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Functional interval observer for discrete-time systems with disturbances

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ABSTRACT

This paper further investigates the design problem of the functional interval observer for discrete-time systems with disturbances. Two methods are given to design functional interval observers. Given monotone system theory, a Luenberger-like functional interval observer is constructed and sufficient conditions are achieved by Sylvester equations in the first method. The second approach is known as the two-step method. An H_∞ functional observer of the system is presented by the H_∞ technique, and then the approach of the zonotope is used to estimate its upper and lower boundaries to improve the estimation accuracy. The above methods are applied to two examples for comparison and to show the correctness.

1. Introduction

State feedback is of great significance to the solutions of many synthesis problems in control systems. However, it is very difficult to obtain complete information of state vectors in actual systems, which makes the state feedback unable to be implemented physically. Due to the huge challenge of state feedback in practical applications, the investigation of state reconfiguration (also known as the observer design problem) has been a topic of research since the 1960s [1–3,50], and many meaningful applications of the observer have been obtained, such as fault estimation [4–6]. The design of the classical Luenberger state observer usually requires that the observer error dynamic systems converge to zero asymptotically. However, due to disturbances, such as clutter in the actual systems, the disturbed error dynamic systems usually only converge to bounded values. In some practical problems, such as estimation of population number, the bounds of the population number need to be given, and thus the definition of interval observer came into being [7]. For the interval observer, it is necessary to ensure that the states of error dynamic systems are non-negative and converge to bounded values, that is, the upper and lower boundaries of the interval can be used to implement real-time tracking of the states of the original systems. Most of the previous works have focused on designing observer gains to make the corresponding errors asymptotically stable (or uniformly bounded (UB)) and positive [8–10]. In general, it is impossible for the error systems to satisfy both asymptotic stability (or uniform boundedness) and positivity. In prior works [11,12], coordinate transformation was used to avoid such defects. By the coordinate transformation method, some interesting results have also been found [13–15]. How-

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ever, the limitation of the coordinate transformation method is that the disturbance attenuation performance of the interval observer may not be well although the cooperativity is guaranteed [16].

Different from a general observer, a functional observer only recovers the linear function of the state variables, which can reduce the order and complexity of the observer construction. In practice, the observation object usually belongs to a linear function of the state variables. For instance, the output of the system is a linear function that needs to be observed when the residual signals are generated in the system detection problem [17,18]. With the development of control theory, the design of a functional observer for uncertain systems has been investigated in many works [19–27]. The idea of decoupling has been introduced [19] while the definition of the unknown-input functional observability was proposed [20,21], in which the complexity of the observer design is reduced. In another work [22], an ϵ-bounded state observer for time-delay systems with bounded disturbances was presented. Moreover, this result was improved and the robust functional observer design problem for discrete time-delay systems with disturbances was further studied [23]. Since then, more works on the robust estimation of observers have been conducted [24,25]. A prior study provided two novel observer design schemes for reconstructing the time-varying fault of nonlinear systems [24]. Trinh et al. proposed an optimal functional state bounding for positive systems with disturbances [25]. Under the framework of polytopic Takagi-Sugeno, Bezzaoucha et al. designed a functional unknown-input observer for fuzzy systems [26]. Based on the former works, a Luenberger-like functional interval observer (FIO) for uncertain linear continuous systems was a recently designed innovation [27], which is the first piece of work on FIO. However, two issues have not been mentioned. First, sufficient conditions were derived by the form of Sylvester equations but the existing conditions of the solutions were not given. Second, relative rough bounds of the upper and lower errors were estimated, and more accurate bounds (or called optimal FIO) were expected.

The approach of set membership estimation is a useful technique that can improve the precision of estimation. Specifically, in the interval estimation problem of set membership, the state vectors of the systems are considered to be a feasible set, and then a proper geometric body is employed to contain the feasible set. Geometric bodies commonly include ellipsoid, ordinary polyhedron, zonotope, and so on. In recent years, many results on the ellipsoid method have been developed [28–30], but the conservatism is obvious. Meanwhile, the computational complexity of the method based on ordinary polyhedron increases exponentially with the magnification of the polyhedron dimension [31]. The zonotope method just depends on simple matrix operation, and the computational burden and conservatism are less than those of the methods based on the ellipsoid and ordinary polyhedron. Therefore, it has aroused the tremendous interest of many researchers and a lot of interesting results have been achieved [32–34]. Moreover, the H_∞ technique, which is a kind of well-known technique of disturbance attenuation, has achieved fruitful results in recent years [35–38]. Among them, the stability analysis was studied as well as the H_∞ performance of linear delay systems [35], and the H_∞ sliding mode controller for stochastic time-delay systems was designed [36]. The H_∞ filtering problem for T-S systems with standard form was also investigated [37]. Then, a T-S descriptor system with time delay was considered and H_∞ sliding mode observer design method was presented [38]. Based on the advantages of these two techniques, an increasing amount of research has focused on a combination of the H_∞ technique and the zonotope method to construct interval observers [39–42], which not only reduced the design constraints of the observers but also improved the estimation accuracy.

