Mathematics

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Contents

Part I

Set Theory

Chapter 1

Sets

1.1 Notions

Definition. Set, Element (Member)
A set is a collection (group, or conglomerate) into a whole of definite, distinct objects of our intuition or our thought. The objects from which a given set is composed are called elements or members of the set. We also say that they belong to that set.

Note. *The human mind possesses the ability to abstract, to think of a variety of different objects as being bound together by some common property, and thus to form a set of objects having that property.*

Note. *By merely defining a set we do not prove its existence. There are properties which do not define sets.*

Definition. Variable
Unspecified, arbitrary sets.

Definition. identical with, equal to
We use the identity sign "=" to express that two variables denote the same set. $X = Y$ if X is the same set as Y.

Definition. System of Sets, Collection of Sets
Elements of the set are sets.

Definition. Property
The property is the ability to think of these objects (as being) together.

Instance 1.1.1. Membership
"... is an element (member) of ...,", "... belongs to ...."

Note. *Denote by $\in$.*

Note. *All other set-theoretic properties can be stated in terms of membership with the help of logical means; identity, logical connectives, and quantifiers.*

Definition. property of, Parameter
A proposition is a property of $X, Y, \ldots Z$ if it holds or does not hold depending on sets (or called parameters) denoted by $X, Y, \ldots Z$.

Definition. Subset, included in
A is a subset of (included in) B if, for every x, $x \in A$ implies $x \in B$.

Note. *Denoted by $A \subseteq B$.*

Definition. Inclusion
The property $\subseteq$ is called inclusion.

Theorem 1.1.1.
$A \subseteq A$.
If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof. Obviously. □

Definition. Proper Subset, properly contained in
If $A \subseteq B$ and $A \subset B$, we say that A is a proper subset of B (A is properly contained in B) and write $A \subset B$.

Definition. Intersection
The intersection of A and B, $A \cap B$, is the set of all x which belong to both A and B.

Theorem 1.1.2.
$\cap emptyset$ would have to be a set of all sets.

Proof. Obviously. □

Definition. Disjoint
A and B are disjoint if $A \cap B = emptyset$.

Definition. Mutually Disjoint
S is a system of mutually disjoint sets if $A \cap B = emptyset$ for all $A, B \in S$ such that $A \neq B$.

Definition. Union
The union of A and B, $A \cup B$, is the set of all x which belong in either A or B (or both).

Definition. Difference
The difference of A and B, $A - B$, is the set of all $x \in A$ which do not belong to B.

Definition. Symmetric Difference
The symmetric difference of A and B, $A \triangle B$, is defined by $A \triangle B = (A - B) \cup (B - A)$.

Definition. Statement
Properties which have no parameters.

***Note*.** *Either true or false.*

***Note*.** *All mathematical theorems are (true) statements.*

Definition. Proposition
Argument.

1.2 Axioms

Axiom. Existence
There exists a set which has no elements.

Axiom. Extensionality
If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

Lemma 1.2.1.
There exists only one set with no elements.

Proof. Obviously. □

Definition. Empty, Vacuous Set
The (unique) set with no elements is called the empty, vacuous set and is denoted $\emptyset$.

***Note*.** *Occasionally refer to $\emptyset$ as the constant.*

Axiom. Comprehension
Let $P(x)$ be a property of x. For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $P(x)$.

***Note*.** *For any sets p, ..., q and any A, there is a set B (depending on p, ..., q and, of course, on A) consisting exactly of all those $x \in A$ for which $P(x, p, \ldots, q)$.*

Lemma 1.2.2.
For every A, there is only one set B such that $x \in B$ if and only if $x \in A$ and $P(x)$.

Proof. Obviously. □

Definition. $\{x \in A \mid P(x)\}$
The set of all $x \in A$ with the property $P(x)$.

***Note*.** *If there is a set A such that, for all x, $P(x)$ implies $x \in A$, then $\{x \in A \mid P(x)\}$ exists, and, moreover, does not depend on A. That means that if A' is another set such that for all x, $P(x)$ implies $x \in A'$, then $\{x \in A' \mid P(x)\} = \{x \in A \mid P(x)\}$.*

Definition. $\{x \mid P(x)\}$
$\{x \mid P(x)\}$ to be the set $\{x \in A \mid P(x)\}$, where A is any set for which $P(x)$ implies $x \in A$.

Note. *$\{x \mid P(x)\}$ is the set of all x with the property $P(x)$.*

Note. *This notation can be used only after it has been proved that some A contains all x with the property P.*

Property 1.2.3.
$\{x \in \emptyset \mid P(x)\} = \emptyset$.

Proof. Obviously. □

Lemma 1.2.4.
If P and Q are sets, then there is a set R such that $x \in R$ if and only if $x \in P$ and $x \in Q$.

Proof. Obviously. □

Definition. Intersection, Operation
We can introduce a name, say $X \cap Y$, and call $X \cap Y$ the intersection of X and Y. $\cap$ as the operation.

Axiom. Pair
For any A and B, there is a set C such that $x \in C$ if and only if $x = A$ or $x = B$.

Note. *$A \in C$ and $B \in C$, and there are no other elements of C.*

Property 1.2.5.
The set C is unique.

Proof. Obviously. □

Definition. Unordered Pair
Unordered pair of A and B as the set having exactly A and B as its elements.

Axiom. Union
For any set S, there exists a set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

Property 1.2.6.
Set U is unique.

Proof. Obviously. □

Definition. Union
U is the union of S.

Note. *Denoted by $\bigcup S$.*

Axiom. Power Set
For any set S, there exists a set P such that $X \in P$ if and only if $X \subseteq S$.

Property 1.2.7.
The set P is uniquely determined.

Proof. Obviously. $\square$

Note. *We call the set of all subsets of S the power set of S and denote it by $\mathcal{P}(S)$.*