A Penalty Alternating Direction Method of Multipliers for Decentralized Composite Optimization

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2. Penalized Approximation Formulation
3. Algorithm Development
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5. Numerical Experiments
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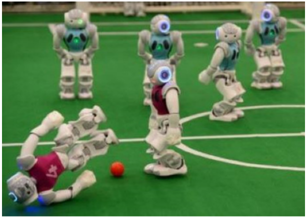
Background

Distributed Network





Background

- Consider the composite optimization problem:

$$\hat{x}^* = \arg\min_{x \in \mathbb{R}^p} \sum_{i=1}^{n} \big(f_i(x) + g_i(x)\big)$$

- Such a finite sum problem is common in machine learning. For example, f_i represents loss function and g_i represents regularization.

Background

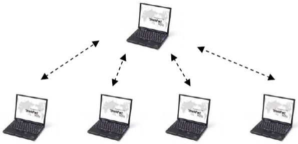


Fig 1. Centralized network

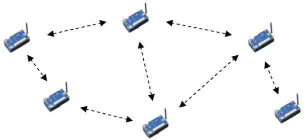


Fig 2. Decentralized network

- Centralized network: master collects/broadcasts x.
- Decentralized network: equivalent formulation

$$\{\hat{x}_i^*\} = \arg\min_{x_i \in \mathbb{R}^p} \sum_{i=1}^{n} \left(f_i(x_i) + g_i(x_i) \right),$$
$$\text{s.t. } x_1 = \cdots = x_n.$$

Each node holds its own x_i and only communicates with its neighbors.

Related Work

- Primal domain
  - Decentralized Gradient Descent (DGD) [Yuan 2016]
  - Second-order methods for the penalized approximation [Mokhtari 2016]
- Dual domain
  - Alternating Direction Method of Multipliers (ADMM) for smooth problem with linear rate [Shi 2014]
  - Proximal decentralized linearized ADMM with ergodic rate $O(1/k)$ [Aybat 2017]
  - The augmented Lagrangian method (ALM) for smooth problem wih linear rate [Jakovetic 2014]
- Other methods
  - PG-EXTRA and NIDS for nonsmooth problem with rate $O(1/k)$ and $o(1/k)$ [Shi 2016, Li 2019]
  - gradient tracking [Lorenzo 2016, Scutari 2019]
  - PGA [Alghunaim 2019] and SONATA [Sun 2019] with linear rate under the assumption that nonsmooth term is common

Outline

Penalized Approximation Formulation

Aggregated variables and functions:

- stack all x_i into matrix $\mathbf{x}$

$$\mathbf{x} \triangleq \begin{pmatrix} - & x_1^T & - \\ & \vdots & \\ - & x_n^T & - \end{pmatrix} \in \mathbb{R}^{n \times p}$$

  ,

- define $f(\mathbf{x}) = \sum_{i=1}^n f_i(x_i)$ and $g(\mathbf{x}) = \sum_{i=1}^n g_i(x_i)$.

Consensus constraint:

- (Assumption 1) Introduce the mixing matrix W with elements $w_{ij} \geq 0$. $w_{ij} = 0$ if and only if $j \notin \mathcal{N}_i \cup \{i\}$. Further, $W^T = W$, $W\mathbf{1}_{n\times 1} = \mathbf{1}_{n\times 1}$ and $null(I - W) = span(\mathbf{1}_{n\times 1})$.
- $x_1 = \cdots = x_n$ is equivalent to $(I - W)\mathbf{x} = 0$. Since $I - W \succeq 0$, $(I - W)^{\frac{1}{2}}\mathbf{x} = 0$.

Penalized Approximation Formulation

Consider the penalized approximation:

$$\hat{x}^* = \arg\min_{x \in \mathbb{R}^p} \sum_{i=1}^{n} \left(f_i(x) + g_i(x)\right)$$

Equivalent

$$\hat{\mathbf{x}}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^{n \times p}} \ f(\mathbf{x}) + g(\mathbf{x}) \quad \text{s.t. } (I - W)^{\frac{1}{2}}\mathbf{x} = 0$$

Penalized Approximation Formulation

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^{n \times p}} \ f(\mathbf{x}) + g(\mathbf{x}) + \frac{1}{2\epsilon}\|(I - W)^{\frac{1}{2}}\mathbf{x}\|_{\mathcal{F}}^2$$

- $\epsilon > 0$ is penalty parameter.
- smaller ϵ brings higher accuracy, i.e. $\mathbf{x}^*$ is close to $\hat{\mathbf{x}}^*$.

