Mathematics

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August 11, 2024

Contents

Chapter 1

Pairs

1.1   Unordered, Ordered Pairs

Notion. *Summary*
Sets define ordered and unordered pairs. Ordered pairs contain coordinates.
Coordinates define equivalent.

Definition.   Unordered Pair
$\{a, b\}$ is a set whose elements are exactly a and b.

Definition.   Ordered Pair
Ordered pair of a and b is denoted by (a, b).

Definition.   Coordinate
a is the first coordinate of the pair, b is the second coordinate.

Definition.
$(a, b) = \{\{a\}, \{a, b\}\}$.

Definition.
For two different sets α and β, $(a, b) = \{\{a, \alpha\}, \{b, \beta\}\}$.

Note. An alternative definition of ordered pairs.

Theorem 1.1.1.
$(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Proof. Obviously.   □

Proposition 1.1.1.1.
Two *ordered pairs* are *equal* if and only if their *first coordinates* are equal and their *second coordinates* are equal.

Proof. Obviously.   □

Note. *Ordered pair should be defined in such a way that two ordered pairs are equal if and only if their first coordinates are equal and their second coordinates are equal.*

Proposition 1.1.1.2.
$(a, b) \neq (b, a)$ if $a \neq b$.

Proof. Obviously.   □

Proposition 1.1.1.3.
If $a \neq b$, (a, b) has two elements, a *singleton* $\{a\}$ and an *unordered pair* $\{a, b\}$. We find the *first coordinate* by looking at the element of $\{a\}$. The *second coordinate* is then the other element of $\{a, b\}$. If $a = b$, then $(a, a) = \{\{a\}\}$ has only one element.

Proof. Obviously.   □

Note. *If $a \neq b$, (a, b) has two elements, a singleton $\{a\}$ and an unordered pair $\{a, b\}$. The first coordinate is the element of $\{a\}$; the second coordinate is the other element of $\{a, b\}$.*

Definition. One-Tuples
$(a) = a$.

Definition. Ordered Triples
$(a, b, c) = ((a, b), c)$.

Definition. Ordered Quadruples
$(a, b, c, d) = (((a, b), c), d)$.

Property 1.1.1.
$(a, b) \in \mathcal{P}(\mathcal{P}(\{a, b\}))$. More generally, if $a \in A$ and $b \in A$, then $(a, b) \in \mathcal{P}(\mathcal{P}(\{a, b\}))$.

Proof. Obviously. □

Property 1.1.2.
$a, b \in \bigcup(a, b)$.

Proof. Obviously. □

Property 1.1.3.
(a, b), (a, b, c), and (a, b, c, d) exist for all a, b, c, and d.

Proof. Obviously. □

Property 1.1.4.
If $(a, b) = (b, a)$, then $a = b$.

Proof. Obviously. □

Property 1.1.5.
$(a, b, c) = (a', b', c')$ implies $a = a'$, $b = b'$, and $c = c'$. $(a, b, c, d) = (a', b', c', d')$ implies $a = a'$, $b = b'$, $c = c'$, and $d = d'$.

Proof. Obviously. □

1.2 Relation

Notion. *Summary*
Undering.

Definition. Binary Relation
A set A is a binary relation if all elements of R are ordered pairs, i.e., if for any $z \in R$ there exist x and y such that $z = (x, y)$. A binary relation A is in A if and only if $R \subseteq A^2$

Definition.   Inverse
Let R be a binary relation. The inverse of R is the set

$$R^{-1} = \{\, z \mid z = (x,y) \wedge \exists\, x, y \left((y,x) \in R\right) \,\}.$$

Definition.   Domain
Let A be a binary relation. The set of all x which are in relation A with some y is called the domain of R and denoted by $\operatorname{dom} R$.

Definition.   Range
Let A be a binary relation. The set of all y such that, for some x, x is in relation R with y is called the range of A, denoted by $\operatorname{ran} R$.

Definition.   Field
Let A be a binary relation. The set $\operatorname{dom} R \cup \operatorname{ran} R$ is called the field of R and is denoted by $\operatorname{field} R$. If $\operatorname{field} R \subseteq X$, we say that R is a relation in X or that R is a relation between elements of X.

Proposition 1.2.0.1.
Both $\operatorname{dom} R$ and $\operatorname{ran} R$ *exist* for any relation R.

Proof. Obviously.   □

Definition.   Image
The image of A under R is the set of all y from the range of R related in R to some element of A; it is denoted by $R[A]$.

Definition.   Inverse Image
The inverse image of B under R is the set of all x from the domain of R related in R to some element of B; it is denoted by $R^{-1}[A]$.

Proposition 1.2.0.2.
$\operatorname{dom} R = \operatorname{ran} R^{-1}$.

Proof. Obviously $\qquad\qquad\qquad\square$

Proposition 1.2.0.3.
The *inverse image of B under R* is equal to the *image of B under R^{-1}*.

Proof. Obviously $\qquad\qquad\qquad\square$

Definition.   Composition
Let R and S be binary relations. The composition of R and S is the relation.

$$S \circ R = \{\, (x, z) \mid \exists\, y \,\big((x, y) \in R \wedge (y, z) \in S\big) \,\}.$$

Note. $(x, z) \in S \circ R$ *means that for some* y, $(x, y) \in R$ *and* $(y, z) \in S$.

Definition.   Cartesian Product
Let A and B be sets. The set of all ordered pairs whose first coordinate is from A and whose second coordinate is from B is called the cartesian product of A and B and denoted $A \times B$.

Note. $A \times B$ *is a relation in which every element of A is related to every element of* B.

Proposition 1.2.0.4.
$A \times B$ *exists*.

Proof. $A \times B = \{\, w \in \mathcal{P}(\mathcal{P}(\{a, b\})) \mid w = (a, b) \wedge \exists\, a, b \,\big(a \in A \wedge b \in B\big) \,\}.$ $\qquad\square$

Proposition 1.2.0.5.
$(A \times B) \times C = A \times B \times C$.

Proof. Obviously.   □

Note. $A \times B \times C = \{\, (a, b, c) \mid a \in A \land b \in B \land c \in C \,\}.$

Definition.   Unary Relation
A unary relation is any set. A unary relation in A is any subset of A.

Definition.   Ternary Relation
A ternary relation is a set of unordered triples. More explicitly, S is a ternary relation if for every $u \in S$, there exist x, y, and z such that $u = (x, y, z)$. If $S \subseteq A^3$, we say that S is a ternary relation in A.

Definition.   Membership Relation
The membership relation on A is defined by

$$\in_A = \{\, (a, b) \mid a, b \in A \land a \in b \,\}.$$

Definition.   Identity Relation
The identity relation on A is defined by

$$Id_A = \{\, (a, b) \mid a, b \in A \land a = b \,\}.$$