A Newton Tracking Algorithm with Exact Linear Convergence Rate for Decentralized Consensus Optimization

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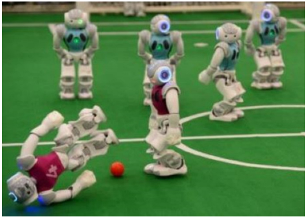
Outline

Outline

Background

Decentralized optimization




Background

Consider the decentralized convex optimization problem

$$x^* = \arg\min_{x \in \mathbb{R}^p} \sum_{i=1}^{n} f_i(x)$$

- $f_i(x)$ is a convex and twice continuously differentiable function.

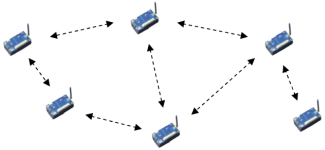


Fig 1. Decentralized network

- Data are distributed across a connected network of n nodes.
- Each node is only allowed to send/receive information to/from its neighboring nodes.
- All the nodes cooperate to get a common variable.

Background

An equivalent decentralized formulation is

$$\{x_i^*\}_{i=1}^n := \arg\min_{\{x_i\}_{i=1}^n} \sum_{i=1}^n f_i(x_i)$$
$$\text{s.t. } x_i = x_j, \forall j \in \mathcal{N}_i, \forall i$$

Two key components for decentralized optimization

- consensus - all nodes must agree on the same state, i.e., $x_1^* = \cdots = x_n^*$.
- optimality - the same state should be the minimizer of the original problem, i.e., $x_1^* = \cdots = x_n^* = x^*$.

Related First-order Work

Primal method

- Gradient methods [Nedic 2009, Yuan 2016]
- Gradient Tracking [Lorenzo 2015, Qu 2017, Nedic 2017,Sun 2019]

Primal-Dual method

- Decentralized Alternating Direction Method of Multipliers (DADMM) [Shi 2014,Chang 2015]
- Decentralized linearized ADMM [Ling 2015]
- Dual Ascent [Maros 2018]

Other method

- EXTRA [Shi 2015]
- NIDS [Li 2019]

Related Second-order Work

Penalized second-order algorithms converge to a neighborhood of an optimal solution

- Network Newton [Mokhtari 2016]
- Distributed asynchronous Newton-based algorithm [Mansoori 2019]

Primal-dual second-order methods achieve exact convergence with linear rates

- DQM [Mokhtari 2016]
- ESOM [Mokhtari 2016]

Second-order methods with superlinear convergence rates under stricter conditions

- Distributed averaged quasi-Newton method for a master-slave network [Soori 2019]
- Polyak's adaptive Newton method running a finite-time set-consensus inner loop [Zhang 2020]

Outline

We make the following assumptions

Assumption 1

Introduce the mixing matrix W with elements $w_{ij} \geq 0$. $w_{ij} = 0$ if and only if $j \notin \mathcal{N}_i \cup \{i\}$. Further, $W^T = W$, $W 1_{n\times 1} = 1_{n\times 1}$ and $null(I - W) = span(1_{n\times 1})$.

Assumption 2

The local objective functions $f_i(x_i)$ are twice differentiable. Hessians $\nabla^2 f_i(x_i)$ are bounded by

$$\mu_f I_p \preceq \nabla^2 f_i(x_i) \preceq L_f I_p.$$

Consider the decentralized optimization problem

$$\{x_i^*\}_{i=1}^n := \arg\min_{\{x_i\}_{i=1}^n} \sum_{i=1}^n f_i(x_i)$$
$$\text{s.t. } x_i = x_j, \forall j \in \mathcal{N}_i, \forall i$$

Global negative Newton direction at $\bar{x}^t \triangleq \frac{1}{n}\sum_{i=1}^n x_i^t$ is

$$u^t \triangleq \left(\frac{1}{n}\sum_{i=1}^n \nabla^2 f_i\left(\bar{x}^t\right)\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n \nabla f_i\left(\bar{x}^t\right)\right)$$

Our idea: Use a local u_i^t to track the negative global Newton direction such that $u_i^t \approx u^t$.

Algorithm Development

The proposed **Newton tracking**

$$
\begin{aligned}
x_i^{t+1} =& x_i^t - u_i^t \\
u_i^{t+1} =& \left(\nabla^2 f_i\left(x_i^{t+1}\right) + \epsilon I_p\right)^{-1} \left[\left(\nabla^2 f_i\left(x_i^t\right) + \epsilon I_p\right) u_i^t + \nabla f_i\left(x_i^{t+1}\right) - \nabla f_i\left(x_i^t\right)\right. \\
& \left. + 2\alpha\left(x_i^{t+1} - \sum_{j \in \mathcal{N}_i} w_{ij} x_j^{t+1}\right) - \alpha\left(x_i^t - \sum_{j \in \mathcal{N}_i} w_{ij} x_j^t\right)\right]
\end{aligned}
$$

with initialization $x_i^0 = 0_p$ and $u_i^0 = \left(\nabla^2 f_i\left(0_p\right) + \epsilon I_p\right)^{-1} \nabla f_i\left(0_p\right)$.