Motivated by the discussions given above, we present two methods to design an FIO for discrete-time systems, and the existing conditions of the corresponding FIO are given. Following the line of Gu et al. [27], a Luenberger-like FIO for linear discrete-time systems is first designed and then the performance of the observer is improved by using the zonotope method. Specifically, in the first method, the Luenberger-like FIO of linear discrete-time systems with disturbances is designed. Sufficient conditions that ensure the existence of the FIO are given by monotone system theory. In the second method, an H_∞ functional observer is designed, and the boundaries of the system are given by the zonotope method. There are two main questions in this paper.

Question 1: Based on the prior work [27], how to design an FIO for linear discrete-time systems with disturbances?

Question 2: How to use the zonotope method to reduce the constraints of the design and improve the estimation accuracy?

This paper mainly has three contributions:

(1) We design an FIO for linear discrete-time systems by using monotone system theory, and the sufficient conditions are presented by Sylvester equations.

(2) The design constraints are reduced and the estimation accuracy is improved by using the zonotope method.

(3) By comparing the two methods, the approach of the zonotope can reduce the constraints of observer design and improve the performance of the proposed FIO.

The remainder of paper is organized as follows: An FIO for the linear discrete-time systems is constructed by monotone system theory in Section 2. In Section 3, an H_∞ functional observer of the system is designed and then the bounds of the states are estimated by the zonotope method. Finally, two examples are illustrated in Section 4, where comparisons of the proposed methods are given and the correctness is demonstrated.

Notations

$x > (\geq)0$ denotes that the values of all elements in vector x are positive (non-negative). For a matrix $A \in R^{m\times n}$, $A > (\geq)0$ represents that all values of its elements are positive (non-negative), and $A^+ = \max\{0, A\}, A^- = \max\{0, -A\}$, then we have $A^+ \geq 0,\ A^- \geq 0$, and $A = A^+ - A^-$. For a real symmetric matrix $A \in R^{n\times n}$, $A \prec 0$ implies that A is a negative definite matrix, $A \succ 0$ implies that A is a positive definite matrix. $\overline{\lambda}(A)$ and $\underline{\lambda}(A)$ stand for the smallest and the largest of real part of eigenvalues of matrix A, respectively. $||\bullet||$ means the Euclidean norm. $\oplus$ denotes the Minkowski sum. $\odot$ refers to the linear mapping.

2. Design of the FIO by monotone system theory

The considered system is described as

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + Df(k), \\ y(k) = Cx(k), \end{cases} \tag{1}$$

where $x(k) \in R^n$, $u(k) \in R^m$, $y(k) \in R^q$ are the state, control input and output, respectively. $f(k) \in R^p$ is the unknown but bounded disturbance, that is, $f^-(k) \le f(k) \le f^+(k)$, where $f^+(k)$ and $f^-(k)$ are the known upper and lower boundaries of $f(k)$. $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{q \times n}$ and $D \in R^{n \times p}$ are given constant matrices. The boundaries of initial state $x(0)$ are determined by $x^+(0)$ and $x^-(0)$. The estimation object of this paper is a linear function:

$$\nu(k) = \Phi x(k), \tag{2}$$

where $\Phi \in R^{\mu \times n}$ is a given constant matrix. For simplicity, we will omit k when necessary. A Luenberger-like FIO for the system (1) is designed in this part and sufficient conditions that ensure the existence of the observer are presented. To achieve these goals, some definitions and lemmas are given firstly.

Definition 1. For system (1), if it satisfies the following initial condition:

$$\hat{x}^-(0) \le x(0) \le \hat{x}^+(0),$$

then the interval observer of system (1) is a pair of bounds $\{\hat{x}^+, \hat{x}^-\}$, and for all $k > 0$, the following holds:

$$\hat{x}^- \le x \le \hat{x}^+.$$

Lemma 1. *[44] Suppose that x satisfies $\hat{x}^- \le x \le \hat{x}^+$, then for any given constant $A \in R^{n \times n}$, the following holds:*

$$A^+\hat{x}^- - A^-\hat{x}^+ \le Ax \le A^+\hat{x}^+ - A^-\hat{x}^-.$$

Remark 1. In accordance with Definition 1, the FIO for system (1) is the pair $\{\bar{\nu}^+, \bar{\nu}^-\}$, for all $k > 0$, the following holds:

$$\bar{\nu}^- \le \nu \le \bar{\nu}^+,$$

under the initial condition

$$\bar{\nu}^-(0) \le \nu(0) \le \bar{\nu}^+(0).$$

Lemma 2. *[45] Consider the system*

$$\begin{cases} x(k+1) = Ex(k) + \iota(k), \\ x(0) = x_0 \ge 0, \end{cases} \tag{3}$$

where E is a given constant matrix and $\iota(k) > 0$ is a nonlinear function. If E is a non-negative matrix, then system (3) is positive.

