Mathematics

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Part I

Logic

Chapter 1

Argument

1.1 Notion

Argument

Definition 1.1. Statement
The use of a sentence that has a definite, fixed truth-value.

Remark. The sentences in an argument must express statements—that is, say something that is either true or false.

Definition 1.2. Sentence Constant
A symbol abbreviating an sentence in natural language, atomic or compound.

Definition 1.3. Statement Variable
A symbol that represents no statement and thus has no truth-value but for which statements can be substituted.

Remark. What does have a truth-value is a statement we substitute for it, The truth-value of the statement substituted for the variable will thus vary according to which statement is substituted.

Definition 1.4. Substitution
An operation performed by putting a statement in place of a variable.

Remark. We obtain statements from statement forms by substitution.

Assumption 1.1.
All statements have a fixed **truth-value**—that is, are either true or false.

Remark. A sentence can be used to make different statements, depending on circumstances.

Definition 1.5. Argument
A set of sentences, one of which, the conclusion, is claimed to be supported by the others, the premises. The **premises** are the reasons given in support of an argument's conclusion. The **conclusion** is the statement in an argument that is argued for on the basis of the argument's premises.

***Remark*.** *Not just any group of sentences makes an argument.*

Definition 1.6. Argument Form
Informally, the logical structure of an argument. Formally, a group of sentence forms, all of whose substitution instances are arguments.

Property

Definition 1.7. (Deductively) Valid
An argument is **valid** if and only if it is not possible for all of its premises to be true and its conclusion false.

Theorem 1.1.1.
If all premises of a valid argument are true, then its conclusion must be true.

Proof. Let P be all premises of a valid argument are true and C be its conclusion must be true.

1.	P	Premise
2.	$\neg(P \wedge \neg Q)$	Definition: valid argument
3.	$\neg P \vee Q$	2
4.	Q	1, 3

□

Corollary 1.1.2.
The conclusion of a valid argument follows logically from the premises.

Proof. Obviously. □

***Note*.** *The truth of the premises of a valid argument guarantees the truth of its conclusion.*

***Remark*.** *If all you know about an argument is that it is valid, that alone tells you nothing about whether the premises are true or the conclusion is true. From the mere fact that an argument is invalid you can draw no conclusion whatsoever about the truth or falsity of the premises or the conclusion.*

Definition 1.8. Inductive Argument
An argument that is not valid but whose premises provide some measure of support for its conclusion.

***Remark*.** *It is possible for the premises of a strong inductive argument to be true and yet the conclusion be false.*

***Note*.** *An inductively strong argument does not guarantee that if its premises are true, then its conclusion also will be true, it does make its conclusion more probable.*

Definition 1.9. Sound Argument
An argument that is valid and has all true premises.

Theorem 1.1.3.
It is not possible that a deductively valid argument be sound, yet have a false conclusion.

Proof. Let S be a deductively valid argument is sound P be all premises are true and C be the conclusion is true.

1.	$S \wedge \neg C$	Assumed Premise
2.	$P \implies C \wedge P$	Definition: Sound
3.	C	2
4.	$(S \wedge \neg C) \wedge C$	1, 3
5.	$S \wedge (\neg C \wedge C)$	4
6.	$\neg(S \wedge \neg C)$	5, Indirect Proof

□

Definition 1.10. Consistent
A set of statements is consistent if and only if it is possible for all the statements to be true.

Theorem 1.1.4.
If an argument is valid, then it is not possible for its premises to be true and its conclusion false.

Proof. Let V be an argument is valid, P be all premises are true and C be the conclusion is true.

1.	$P \implies Q$	Definition: Valid Argument
2.	$\neg P \vee Q$	1
3.	$\neg(P \wedge \neg Q)$	2

□

Corollary 1.1.5.
An argument is valid if and only if it is inconsistent to say that all its premises are true and its conclusion is false.

Proof. Obviously. □

Chapter 2

Sentential Logic

2.1 Notion

Sentence

Definition 2.1. Sentence
A grammatically well-formed unit of natural language that can be used to make a statement, often different statements on different occasions.

Definition 2.2. Compound Sentence
A sentence containing at least one sentence connective.

Definition 2.3. Component Sentence
The sentence or sentences operated on by a truth-functional operator.

Definition 2.4. Sentence Connective
A term or phrase used to make a larger sentence from two smaller ones. e.g., and, or, if ... then, if and only if, not

Definition 2.5. Atomic (Simple) Sentence
A sentence is a sentence that contains no sentence connectives.

Definition 2.6. Compound Sentence
A sentence is a sentence containing at least one sentence connective.

Remark. Compound Sentences are sentences built from shorter sentences by means of sentence connectives.

Definition 2.7. Sentence Form
An expression containing only a sentential variable or sentential variables and logical connectives.

