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INTERVAL ESTIMATION METHODS OF FAULT ESTIMATION FOR DISCRETE-TIME SWITCHED SYSTEMS

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Abstract. This paper investigates the fault estimation problem for discrete-time switched systems. Firstly, the sensor fault is regarded as an augmented state, and then the original system is converted into a singular system. After that, based on the obtained singular system, a robust augmented state observer is designed using L_∞ technique. Given above robust augmented state observer, an interval observer based on the zonotope method is established to complete the fault interval estimation. Finally, a circuit system is simulated to illustrate the efficiency of the proposed method.

1. **Introduction.** Due to the influence of aging in operating conditions and devices, the occurence of fault is inevitable in practice. Thus, to guarantee the safety and reliability of control systems, the investigation on fault diagnosis is especially important. The tasks of fault diagnosis mainly include fault detection [22], fault isolation [29] and fault estimation [35]. The purpose of fault detection and fault isolation is to determine whether the systems have faults and the locations of the faults, respectively, while the purpose of fault estimation is to obtain amplitude information of faults and reduce the impact of fault variation. Up to now, model-based fault diagnosis has been mostly studied and a lot of works based on model method have been proposed, including sliding mode observer [30], Kalman filter [10], adaptive observer [32], and so on. However, the works mentioned above were mainly devoted to fault detection and fault isolation. Since the amplitude information of the faults of the systems is the main basis of active fault tolerant control, more and more attention has been paid to model-based fault estimation during the past decades [24–26, 31, 37].

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Switched systems, as an important class of hybrid systems, can model many practical systems such as network systems [28], circuit systems [19], vehicle dynamics [18], etc. Therefore, the investigation on switched system has been paid much attention, the topic includes stabilization [14], filter design [2] and fault estimation [20], and so on. The observer based fault estimation method for switched systems has been concerned in the works herein, such as [11, 34]. Considering the uncertainties in practical systems, the interval observers were proved to be very useful in the estimation of faults for uncertain systems [27]. The interval observer was designed for switched systems by the monotone theory for the first time [13]. By using the coordinate transformation method, the interval observer was designed for switched systems with unknown input [21], and then the optimal interval observer design as well as fault estimation for T-S fuzzy systems were both studied in [12]. Besides, there also exists many other works on interval observer for switched systems such as [7, 9].

On the other hand, the set membership estimation is a powerful method of state estimation using appropriate geometric body (e.g. ellipse set [8], minimum-volume parallelotope [5], zonotope [1]. Compared with other geometric bodies, zonotopes do not only reduce the constraints of design, but also improve the accuracy of estimation [4, 38]. By the help of zonotope technique, further results on interval estimation for switched systems can be found in [15–17, 33] and so on. [16] presented an interval observer design method for switched systems, [17] considered the case where the switched law is asynchronous and gave the interval observer framework. By using the reachable set analysis method, an optimal interval observer was designed in [33]. The functional interval observer for a kind of singular switched systems was given and the advantage was also proved in [15]. However, the sensor fault estimation for switched systems based on interval observer has been reported rarely.

Based on the above discussion, the problem of interval fault estimation is analyzed for discrete-time switched systems in this paper. Firstly, the system fault is treated as an augmented state and the original system is converted into a singular system. Then the L_∞ technique is applied to design the robust observer. Moreover, the upper and lower boundaries of the system fault are recovered by the zonotope method. The main contributions are threefold:
(1) The L_∞ observer for switched systems with average dwell time (ADT) is designed, and then the upper and lower bounds of the sensor fault are estimated by using zonotopes.
(2) The proposed method does not require that the coefficient matrices of error systems conform to be positive, which relaxes the design constraints of interval observers.
(3) Compared with the traditional time-varying coordinate transformation method in [13], the designed method improves the accuracy of the fault estimation.

In this paper, problem statement and some preliminaries are given in Section 2. In Section 3, the optimal robust augmented state observer is designed by using L_∞ formalism. To complete the fault interval estimation, the boundaries of fault are recovered by the zonotope method in Section 4. Finally, in Section 5, an example of circuit system is simulated to show the effectiveness of the proposed method.

Notations Z^+ is the set of all real (positive) numbers. I_n refers to the identity matrix with $n \times n$ dimension, and 0_s refers to the zero matrix with s dimension. The norm L_2 of a vector x is denoted by $\|x\|_2$, i.e., $\|x\|_2 = \sqrt{x^T x}$. The notation

$\|x\|_\infty$ refers to the L_∞ norm of x, i.e., $\sup_k \|x(k)\|$. $*$ represents an ellipsis for the term that is introduced by symmetry. The notation $P \succ 0 (\prec 0)$ means that P is a positive (negative) definite matrix. $\underline{\lambda}(Q)(\overline{\lambda}(Q))$ represents the minimum (maximum) eigenvalue of the matrix Q.