Lemma 3. *[17,46] Suppose that the matrix E is a Schur matrix, then system (3) is known as input-state stable (ISS), and for all $k \ge 0$, the following hold:*
(1) if $\iota(k)$ is UB, then $x(k)$ is also UB;
(2) if $\iota(k) \to 0$, then $x(k) \to 0$.

Definition 2. [9] The system

$$g(k+1) = F(g(k), u(k), y(k), f^+(k), f^-(k)), \tag{4}$$

with outputs

$$\begin{cases} \bar{\nu}^+(k) = H_1(k, g(k), y(k)), \\ \bar{\nu}^-(k) = H_2(k, g(k), y(k)), \end{cases} \tag{5}$$

is called an FIO of the linear function $\nu(k)$ defined in (2) if
(1) system (4) is ISS;
(2) the state ν and outputs $\bar{\nu}^+$, $\bar{\nu}^-$ satisfy the asymptotic relation $\bar{\nu}^- \le \nu \le \bar{\nu}^+$ for all $k \ge 0$;
(3) if $||f^+ - f^-||$ is UB then $||\bar{\nu}^+ - \bar{\nu}^-||$ is UB, and if $||f^+ - f^-|| \to 0$ then $||\bar{\nu}^+ - \bar{\nu}^-|| \to 0$.

Remark 2. In Definition 2, the requirement of the ISS can guarantee that system (4) is UB or asymptotically stable.

Lemma 4. *[47](Hamilton-Cayley theorem) Given matrix $\Upsilon \in R^{n \times n}$, and $f(\lambda) = |\lambda I - \Upsilon| = \lambda^n + m_{n-1}\lambda^{n-1} + \cdots + m_1\lambda + m_0$ is the characteristic polynomial of Υ, then $f(\Upsilon) = \Upsilon^n + m_{n-1}\Upsilon^{n-1} + \cdots + m_1\Upsilon + m_0 I = 0$.*

Lemma 5. *If the eigenvalues of matrix $\Gamma \in R^{\mu \times \mu}$ are not zero and different from that of matrix $A \in R^{n \times n}$, then there exists a matrix $\Pi \in R^{\mu \times n}$ such that the Sylvester equation $\Pi A - \Gamma\Pi = \Sigma C$ holds, where $\Sigma \in R^{\mu \times q}$ is a given matrix. Moreover, Π can be determined by*

$$\Pi = -[\alpha(\Gamma)]^{-1} UNV,$$

where

$$U = \begin{bmatrix} \Sigma & \Gamma\Sigma & \cdots & \Gamma^{n-1}\Sigma \end{bmatrix}, \quad N = \begin{bmatrix} m_1 I & m_2 I & \cdots & m_{n-1} I & I \\ m_2 I & m_3 I & \cdots & I & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-1} I & I & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

$$\alpha(\Gamma) = \Gamma^n + m_{n-1}\Gamma^{n-1} + \cdots + m_1\Gamma + m_0 I.$$

Proof. Firstly, the characteristic polynomial of matrix A can be described as

$$\alpha(\lambda) = det(\lambda I - A) = \lambda^n + m_{n-1}\lambda^{n-1} + \cdots + m_1\lambda + m_0. \tag{6}$$

From Lemma 4, $\alpha(A) = 0$. Denote the eigenvalues of Γ by λ_i $(i = 1, 2, \cdots, \mu)$, and then the eigenvalues of $\alpha(\Gamma)$ are $\alpha(\lambda_i)$ $(i = 1, 2, \cdots, \mu)$. Since matrices A and Γ do not have the same eigenvalues, then one obtains $\alpha(\lambda_i) \neq 0$ $(i = 1, 2, \cdots, \mu)$, and matrix $\alpha(\Gamma)$ is a nonsingular matrix. It follows from the Sylvester equation $\Pi A - \Gamma\Pi = \Sigma C$ that