Definition 2.8. Sentential Logic
The logic that deals with relationships holding between sentences, atomic or compound, without dealing with the interior structure of atomic sentences.

Note. *Sentence logic can be developed without considering the interior structure of atomic sentences.*

Definition 2.9. Symbolic Logic
The modern logic that includes sentential logic and predicate logic.

Truth

Definition 2.10. Truth-Value
There are two truth-values—namely, true and false.

Remark. *Every sentence in sentential logic has a definite truth-value, and the truth-value of every compound sentence is a function of the truth-value of its component (or atomic) sentences.*

Definition 2.11. Truth Condition
The conditions under which a statement is true are called its truth conditions.

Definition 2.12. Truth-Function
A function that takes one or more truth-values as its input and returns a single truth-value as its output.

Definition 2.13. Truth-Functional Operator
An operator is truth-functional if the truth-values of the sentences formed by its use are determined by the truth-values of the sentences it connects. Similarly, sentence forms constructed by means of truth-functional sentence connectives are such that the truth-values of their substitution instances are determined by the truth-values of their component sentences.

Remark. *Our system of logic has five truth-functional operators. One operator, not, takes only one input; the other four, and, or, if ... then, and if and only if, take two.*

Remark. *The meaning of a logical operator is given by its truth conditions.*

Definition 2.14. Truth Table
A table giving the truth-values of all possible substitution instances of a given sentence form, in terms of the possible truth-values of the component sentences of these substitution instances.

Remark. *A statement does not have a truth table; a statement has a line in a truth table, depending on the truth-value of its component statements. Truth tables are not given for statements but only for statement forms.*

Connective

Definition 2.15. Scope
The scope of an operator is the component sentence or sentences that the operator operates on. The negation operator operates on a single component sentence. All the other operators operate on two component sentences.

Remark. ***Parentheses*** *are used to indicate the scope of each logical operator in any sentence.*

Remark. *The scope of the negation operator is always the shortest complete sentence that follows it.*

Definition 2.16. Main Connective
The sentence connective that has the greatest scope. The main connective of a sentence is the truth-functional connective whose scope encompasses the entire remainder of the sentence.

Definition 2.17. Well-Formed
A well-formed statement is one constructed according to the formation rules of a language. Sometimes such a statement is called a well-formed formula, abbreviation wff.

Remark. *No sentence is legitimate–that is, well-formed–unless it is clear which operator is the main operator for the sentence and which operators have which component sentences within their scope.*

Definition 2.18. Negation
A sentence whose main connective is "not," "no," or a similar term.

Definition 2.19. Negation Operator
Let T and F be truth-value true and false seperately. The negation operator is a truth-functional connective defined as the set $\{(T, F), (F, T)\}$.

Note. *It operates only on individual sentences.*

Definition 2.20. Conjunction
In sentential logic, a compound sentence (or sentence form) whose main connective is conjunction connective. **Conjunction connective** is "and," "and on the contrary," "also," "although," "both . . . and," "but," "despite (in spite of) the fact that," "however," "on the other hand," "still," "while," "yet," or a similar term. **Conjunct** is one of the sentences joined together by conjunction connective.

Definition 2.21. Conjunction Operator
Let T and F be truth-value true and false seperately. The conjunction operator is a truth-functional connective defined as the set $\{((T, T), T)$, $((T, F), F)$, $((F, T), F)$, $((F, F), F)\}$.

Proposition 2.1.1.
The conjunction operator is commutative.

Proof. Let P, Q be statements.

1.	$P \cdot Q$	Assumed Premise
2.	P	1
3.	$Q \cdot P$	2
4.	$(P \cdot Q) \implies (Q \cdot P)$	1, 3, Conditional Proof
5.	$Q \cdot P$	Assumed Premise
6.	P	5
7.	$P \cdot Q$	6
8.	$(Q \cdot P) \implies (P \cdot Q)$	5, 7, Conditional Proof
9.	$(Q \cdot P) \iff (P \cdot Q)$	4, 8

$\square$

Remark. *p and q have the same truth-values in every line of the truth table.*

Definition 2.22. Disjunctions
A compound sentence whose main connective is an "or." **Disjunctions connective** is "or," "either . . . or," or a similar term. **Disjunct** is either of the component sentences in a disjunction.

Remark. *There are two different senses of the connective "or" in common use.*

Definition 2.23. Disjunction Operator
Let T and F be truth-value true and false seperately. The disjunction operator is a truth-functional connective defined as the set $\{((T,T),T)$, $((T,F),T)$, $((F,T),T)$, $((F,F),F)\}$.

Definition 2.24. Exclusive "Or"
One and only one of the two disjuncts is true, not both. Or, at least one disjunct is false.