DGD to Solve Penalized Problem

- When $g(\mathbf{x}) = 0$, penalized problem can be solved by GD:

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \gamma\big(\nabla f(\mathbf{x}^k) + \frac{1}{\epsilon}(I - W)\mathbf{x}^k\big),$$

  where $\gamma > 0$ is the step size.
- GD is not efficient, why?
  - When ϵ is very small, the Lipschitz constant is in the order of $O(\frac{1}{\epsilon})$. To converge, γ must be in the order of $O(\epsilon)$.
  - Setting $\gamma = \epsilon$ recovers the DGD update $\mathbf{x}^{k+1} = W\mathbf{x}^k - \epsilon\nabla f(\mathbf{x}^k)$.
  - Proximal DGD faces with the same problem.
- To tackle the unfavorable accuracy-speed tradeoff, we use ADMM-based method.

Outline

Introduce an auxiliary variable **z**:

$$\hat{x}^* = \arg\min_{x \in \mathbb{R}^p} \sum_{i=1}^{n} \left(f_i(x) + g_i(x)\right)$$

$\xleftrightarrow{\textit{Equivalent}}$

$$\hat{\mathbf{x}}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^{n \times p}} \; f(\mathbf{x}) + g(\mathbf{x})$$
$$\text{s.t. } (I - W)^{\frac{1}{2}}\mathbf{x} = 0$$

$\Big\downarrow$ ***Penalized Approximation Formulation***

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^{n \times p}} \; f(\mathbf{x}) + g(\mathbf{x}) + \frac{1}{2\epsilon}\|(I - W)^{\frac{1}{2}}\mathbf{x}\|_{\mathcal{F}}^2$$

$\Big\updownarrow$ ***Equivalent***

$$(\mathbf{x}^*, \mathbf{z}^*) = \arg\min_{\mathbf{x}, \mathbf{z} \in \mathbb{R}^{n \times p}} \; f(\mathbf{x}) + g(\mathbf{x}) + \frac{1}{2\epsilon}\|\mathbf{z}\|_{\mathcal{F}}^2$$
$$\text{s.t. } (I - W)^{\frac{1}{2}}\mathbf{x} = \mathbf{z}$$

Algorithm Development

- The augmented Lagrangian function is $L_\alpha(\mathbf{x}, \mathbf{z}, \Pi) = f(\mathbf{x}) + g(\mathbf{x}) + \frac{1}{2\epsilon}\|\mathbf{z}\|_{\mathcal{F}}^2 + \left\langle \Pi, (I - W)^{\frac{1}{2}}\mathbf{x} - \mathbf{z} \right\rangle + \frac{\alpha}{2}\|(I - W)^{\frac{1}{2}}\mathbf{x} - \mathbf{z}\|_{\mathcal{F}}^2$.
  - $\Pi \in \mathbb{R}^{n \times p}$ is the Lagrange multiplier (also called dual variable).
  - $\alpha > 0$ is a parameter.
- Traditional ADMM iterates as

$$
\begin{aligned}
\mathbf{x}^{k+1} &= \arg\min_{x} L_\alpha(\mathbf{x}, \mathbf{z}^k, \Pi^k), \\
\mathbf{z}^{k+1} &= \arg\min_{z} L_\alpha(\mathbf{x}^{k+1}, \mathbf{z}, \Pi^k) \\
&= \frac{1}{\alpha + \frac{1}{\epsilon}}\left[\Pi^k + \alpha(I - W)^{\frac{1}{2}}\mathbf{x}^{k+1}\right], \\
\Pi^{k+1} &= \Pi^k + \alpha\left[(I - W)^{\frac{1}{2}}\mathbf{x}^{k+1} - \mathbf{z}^{k+1}\right].
\end{aligned}
$$

- $\mathbf{x}^{k+1}$ does not have a closed form solution $\to$ *linearization*

Algorithm Development

- Separate the augmented Lagrangian function as

$$L_\alpha(\mathbf{x}, \mathbf{z}^k, \Pi^k) \triangleq \underbrace{g(\mathbf{x})}_{\text{nonsmooth}} + \underbrace{\tilde{L}_\alpha(\mathbf{x}, \mathbf{z}^k, \Pi^k)}_{\text{smooth}}.$$

- Replace the smooth part by a quadratic approximation

$$L_\alpha(\mathbf{x}, \mathbf{z}^k, \Pi^k) \approx \underbrace{g(\mathbf{x})}_{\text{nonsmooth}}$$
$$+ \underbrace{\tilde{L}_\alpha(\mathbf{x}^k, \mathbf{z}^k, \Pi^k) + \left\langle \nabla_{\mathbf{x}} \tilde{L}_\alpha(\mathbf{x}^k, \mathbf{z}^k, \Pi^k), \mathbf{x} - \mathbf{x}^k \right\rangle + \frac{1}{2c} \|\mathbf{x} - \mathbf{x}^k\|_{\mathcal{F}}^2}_{\text{quadratic approximation}}.$$

- The primal update is the proximal mapping of g defined as $\text{prox}_{cg}(y) \triangleq \text{argmin}_x \{g(x) + \frac{1}{2c}\|x - y\|^2\}$, where $c > 0$ is a scalar.

