

Mathematics

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Chapter 1

Pairs

1.1 Unordered, Ordered Pairs

Notion. *Summary*

*Sets define ordered and unordered pairs. Ordered pairs contain coordinates.
Coordinates define equivalent.*

Definition. Unordered Pair

$\{a, b\}$ is a set whose elements are exactly a and b .

Definition. Ordered Pair

Ordered pair of a and b is denoted by (a, b) .

Definition. Coordinate

a is the first coordinate of the pair, b is the second coordinate.

Definition.

$(a, b) = \{\{a\}, \{a, b\}\}.$

Definition.

For two different sets α and β , $(a, b) = \{\{a, \alpha\}, \{b, \beta\}\}.$

Note. An alternative definition of ordered pairs.

Theorem 1.1.1.

$(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Proof. Obviously. □

Proposition 1.1.1.1.

Two *ordered pairs* are *equal* if and only if their *first coordinates* are equal and their *second coordinates* are equal.

Proof. Obviously. □

Note. Ordered pair should be defined in such a way that two ordered pairs are equal if and only if their first coordinates are equal and their second coordinates are equal.

Proposition 1.1.1.2.

$(a, b) \neq (b, a)$ if $a \neq b$.

Proof. Obviously. □

Proposition 1.1.1.3.

If $a \neq b$, (a, b) has two elements, a *singleton* $\{a\}$ and an *unordered pair* $\{a, b\}$. We find the *first coordinate* by looking at the element of $\{a\}$. The *second coordinate* is then the other element of $\{a, b\}$. If $a = b$, then $(a, a) = \{\{a\}\}$ has only one element.

Proof. Obviously. □

Note. If $a \neq b$, (a, b) has two elements, a singleton $\{a\}$ and an unordered pair $\{a, b\}$. The first coordinate is the element of $\{a\}$; the second coordinate is the other element of $\{a, b\}$.

Definition. One-Tuples

$(a) = a$.

Definition. Ordered Triples

$(a, b, c) = ((a, b), c)$.

Definition. Ordered Quadruples

$(a, b, c, d) = (((a, b), c), d)$.

Property 1.1.1.

$(a, b) \in \mathcal{P}(\mathcal{P}(\{a, b\}))$. More generally, if $a \in A$ and $b \in A$, then $(a, b) \in \mathcal{P}(\mathcal{P}(\{a, b\}))$.

Proof. Obviously. □

Property 1.1.2.

$a, b \in \bigcup(a, b)$.

Proof. Obviously. □

Property 1.1.3.

(a, b) , (a, b, c) , and (a, b, c, d) exist for all a, b, c , and d .

Proof. Obviously. □

Property 1.1.4.

If $(a, b) = (b, a)$, then $a = b$.

Proof. Obviously. □

Property 1.1.5.

$(a, b, c) = (a', b', c')$ implies $a = a'$, $b = b'$, and $c = c'$. $(a, b, c, d) = (a', b', c', d')$ implies $a = a'$, $b = b'$, $c = c'$, and $d = d'$.

Proof. Obviously. □

1.2 Relation

Notion. Summary

Underling.

Definition. Binary Relation

A set A is a binary relation if all elements of R are ordered pairs, i.e., if for any $z \in R$ there exist x and y such that $z = (x, y)$.

Note. Relations between objects of two sorts called binary relations.

Note. A binary relation is determined by specifying all ordered pairs of objects in that relation; it does not matter by what property the set of these ordered pairs is described.

Definition. Inverse

Let R be a binary relation. The inverse of R is the set

$$R^{-1} = \{ z \mid z = (x, y) \wedge \exists x, y ((y, x) \in R) \}.$$

Definition. Domain

Let A be a binary relation. The set of all x which are in relation A with some y is called the domain of R and denoted by $\text{dom } R$.

Note. $\text{dom } R$ is the set of all first coordinates of ordered pairs in R .

Definition. Range

Let A be a binary relation. The set of all y such that, for some x , x is in relation R with y is called the range of A , denoted by $\text{ran } R$.

Note. $\text{ran } R$ is the set of all second coordinates of ordered pairs in R .

Definition. Field

Let A be a binary relation. The set $\text{dom } R \cup \text{ran } R$ is called the field of R and is denoted by $\text{field } R$. If $\text{field } R \subseteq X$, we say that R is a relation in X or that R is a relation between elements of X .

Proposition 1.2.0.1.

Both $\text{dom } R$ and $\text{ran } R$ exist for any relation R .

Proof. Obviously. □

Definition. Image

The image of A under R is the set of all y from the range of R related in R to some element of A ; it is denoted by $R[A]$.

Definition. Inverse Image

The inverse image of B under R is the set of all x from the domain of R related in R to some element of B ; it is denoted by $R^{-1}[A]$.

Proposition 1.2.0.2.

$\text{dom } R = \text{ran } R^{-1}$.

Proof. Obviously

□

Proposition 1.2.0.3.

The inverse image of B under R is equal to the image of B under R^{-1} .

Proof. Obviously

□

Definition. Composition

Let R and S be binary relations. The composition of R and S is the relation.

$$S \circ R = \{ (x, z) \mid \exists y ((x, y) \in R \wedge (y, z) \in S) \}$$

.

Note. $(x, z) \in S \circ R$ means that for some y , $(x, y) \in R$ and $(y, z) \in S$.