Mathematics

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Chapter 1

Pairs

1.1 Unordered, Ordered Pairs

Notion. *Summary*
Set Derivations: Ordered and Unordered Pairs.
Set Structures: Coordinates.
Coordinate Derications: Equivalent of Sets.

Definition. Unordered Pair
$\{a, b\}$ is a set whose elements are exactly a and b.

Definition. Ordered Pair
Ordered pair of a and b is denoted by (a, b).

Definition. Coordinate
a is the first coordinate of the pair, b is the second coordinate.

Definition.
$(a, b) = \{\{a\}, \{a, b\}\}$.

Definition.
For two different sets α and β, $(a, b) = \{\{a, \alpha\}, \{b, \beta\}\}$.

Note. *An alternative definition of ordered pairs.*

Theorem 1.1.1.
$(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Proof. Obviously. □

Proposition 1.1.1.1.
Two *ordered pairs* are *equal* if and only if their *first coordinates* are equal and their *second coordinates* are equal.

Proof. Obviously. □

Note. Ordered pair should be defined in such a way that two ordered pairs are equal if and only if their first coordinates are equal and their second coordinates are equal.

Proposition 1.1.1.2.
$(a, b) \neq (b, a)$ if $a \neq b$.

Proof. Obviously. □

Proposition 1.1.1.3.
If $a \neq b$, (a, b) has two elements, a *singleton* $\{a\}$ and an *unordered pair* $\{a, b\}$. We find the *first coordinate* by looking at the element of $\{a\}$. The *second coordinate* is then the other element of $\{a, b\}$. If $a = b$, then $(a, a) = \{\{a\}\}$ has only one element.

Proof. Obviously. □

Note. If $a \neq b$, (a, b) has two elements, a singleton $\{a\}$ and an unordered pair $\{a, b\}$. The first coordinate is the element of $\{a\}$; the second coordinate is the other element of $\{a, b\}$.

Definition. One-Tuples
$(a) = a$.

Definition. Ordered Triples
$(a, b, c) = ((a, b), c)$.

Definition. Ordered Quadruples
$(a, b, c, d) = (((a, b), c), d)$.

Property 1.1.1.
$(a, b) \in \mathcal{P}(\mathcal{P}(\{a, b\}))$. More generally, if $a \in A$ and $b \in A$, then $(a, b) \in \mathcal{P}(\mathcal{P}(\{a, b\}))$.

Proof. Obviously. □

Property 1.1.2.
$a,\ b \in \bigcup(a, b)$.

Proof. Obviously. □

Property 1.1.3.
(a, b), (a, b, c), and (a, b, c, d) exist for all a, b, c, and d.

Proof. Obviously. □

Property 1.1.4.
If $(a, b) = (b, a)$, then $a = b$.

Proof. Obviously. □

Property 1.1.5.
$(a, b, c) = (a', b', c')$ implies $a = a'$, $b = b'$, and $c = c'$. $(a, b, c, d) = (a', b', c', d')$ implies $a = a'$, $b = b'$, $c = c'$, and $d = d'$.

Proof. Obviously. □

1.2 Relation

Notion. *Summary*
Binary Relation Structures: Domain, Range, Field, Image.
Binary Relation Operators: Inverse, Composition, Cartesian Product,
Binary Relation Instances: Membership Relation, Identity Relation.
Binary Relation Generalizations: Unary Relation, Ternary Relation.

Definition. Binary Relation
A set A is a binary relation if all elements of R are ordered pairs, i.e., if for any $z \in R$ there exist x and y such that $z = (x, y)$. A binary relation A is in A if and only if $R \subseteq A^2$

Note. *Relations between objects of two sorts called binary relations.*

Note. *A binary relation is determined by specifying all ordered pairs of objects in that relation; it does not matter by what property the set of these ordered pairs is described.*

Definition. Domain
Let A be a binary relation. The set of all x which are in relation A with some y is called the domain of R and denoted by $\operatorname{dom} R$.

Note. $\operatorname{dom} R$ *is the set of all first coordinates of ordered pairs in* R.

Definition. Range
Let A be a binary relation. The set of all y such that, for some x, x is in relation R with y is called the range of A, denoted by $\operatorname{ran} R$.

Note. $\operatorname{ran} R$ *is the set of all second coordinates of ordered pairs in* R.

Definition. Field
Let A be a binary relation. The set $\operatorname{dom} R \cup \operatorname{ran} R$ is called the field of R and is denoted by $\operatorname{field} R$. If $\operatorname{field} R \subseteq X$, we say that R is a relation in X or that R is a relation between elements of X.

Proposition 1.2.0.1.
Both $\operatorname{dom} R$ and $\operatorname{ran} R$ *exist* for any relation R.

Proof. Obviously. □

Definition. Image
The image of A under R is the set of all y from the range of R related in R to some element of A; it is denoted by $R[A]$.

Definition. Inverse Image
The inverse image of B under R is the set of all x from the domain of R related in R to some element of B; it is denoted by $R^{-1}[A]$.

Proposition 1.2.0.2.
$\operatorname{dom} R = \operatorname{ran} R^{-1}$.

Proof. Obviously □

Proposition 1.2.0.3.
The *inverse image of B under R* is equal to the *image of B under R^{-1}*.