- The global Hessian $\frac{1}{n}\sum_{i=1}^n \nabla^2 f_i\left(\bar{x}^{t+1}\right)$ is replaced by the regularized local Hessian $\nabla^2 f_i\left(x_i^{t+1}\right) + \epsilon I_p$.
- The global gradient $\frac{1}{n}\sum_{i=1}^n \nabla f_i\left(\bar{x}^t\right)$ is replaced by three terms that are locally computable.

Why $u_i^t \approx u^t$?

Rewrite Newton tracking

$$\left(\nabla^2 f_i\left(x_i^{t+1}\right)+\epsilon I_p\right) u_i^{t+1} = \Big[\left(\nabla^2 f_i\left(x_i^t\right)+\epsilon I_p\right) u_i^t + \nabla f_i\left(x_i^{t+1}\right)-\nabla f_i\left(x_i^t\right) + 2\alpha\Big(x_i^{t+1}-\sum_{j\in\mathcal{N}_i} w_{ij}x_j^{t+1}\Big)-\alpha\Big(x_i^t-\sum_{j\in\mathcal{N}_i} w_{ij}x_j^t\Big)\Big]$$

Sum up over $i = 1 \ldots n$ and invoke the double stochasticity of W

$$\sum_{i=1}^{n}\left(\nabla^2 f_i\left(x_i^{t+1}\right)+\epsilon I_p\right) u_i^{t+1} = \sum_{i=1}^{n}\Big[\left(\nabla^2 f_i\left(x_i^t\right)+\epsilon I_p\right) u_i^t + \nabla f_i\left(x_i^{t+1}\right)-\nabla f_i\left(x_i^t\right)\Big]$$

With $\sum_{i=1}^n \nabla f_i\left(x_i^0\right) = \sum_{i=1}^n \left(\nabla^2 f_i\left(x_i^0\right)+\epsilon I_p\right) u_i^0$, sum up over time t

$$\sum_{i=1}^{n}\left(\nabla^2 f_i\left(x_i^t\right)+\epsilon I_p\right) u_i^t = \sum_{i=1}^{n}\nabla f_i\left(x_i^t\right)$$

Thus, when x_i^t is close to $\bar{x}^t$, u_i^t tracks a regularized Newton direction.

Newton Tracking

Newton tracking can be written in a compact form

$$
\begin{aligned}
\mathrm{x}^{t+1} =& \mathrm{x}^{t} - \mathrm{u}^{t} \\
\mathrm{u}^{t+1} =& \left(\mathrm{H}^{t+1}\right)^{-1}\Big[\mathrm{H}^{t}\mathrm{u}^{t} + \nabla f\left(\mathrm{x}^{t+1}\right) - \nabla f\left(\mathrm{x}^{t}\right) \\
& +\alpha(\mathrm{I}-\mathrm{W})\left(2\mathrm{x}^{t+1}-\mathrm{x}^{t}\right)\Big]
\end{aligned}
$$

- x (or u) stacks local variables such as $\mathrm{x} \triangleq [x_1; \ldots; x_n] \in \mathbb{R}^{np}$
- $\mathrm{W} \triangleq W \otimes I_p \in \mathbb{R}^{np \times np}$
- $f(\mathrm{x}) = f(x_1, \cdots, x_n) = \sum_{i=1}^{n} f_i(x_i)$.
- $\nabla f(\mathrm{x}) = [\nabla f_1(x_1); \ldots; \nabla f_n(x_n)] \in \mathbb{R}^{np}$
- $\mathrm{H} \in \mathbb{R}^{np \times np}$ is a block diagonal matrix whose i-th diagonal block is $\nabla^2 f_i(x) + I_p$