2. **Problem statement.**

Consider the following switched system:

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) + D_{\sigma(k)}w(k), \\ y(k) = C_{\sigma(k)}x(k) + F_{\sigma(k)}f(k), \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the system state, $u(k) \in R^m$ is the control input, $w(k) \in R^p$ is an external disturbance, $y(k) \in R^q$ is the measurable output vector of the system, and $f(k) \in R^s$ is a fault. $\sigma(k)$ is the piecewise time consist function taking the values from the set $\mathcal{Z} = \{1, 2, \cdots, N\}$. For any $\sigma(k) = i \in \mathcal{Z}$, $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$, $C_i \in R^{q\times n}$, $D_i \in R^{n\times p}$, $F_i \in R^{q\times s}$ are known matrices. For simplicity, we omit k when necessary.

According to [9, 16, 19], (1) is a general form of many systems, such as robot systems, power electronic systems, traffic systems and so on. Especially, we consider the interval estimation of sensor fault since the fault $f(k)$ may occur in the measurement channel.

Assumption 1. The disturbance $w(k)$ is bounded as:

$$\|w(k)\|_2 \le \|w\|_\infty,$$

where $\|w\|_\infty > 0$ is a constant.

Remark 1. If Assumption 1 holds, then the external disturbance $w(k)$ of the original system shows a peak-to-peak performance, i.e., $w^T(k)w(k) < c$, for some constant $c > 0$. The condition on disturbance signal with bounded peak value is also known as L_∞ performance. Different from H_∞ performance of disturbance which requires the energy performance of the system disturbance on the whole time domain, i.e, $\sum_{k=0}^{\infty} w^T(k)w(k) < \eta$, for some constant $\eta > 0$. In some cases, L_∞ performance is more applicable than H_∞ performance in terms of residual evaluation.

To estimate the fault $f(k)$ in system (1), we denote that

$$\overline{x}(k) = \begin{bmatrix} x(k) \\ f(k) \end{bmatrix}, \tag{2}$$

then the following augmented system is achieved:

$$\begin{cases} E\overline{x}(k+1) = \overline{A}_i\overline{x}(k) + \overline{B}_iu(k) + \overline{D}_iw(k), \\ y(k) = \overline{C}_i\overline{x}(k), \end{cases} \tag{3}$$

where

$$E_i = \begin{bmatrix} I_n & 0 \\ 0 & 0_s \end{bmatrix}, \overline{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0_s \end{bmatrix},$$
$$\overline{B}_i = \begin{bmatrix} B_i \\ 0_s \end{bmatrix}, \overline{D}_i = \begin{bmatrix} D_i \\ 0_s \end{bmatrix}, \overline{C}_i = \begin{bmatrix} C_i & F_i \end{bmatrix}.$$

If matrices E_i and $\overline{C}_i$ satisfy the rank condition: $rank \begin{bmatrix} E_i \\ \overline{C}_i \end{bmatrix} = n+s$, then there exist a set of matrices $T_i \in R^{(n+s)\times(n+s)}$ and $N_i \in R^{(n+s)\times q}$ such that

$$T_i E_i + N_i \overline{C}_i = I_{n+s}, \tag{4}$$

then, system (3) is equivalent to:

$$\begin{aligned} \overline{x}(k+1) &= (T_i E_i + N_i \overline{C}_i)\overline{x}(k+1) \\ &= T_i \overline{A}_i \overline{x}(k) + T_i \overline{B}_i u(k) + T_i \overline{D}_i w(k) + N_i y(k+1). \end{aligned} \tag{5}$$

Lemma 2.1. *[23] For matrices $A \in R^{a\times b}$, $B \in R^{b\times c}$, $C \in R^{a\times c}$, if $rank(B) = c$, then general solutions of the matrix equation $AB = C$ are as follows:*

$$A = CB^{\dagger} + \Xi(I_b - BB^{\dagger}), \tag{6}$$

where $\Xi \in R^{a\times b}$ is an arbitrary matrix, $B^{\dagger}$ is the generalized inverse matrix of B.

Remark 2. Note that equation (4) satisfies:

$$Rank \begin{bmatrix} E_i \\ \overline{C}_i \end{bmatrix} = Rank \begin{bmatrix} I_n & 0 \\ 0 & 0_s \\ C_i & F_i \end{bmatrix} = n+s. \tag{7}$$

By using Lemma 2.1, we can get

$$\begin{bmatrix} T_i & N_i \end{bmatrix} = \begin{bmatrix} E_i \\ \overline{C}_i \end{bmatrix}^{\dagger} + \Xi(I_{n+s+q} - \begin{bmatrix} E_i \\ \overline{C}_i \end{bmatrix} \begin{bmatrix} E_i \\ \overline{C}_i \end{bmatrix}^{\dagger}), \tag{8}$$

where $\Xi \in R^{(n+s)\times(n+s+q)}$ is an arbitrary matrix.