Proof. Obviously □

Definition. Inverse
Let R be a binary relation. The inverse of R is the set

$$R^{-1} = \{\, z \mid z = (x, y) \land \exists\, x, y \left((y, x) \in R\right) \}.$$

Definition. Composition
Let R and S be binary relations. The composition of R and S is the relation.

$$S \circ R = \{\, (x, z) \mid \exists\, y \left((x, y) \in R \land (y, z) \in S\right) \}.$$

Note. *$(x, z) \in S \circ R$ means that for some y, $(x, y) \in R$ and $(y, z) \in S$.*

Definition. Cartesian Product
Let A and B be sets. The set of all ordered pairs whose first coordinate is from A and whose second coordinate is from B is called the cartesian product of A and B and denoted $A \times B$.

Note. *$A \times B$ is a relation in which every element of A is related to every element of B.*

Proposition 1.2.0.4.
$A \times B$ exists.

Proof. $A \times B = \{ w \in \mathcal{P}(\mathcal{P}(\{a,b\})) \mid w = (a,b) \wedge \exists\, a, b \left(a \in A \wedge b \in B\right) \}$. □

Proposition 1.2.0.5.
$(A \times B) \times C = A \times B \times C$.

Proof. Obviously. □

Note. $A \times B \times C = \{ (a,b,c) \mid a \in A \wedge b \in B \wedge c \in C \}$.

Definition. Unary Relation
A unary relation is any set. A unary relation in A is any subset of A.

Definition. Ternary Relation
A ternary relation is a set of unordered triples. More explicitly, S is a ternary relation if for every $u \in S$, there exist x, y, and z such that $u = (x,y,z)$. If $S \subseteq A^3$, we say that S is a ternary relation in A.

Definition. Membership Relation
The membership relation on A is defined by

$$\in_A = \{ (a,b) \mid a, b \in A \wedge a \in b \}.$$

Definition. Identity Relation
The identity relation on A is defined by

$$\mathrm{Id}_A = \{ (a,b) \mid a, b \in A \wedge a = b \}.$$

Property 1.2.1.
Let A be a binary relation. $\operatorname{dom} R$ and $\operatorname{ran} R$ *exist*.

Proof. $(x,y) \in R$ implies $x, y \in \bigcup(\bigcup R)$. □

Remark. *Union flatten a set. Powerset structure a set.*

Property 1.2.2.
R^{-1} *exist*.

Proof. $R^{-1} \subseteq \operatorname{dom} R \times \operatorname{ran} R$. □

Property 1.2.3.
$S \circ R$ *exist*.

Proof. $S \circ R \subseteq \operatorname{dom} R \times \operatorname{ran} S$. □

Property 1.2.4.
For any three binary relations R, S, and T, $T \circ (S \circ R) = (T \circ S) \circ R$.

Proof. $S \circ R \subseteq \operatorname{dom} R \times \operatorname{ran} S$. □

Property 1.2.5.
$A \times B \times C$ *exist*.

Proof. Obviously. □

Property 1.2.6.
Let R be a binary relation and A and B sets.

1. $R[A \cup B] = R[A] \cup R[B]$
2. $R[A \cap B] \subseteq R[A] \cap R[B]$
3. $R[A - B] \supseteq R[A] - R[B]$
4. $R^{-1}[A \cup B] = R^{-1}[A] \cup R^{-1}[B]$
5. $R^{-1}[A \cap B] \subseteq R^{-1}[A] \cap R^{-1}[B]$
6. $R^{-1}[A - B] \supseteq R^{-1}[A] - R^{-1}[B]$
7. $R^{-1}[A - B] \supseteq R^{-1}[A] - R^{-1}[B]$
8. $R^{-1}[R[A]] \supseteq A \cap \operatorname{dom} R$
9. $R[R^{-1}[B]] \supseteq B \cap \operatorname{ran} R$

Proof. Obviously. □

Property 1.2.7.
Let $R \subseteq X \times Y$.

1. $R[X] = \operatorname{ran} R$.

2. $R^{-1}[X] = \operatorname{dom} R$.
3. If $a \notin \operatorname{dom} R$, $R[\{a\}] = \emptyset$.
4. If $b \notin \operatorname{ran} R$, $R^{-1}[\{b\}] = \emptyset$.
5. $\operatorname{dom} R = \operatorname{ran} R^{-1}$.
6. $\operatorname{ran} R = \operatorname{dom} R^{-1}$.
7. $(R^{-1})^{-1} = R$.
8. $R^{-1} \circ R \supseteq \mathrm{Id}_{\operatorname{dom} R}$.
9. $R \circ R^{-1} \supseteq \mathrm{Id}_{\operatorname{ran} R}$.

Proof. Obviously. □

Property 1.2.8.

1. $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.
2. $(A_1 \cup A_2) \times B = (A_1 \times B) \cup (A_2 \times B)$.
3. $(A_1 \cap A_2) \times B = (A_1 \times B) \cap (A_2 \times B)$.
4. $(A_1 - A_2) \times B = (A_1 \times B) - (A_2 \times B)$.
5. $(A_1 \triangle A_2) \times B = (A_1 \times B) \triangle (A_2 \times B)$.

Proof. Obviously. □