$$\begin{aligned} \Pi A^2 - \Gamma^2\Pi &= (\Gamma\Pi + \Sigma C)A - \Gamma^2\Pi \\ &= \Gamma(\Pi A - \Gamma\Pi) + \Sigma CA \\ &= \Sigma CA + \Gamma\Sigma C, \\ \Pi A^3 - \Gamma^3\Pi &= \left(\Pi A^2 - \Gamma^2\Pi\right)\mathrm{A} + \Gamma^2(\Pi A - \Gamma\Pi) \\ &= \Sigma CA^2 + \Gamma\Sigma CA + \Gamma^2\Sigma C, \\ &\cdots\cdots \\ \Pi A^{\mathrm{n}} - \Gamma^{\mathrm{n}}\Pi &= \left(\Pi A^{\mathrm{n}-1} - \Gamma^{\mathrm{n}-1}\Pi\right)\mathrm{A} + \Gamma^{\mathrm{n}-1}(\Pi A - \Gamma\Pi) \\ &= \Sigma CA^{\mathrm{n}-1} + \Gamma\Sigma CA^{\mathrm{n}-2} + \cdots + \Gamma^{\mathrm{n}-2}\Sigma CA + \Gamma^{\mathrm{n}-1}\Sigma C. \end{aligned} \tag{7}$$

Thus,

$$\begin{aligned} &\Pi I - I\Pi = 0, \\ &\Pi A - \Gamma\Pi = \Sigma C, \\ &\Pi A^2 - \Gamma^2\Pi = \Sigma CA + \Gamma\Sigma C, \\ &\Pi A^3 - \Gamma^3\Pi = \Sigma CA^2 + \Gamma\Sigma CA + \Gamma^2\Sigma C, \\ &\cdots\cdots \\ &\Pi A^n - \Gamma^n\Pi = \Sigma CA^{n-1} + \Gamma\Sigma CA^{n-2} + \cdots + \Gamma^{n-2}\Sigma CA + \Gamma^{n-1}\Sigma C. \end{aligned} \tag{8}$$

In (8), the first equation is multiplied by m_0, the next equation is multiplied by m_1, $\cdots$, the penultimate equation is multiplied by m_{n-1}, and the last equation is multiplied by 1. By adding all the equations together, the following can be obtained:

$$\begin{aligned} &\Pi\left(m_0 I + m_1 A + \cdots + m_{n-1}A^{n-1} + A^n\right) - \left(m_0 I + m_1\Gamma + \cdots + m_{n-1}\Gamma^{n-1} + \Gamma^n\right)\Pi \\ &\quad = \mathrm{m}_1\Sigma\mathrm{C} + \mathrm{m}_2(\Sigma\mathrm{CA} + \Gamma\Sigma\mathrm{C}) + \mathrm{m}_3\left(\Sigma\mathrm{CA}^2 + \Gamma\Sigma\mathrm{CA} + \Gamma^2\Sigma\mathrm{C}\right) + \cdots + \Sigma\mathrm{CA}^{\mathrm{n}-1} + \Gamma\Sigma\mathrm{CA}^{\mathrm{n}-2} + \cdots \\ &\qquad + \Gamma^{\mathrm{n}-2}\Sigma CA + \Gamma^{\mathrm{n}-1}\Sigma C, \end{aligned} \tag{9}$$

which is equivalent to

$$\Pi\alpha(A) - \alpha(\Gamma)\Pi = \begin{bmatrix} \Sigma & \Gamma\Sigma & \cdots & \Gamma^{n-1}\Sigma \end{bmatrix} \begin{bmatrix} m_1 I & m_2 I & \cdots & m_{n-1} I & I \\ m_2 I & m_3 I & \cdots & I & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-1} I & I & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{10}$$

Given the expressions of U, N, and V, we have $\Pi\alpha(A) - \alpha(\Gamma)\Pi = UNV$. Since $\alpha(A) = 0$ and $\alpha(\Gamma)$ is nonsingular, then $\Pi = -[\alpha(\Gamma)]^{-1}UNV$. □

Lemma 6. *[28] For the equation $\Phi = \Psi\Pi + LC$, where $\Phi \in R^{\mu\times n}$ and $\Pi \in R^{\mu\times n}$ are arbitrary matrices. If $\Psi \in R^{\mu\times\mu}$ satisfies $rank\begin{bmatrix} C \\ \Phi - \Psi\Pi \end{bmatrix} = rank\ C = q$, then*

$$L = (\Phi - \Psi\Pi)C^{\dagger}, \tag{11}$$

where $C\dagger \in R^{n\times q}$ is a generalized inverse of C.