Definition 2.2. If system (1) satisfies the following initial condition:

$$\hat{x}^-(0) \le x(0) \le \hat{x}^+(0), \tag{9}$$

then, the interval observer of system (1) is a pair of boundaries $\{\hat{x}^-, \hat{x}^+\}$ and for all $k > 0$, the following holds:

$$\hat{x}^-(k) \le x(k) \le \hat{x}^+(k).$$

For system (5), the following observer is constructed as:

$$\hat{\overline{x}}(k+1) = T_i \overline{A}_i \hat{\overline{x}}(k) + T_i \overline{B}_i u(k) + L_i(y(k) - \overline{C}_i \hat{\overline{x}}(k)) + N_i y(k+1), \tag{10}$$

where $\hat{\overline{x}}(k) \in R^{n+s}$, and $L_i \in R^{(n+s)\times q}$ are observer gains to be determined. Then, the error dynamic of between system (5) and observer (10) is defined as:

$$\begin{aligned} e(k+1) &= \overline{x}(k+1) - \hat{\overline{x}}(k+1) \\ &= (T_i \overline{A}_i - L_i \overline{C}_i)e(k) + T_i \overline{D}_i w(k). \end{aligned} \tag{11}$$

Definition 2.3. If the error system (11) is stable and satisfies the following L_∞ condition:

$$\|e(k)\|_2 \le \varrho\sqrt{\|w\|_\infty^2 + V(0)a^k}, \tag{12}$$

where $\varrho > 0$ and $0 < a < 1$, $V(0) = e^T(0)P_\sigma e(0)$, and $P_\sigma \succ 0$. Then the observer (10) is an L_∞ observer of system (5).

Some definitions and properties of zonotope are given in the sequel.

Definition 2.4. Consider a unitary interval $\mathbb{Z} = [-1, 1]$, which is called a box. For a vector $p \in R^n$ and a matrix $H \in R^{n\times t}$, the ξ order zonotope is as follows:

$$p \oplus H\mathbb{Z}^{\xi} = \{p + Hz, z \in \mathbb{Z}^{\xi}\},$$

where $\mathbb{Z}^{\xi}$ represents a box of ξ unitary intervals, $\oplus$ represents the the Minkowski sum of two sets. In this paper, the ξ order zonotope is denoted by $\langle p, H\rangle$.

Lemma 2.5. *[6] For the zonotope $\langle p, H\rangle$, where $p \in R^n, H \in R^{n\times t}$, the following properties hold:*
(1) $\langle p_1, H_1\rangle \oplus \langle p_2, H_2\rangle = \langle p_1 + p_2, \begin{bmatrix} H_1 & H_2 \end{bmatrix}\rangle$,
(2) $L \odot \langle p, H\rangle = \langle Lp, LH\rangle$,
(3) $\langle p, H\rangle \subseteq \langle p, \bar{H}\rangle$,
where $\bar{H} \in R^{n\times n}$ is a diagonal matrix and the diagonal elements of $\bar{H}$ are $\bar{H}_{i,i} = \sum_{j=1}^{t} \mid H_{i,j} \mid,\ i = 1, ..., n$.

Lemma 2.6. *[1] For a given zonotope $\langle p, \mathbb{H}\rangle \subset R^n$, the following holds*

$$\langle p, H\rangle \subseteq p \oplus R_q(\hat{\mathbb{H}}(k))\mathbb{Z}^t,\ n \le q \le s,$$

where $R_q(\hat{\mathbb{H}}(k)) = [\mathbb{H}_T\ Q]$, and $\mathbb{H}_T$ is made up of the first $q-n$ columns vector of $\hat{\mathbb{H}}(k)$, $Q \in R^n$ is a diagonal matrix with $Q_{i,i} = \sum_{j=q-n+1}^{t} \mid \hat{\mathbb{H}}(k)_{i,j} \mid$, $i = 1, 2, \cdots, n$. And the number of the column of matrix $\hat{\mathbb{H}}(k)$ is t.

Assumption 2. The initial condition of (1) is unknown but bounded:

$$\begin{cases} x_0 \in \langle p_0, X_0\rangle, \\ w(k) \in \langle \mathbf{0}, \mathbb{W}_0\rangle, \end{cases}$$

where p_0, X_0, $\mathbb{W}_0$ are known matrices.

Definition 2.7. [36] Consider the time interval $[k_1, k_2)$ of the switched system, and let $N_\sigma(k_1, k_2)$ be the switching number of $\sigma(k)$ during the interval $[k_1, k_2)$. If the following condition holds:

$$N_\sigma(k_1, k_2) \le N_0 + \frac{(k_2 - k_1)}{\tau^*},$$

where $N_0 \ge 0$, then $\tau^* > 0$ is the ADT of σ. In this paper, we choose $N_0 = 0$.

