Mathematics

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Contents

Chapter 1

Pairs

1.1 Function

Notion. *Summary*
Function Structures: Domain, Range, Restriction, Extension.
Function Operators: Equivalent, Inverse, Composition.
Function Instances: Injective, Invertable, Compatible.
Function Generalizations: Index System of Sets.

Definition. Function
A binary relation F is a function if and only if for every a from dom F there is exactly one b such that aFb. This unique b is called the value of F at a. $F(a)$ is not defined if $a \notin domainF$.

Note. *A binary relation F is called a* function *(or* mapping, correspondence*) if* aFb_1 and aFb_2 imply $b_1 = b_2$ *for any a, b_1, and b_2.*

Note. Function *is a* procedure, *a* rule, assigning *to* any object a from the domain of the function a unique object b, the value of the function at a.

Note. *A* function *represents* a special type of relation, *a relation where every* object a from the domain *is* related to *precisely* one object in the range, *namely, to* the value of the function at a.

Proposition 1.1.1.
Let F and G be functions. *$F = G$ if and only if* $\operatorname{dom} F = \operatorname{dom} G$ *and* $F(x) = G(x)$ for all $x \in \operatorname{dom} F$.

Proof. The Axiom of Extensionality. □

Definition.
Let F be a function and A and B sets.

1. F is a function on A if $\operatorname{dom} F = A$.
2. F is a function into B if $\operatorname{ran} F \subseteq B$.
3. F is a function onto B if $\operatorname{ran} F = B$.
4. The restriction of the function F to A is the function $F \restriction A = \{ (a, b) \in F \mid a \in A \}$.
5. If G is a restriction of F to some A, we say that F is an extension of G.

Note. *Concepts of domain, range, image, inverse image, inverse, and composition can be applied to functions.*

Proposition 1.1.2.
Let f and g be functions. Then *$g \circ f$ is a function*, *$g \circ f$ is defined at x* if and only if *f is defined at x and g is defined at $f(x)$*.

1. $\operatorname{dom}(g \circ f) = \operatorname{dom} f \cap f^{-1}[\operatorname{dom} g]$.
2. $(g \circ f)(x) = g(f(x))$ for all $x \in \operatorname{dom}(g \circ f)$.

Proof. Obviously. □

Definition. Invertiable
A function f is invertible if f^{-1} is a function.

Proposition 1.1.3.
If f is a function, its *inverse f^{-1} is a relation*, but it *may not be a function*.

Proof. Obviously. □

Definition.
A function f is called one-to-one or injective if $a_1 \in \operatorname{dom} f$, $a_2 \in \operatorname{dom} f$, and $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. In other words if $a_1 \in \operatorname{dom} f$, $a_2 \in \operatorname{dom} f$, and $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

Note. *A one-to-one function attains different values for different elements from its domain.*

Proposition 1.1.4.
A function is *invertible* if and only if it is *one-to-one*.

Proof. Let f be invertible; then f^{-1} is a function. □

Proposition 1.1.5.
If f *is invertible*, then f^{-1} *is also invertible*.

Proof. $(f^{-1})^{-1} = f$. □

Definition.
Function f and g are called compatible if $f(x) = g(x)$ for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

Definition.
A set of functions F is called a compatible system of functions if any two functions f and g from F are compatible.

Proposition 1.1.6.
Functions f *and* g *are compatible* if and only if $f \cup g$ *is a function*. Functions f *and* g *are compatible* if and only if $f \restriction (\operatorname{dom} f \cap \operatorname{dom} g) = g \restriction (\operatorname{dom} f \cap \operatorname{dom} g)$.

Proof. Obviously. □

Theorem 1.1.1.
If F is a *compatible system of functions*, the $\bigcup F$ is a *function* with $\operatorname{dom}(\bigcup F) = \bigcup\{\, \operatorname{dom} f \mid f \in F \,\}$. The function $\bigcup F$ *extends all* $f \in F$.

Proof. Obviously. □

Definition.
Let A and B be sets. The set of all functions on A into B is denoted B^A.

Proposition 1.1.7.
B^A *exists*.

Proof. $B^A \subseteq \mathcal{P}(A \times B)$. □

Definition. Indexed System of Sets
Let $S = \langle S_i \mid i \in I \rangle$ be a function with domain I, the values S_i are arbitrary sets. We call the function $\langle S_i \mid i \in I \rangle$ an indexed system of sets, whenever we wish to stress that the values of S are sets.

Definition. be Indexed by
$A = \{ S_i \mid i \in I \} = \operatorname{ran} S$, where S is a function on I.

Proposition 1.1.8.
$\bigcup A$ and $\bigcap A$ were defined for any system of sets $A (A \neq \emptyset$ in case of intersection).
$\bigcup A = \bigcup \{ S_i \mid i \in I \} = \bigcup\limits_{i \in I} S_i$; $\bigcap A = \bigcap \{ S_i \mid i \in I \} = \bigcap\limits_{i \in I} S_i$.

Proof. Obviously. □

Property 1.1.1.
Every system of sets A can be indexed by a function.

Proof. Obviously. □

Definition. Product of the Indexed System
Let $S = \langle S_i \mid i \in I \rangle$ and f be a function on I. $\prod S = \{ f \mid \forall i \in I \left(f_i \in S_i \right) \}$

Proposition 1.1.9.
$\prod_{i \in I} S_i$ *exists*.

Proof. $\prod_{i \in I} S_i \subseteq \mathcal{P}((I \times \bigcup_{i \in I} S_i))$ □

Definition. Exponentiation of Sets
If the indexed system S is such that $S_i = B$ for all $i \in I$, then $\prod_{i \in I} S_i = B^I$

Note. *The "exponentiation" of sets is related to "multiplication" of sets in the same way as similar operations on numbers are related.*