2. Electric polarization

> Matter is not free space, it contains atoms

2.1 Bound charges

Electric field can induce a dipole moment in a neutral atom

Example 4.1. A primitive model for an atom consists of a point nucleus ($+q$) surrounded by a uniformly charged spherical cloud ($-q$) of radius a (Fig. 4.1). Calculate the atomic polarizability of such an atom.

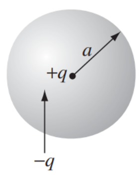


FIGURE 4.1

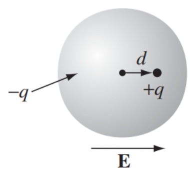


FIGURE 4.2

And the electric moment is defined as $\mathbf{p} = \alpha \mathbf{E}$ where α is the **polarizability** of the atom. A list of polarizability can be seen from the handout.

If, for example, the outside E ~ 10^6 V/m, then the displacement would be about 10^{-15} m, which is much less the size of an atom.

Also, there are polar molecules which already have a dipole moment, but they are randomized because of the present of inner energy $K_B T$

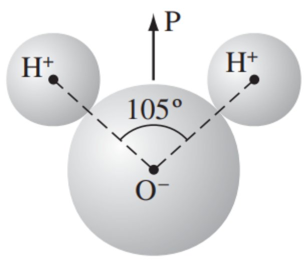


FIGURE 4.4

However, an E field can line them up

For a bok of aligned dipoles (per unit volume), define the polarization $\mathbf{P}$ as the dipole moment per unit volume as $\mathbf{P} = n\mathbf{p}$

If they are not aligned, then the polarization is zero.

> Question: what is the electric potential produced by a box of dipoles?

$$
\begin{aligned}
V(\mathbf{R}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r})\cdot\mathbf{R}}{|R|^3} d^3\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int \mathbf{P}(\mathbf{r})\cdot\nabla\frac{1}{R} d\tau' \\
\nabla\frac{1}{R} &= -\frac{\mathbf{R}}{R^3} \\
\nabla\frac{1}{|r-r'|} &= -\frac{r-r'}{|r-r'|^3} \\
\nabla'\frac{1}{|r-r'|} &= \frac{r-r\backslash}{|r-r'|^3}
\end{aligned}
$$

Note that $\nabla\cdot(f\mathbf{F}) = f\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla f$

$$
\begin{aligned}
\Rightarrow V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int \nabla'\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{R}|} d\tau' - \frac{1}{4\pi\epsilon_0}\int \frac{1}{R}\nabla\cdot\mathbf{P}(\mathbf{r}') d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int_{s'} \frac{\sigma_b dS'}{R} + \frac{1}{4\pi\epsilon_0}\int_{\tau'} \frac{\rho_b d\tau'}{R}
\end{aligned}
$$

where $\sigma_b = \mathbf{P}\cdot\mathbf{n}$ is the surface bounded charge density and $\rho_b = -\nabla\cdot\mathbf{P}$ is the volume bound charge density (the minus sign is because $\mathbf{P}$ is defined as the dipole moment per unit volume).

The derivation can be seen from Griffiths' book page 176.

Remember to drop primes!

In bulk, Gauss's Law is $\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$, but $\rho = \rho_{free} + \rho_{bound}$ remember that $\rho_{bound} = -\nabla\cdot\mathbf{P}$ and hence $\nabla\cdot\mathbf{E} = \frac{\rho_{free}}{\epsilon_0} - \frac{\nabla\cdot\mathbf{P}}{\epsilon_0}$

We define electric displacement

$$
\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}
$$

$$
\Rightarrow \nabla\cdot\mathbf{D} = \rho_{free}
$$

That is much easier, since it only depends on free charges.

The integral form is

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{free}$$

Linear dielectrics

For a linear dielectric, $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ where χ_e is the electric susceptibility, which is dimensionless.

$$\Rightarrow \mathbf{D} = \epsilon_0 \mathbf{E} + P = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

$(1 + \chi_e)$ is the relative permitivity ϵ_r and it is dimensionless.