Lemma 2.8. *[3] For a given symmetric matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$, the following three inequalities are equivalent:*
(1) $M \prec 0$,
(2) $M_{11} \prec 0,\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} \prec 0$,
(3) $M_{22} \prec 0,\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T \prec 0$.

3. L_∞ **observer design.** In this section, the L_∞ technique is applied to design an observer for system (5). Sufficient conditions with ADT are then given.

Theorem 3.1. *Given $0 < \varepsilon < 1$ and $\mu > 1$. If there exist a constant $\gamma_2 \neq 0$ and matrices $P_i, P_j \in R^{(n+s)\times(n+s)} \succ 0$ and for all $(\sigma(k_l) = i,\ \sigma(k_l - 1) = j) \in S, j \neq i$,*

$$\begin{aligned} &(a)\ G = \begin{bmatrix} -\varepsilon P_i & * & * \\ 0 & -\gamma_2^2 I & * \\ P_i T_i \overline{A}_i - W_i \overline{C}_i & P_i T_i \overline{D}_i & -P_i \end{bmatrix} \prec 0, \\ &(b)\ P_i \prec \mu P_j, i, j \in \mathcal{Z}, \\ &(c)\ \frac{1}{\tau^*} ln\mu + ln\varepsilon < 0, \end{aligned} \tag{13}$$

where $W_i = P_i L_i$, then the observer (10) is the L_∞ observer of system (5).

Proof. Define $\Theta_{11} = \begin{bmatrix} -\varepsilon P_i & 0 \\ 0 & -\gamma_2^2 I \end{bmatrix}$, $\Theta_{12}^T = \begin{bmatrix} P_i T_i \overline{A}_i - W_i \overline{C}_i & P_i T_i \overline{D}_i \end{bmatrix}$, $\Theta_{22} = -P_i$. Since $\Theta_{22} \prec 0$, by using Lemma 2.8, we can get from (13)(a) that

$$\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{12}^T \prec 0. \tag{14}$$

Substituting $W_i = P_i L_i$ into (14), we obtain

$$G_2 = \begin{bmatrix} (T_i\overline{A}_i - L_i\overline{C}_i)^T P_i (T_i\overline{A}_i - L_i\overline{C}_i) - \varepsilon P_i & * \\ \overline{D}_i^T T_i^T P_i (T_i\overline{A}_i - L_i\overline{C}_i) & \overline{D}_i^T T_i^T P_i T_i \overline{D}_i - \gamma_2^2 I \end{bmatrix} \prec 0. \tag{15}$$

It follows from (15) that

$$\zeta^T G_2 \zeta < 0, \tag{16}$$

where $\zeta^T = \begin{bmatrix} e^T(k) & w^T(k) \end{bmatrix}$. Thus, (16) implies that

$$\begin{aligned} &e^T(k)[(T_i\overline{A}_i - L_i\overline{C}_i)^T P_i (T_i\overline{A}_i - L_i\overline{C}_i) - \varepsilon P_i] e(k) \\ &+ 2w^T(k)\overline{D}_i^T T_i^T P_i (T_i\overline{A}_i - L_i\overline{C}_i) e(k) + w^T(k)(\overline{D}_i^T T_i^T P_i T_i \overline{D}_i - \gamma_2^2 I) w(k) < 0. \end{aligned} \tag{17}$$

We choose the following Lyapunov function:

$$V_i(k) = e^T(k) P_i e(k). \tag{18}$$

By calculating the difference of V_i, we obtain

$$\begin{aligned} \triangle V_i &= V_i(k+1) - V_i(k) \\ &= e^T(k)[(T_i\overline{A}_i - L_i\overline{C}_i)^T P_i (T_i\overline{A}_i - L_i\overline{C}_i) - P_i] e(k) + 2w^T(k) \\ &\overline{D}_i^T T_i^T P_i (T_i\overline{A}_i - L_i\overline{C}_i) e(k) + w^T(k) \overline{D}_i^T T_i^T P_i T_i \overline{D}_i w(k). \end{aligned} \tag{19}$$

Further simplifying (19), one can get

$$\triangle V_i < (\varepsilon - 1) e^T(k) P_i e(k) + \gamma_2^2 w^T(k) w(k), \tag{20}$$

which means that

$$V_i(k+1) < \varepsilon V_i(k) + \gamma_2^2 w^T(k) w(k). \tag{21}$$

For the interval $[k_l, k)$, by repeating (21), we have

$$V_i(k) < \varepsilon^{k-k_l} V_i(k_l) + \gamma_2^2 \sum_{\psi=0}^{k-k_l-1} \varepsilon^{\psi} w^T(k-1-\psi) w(k-1-\psi). \tag{22}$$