Algorithm Development

- The proposed penalty ADMM (PAD) updates as

$$
\begin{aligned}
\mathbf{x}^{k+1} &= \mathrm{prox}_{cg}\left(\mathbf{x}^k - c\left[\nabla f(\mathbf{x}^k) + \alpha(I-W)^{\frac{1}{2}}((I-W)^{\frac{1}{2}}\mathbf{x}^k - \mathbf{z}^k + \frac{\Pi^k}{\alpha})\right]\right),\\
\mathbf{z}^{k+1} &= \frac{1}{\alpha + \frac{1}{\epsilon}}\left[\Pi^k + \alpha(I-W)^{\frac{1}{2}}\mathbf{x}^{k+1}\right],\\
\Pi^{k+1} &= \Pi^k + \alpha\left[(I-W)^{\frac{1}{2}}\mathbf{x}^{k+1} - \mathbf{z}^{k+1}\right].
\end{aligned}
$$

- Simplify updates by introducing $\bar{\Pi}^k = (I-W)^{\frac{1}{2}}\Pi^k$ and $\bar{\mathbf{z}}^k = (I-W)^{\frac{1}{2}}\mathbf{z}^k$.

Algorithm 1 PAD run by agent i

Require: Choose the parameters ϵ, α and c. Initialize the local variables to x_i^0, $\bar{z}_i{}^0$ and $\bar{\pi}_i^0$.

1: **for** $k = 1, 2, \cdots$ **do**

2: Update local variable x_i^{k+1} by

$$x_i^{k+1} = \mathrm{prox}_{cg_i}\left(x_i^k - c\left[\nabla f_i(x_i^k) + \alpha(x_i^k - \sum_{j\in\mathcal{N}_i} w_{ij}x_j^k - \bar{z}_i^k) + \bar{\pi}_i^k\right]\right).$$

3: Transmit x_i^{k+1}/ receive x_j^{k+1} from neighbors $j \in \mathcal{N}_i$.

4: Update local auxiliary variable $\bar{z}_i^{k+1}$ by

$$\bar{z}_i^{k+1} = \frac{1}{\alpha + \frac{1}{\epsilon}}\left[\bar{\pi}_i^k + \alpha(x_i^{k+1} - \sum_{j\in\mathcal{N}_i} w_{ij}x_j^{k+1})\right].$$

5: Update local dual variable $\bar{\pi}_i^{k+1}$ by

$$\bar{\pi}_i^{k+1} = \bar{\pi}_i^k + \alpha\left[(x_i^{k+1} - \sum_{j\in\mathcal{N}_i} w_{ij}x_j^{k+1}) - \bar{z}_i^{k+1}\right].$$

6: **end for**

Outline

Convergence and Rate of Convergence

For convergence, assume

- (Assumption 2) Each $g_i(x)$ is convex and nonsmooth. The proximal mapping of $g_i(x)$ can be computed easily.
- (Assumption 3) Each $f_i(x)$ is convex and differentiable with a Lipschitz continuous gradient such that

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_f \|x - y\|, \ \forall x, y \in \mathbb{R}^p,$$

where $L_f > 0$ is the Lipschitz constant.

For linear convergence rate, further assume

- (Assumption 4) Each $f_i(x)$ is strongly convex such that

$$\left\langle x - y, \nabla f_i(x) - \nabla f_i(y) \right\rangle \geq \mu_f \|x - y\|^2, \ \forall x, y \in \mathbb{R}^p,$$

where $\mu_f > 0$ is the strong convexity constant.

Convergence and Rate of Convergence

Theorem 1 (Convergence)

Under Assumptions 1, 2 and 3, if the parameters α and c are chosen such that $c < \frac{1}{L_f + \alpha\lambda_{max}(I-W)}$, then PAD from any initial points converges to the optimal solution $\mathbf{x}^*$ of the penalized approximation problem.

Theorem 2 (Convergence rate)

Under Assumption 1-4, if the parameters α and c are chosen such that $c < \frac{1}{\frac{L_f^2}{\mu_f} + \alpha\lambda_{max}(I-W)}$, then PAD from any initial points converges linearly to the optimal solution $\mathbf{x}^*$ of the penalized approximation problem.

- Since $L_f/\mu_f > 1$, the bound of Theorem 2 is smaller.
- Step sizes do not rely on ϵ. A sufficiently small ϵ can be used so that the penalized approximation error is negligible.