The Luenberger-like FIO of system (1) is described as

$$\begin{cases} \overline{g}^{+}(k+1) = \Gamma\overline{g}^{+}(k) + \Sigma y(k) + \Xi u(k) + (\Pi D)^{+}f^{+}(k) - (\Pi D)^{-}f^{-}(k), \\ \overline{g}^{-}(k+1) = \Gamma\overline{g}^{-}(k) + \Sigma y(k) + \Xi u(k) + (\Pi D)^{+}f^{-}(k) - (\Pi D)^{-}f^{+}(k), \\ \overline{v}^{+}(k) = \Psi^{+}\overline{g}^{+}(k) - \Psi^{-}\overline{g}^{-}(k) + Ly(k), \\ \overline{v}^{-}(k) = \Psi^{+}\overline{g}^{-}(k) - \Psi^{-}\overline{g}^{+}(k) + Ly(k), \end{cases} \tag{12}$$

where $\bar{g}^{+}(k)$ and $\bar{g}^{-}(k) \in R^{\mu}$, and the observer gains $\Gamma \in R^{\mu\times\mu}$, $\Sigma \in R^{\mu\times q}$, $\Xi \in R^{\mu\times m}$, $L \in R^{\mu\times q}$, $\Pi \in R^{\mu\times n}$ and $\Psi \in R^{\mu\times\mu}$ will be determined later. $\bar{v}^{+}$ and $\bar{v}^{-}$ are the estimated bounds of v.

Remark 3. Different from the general interval observer [8–15], the FIO only recovers the linear function of the state variables, which can reduce the order and complexity of the observer construction.

Theorem 1. *If the corresponding coefficient matrices of system (12) satisfy the equations*

$$\Xi = \Pi B, \tag{13}$$

$$\Pi A - \Gamma\Pi = \Sigma C, \tag{14}$$

$$\Phi = \Psi\Pi + LC, \tag{15}$$

and Γ is a Schur and non-negative matrix, then system (12) is the FIO of system (1).

Proof. Denote

$$\begin{cases} \epsilon^{+}(k+1) = \overline{g}^{+}(k+1) - \Pi x(k+1), \\ \epsilon^{-}(k+1) = \Pi x(k+1) - \overline{g}^{-}(k+1), \end{cases} \tag{16}$$

and

$$\begin{cases} e^{+}(k) = \overline{v}^{+}(k) - \Phi x(k), \\ e^{-}(k) = \Phi x(k) - \overline{v}^{-}(k). \end{cases} \tag{17}$$

Then, we can get

$$\begin{aligned} \epsilon^{+}(k+1) &= \overline{g}^{+}(k+1) - \Pi x(k+1) \\ &= \Gamma\overline{g}^{+} + \Sigma y + \Xi u + (\Pi D)^{+}f^{+} - (\Pi D)^{-}f^{-} - \Pi Ax - \Pi Bu - \Pi Df \\ &= \Gamma\left(\overline{g}^{+} - \Pi x\right) + \Gamma\Pi x + \Sigma Cx + \Xi u + (\Pi D)^{+}f^{+} - (\Pi D)^{-}f^{-} - \Pi Ax - \Pi Bu - \Pi Df \\ &= \Gamma\epsilon^{+} + (\Gamma\Pi + \Sigma C - \Pi A)x + (\Xi - \Pi B)u + (\Pi D)^{+}f^{+} - (\Pi D)^{-}f^{-} - \Pi Df, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \epsilon^{-}(k+1) &= \Pi x(k+1) - \overline{g}^{-}(k+1) \\ &= \Pi Ax + \Pi Bu + \Pi Df - \Gamma\overline{g}^{-} - \Sigma y - \Xi u - (\Pi D)^{+}f^{-} + (\Pi D)^{-}f^{+} \\ &= \Gamma\left(\Pi x - \overline{g}^{-}\right) - \Gamma\Pi x - \Sigma Cx - \Xi u - (\Pi D)^{+}f^{-} + (\Pi D)^{-}f^{+} + \Pi Ax + \Pi Bu + \Pi Df \\ &= \Gamma\epsilon^{-} + (\Pi A - \Gamma\Pi - \Sigma C)x + (\Pi B - \Xi)u - (\Pi D)^{+}f^{-} + (\Pi D)^{-}f^{+} + \Pi Df, \end{aligned} \tag{19}$$

where $\Gamma \in R^{\mu\times\mu}$, $\Sigma \in R^{\mu\times q}$, $\Xi \in R^{\mu\times m}$, and $\Pi \in R^{\mu\times n}$ are the observer gains defined in (12). Similarly, by the derivation of (17), we can get