Connection with Gradient Tracking

Gradient tracking proceeds as

$$\begin{aligned}\mathrm{x}^{t+1} &= \mathrm{W}\mathrm{x}^{t} - \alpha \mathrm{y}^{t} \\ \mathrm{y}^{t+1} &= \mathrm{W}\mathrm{y}^{t} + \nabla f\left(\mathrm{x}^{t+1}\right) - \nabla f\left(\mathrm{x}^{t}\right)\end{aligned}$$

with initialization $\mathrm{y}^{0} = \nabla f\left(\mathrm{x}^{0}\right)$.
To see the connection with Newton tracking, we rewrite

$$\begin{aligned}\mathrm{x}^{t+1} =& \mathrm{x}^{t} - \mathrm{r}^{t} \\ \mathrm{r}^{t+1} =& \color{red}{\mathrm{W}\mathrm{r}^{t}} + \color{blue}{\alpha\left[\nabla f\left(\mathrm{x}^{t+1}\right) - \nabla f\left(\mathrm{x}^{t}\right)\right]} + \color{teal}{(\mathrm{I}-\mathrm{W})\left(\mathrm{x}^{t+1} - \mathrm{W}\mathrm{x}^{t}\right)}\end{aligned}$$

with $\mathrm{r}^{t} = (\mathrm{I}-\mathrm{W})\mathrm{x}^{t} + \alpha \mathrm{y}^{t} \in \mathbb{R}^{np}$ and $\sum_{i=1}^{n} r_i^0 = \alpha \sum_{i=1}^{n} \nabla f_i\left(x_i^0\right)$. Sum up over node i and time t

$$\sum_{i=1}^{n} r_i^t = \alpha \sum_{i=1}^{n} \nabla f_i\left(x_i^t\right)$$

Thus, when x_i^t is close to $\bar{x}^t$, r_i^t tracks a scaled gradient direction.

Connection with Primal-dual Algorithms

Under Assumption 1, we have $(\mathrm{I}-\mathrm{W})^{\frac{1}{2}}\mathrm{x}=0$ if and only if $x_1=\cdots=x_n$. Thus, the original problem is equivalent to

$$\mathrm{x}^* \triangleq \arg\min_{\mathrm{x}} f(\mathrm{x}) \quad \text{s.t. } (\mathrm{I}-\mathrm{W})^{\frac{1}{2}}\mathrm{x}=0.$$

The augmented Lagrangian is

$$L(\mathrm{x},\mathrm{v})=f(\mathrm{x})+\left\langle \mathrm{v},(\mathrm{I}-\mathrm{W})^{\frac{1}{2}}\mathrm{x}\right\rangle+\frac{\alpha}{2}\mathrm{x}^T(\mathrm{I}-\mathrm{W})\mathrm{x}.$$

We use a quadratic approximation of f and a linear approximation of $\mathrm{x}\mapsto\frac{\alpha}{2}\mathrm{x}^T(\mathrm{I}-\mathrm{W})\mathrm{x}$. The update of x^{t+1} is given by the solution of

$$\begin{aligned}\min_{\mathrm{x}}&\left\langle \nabla f\left(\mathrm{x}^t\right)+(\mathrm{I}-\mathrm{W})^{\frac{1}{2}}\mathrm{v}^t+\alpha(\mathrm{I}-\mathrm{W})\mathrm{x}^t,\mathrm{x}-\mathrm{x}^t\right\rangle\\&+\frac{1}{2}\left(\mathrm{x}-\mathrm{x}^t\right)^T\nabla^2 f\left(\mathrm{x}^t\right)\left(\mathrm{x}-\mathrm{x}^t\right)+\frac{\epsilon}{2}\left\|\mathrm{x}-\mathrm{x}^t\right\|^2.\end{aligned}$$

Connection with Primal-dual Algorithms

Thus, the updates of x^{t+1} and v^{t+1} are

$$\mathsf{x}^{t+1} = \mathsf{x}^t - \left(\mathsf{H}^t\right)^{-1}\left[\nabla f\left(\mathsf{x}^t\right) + (\mathsf{I}-\mathsf{W})^{\frac{1}{2}}\mathsf{v}^t + \alpha(\mathsf{I}-\mathsf{W})\mathsf{x}^t\right]$$
$$\mathsf{v}^{t+1} = \mathsf{v}^t + \alpha(\mathsf{I}-\mathsf{W})^{\frac{1}{2}}\mathsf{x}^{t+1}$$

where we set $x^0 = 0$ and $v^0 = 0$. By manipulation, we get

$$\begin{aligned}\mathsf{x}^{t+1} =& \mathsf{x}^t - \left(\mathsf{H}^t\right)^{-1}\mathsf{q}^t\\ \mathsf{q}^{t+1} =& \mathsf{q}^t + \nabla f\left(\mathsf{x}^{t+1}\right) - \nabla f\left(\mathsf{x}^t\right) + \alpha(\mathsf{I}-\mathsf{W})\left(2\mathsf{x}^{t+1} - \mathsf{x}^t\right),\end{aligned}$$

which is equivalent to Newton tracking in the sense that $\mathsf{u}^t = \left(\mathsf{H}^t\right)^{-1}\mathsf{q}^t$.