Example:

Field from a point charge in a linear dielectric:

$$\int_s \mathbf{D} \cdot d\mathbf{s} = Q_{free}$$

$$\Rightarrow \mathbf{D} = \frac{Q_f}{4\pi r^2} \hat{\mathbf{r}}$$

$$\Rightarrow \mathbf{E} = \frac{Q_f}{4\pi \epsilon_0 r^2} \hat{\mathbf{r}}$$

> Recap of lecture 5:
>
> Polarization: $\overrightarrow{P} = n\overrightarrow{p}$
>
> Bound charges: $\rho_b = -\nabla \cdot \overrightarrow{P}$, $\sigma_b = \overrightarrow{P} \cdot \hat{n}$
>
> Electric displacement: $\overrightarrow{D} = \epsilon_0 \overrightarrow{E} + \overrightarrow{P} = \epsilon_0 \epsilon_r \overrightarrow{E}$
>
> Maxwell: $\nabla \cdot \overrightarrow{D} = \rho_f$
>
> Linear dielectric: $\overrightarrow{P} = \epsilon_0 \chi_e \overrightarrow{E}$
>
> Where χ_e is the electric susceptibility = $\epsilon_r - 1$

Question Has D solved everything?
-- No. The div equation is easier, but...

> Reminder: $\overrightarrow{\nabla} \cdot \overrightarrow{E} = \frac{\rho_{tot}}{\epsilon_0}$
> $\overrightarrow{\nabla} \cdot \overrightarrow{D} = \rho_f$
>
> And the equation for $\overrightarrow{D}$ is $\epsilon_0 \overrightarrow{E} + \overrightarrow{P} = \epsilon_0 \epsilon_r \overrightarrow{E}$

$$\overrightarrow{\nabla} \times \overrightarrow{E} = 0$$

$$\overrightarrow{\nabla} \times \overrightarrow{D} = \epsilon \overrightarrow{\nabla} \times \overrightarrow{E} + \overrightarrow{\nabla} \times \overrightarrow{P} \neq 0$$

Boundrary conditions

$E_{//}$ is continuous

$$\overrightarrow{\nabla} \times \overrightarrow{E} = 0$$

$$\oint \overrightarrow{E} \cdot \overrightarrow{dl} = 0$$

But $D_{//}$ is not continuous (if $\nabla \times \overrightarrow{P} \neq 0$ at interface)

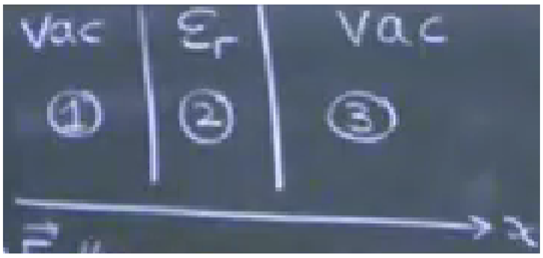
$D_\perp$ is continuous

$$\nabla \cdot \overrightarrow{D} = \rho_f$$

$$\oint \overrightarrow{D} \cdot \overrightarrow{da} = Q_f$$

but $E_\perp$ may not be continuous if we have bound charges at the interface

Example



$$\Rightarrow E_0 // x$$

$$E_1 = E_3 = E_0$$

$D_\perp$ is continuous

$$\Rightarrow D_1 = D_3 = \epsilon_0 E_0 = D_2$$

$D_2 = \epsilon_0 \epsilon_r E_2$

$$\Rightarrow E_2 = \frac{E_0}{\epsilon_r}$$

So there is a smaller E field in the dielectric

$$\begin{aligned} P_2 &= \epsilon_0(\epsilon_r - 1)E_2 \\ &= \frac{\epsilon_0}{\epsilon_r}(\epsilon_r - 1)E_0 \end{aligned}$$

$(P_1 = P_3 = 0)$

Since P_2 is constant, $\nabla \times \overrightarrow{P} = 0 \Rightarrow \rho_b = 0$

$z = o, \sigma_b = \mathbf{P} \cdot \mathbf{n} = \frac{\epsilon_0}{\epsilon_r}(\epsilon_r - 1)E_0$
$z = d, \sigma_b = \mathbf{P} \cdot \mathbf{n} = \frac{\epsilon_0}{\epsilon_r}(\epsilon_r - 1)E_0$

Example:

There is dielectric sphere, with radius a and relative permittiaty \epsilon_r. The sphere is in a uniform electric field $\mathbf{E_0}$. Find V.

