Steady-State Diffusion with Degradation in 2D: Multiple Finite-Radius Cells

1 Introduction

We consider a 2D model in which one or more cancer cells secrete a chemokine. The chemokine diffuses in the surrounding tissue and degrades at a constant rate. We start by reviewing the standard derivation using Fick's laws and show how a point-source approach leads to a logarithmic singularity at the cell center. We then remove this singularity by modeling each cell with a finite radius.

1.1 Fick's Laws and Degradation

Fick's First Law. Diffusion flux $\mathbf{J}$ is proportional to the negative gradient of the concentration:

$$\mathbf{J} = -D\nabla c,$$

where D is the diffusion coefficient, and $c(x, y)$ is the concentration.

Fick's Second Law. Applying mass conservation:

$$\frac{\partial c}{\partial t} = -\nabla \cdot \mathbf{J} = D\Big(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\Big).$$

If chemokines degrade at a constant rate δ, we add a linear sink term:

$$\frac{\partial c}{\partial t} = D\Big(\nabla^2 c\Big) - \delta\, c.$$

The **steady-state** condition $\partial c/\partial t = 0$ yields

$$D\,\nabla^2 c = \delta\, c \quad \Longrightarrow \quad \nabla^2 c - \lambda^2\, c = 0, \quad \text{where } \lambda^2 = \frac{\delta}{D}.$$

2 Radial Form and Modified Bessel Functions

2.1 Radial Symmetry in 2D

If we assume a single cancer cell in the plane at the origin, we might switch to polar coordinates (r, θ) and assume no angular dependence:

$$\nabla^2 c = \frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\,\frac{\partial c}{\partial r}.$$

Thus,

$$\frac{d^2 c}{dr^2} + \frac{1}{r}\frac{dc}{dr} - \lambda^2 c = 0.$$

2.2 Substitution $\rho = \lambda r$

Defining $\rho = \lambda r$, we transform the ODE into

$$\frac{d^2c}{d\rho^2} + \frac{1}{\rho}\frac{dc}{d\rho} - c = 0,$$

the *modified Bessel equation of order 0*, whose general solution is

$$c(r) = A\, I_0(\lambda r) + B\, K_0(\lambda r).$$

The function I_0 grows at large r, while K_0 decays at infinity but diverges (logarithmically) at $r = 0$.

2.3 Point-Source Singularity and the Need for a Fix

To force $c(r) \to 0$ as $r \to \infty$, we set $A = 0$ and keep $c(r) = B\, K_0(\lambda r)$. That solution has a $\log(r)$-type blow-up as $r \to 0$. Physically, in a 2D infinite domain, a true point source leads to an infinite concentration at the origin, which can be unrealistic for modeling a finite-sized cell.

3 Single Cell with Finite Radius

3.1 Inner–Outer Piecewise Approach

Let the cancer cell occupy a disk $0 \le r \le a$. Inside:

$$c_{\text{in}}(r) = c_{\text{cell}} \quad \text{(constant, or solve a separate PDE)}.$$

Outside ($r \ge a$), solve $\nabla^2 c - \lambda^2 c = 0$ in radial form:

$$c_{\text{out}}(r) = A\, I_0(\lambda r) + B\, K_0(\lambda r).$$

For $c_{\text{out}}(r) \to 0$ as $r \to \infty$, set $A = 0$.

3.2 Matching at $r = a$

1. **Continuity**: $c_{\text{cell}} = c_{\text{out}}(a) = B\, K_0(\lambda a)$,
2. **Hence** $B = \dfrac{c_{\text{cell}}}{K_0(\lambda a)}$.

Thus, for $r \ge a$:

$$c_{\text{out}}(r) = c_{\text{cell}}\, \frac{K_0(\lambda r)}{K_0(\lambda a)}.$$

No logarithmic singularity occurs at $r = 0$ because we only use K_0 for $r \ge a$; inside, the concentration is finite (c_{cell}). The *piecewise* solution is

$$c(r) = \begin{cases} c_{\text{cell}}, & r < a, \\ c_{\text{cell}}\, \dfrac{K_0(\lambda r)}{K_0(\lambda a)}, & r \ge a. \end{cases}$$

4 Multiple Finite-Radius Cells in 2D

Let there be N cells, each with:

- Center (x_i, y_i),
- Radius a_i,
- Intracellular concentration $c_{\text{cell},i}$ (or a more detailed PDE).

Define $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$. Then the piecewise solution for cell i is:

$$c_i(x, y) = \begin{cases} c_{\text{cell},i}, & r_i \le a_i, \\ c_{\text{cell},i} \dfrac{K_0(\lambda\, r_i)}{K_0(\lambda\, a_i)}, & r_i > a_i. \end{cases}$$

The **total** concentration is the sum of all cells:

$$c_{\text{total}}(x, y) = \sum_{i=1}^{N} c_i(x, y).$$

No cell is treated as a mathematical point, so each cell region $r_i < a_i$ uses a finite (or separate PDE) solution, avoiding the log blow-up.

4.1 Gradient Computation

We can find the gradient $\nabla c_{\text{total}}(x, y)$ by summing partial derivatives. For each cell i:

1. **Inside Cell** i ($r_i \le a_i$): c_i is constant ($c_{\text{cell},i}$), so
$$\nabla c_i = (0, 0).$$

2. **Outside Cell** i ($r_i > a_i$):
$$c_i(r_i) = c_{\text{cell},i} \frac{K_0(\lambda r_i)}{K_0(\lambda a_i)}.$$
Compute partials using chain rule. For example, w.r.t. x:
$$\frac{\partial c_i}{\partial x} = c_{\text{cell},i} \frac{1}{K_0(\lambda a_i)} \left[\frac{\partial}{\partial x} K_0(\lambda r_i) \right].$$
Recall
$$\frac{d}{dz} K_0(z) = -K_1(z), \quad \frac{\partial r_i}{\partial x} = \frac{x - x_i}{r_i}.$$
Hence
$$\frac{\partial}{\partial x} K_0(\lambda r_i) = -\lambda\, K_1(\lambda r_i) \frac{x - x_i}{r_i}.$$
Thus
$$\frac{\partial c_i}{\partial x} = -\, c_{\text{cell},i} \frac{\lambda}{K_0(\lambda a_i)} K_1(\lambda r_i) \frac{x - x_i}{r_i}.$$
Similarly for $\partial/\partial y$. Summing over all i yields $\nabla c_{\text{total}}(x, y)$.

Hence each cell contributes a *non-singular* gradient term only outside its radius. Inside, the gradient is zero (for the constant model), or it could be computed from a different PDE if you used one.

5 Conclusion

By giving each cell a finite radius and using a piecewise approach, we avoid the infinite concentration at $r = 0$ that arises for a point source in 2D. Each cell's "outer" solution for $r_i > a_i$ is still $B_i\, K_0(\lambda\, r_i)$ (decaying at infinity), matched to a finite value inside $r_i \leq a_i$. Summing these piecewise solutions for multiple cells produces the total concentration, and summing their partial derivatives yields a gradient *without* any $\ln(r)$ divergences at each cell center.