Since $k_l - 1 = j$, together with inequalities (13)(b), we can get

$$\begin{aligned} V_i(k) &< \varepsilon^{k-k_l} V_i(k_l) + \gamma_2^2 \sum_{\psi=0}^{k-k_l-1} \varepsilon^{\psi} w^T(k-1-\psi) w(k-1-\psi) \\ &< \varepsilon^{k-k_l+1} \mu V_j(k_l - 1) + \varepsilon^{k-k_l} \mu \gamma_2^2 w^T(k_l-1) w(k_l-1) \\ &+ \gamma_2^2 \sum_{\psi=0}^{k-k_l-1} \varepsilon^{\psi} w^T(k-1-\psi) w(k-1-\psi). \end{aligned} \tag{23}$$

Repeating (23) yields

$$V_i(k) < \varepsilon^k \mu^{N_\sigma(0,k)} V_{\sigma(0)}(0) + \gamma_2^2 \sum_{\psi=0}^{k-1} \varepsilon^{\psi} \mu^{N_\sigma(k-\psi,k)} w^T(k-1-\psi) w(k-1-\psi). \tag{24}$$

In view of $\mu > 1$ and the fact that $N_\sigma(0,k) \geq N_\sigma(k-\psi,k)$, we obtain

$$\begin{aligned} &\gamma_2^2 \sum_{\psi=0}^{k-1} \varepsilon^{\psi} \mu^{N_\sigma(k-\psi,k)} w^T(k-1-\psi) w(k-1-\psi) \\ &\leq \gamma_2^2 \sum_{\psi=0}^{k-1} \varepsilon^{\psi} \mu^{N_\sigma(0,k)} w^T(k-1-\psi) w(k-1-\psi). \end{aligned} \tag{25}$$

It follows from (24) and (25) that

$$V_i(k) < \mu^{N_{\sigma(0,k)}} [\varepsilon^k V_{\sigma(0)}(0) + \gamma_2^2 \sum_{\psi=0}^{k-1} \varepsilon^{\psi} w^T(k-1-\psi) w(k-1-\psi)]. \tag{26}$$

Since $V_i(k) \geq \underline{\lambda}(P_i) \|e(k)\|_2^2$, the estimation error $e(k)$ satisfies

$$\begin{aligned} \|e(k)\|_2^2 &\leq \frac{1}{\underline{\lambda}(P_i)} V_i(k) \\ &\leq \frac{\mu^{N_{\sigma(0,k)}}}{\underline{\lambda}(P_i)} (\varepsilon^k V_{\sigma(0)}(0) + \gamma_2^2 \sum_{\psi=0}^{k-1} \varepsilon^{\psi} \|w(k)\|_\infty^2). \end{aligned} \tag{27}$$

By Definition 2.3, the proof is completed. □

Remark 3. In order to improve the performance of L_∞ observer, γ_2 should be chosen as small as possible. Then, the optimal L_∞ problem is formulated as:

$$\begin{cases} min \quad \gamma_2^2 \\ subject\ to: \\ \begin{bmatrix} -\varepsilon P_i & * & * \\ 0 & -\gamma_2^2 I & * \\ P_i T_i \overline{A}_i - W_i \overline{C}_i & P_i T_i \overline{D}_i & -P_i \end{bmatrix} \prec 0, \\ P_i \prec \mu P_j. \end{cases} \tag{28}$$

4. **Zonotope estimation.** In this section, the interval estimation of the fault $f(k)$ is presented based on the designed observer (10) in the former section. We first give the following theorem.

Theorem 4.1. *Let Assumption 2 hold, then the state $\overline{x}(k)$can be enclosed by a zonotope $\overline{X}(k)$, and $\overline{X}(k)$ is determined by*

$$\overline{X}(k) = \langle \hat{\overline{x}}(k), \hat{\mathbb{H}}(k) \rangle,$$

where $\hat{\overline{x}}(k)$ is given by (10), $\hat{\mathbb{H}}(k)$ has the recursive formulation form as follows:

$$\hat{\mathbb{H}}(k+1) = [(T_i\overline{A}_i - L_i\overline{C}_i)R_q(\hat{\mathbb{H}}(k)) \quad T_i\overline{D}_iW_i],$$

where $R_q(\hat{\mathbb{H}}(k))$ is defined in Lemma 2.6.