Definition 2.25. Exclusive Disjunction
A compound sentence whose main connective is an exclusive "or."

Definition 2.26. Inclusive (Nonexclusive) "Or", or And/Or
At least one of the two disjuncts is true, but leaves open the possibility that both disjuncts are true. Or, at least one of its disjuncts is true.

Remark. *A sentence whose major connective is an exclusive "or" asserts more than it would if the "or" were inclusive. The inclusive "or" is only part of the meaning of the exclusive "or."*

Definition 2.27. Conditional, or Hypothetical
A compound sentence that expresses an "If *dots* then" relationship between its component sentences. The sentence follows "if" in nan "if . . . then" statements is called **antecedent**, and the sentence follows the "then" is called **consequent**. The consequent is **necessary condition** for the antecedent; the antecedent expresses a sufficient condition.

Remark. *Let P, Q be statements. P only if Q is logically equivalent to $\neg P \implies \neg Q$.*

Note. *A material conditional is essentially a statement that the truth of the antecedent is sufficient for the truth of the consequent and that the truth of the consequent is necessary for the truth of the antecedent. The consequent alone may not be sufficient to guarantee the consequent being true.*

Definition 2.28. Material Implication
Let T and F be truth-value true and false seperately. The conditional operator is a truth-functional connective (or truth-function) defined as the set $\{((T,T),T)$, $((T,F),F)$, $((F,T),T)$, $((F,F),F)\}$.

Definition 2.29. Material Conditional
A compound statement whose main connective is material implication. A compound statement expresses an "If . . . then" relationship between its component sentences.

Remark. *Conditional sentences differ with respect to the kind of connection they express between anteced- ent and consequent. The connection between antecedent and consequent is logical, if the antecedent is true implies the consequent must be; the connection is causal if the conditions stated in the antecedent cause the conditions stated in the consequent.*

Note. *The general form of a conditional sentence is "if (antecedent), then (consequent)."*

Definition 2.30. Materially Equivalent
Let P, Q be statements. P, Q are said to be **materially equivalent** (or P if and only if Q) when P, Q have the same truth-value.

Definition 2.31. Logically Equivalent
A sentence p is logically equivalent to a sentence q if and only if $p \equiv q$ is a tautology.

Definition 2.32. Material Equivalence Operator
Let T and F be truth-value true and false seperately. The material equivalence operator is a truth-functional connective (or truth-function) defined as the set $\{((T,T),T)$, $((T,F),F)$, $((F,T),F)$, $((F,F),T)\}$.

Definition 2.33. Material Equivalences (or Material Biconditionals, Biconditionals)
A compound statement whose main sentence connective is material equivalence operator. A compound statement expresses an "if and only if" relationship between two component sentences.

Remark. *Material Equivalences are themselves equivalent to two-directional material conditionals.*

Algorithm 2.1. Test of the Truth-Functional Property of a Connective
If a connective is truth-functional, you will never be able to describe situations where in one case the sentence is true and in the other the sentence is false and yet the truth-value of the component sentence or sentences is the same in both cases.

Notion. Many sentences contain more than one logical operator. e.g., Not . . . Both, Neither . . . Nor, Unless. Let P, Q be statements. Not both P and Q is symbolized $\neg(P \wedge Q)$; Neither P nor Q is symbolized $\neg P \wedge \neg Q$; P unless Q is symbolized $\neg Q \implies P$ (or $P \vee Q$).

Claim 2.1.2.
$\neg Q \implies P$ is logically equivalent to $P \vee Q$.

Proof. Let P, Q be statements.

1.	$\neg Q \implies P$	Assumed Premise
2.	$\neg\neg Q \vee P$	1
3.	$Q \vee P$	2
4.	$P \vee Q$	3
5.	$(\neg Q \implies P) \implies (P \vee Q)$	1, 4, Conditional Proof
6.	$P \vee Q$	Assumed Premise
7.	$Q \vee P$	6.
8.	$\neg\neg Q \vee P$	7.
9.	$\neg Q \implies P$	8.
10.	$(P \vee Q) \implies (\neg Q \implies P)$	6, 9, Conditional Proof
11.	$(P \vee Q) \iff (\neg Q \implies P)$	5, 10

□

Symbolization

Algorithm 2.2. Test of a Proposed Symbolization
If it is possible for your proposed symbolization and the sentence in natural language would have different truth-values under the same circumstances, the proposed symbolization is not an adequate translation. Thatt is, do the symbolization and the sentence in natural language have the same truth conditions.

Language

Definition 2.34. Metalanguage
The language in which we talk about another language.

Definition 2.35. Object Language
The language we are studying.

Remark. *The natural language is our metalanguage, and sentential logic is our object language.*

Part II

Set Theory