sqrt{\frac{\mu_0\sigma\omega}{2}}$$

$$\nabla\times\mathbf{E} = -\dot{\mathbf{B}}$$

$$-\frac{\partial B_y}{\partial t} = (\nabla\times\mathbf{E})_y = \frac{\partial E_x}{\partial z}$$

$$i\omega B_0 == i(k+i\kappa)E_0$$

$$\begin{aligned}
Z = \frac{E_0}{H_0} &= \frac{E_0}{B_0/\mu_0} = \frac{\mu_0\omega}{k+i\kappa} \\
&= \sqrt{\frac{2\mu_0\omega}{\sigma}}\frac{1}{1+i} \\
\Rightarrow Z &= \sqrt{\frac{\mu_0\omega}{2\sigma}}(1-i)
\end{aligned}$$

Reflection from a meatal surface

air	metal
$Z_1 = \sqrt{\frac{\mu_0}{\epsilon_0}}$	$Z_2 = \sqrt{\frac{\mu_0 \omega}{2\sigma}}(1-i)$

$|z_2| \ll |Z_1|$ for a good conductor because $\sigma \gg \epsilon_0 \epsilon_r \omega$

write $\alpha = \frac{\sqrt{\frac{\mu_0 \omega}{2\sigma}}}{\sqrt{\frac{\mu_0}{\epsilon_0}}} = \sqrt{\frac{\omega \epsilon_0}{2\sigma}} \ll 1$

$$
\begin{aligned}
r = \frac{E_r}{E_i} &\quad \text{normal incidence} \\
= \frac{Z_2 - Z_1}{Z_2 + Z_1} \\
= \frac{Z_2/Z_1 - 1}{Z_2/Z - 1 + 1} \\
= \frac{\alpha(1-i) - 1}{\alpha(1-i) + 1} \\
\approx \frac{1 - 2\alpha + \cdots}{1 + 2\alpha + \cdots} \\
= 1 - 4\alpha + O(\alpha^2) \\
= 1 - \frac{2\omega\delta}{c} + O(\alpha^2) \\
= 1 - \frac{4\pi\delta}{\lambda} + O(\alpha^2)
\end{aligned}
$$

where $\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$ is the skin depth

$\Rightarrow$ most of the EM wave intensity is reflected

$\Rightarrow$ metals are shiny!