Proof. Assume that $e(k) \in \hat{\Phi}(k) = \langle 0, \hat{\mathbb{H}}(k) \rangle$, since $\overline{x}(k) = \hat{\overline{x}}(k) + e(k)$, we can get

$$\overline{x}(k) \in \overline{X}(k) = \hat{\overline{x}}(k) \oplus \hat{\Phi}(k) = \langle \hat{\overline{x}}(k), \hat{\mathbb{H}}(k) \rangle.$$

It deduces from (11) that

$$\begin{aligned} e(k+1) \in \Phi(k+1) &= \langle 0, \mathbb{H}(k) \rangle \\ &= (T_i\overline{A}_i - L_i\overline{C}_i) \odot \langle 0, \hat{\mathbb{H}}(k) \rangle \oplus T_i\overline{D}_i \odot \langle 0, W_i \rangle. \end{aligned}$$

From Lemma 2.6, we have $e(k) \in \langle 0, \mathbb{H}(k) \rangle \subseteq \langle 0, R_q(\hat{\mathbb{H}}(k)) \rangle$, then

$$\begin{aligned} e(k+1) &\in \langle 0, (T_i\overline{A}_i - L_i\overline{C}_i)R_q(\hat{\mathbb{H}}(k)) \rangle \oplus \langle 0, T_i\overline{D}_iW_i \rangle \\ &\subset \langle 0, \left[(T_i\overline{A}_i - L_i\overline{C}_i)R_q(\hat{\mathbb{H}}(k)) \quad T_i\overline{D}_iW_i\right] \rangle, \end{aligned}$$

which means that

$$\Phi(k+1) = \langle 0, \left[(T_i\overline{A}_i - L_i\overline{C}_i)R_q(\hat{\mathbb{H}}(k)) \quad T_i\overline{D}_iW_i\right] \rangle.$$

Define that $\hat{\Phi}(k+1) = \langle 0, \hat{\mathbb{H}}(k+1) \rangle$, then

$$\hat{\mathbb{H}}(k+1) = \left[(T_i\overline{A}_i - L_i\overline{C}_i)R_q(\hat{\mathbb{H}}(k)) \quad T_i\overline{D}_iW_i\right].$$

□

Remark 4. The system fault f of system (1) is bounded by f^+ and f^-:

$$\begin{cases} f_i^+ = \hat{\overline{x}}_{i+n} + \sum_{j=1}^{t} \mid \hat{H}(k)_{i,j} \mid, \ i = 1, 2, \cdots, s, \\ f_i^- = \hat{\overline{x}}_{i+n} - \sum_{j=1}^{t} \mid \hat{H}(k)_{i,j} \mid, \ i = 1, 2, \cdots, s. \end{cases} \tag{29}$$

Then, we can use the following algorithm to estimate the fault of the original system by zonotope method.

Remark 5. By using ADT, we consider the system mode in the time interval $[k_m, k_{m+l}]$ which is invariable. So, one can deduce the error system (11). Then, to calculate the zonotope of the error dynamics, it is essential to iterate (11) from $k = 0$ to $k = K$. In the time interval $[0,\ K)$, the system mode will switch to any other mode. To solve the problem, we divide the whole time interval $[0,\ K)$ into many small intervals, and the switched system would remain in a single mode in these time intervals.

Algorithm 1 Algorithm for fault estimation by zonotope approach.

Input: $\mathbb{W}_0$, $\hat{\mathbb{H}}(0)$
Output: f^+, f^-
Given the initial conditions:
$\hat{\mathbb{H}}(0) = \hat{\mathbb{H}}_0$, $w(k) \in \langle \mathbf{0}, \mathbb{W}_0 \rangle$;
for $k \geq 0$ **do**
  $\hat{\overline{x}}(k+1) = T_i\overline{A}_i\hat{\overline{x}}(k) + T_i\overline{B}_iu(k) + L_i(y(k) - \overline{C}_i\hat{\overline{x}}(k)) + N_iy(k+1)$
  $e(k) \in \hat{\Phi}(k) = \langle 0, \hat{\mathbb{H}}(k) \rangle$
  $\overline{x}(k) \in \overline{X}(k) = \hat{\overline{x}}(k) \oplus \hat{\Phi}(k) = \langle \hat{\overline{x}}(k), \hat{\mathbb{H}}(k) \rangle$
  $e(k+1) \in \Phi(k+1) = (T_i\overline{A}_i - L_i\overline{C}_i) \odot \langle 0, \hat{\mathbb{H}}(k) \rangle \oplus T_i\overline{D}_i \odot \langle 0, W_i \rangle$
  $\Phi(k+1) = \langle 0, \left[(T_i\overline{A}_i - L_i\overline{C}_i)R_q(\hat{\mathbb{H}}(k)) \quad T_i\overline{D}_iW_i \right] \rangle.$
  $e(k) \in \langle 0, \mathbb{H}(k) \rangle \subseteq \langle 0, \hat{\mathbb{H}}(k) \rangle$
  $e(k+1) \in \hat{\Phi}(k+1) = \langle 0, \hat{\mathbb{H}}(k+1) \rangle$
  $\hat{\mathbb{H}}(k+1) = \left[(T_i\overline{A}_i - L_i\overline{C}_i)R_q(\hat{\mathbb{H}}(k)) \quad T_i\overline{D}_iW_i \right]$
  $f_i^+ = \hat{\overline{x}}_{i+n} + \sum_{j=1}^{t} \mid \hat{\mathbb{H}}(k)_{i,j} \mid$
  $f_i^- = \hat{\overline{x}}_{i+n} - \sum_{j=1}^{t} \mid \hat{\mathbb{H}}(k)_{i,j} \mid$;
end for

