Mathematics

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August 2023

Contents

Part I

Set Theory

Chapter 1

Sets

1.1 Notions

Unter einer Menge verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten in unserer Anschauung oder unseres Denkens (welche die Elemente von M genannt werden) zu einem ganzen.

Definition. Set, Element (Member), belong to
A set is a collection of objects of our intuition. The objects are elements (members) of the set. We say that they belong to that set.

Property defines a set.

Definition. Property
The notion of thinking of objects as being together.

Definition. property of, depends on, Parameter
A proposition is a property of $X, Y, \ldots Z$ if it holds or does not hold depending on sets (or called parameters) $X, Y, \ldots Z$.

Note. *A variety of objects are bound by some common property, and form a set of objects having that property.*

Note. *By merely defining a set, we do not prove its existence. There are properties which do not define sets.*

Definition. Statement
Properties which have no parameters.

Note. *Either true or false.*

Note. *All mathematical theorems are (true) statements.*

Instance 1.1.1. Membership, belongs to
"$\ldots$ is an element (member) of $\ldots$,", "$\ldots$ belongs to $\ldots$". Denote by $\in$.

Note. *Basic set-theoretic property.*

Definition. Proposition
Argument.

Unspecified sets.

Definition. Variable
Unspecified, arbitrary sets.

Definition. same set as, identical with, equal to
$X = Y$ if X is the same set as (is identical with, is equal to) Y.

All other set-theoretic properties can be stated in terms of membership with the help of logical means; identity, logical connectives, and quantifiers.

Definition. Logical Connectives
Expressions like "not ...," " ... and .... ," "if ..., then ...," and " ... if and only if ...."

Definition. Quantifiers
"for all" ("for every") and "there is" ("there exists").

Definition. System of Sets, Collection of Sets
Elements of the set are sets.

1.2 Axioms

Axiom. Existence
There exists a set which has no elements.

Lemma 1.2.1.
There exists only one set with no elements.

Proof. Obviously. □

Definition. Empty, Vacuous Set
The unique set with no elements is called the empty, vacuous set and denoted $\emptyset$.

Note. *Occasionally refer to $\emptyset$ as the constant.*

Axiom. Extensionality
If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

Axiom. Comprehension
Let $P(x)$ be a property of x. For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $P(x)$.

Lemma 1.2.2. Intersection
If P and Q are sets, then there is a unique set R such that $x \in R$ if and only if $x \in P$ and $x \in Q$.

Proof. Obviously. □

Definition. Intersection, Operation
In intersection lemma, R is called the intersection of X and Y and denoted as $X \cap Y$. $\cap$ as the operation.

Lemma 1.2.3.
For every A, there is only one set B such that $x \in B$ if and only if $x \in A$ and $P(x)$.

Proof. Obviously. □

Note. *For any sets $p, \ldots, q$ and any A, there is a set B (depending on $p, \ldots, q$ and, of course, on A) consisting of all $x \in A$ for which $P(x, p, \ldots, q)$.*

Definition. $\{x \in A \mid P(x)\}$
The set of all $x \in A$ with the property $P(x)$.

Note. *If there is a set A such that, for all x, $P(x)$ implies $x \in A$, then $\{x \in A \mid P(x)\}$ exists, and, moreover, does not depend on A. That is, if A' is a set such that for all x, $P(x)$ implies $x \in A'$, then $\{x \in A' \mid P(x)\} = \{x \in A \mid P(x)\}$.*

Definition. $\{x \mid P(x)\}$
$\{x \mid P(x)\}$ to be the set $\{x \in A \mid P(x)\}$, where A is any set for which $P(x)$ implies $x \in A$.

Note. *$\{x \mid P(x)\}$ is the set of all x with the property $P(x)$.*

Note. *This notation can be used only after it has been proved that some A contains all x with the property P.*

Property 1.2.4.
$\{x \in \emptyset \mid P(x)\} = \emptyset$.

Proof. Obviously. □

Axiom. Pair
For any A and B, there is a set C such that $x \in C$ if and only if $x = A$ or $x = B$.

Note. *$A \in C$ and $B \in C$, and there are no other elements of C.*

Property 1.2.5.
The set C is unique.

Proof. Obviously. □

Definition. Unordered Pair
Unordered pair of A and B is the set having exactly A and B as its elements and denoted as $\{A, B\}$.

Note. *If $A = B$, we write $\{A\}$ instead of $\{A, A\}$.*

Axiom. Union
For any set S, there exists a set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

Property 1.2.6. Union
Set U is unique.

Proof. Obviously. $\square$

Definition. Union
U in union property is the union of S and denoted by $\bigcup S$.

Note. *The union of a system of sets S is then a set of precisely those x which belong to some set from the system S.*

Axiom. Power Set
For any set S, there exists a set P such that $X \in P$ if and only if $X \subseteq S$.

Property 1.2.7.
The set P is uniquely determined.

Proof. Obviously. $\square$

Note. *We The set of all subsets of S is called the power set of S and denote it by $\mathcal{P}(S)$.*

1.3 Elementary Operations

Definition. Subset, included in
A is a subset of (included in) B if, for every x, $x \in A$ implies $x \in B$ and denoted by $A \subseteq B$.

Property 1.3.1.
$A \subseteq B \wedge B \subseteq C \implies A \subseteq C$.

Proof. Obviously. $\square$

Definition. Inclusion
The property $\subseteq$.

Theorem 1.3.2.
$A \subseteq A$.
If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof. Obviously. $\square$

Definition. Proper Subset, properly contained in
If $A \subseteq B$ and $A \neq B$, A is a proper subset of B (A is properly contained in B) and denoted by $A \subset B$.

Definition. Intersection
The intersection of A and B, $A \cap B$, is the set of all x which belong to both A and B. The intersection $\bigcap S$ of a nonempty system of sets S: $x \in \bigcap S$ if and only if $x \in A$ for all $A \in S$.

Property 1.3.3.
$\bigcap \emptyset$ would have to be a set of all sets.

Proof. Every x belongs to all $\in \emptyset$. □

Theorem 1.3.4.
$\cap emptyset$ would have to be a set of all sets.

Proof. Obviously. □

Definition. Disjoint
A and B are disjoint if $A \cap B = emptyset$.

Definition. Mutually Disjoint
S is a system of mutually disjoint sets if $A \cap B = emptyset$ for all $A, B \in S$ such that $A \neq B$.

Definition. Union
The union of A and B, $A \cup B$, is the set of all x which belong in either A or B (or both).

Definition. Difference
The difference of A and B, $A - B$, is the set of all $x \in A$ which do not belong to B.

Definition. Symmetric Difference
The symmetric difference of A and B, $A \triangle B$, is defined by $A \triangle B = (A - B) \cup (B - A)$.

Property 1.3.5. Commutativity
$A \cap B = B \cap A$, $A \cup B = B \cup A$, $A \triangle B = B \triangle A$

Proof. Obviously □

Property 1.3.6. Associativity
$(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$,

Proof. Obviously □

Note. *We do not need parentheses for the intersection, union and for more than three sets.*

Property 1.3.7. Distributivity
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. Obviously □

Property 1.3.8. DeMorganlaws
$C - (A \cap B) = (C - A) \cup (C - B)$, $C - (A \cup B) = (C - A) \cap (C - B)$

Proof. Obviously $\square$

Property 1.3.9.
$A - B = \emptyset$ if and only if $A \subseteq B$.

Proof. Obviously. $\square$

Property 1.3.10.
$A \triangle A = \emptyset$

Proof. Obviously. $\square$