$$\begin{aligned} e^{+}(k) &= \overline{v}^{+}(k) - \Phi x(k) \\ &= \Psi^{+}\overline{g}^{+} - \Psi^{-}\overline{g}^{-} + Ly - \Phi x \\ &= \Psi^{+}\overline{g}^{+} - \Psi^{+}\Pi x - \Psi^{-}\overline{g}^{-} + \Psi^{-}\Pi x + \Psi^{+}\Pi x - \Psi^{-}\Pi x + Ly - \Phi x \\ &= \Psi^{+}\left(\overline{g}^{+} - \Pi x\right) + \Psi^{-}\left(\Pi x - \overline{g}^{-}\right) + \left(\Psi^{+} - \Psi^{-}\right)\Pi x + LCx - \Phi x \\ &= \Psi^{+}\epsilon^{+} + \Psi^{-}\epsilon^{-} + (LC + \Psi\Pi - \Phi)x, \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
e^-(k) &= \Phi x(k) - \overline{v}^-(k) \\
&= \Phi x - \Psi^+ \overline{g}^- + \Psi^- \overline{g}^+ - Ly \\
&= \Phi x - \Psi^+ \overline{g}^- + \Psi^+ \Pi x + \Psi^- \overline{g}^+ - \Psi^- \Pi x - \Psi^+ \Pi x + \Psi^- \Pi x - Ly \\
&= \Psi^- \left(\overline{g}^+ - \Pi x\right) + \Psi^+ \left(\Pi x - \overline{g}^-\right) - \left(\Psi^+ - \Psi^-\right) \Pi x - LCx + \Phi x \\
&= \Psi^- \epsilon^+ + \Psi^+ \epsilon^- + (\Phi - LC - \Psi \Pi) x,
\end{aligned} \tag{21}
$$

where $L \in R^{\mu \times q}$ and $\Psi \in R^{\mu \times \mu}$ are the observer gains defined in (12), and $\Phi \in R^{\mu \times n}$ is defined in the linear function (2). Taking (13)-(15) into the last lines of (18)-(21), it can be simplified as

$$
\begin{cases}
\epsilon^+(k+1) = \Gamma \epsilon^+(k) + \Lambda^+(k), \\
\epsilon^-(k+1) = \Gamma \epsilon^-(k) + \Lambda^-(k),
\end{cases} \tag{22}
$$

$$
\begin{cases}
e^+(k) = \Psi^+ \epsilon^+(k) + \Psi^- \epsilon^-(k), \\
e^-(k) = \Psi^- \epsilon^+(k) + \Psi^+ \epsilon^-(k),
\end{cases} \tag{23}
$$

where $\Lambda^+ = (\Pi D)^+ f^+ - (\Pi D)^- f^- - \Pi D f$, and $\Lambda^- = \Pi D f - (\Pi D)^+ f^- + (\Pi D)^- f^+$. According to Lemma 1, $\Lambda^+ \geq 0$, $\Lambda^- \geq 0$. From Lemma 3, since Γ is a Schur matrix, system (12) is ISS. In addition, Γ is also a non-negative matrix, and together with $\Lambda^+ \geq 0$, $\Lambda^- \geq 0$, we obtain $\epsilon^+ \geq 0$, and $\epsilon^- \geq 0$ by Lemma 2. By the definition of e, then

$$
\bar{v}^- \leq v = \Phi x \leq \bar{v}^+. \tag{24}
$$

Denote $\epsilon_l(k+1) = \bar{g}^+(k+1) - \bar{g}^-(k+1)$ and $e_l(k) = \bar{v}^+(k) - \bar{v}^-(k)$, and we have

$$
\begin{aligned}
\epsilon_l(k+1) &= \overline{g}^+(k+1) - \overline{g}^-(k+1) \\
&= \Gamma \epsilon_l + (\Pi D)^+ f^+ - (\Pi D)^- f^- - (\Pi D)^+ f^- + (\Pi D)^- f^+ \\
&= \Gamma \epsilon_l + \left[(\Pi D)^+ + (\Pi D)^-\right]\left(f^+ - f^-\right),
\end{aligned} \tag{25}
$$

and

$$
e_l(k) = \overline{v}^+(k) - \overline{v}^-(k) = \left(\Psi^+ + \Psi^-\right)\left(\overline{g}^+ - \overline{g}^-\right) = \left(\Psi^+ + \Psi^-\right)\epsilon_l. \tag{26}
$$

According to Lemma 3, if $||f^+ - f^-||$ is UB for all $k \geq 0$, then $||\epsilon_l||$ is also UB, which means that $||e_l|| = ||\bar{v}^+ - \bar{v}^-||$ is also UB. Similarly, if $||f^+ - f^-|| \to 0$ for all $k \geq 0$, then $||v^+ - v^-|| \to 0$. □

Remark 4. An example is given to explain how to solve Eqs. (13)–(15). The system matrices A, B, C are supposed to possess the following observable canonical form:

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.
$$

(i) For matrix A, we have $\alpha(\lambda) = det(\lambda I - A) = \lambda^3 + 3\lambda^2 + 2\lambda + 1$. If $\Gamma = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, we obtain $\alpha(\Gamma) = \Gamma^3 + 3\Gamma^2 + 2\Gamma + I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Given $\Sigma = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, according to Lemma 5,

$$
U = \begin{bmatrix} \Sigma & \Gamma \Sigma & \Gamma^2 \Sigma \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -4 \end{bmatrix},\ N = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},
$$

and

$$
V = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

So, one can get that $\Pi = -[\alpha(\Gamma)]^{-1} UNV = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