Outline

Numerical Experiments

1 Decentralized Logistic Regression

- $\min_x \frac{1}{n}\sum_{i=1}^n \left\{ \sum_{j=1}^{m_i} \ln\left(1+\exp(-(M_{(i)j}x)y_{(i)j})\right)\right\}$.
- $n = 30$, connectivity ratio $= 0.5$, $p = 10$ and $m_i = 5, \forall i$.
- Relative error $\|\mathbf{x}^k - \hat{\mathbf{x}}^*\|_{\mathcal{F}}/\|\mathbf{x}^0 - \hat{\mathbf{x}}^*\|_{\mathcal{F}}$.

Decentralized Logistic Regression

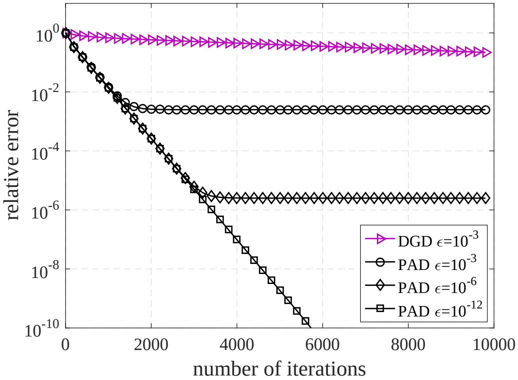


Fig 3. Relative error of DGD (with $\epsilon = 10^{-3}$) and PAD (with $\epsilon = 10^{-3}$, $\epsilon = 10^{-6}$ and $\epsilon = 10^{-12}$, respectively). For DGD, its step size is $\gamma = \epsilon$. For PAD, $\alpha = 0.6$ and $c = 0.032$ according to the condition in Theorem 1.

2 Decentralized Quadratic Programming

- $\min_x \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} x^T Q_i x + h_i^T x,\ s.t. a_i^T x \leq b_i, \forall i.$
- $g_i(x) = \begin{cases} 0 & \text{if } a_i^T x \leq b_i \\ +\infty & \text{otherwise} \end{cases}$
- $n = 10$, connectivity ratio $= 0.4$ and $p = 50$.
- Relative error $\|\mathbf{x}^k - \hat{\mathbf{x}}^*\|_{\mathcal{F}} / \|\mathbf{x}^0 - \hat{\mathbf{x}}^*\|_{\mathcal{F}}$.

Decentralized Quadratic Programming

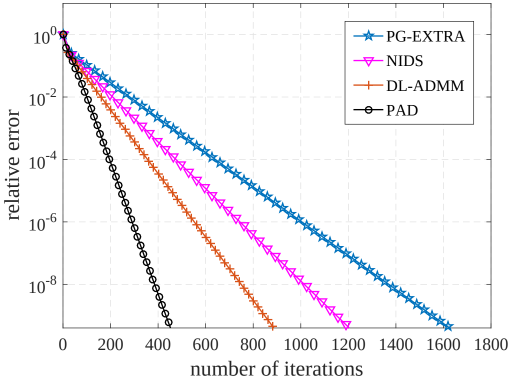


Fig 4. Relative error of PG-EXTRA, NIDS, DL-ADMM and PAD. For PAD, $\epsilon = 10^{-12}$, $\alpha = 1.2$ and $c = 0.2$ as suggested in Theorem 2. For PG-EXTRA, $c = 2\lambda_{\min}((I + W)/2)/L_f$. For NIDS, $c = 1.9/L_f$. The parameters of DL-ADMM are hand-optimized.

3 Application in Classification for Breast Cancer Data

- $\min_x \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\ln\left(1+\exp\left(-\left(M_{(i)j}x\right)y_{(i)j}\right)\right)+\frac{1}{n}\sum_{i=1}^{n}\lambda_i\|x\|_1.$
- $n=50$, connectivity ratio $=0.5$, $p=10$ and $\lambda_i=\frac{0.1}{n}$. Each node i holds $m_i=10$ training samples.
- Percentage of correct predictions.

Application in Classification for Breast Cancer Data

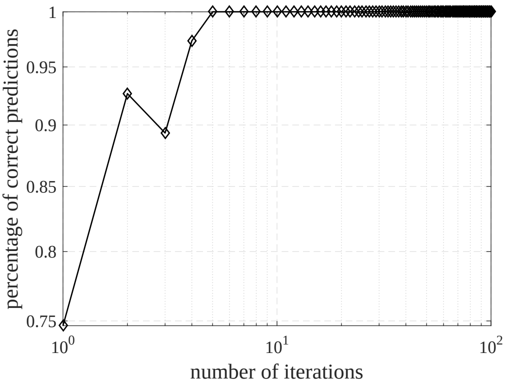


Fig 5. Percentage of correct predictions of PAD. We set $\epsilon = 10^{-12}$, $\alpha = 0.2$ and $c = 0.9$ by hand-optimization.

Outline

Conclusions

- Consider a penalized approximation of the decentralized composite problem. The penalty parameter can be very small.
- By linearization, PAD has low computational costs.
- PAD is provably convergent under convexity, and linearly convergent under strong convexity.

Thank you !