Mathematics

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Part I

Logic

Chapter 1

Argument

1.1 Notion

Definition 1.1. Sentence
Undefined object.

Definition 1.2. Statement
The use of a sentence that has a definite, fixed truth-value. The **sentences** in an argument must express statements—that is, say something that is either true or false.

Definition 1.3. Sentence Constant
A symbol abbreviating an sentence in natural language, atomic or compound.

Definition 1.4. Statement Variable
A symbol that represents no statement and thus has no truth-value but for which statements can be substituted; what does have a truth-value is a statement we substitute for it, The truth-value of the statement substituted for the variable will thus vary according to which statement is substituted.

Definition 1.5. Substitution
An operation performed by putting a statement in place of a variable.

Remark. *We obtain statements from statement forms by substitution.*

Assumption 1.1.
All statements have a fixed **truth-value**—that is, are either true or false.

Remark. *A sentence can be used to make different statements, depending on circumstances.*

Definition 1.6. Argument
A set of sentences, one of which, the conclusion, is claimed to be supported by the others, the premises. The **premises** are the reasons given in support of an argument's conclusion. The **conclusion** is the statement in an argument that is argued for on the basis of the argument's premises.

Remark. *Not just any group of sentences makes an argument.*

Definition 1.7. Argument Form
Informally, the logical structure of an argument. Formally, a group of sentence forms, all of whose substitution instances are arguments.

Definition 1.8. (Deductively) Valid
An argument is **valid** if and only if it is not possible for all of its premises to be true and its conclusion false.

Theorem 1.1.1.
If all premises of a valid argument are true, then its conclusion must be true.

Proof. Let P be all premises of a valid argument are true and C be its conclusion must be true.

1.	P	Premise
2.	$\neg(P \wedge \neg Q)$	Definition: valid argument
3.	$\neg P \vee Q$	2
4.	Q	1, 3

□

Corollary 1.1.2.
The conclusion of a valid argument follows logically from the premises.

Proof. Obviously. □

Note. *The truth of the premises of a valid argument guarantees the truth of its conclusion.*

Remark. *If all you know about an argument is that it is valid, that alone tells you nothing about whether the premises are true or the conclusion is true. From the mere fact that an argument is invalid you can draw no conclusion whatsoever about the truth or falsity of the premises or the conclusion.*

Definition 1.9. Inductive Argument
An argument that is not valid but whose premises provide some measure of support for its conclusion.

Remark. *It is possible for the premises of a strong inductive argument to be true and yet the conclusion be false.*

Note. *An inductively strong argument does not guarantee that if its premises are true, then its conclusion also will be true, it does make its conclusion more probable.*

Definition 1.10. Sound Argument
An argument that is valid and has all true premises.

Theorem 1.1.3.
It is not possible that a deductively valid argument be sound, yet have a false conclusion.

Proof. Let S be a deductively valid argument is sound P be all premises are true and C be the conclusion is true.

1.	$S \wedge \neg C$	Assumed Premise
2.	$P \implies C \wedge P$	Definition: Sound
3.	C	2
4.	$(S \wedge \neg C) \wedge C$	1, 3
5.	$S \wedge (\neg C \wedge C)$	4
6.	$\neg(S \wedge \neg C)$	5, Indirect Proof

□

Definition 1.11. Consistent
A set of statements is consistent if and only if it is possible for all the statements to be true.

Theorem 1.1.4.
If an argument is valid, then it is not possible for its premises to be true and its conclusion false.

Proof. Let V be an argument is valid, P be all premises are true and C be the conclusion is true.

1.	$P \implies Q$	Definition: Valid Argument
2.	$\neg P \vee Q$	1
3.	$\neg(P \wedge \neg Q)$	2

□

Corollary 1.1.5.
An argument is valid if and only if it is inconsistent to say that all its premises are true and its conclusion is false.

Proof. Obviously. □

Chapter 2

Sentential Logic

2.1 Notion

Definition 2.1. Sentence Connective
A term or phrase used to make a larger sentence from two smaller ones. e.g., and, or, if ... then, if and only if, not

Definition 2.2. Atomic(Simple) Sentence, Compound Sentence
Atomic sentence is a sentence that contains no sentence connectives; **Compound** sentence is a sentence containing at least one sentence connective.

Remark. *Compound Sentences are sentences built from shorter sentences by means of sentence connectives.*

Note. ***Sentence logic*** *can be developed without considering the interior structure of atomic sentences.* ***Symbolic logic*** *is the modern logic that includes sentential logic and predicate logic.*

Definition 2.3. Truth-Value
There are two truth-values—namely, true and false.

Remark. *Every sentence in sentential logic has a definite truth-value, and the truth-value of every compound sentence is a function of the truth-value of its component (or atomic) sentences.*

Definition 2.4. Truth Condition
The conditions under which a statement is true are called its truth conditions.

Definition 2.5. Truth-Function
A function that takes one or more truth-values as its input and returns a single truth-value as its output.

Definition 2.6. Truth-Functional Operator
An operator is truth-functional if the truth-values of the sentences formed by its use are determined by the truth-values of the sentences it connects. Similarly,

sentence forms constructed by means of truth-functional sentence connectives are such that the truth-values of their substitution instances are determined by the truth-values of their component sentences.

Remark. *Our system of logic has five truth-functional operators. One operator, not, takes only one input; the other four, and, or, if ... then, and if and only if, take two.*

Remark. *The meaning of a logical operator is given by its truth conditions.*

Definition 2.7. Truth Table
A table giving the truth-values of all possible substitution instances of a given sentence form, in terms of the possible truth-values of the component sentences of these substitution instances.

Remark. *A statement does not have a truth table; a statement has a line in a truth table, depending on the truth-value of its component statements. Truth tables are not given for statements but only for statement forms.*

Definition 2.8. Conjunction
In sentential logic, a compound sentence (or sentence form) whose main connective is dot connective. **dot** connective is "and," "and on the contrary," "also," "although," "both ... and," "but," "despite (in spite of) the fact that," "however," "on the other hand," "still," "while," "yet," or a similar term. **Conjunct** is one of the sentences joined together by dot connective.

Definition 2.9. Dot Operator
Let T and F be truth-value true and false seperately. The dot operator is the set $\{((T,T),T),((T,F),F),((F,T),F),((F,F),F)\}$.

Proposition 2.1.1.
The dot is commutative.

Proof. Let P, Q be statement.

1.	$P \cdot Q$	Assumed Premise
2.	P	1
3.	$Q \cdot P$	2
4.	$(P \cdot Q) \implies (Q \cdot P)$	1, 3, Conditional Proof
5.	$Q \cdot P$	Assumed Premise
6.	P	5
7.	$P \cdot Q$	6
8.	$(Q \cdot P) \implies (P \cdot Q)$	5, 7, Conditional Proof

□

Definition 2.10. Logically Equivalent
A sentence p is logically equivalent to a sentence q if and only if $p \equiv q$ is a tautology.

Remark. *p and q have the same truth-values in every line of the truth table.*

Algorithm 2.1. Test of a Proposed Symbolization
If it is possible for your proposed symbolization and the sentence in natural language would have different truth-values under the same circumstances, the proposed symbolization is not an adequate translation.

Algorithm 2.2. Test of the Truth-Functional Property of a Connective
If a connective is truth-functional, you will never be able to describe situations where in one case the sentence is true and in the other the sentence is false and yet the truth-value of the component sentence or sentences is the same in both cases.

Part II

Set Theory