Remark 6. Different from [15], the interval fault estimation for switched systems is studied in this paper. The main approach used is the L_∞ technique combined with zonotope method, and the advantage of the proposed approach lies in the tight interval estimation with less restrictive conditions, which is demonstrated by the following simulation example.

5. **Example.** The circuit system shown in Fig.1 is used to verify the validity of the proposed method. The mathematical model is as follows:

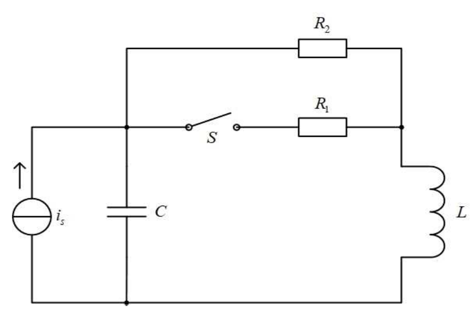

FIGURE 1. Circuit system diagram

$$\begin{cases} \dot{i}_l(t) = \dfrac{1}{L}u_c(t) - [S(t)\dfrac{R_1R_2}{R_1+R_2} + (1-S(t))R_2]\dfrac{i_l(t)}{L}, \\ \dot{u}_c(t) = \dfrac{1}{C}i_s(t) - \dfrac{1}{C}i_l(t), \end{cases}$$

where $i_l(t)$ is the current through the inductor, $u_c(t)$ is the capacitor voltage, $i_s(t)$ is the input-current, and the switching signal has the form: $S(t) = \begin{cases} 0, & switching\ off \\ 1, & switching\ on \end{cases}$.

Denote that $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} i_l(t) \\ u_c(t) \end{bmatrix}$, $u(t) = i_s(t)$, $y(t) = i_l(t)$. By discretizing the above system, we have

$$\begin{cases} x(k+1) = A_i x(k) + B_i u(k) + D_i w(k), \\ y(k) = C_i x(k) + F_i f(k). \end{cases}$$

In this paper, the input vector $u(k) = 2\cos(k)$. Assume that $w(k) = \begin{cases} 0.05sin(k), & k < 10, \\ 0, & k \geq 10, \end{cases}$ and the output fault is given by

$$f(k) = \begin{cases} 0, & k < 40, \\ 2\sin(0.2k), & k \geq 40. \end{cases}$$

By a direct computation, we have

$$\sqrt{w^T(k)w(k)} = \begin{cases} 0.05|\sin k|, & k < 10, \\ 0, & k \geq 10. \end{cases}$$

Thus, $||w(k)||_2 \leq 0.05$ for any $k > 0$. It means that Assumption 1 holds.

Consider the relevant parameters of the system as $R_1 = 1.418\Omega$, $R_2 = 2.25\Omega$, $L = 0.98H$, $C = 0.95F$, then the correlation matrix of the discrete system can be obtained

$$A_1 = \begin{bmatrix} 0.1122 & 1.0204 \\ -1.0526 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1.0526 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F_1 = 1.1.$$

$$A_2 = \begin{bmatrix} -1.2959 & 1.0204 \\ -1.0526 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1.0526 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F_2 = 1.1.$$

The matrix parameters of the generalized system can be constructed as

$$\overline{A}_1 = \begin{bmatrix} 0.1122 & 1.0204 & 0 \\ -1.0526 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{B}_1 = \begin{bmatrix} 0 \\ 1.0526 \\ 0 \end{bmatrix},$$

$$\overline{C}_1 = \begin{bmatrix} 1 & 0 & 1.1 \end{bmatrix}, \quad \overline{D}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\overline{A}_2 = \begin{bmatrix} -1.2959 & 1.0204 & 0 \\ -1.0526 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 \\ 1.0526 \\ 0 \end{bmatrix},$$

$$\overline{C}_2 = \begin{bmatrix} 1 & 0 & 1.1 \end{bmatrix}, \quad \overline{D}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