> We got to use Laplace's equation

General solution to Laplace's equation in spherical coordinates (with azimuthal symmetry) is

$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r}^{l+1} \right) P_l(\cos\theta)$

where P_l is the Legendre polynomial.

Solve separatively for r > a and r < a

As $r \to \infty, V \to -E_0 r \cos\theta$
As $r \to 0, V$ should be finite (not blow up)
$\Rightarrow B_l = 0$ for r<a

case r<a:
$V = V_{in} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

And for r>a:
$V = V_{out} = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

As for r = a, $E_{//}, D_\perp, V$ are continuous

$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 a \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta)$

$\Rightarrow A_l = \begin{cases} \frac{B_l}{a^{l+1}} & l \neq 1 \\ -E_0 + \frac{B_1}{a^2} & l = 1 \end{cases}$

$E_0 = -\frac{1}{r}\frac{\partial V}{\partial \theta}$ is continuous

$\to$ generates same expressions

That is not terribly useful, so we will try another one

$D_\perp = -\epsilon_0 \epsilon_r \frac{\partial V}{\partial r}$ is continuous

$\Rightarrow -\epsilon_0 \epsilon_r \frac{\partial V}{\partial r}|_{r=a} = -\epsilon_0 \frac{\partial V_{out}}{\partial r}|_{r=a}$

$\epsilon_r \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) = -E_0 \cos\theta + \sum_{l=0}^{\infty} -(l+1) \frac{B_l}{a^{l+2}} P_l(\cos\theta)$

$A_l = \begin{cases} -\frac{1}{\epsilon_r}\frac{l+1}{l}\frac{B_l}{a^{2l+1}} & l \neq 1 \\ \frac{1}{\epsilon_r}(-E_0 - \frac{2B_l}{a^3}) & l = 1 \end{cases}$

Hence, we got two expressions for A in terms of l. They cant both be right unless $A_l = B_l = 0$ for $l \neq 1$

On the other hand, we have

$A_1 = \underbrace{-E_0 + \frac{B_1}{a^3}}_{①} = \underbrace{\frac{1}{\epsilon_r}\left(-E_0 - \frac{2B_0}{a^3}\right)}_{②}$

$\epsilon_r(① - ②) = 0$

Hence

$(1-\epsilon_r)E_0+(\epsilon_r+2)\frac{B_1}{a^3}=0$

$\Rightarrow B_1=a^3E_0\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)$

$\rightarrow A_1=-\frac{3E_0}{\epsilon_r+2}$

Now, put everything together

$$V=\begin{cases}\underbrace{\frac{3}{\epsilon_r+2}E_0r\cos\theta}_{\text{uniform field}} & r<a\\ \underbrace{-E_0r\cos\theta}_{\text{uniform field}}+\underbrace{\left(\frac{E_r-1}{E_r+2}\,\frac{E_0a^3\cos\theta}{r^2}\right)}_{\text{dipole field}} & r>a\end{cases}$$

Interestingly, we all have uniform fields applied inside and outside. And if in vacuum, we will have uniform field outside (dipole field vanishes)

P inside sphere:

$P=\epsilon_0(\epsilon_r-1)E_{in}$

where E_in is $\frac{3E_0}{\epsilon_r+2}$

Dipole moment of sphere

$p=\frac{4}{3}\pi a^3P=\frac{4\pi a^3\epsilon_0(\epsilon_r-1)E_0}{\epsilon_r+2}$

Polarizability

$$\begin{aligned}\alpha&=\frac{P}{E_0}\\&=4\pi a^3\epsilon_0\frac{\epsilon_r-1}{\epsilon_r+2}\end{aligned}$$

Bound charge on the surface

$$\begin{aligned}\sigma_b&=\mathbf{P}\cdot\hat{\mathbf{n}}\\&=P\cos\theta\\&=\frac{3\epsilon_0(\epsilon_r-1)}{\epsilon_r+2}E_0\cos\theta\end{aligned}$$