(ii) Taking $\Pi = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ into Eq. (13), we obtain $\Xi = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(iii) Given $\Phi = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$, by Lemma 6, we assume $\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}$ and $rank\begin{bmatrix} C \\ \Phi - \Psi\Pi \end{bmatrix} = rank\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2\psi_{11} - \psi_{12} & 1 + \psi_{11} - \psi_{12} \\ 0 & 2 + 2\psi_{21} - \psi_{22} & \psi_{21} - \psi_{22} \end{bmatrix} = rank\, C = 1$. So, the following equations are derived:

$$\begin{cases} 2\psi_{11} - \psi_{12} = 0, \\ 1 + \psi_{11} - \psi_{12} = 0, \\ 2 + 2\psi_{21} - \psi_{22} = 0, \\ \psi_{21} - \psi_{22} = 0. \end{cases} \tag{27}$$

Solving (27) results in $\Psi = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix}$. In view of $C^{\dagger} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, taking the value of Ψ into Eq. (11) yields $L = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Furthermore, if the system matrices A, B and C are in more general form, we can use SCILAB [47] to find the feasible solutions from the constraints of linear matrix equalities (13)-(15).

Remark 5. The steps for designing a Luenberger-like FIO are given in the algorithm below.

Algorithm 1 Algorithm for interval estimation by monotone system theory.

Input: $\Phi,\ \Gamma,\ \Sigma,\ \nu(0),\ \bar{g}^+(0),\ \bar{g}^-(0),\ f^+(k),\ f^-(k)$
Output: $\bar{\nu}^+,\ \bar{\nu}^-$
Given the values of matrices Γ and Σ;
do
$\Pi A - \Gamma\Pi = \Sigma C$;
Output: Π;
do
$\Xi = \Pi B$;
Output: Ξ;
Given the value of matrix Φ;
do
$\Phi = \Psi\Pi + LC$;
Output: $\Psi,\ L$;
Given initial values:
$\nu(0) = \Phi x_0,\ \bar{g}^+(0) = \bar{g}_0^+,\ \bar{g}^-(0) = \bar{g}_0^-,\ f^+(k) = f^+,\ f^-(k) = f^-$;
for $k \geq 0$ **do**
  $\bar{g}^+(k+1) = \Gamma\bar{g}^+(k) + \Sigma y(k) + \Xi u(k) + (\Pi D)^+ f^+(k) - (\Pi D)^- f^-(k)$
  $\bar{g}^-(k+1) = \Gamma\bar{g}^-(k) + \Sigma y(k) + \Xi u(k) + (\Pi D)^+ f^-(k) - (\Pi D)^- f^+(k)$
  $\bar{\nu}^+ = \Psi^+\bar{g}^+ - \Psi^-\bar{g}^- + Ly$
  $\bar{\nu}^- = \Psi^+\bar{g}^- - \Psi^-\bar{g}^+ + Ly$;
end for

Remark 6. The specific method of solving the Sylvester equations has been proposed [27], but the existing conditions of the solutions of these equations have not been given. In Lemma 5, sufficient conditions for the existence of solutions are given. Moreover, a simple method to solve the Sylvester equations is also introduced.

Corollary 1. *Consider system (1) with $u(k) = 0$, that is,*

$$\begin{cases} x(k+1) = Ax(k) + Df(k), \\ y(k) = Cx(k), \end{cases} \tag{28}$$

and the estimation object is

$$\nu(k) = \Phi x(k),$$

where $\Phi \in R^{\mu \times n}$ is defined in (2). Then, the Luenberger-like FIO of system (28) is described as

$$\begin{cases} \tilde{g}^+(k+1) = \Gamma\tilde{g}^+(k) + \Sigma y(k) + (\Pi D)^+ f^+(k) - (\Pi D)^- f^-(k), \\ \tilde{g}^-(k+1) = \Gamma\tilde{g}^-(k) + \Sigma y(k) + (\Pi D)^+ f^-(k) - (\Pi D)^- f^+(k), \\ \tilde{\nu}^+(k) = \Psi^+\tilde{g}^+(k) - \Psi^-\tilde{g}^-(k) + Ly(k), \\ \tilde{\nu}^-(k) = \Psi^+\tilde{g}^-(k) - \Psi^-\tilde{g}^+(k) + Ly(k). \end{cases} \tag{29}$$

If the corresponding coefficient matrices of system (29) satisfy the equations

$$\Pi A - \Gamma\Pi = \Sigma C,$$
$$\Phi = \Psi\Pi + LC,$$

and Γ *is a Schur and non-negative matrix, then system (29) is the FIO of system (28).*

Proof. The proof process is the same as Theorem 1, and thus it is omitted here. □

Remark 7. When using monotone system theory to design the observer, it needs to be ensured that matrix Γ is both a Schur matrix and a non-negative matrix. It is often impossible for these two conditions to be satisfied at the same time. Besides, the interval width of the estimated trajectories cannot be guaranteed. To reduce the constraints, the approach of the zonotope is used to improve the design conditions as well as the performance of the FIO in the next section.