Outline

Theorem 1

Under Assumptions 1 and 2, suppose that the parameters ϵ and α satisfy $\epsilon - \alpha\lambda_{\max}(\mathrm{I}-\mathrm{W}) > \frac{4L_f^2}{\mu_f}$. Then Newton tracking starting with $x_i^0 = 0_p$ and $u_i^0 = \left(\nabla^2 f_i(0_p) + \epsilon I_p\right)^{-1} \nabla f_i(0_p)$ satisfies

$$\left\|\zeta^{t+1} - \zeta^*\right\|_{\mathrm{G}}^2 \leq \frac{1}{1+\delta'} \left\|\zeta^t - \zeta^*\right\|_{\mathrm{G}}^2,$$

where $\delta' > 0$.

- Define $\zeta^t = \begin{bmatrix} \mathrm{x}^t \\ \mathrm{v}^t \end{bmatrix}, \zeta^* = \begin{bmatrix} \mathrm{x}^* \\ \mathrm{v}^* \end{bmatrix}, \mathrm{G} = \begin{bmatrix} \epsilon\mathrm{I} - \alpha(\mathrm{I}-\mathrm{W}) & 0 \\ 0 & \frac{1}{\alpha}\mathrm{I} \end{bmatrix}$.
- Theorem 1 shows that the sequence $\{\left\|\zeta^{t+1} - \zeta^*\right\|_{\mathrm{G}}^2\}_t$ converges linearly with the factor $\frac{1}{1+\delta'}$.

Outline

Decentralized logistic regression problem

$$\min_{x \in \mathbb{R}^p} \frac{\rho}{2}\|x\|^2 + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \ln\left(1 + \exp\left(-\left(\mathrm{o}_{ij}^T x\right) \mathrm{p}_{ij}\right)\right)$$

- Node i has access to m_i training samples, $(\mathrm{o}_{ij}, \mathrm{p}_{ij}) \in \mathbb{R}^p \times \{-1, +1\}$.
- Relative error $\|\mathrm{x}^t - \hat{\mathrm{x}}^*\| \,/\, \left\|\mathrm{x}^0 - \hat{\mathrm{x}}^*\right\|$.

Comparison with Related Methods

- $n = 10$, connectivity ratio=0.5, $m_i = 12$, $p = 8$, $\rho = 0.001$

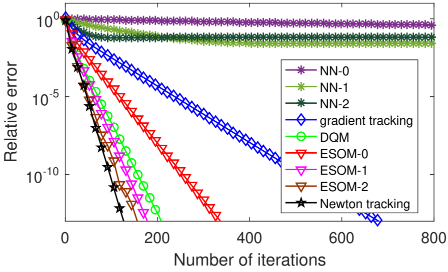


Fig 2. Relative errors of Newton tracking, gradient tracking, NN-K, ESOM-K, and DQM versus number of iterations.

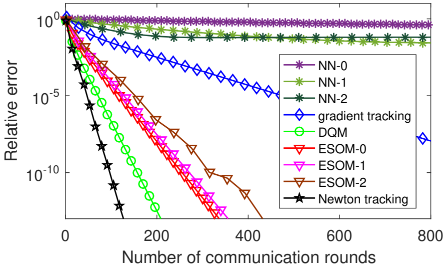


Fig 3. Relative errors of Newton tracking, gradient tracking, NN-K, ESOM-K, and DQM versus number of communication rounds.

Effect of Network Topology

- Four different topologies including line graph, cycle graph, random graphs, and complete graph
- $n = 10$, $m_i = 12$, $p = 8$, connectivity ratio=0.5, $\rho = 0.001$

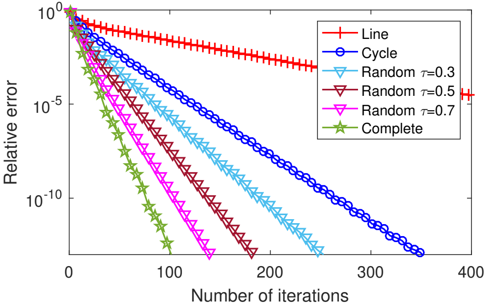


Fig 4. Relative errors of Newton tracking for different topologies

Outline

Conclusions

To summarize

- We propose Newton tracking, in which each node updates its local variable along a modified local Newton direction.
- Newton tracking employs a fixed step size and yet can still be proven to converge to an exact solution.
- We give the connections between Newton tracking and several existing methods, including gradient tracking and primal-dual algorithms.
- Newton tracking is linearly convergent under the strong convexity assumption.

Thank you !