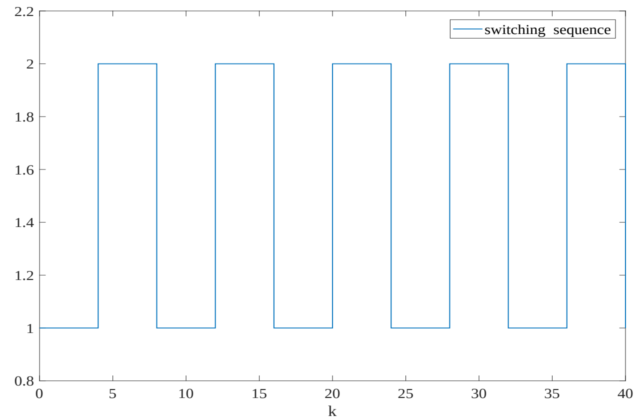

FIGURE 2. Switching sequence

In the simulation, we choose the matrix Ξ as

$$\Xi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Thus, it follows from (8) that

$$T_1 = T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.1926 & 0 & 1 \end{bmatrix}, \quad N_1 = N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1926 & 0.5254 \end{bmatrix}.$$

Solving (28) yields

$$P_1 = \begin{bmatrix} 13.0833 & -0.0110 & 67.8758 \\ -0.0110 & 0.0157 & 0.0021 \\ 67.8758 & 0.0021 & 352.4291 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 13.0833 & -0.0110 & 67.8758 \\ -0.0110 & 0.0157 & 0.0021 \\ 67.8758 & 0.0021 & 352.4291 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.8091 & -0.6820 \\ -0.3407 & 0.1595 \\ -0.1472 & 0.1252 \end{bmatrix}, L_2 = \begin{bmatrix} 0.6062 & 0.3558 \\ -0.3487 & 0.1654 \\ 0.1156 & -0.0675 \end{bmatrix}.$$

To illustrate the superiority of the proposed method, the method in [13] is compared with the above method. From [13], we choose that

$$R_1(k) = \begin{bmatrix} -4.4708 & 6.0945 & 0.8609 \\ 5.2185 \cdot (-1)^k & -5.4430 \cdot (-1)^k & -1.0049 \cdot (-1)^k \\ 0.1891 & 0 & 0.9819 \end{bmatrix},$$

$$R_2(k) = \begin{bmatrix} -4.4708 & 6.0945 & 0.8607 \\ 5.2185 \cdot (-1)^k & -5.4430 \cdot (-1)^k & -1.0050 \cdot (-1)^k \\ 0.1890 & 0 & 0.9820 \end{bmatrix},$$

then we find that

$$R_1(k+1)(TA_1 - L_1C)R^{-1}(k) = \begin{bmatrix} 0.2235 & 0 & 0 \\ 0 & 0.0144 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R_2(k+1)(TA_2 - L_2C)R^{-1}(k) = \begin{bmatrix} 0.2235 & 0 & 0 \\ 0 & 0.0144 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Fig.2 shows the switching sequence, and the fault estimation of the proposed method is shown in Fig.3. We can get a comparison of time-varying coordinate transformation method and L_∞ technique in Fig.3. It indicates that fault estimation method based on L_∞ technique has a higher degree of accuracy than time-varying coordinate transformation method.

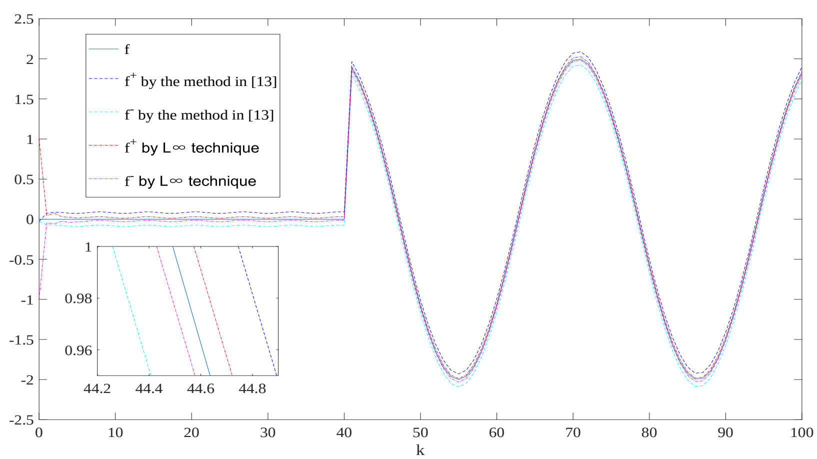

FIGURE 3. Comparison of time-varying coordinate transformation and L_∞ technique

6. **Conclusion.** The fault estimation method for discrete-time switched systems is proposed in this paper. A robust augmented state observer is designed by using L_∞ technique, which relaxes the design constraints of fault interval estimation. The zonotope method is used to estimate the upper and lower bounds of fault with better accuracy. Moreover, the correctness and validity of the design method are demonstrated by a simulation example of circuit system. In the nearly future, we will investigate the fault estimation based on interval observer for nonlinear switched systems.

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