3. Design of the FIO by the zonotope method

3.1. Design of the H_∞ functional observer

An H_∞ functional observer for system (1) is designed as follows:

$$\hat{g}(k+1) = \Gamma\hat{g}(k) + \Sigma y(k) + \Xi u(k), \tag{30}$$

where $\hat{g}(k) \in R^{\mu}$ and the observer gains $\Gamma \in R^{\mu\times\mu}$, $\Sigma \in R^{\mu\times q}$, and $\Xi \in R^{\mu\times m}$ will be determined in the sequel. The error is given by

$$e(k) = \Phi x(k) - \hat{g}(k), \tag{31}$$

where $\Phi \in R^{\mu\times n}$ is defined in (2).

Definition 3. Observer (30) is said to be the H_∞ functional observer for linear function (2), if
(1) $f(k) = 0$, then error (31) converges to zero asymptotically,
(2) $f(k) \neq 0$, then under initial condition $e(0) = 0$, we have

$$\sum_{k=0}^{\infty} e^T e < \gamma \sum_{k=0}^{\infty} f^T f, \tag{32}$$

where $\gamma > 0$ is the disturbance attenuation level.

Lemma 7. *[49] (Schur complements) Consider a given symmetrical matrix* $\Theta = \begin{bmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{12}^T & \Theta_{22}\end{bmatrix}$, *and the following formulas are equivalent:*

$$(1)\ \Theta \prec 0,$$
$$(2)\ \Theta_{11} \prec 0, \Theta_{22} - \Theta_{12}^T\Theta_{11}^{-1}\Theta_{12} \prec 0,$$
$$(3)\ \Theta_{22} \prec 0, \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{12}^T \prec 0.$$

Theorem 2. *Let* $0 < \xi < 1$ *be a given constant. If there exists a positive constant* γ *and matrices* $P \in R^{\mu\times\mu} \succ 0$, $R \in R^{\mu\times\mu}$ *such that*

$$\begin{bmatrix} -\xi P + I & * & * \\ D^T\Phi^T R & D^T\Phi^T P\Phi D - \gamma I & * \\ R & 0 & -P \end{bmatrix} \prec 0, \tag{33}$$

$$\Phi A = \Gamma\Phi + \Sigma C, \tag{34}$$

$$\Xi = \Phi B, \tag{35}$$

then system (30) is the H_∞ *functional observer of system (1), and the observer gain* Γ *is determined by* $\Gamma = P^{-1}R$.

Proof. The inequality (33) is rewritten as follows:

$$\Theta = \begin{bmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{12}^T & \Theta_{22}\end{bmatrix} \prec 0, \tag{36}$$

where $\Theta_{11} = \begin{bmatrix} -\xi P + I & * \\ D^T\Phi^T R & D^T\Phi^T P\Phi D - \gamma I\end{bmatrix}$, $\Theta_{12} = \begin{bmatrix} R^T \\ 0_{p\times\mu}\end{bmatrix}$, and $\Theta_{22} = -P$. Since $\Theta \prec 0$ and $\Theta_{22} \prec 0$, according to Lemma 7, we have

$$\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{12}^T \prec 0. \tag{37}$$

Substituting $R = P\Gamma$ into (37) yields

$$\begin{bmatrix} \Gamma^T P\Gamma - \xi P + I & * \\ D^T\Phi^T P\Gamma & D^T\Phi^T P\Phi D - \gamma I \end{bmatrix} \prec 0. \tag{38}$$

Choose the following Lyapunov candidate

$$V(k) = e^T(k)Pe(k). \tag{39}$$

From Definition 3, the proof can be divided into two steps.

(i) $f(k) = 0$, under the derivation of error (31), the following is obtained:

$$\begin{aligned} e(k+1) &= \Phi x(k+1) - \hat{g}(k+1) \\ &= \Phi Ax + \Phi Bu - \Gamma\hat{g} - \Sigma y - \Xi u \\ &= \Phi Ax + \Gamma\Phi x - \Gamma\Phi x - \Gamma\hat{g} - \Sigma Cx - \Xi u + \Phi Bu \\ &= \Gamma\left(\Phi x - \hat{g}\right) + (\Phi A - \Gamma\Phi - \Sigma C)x + (\Phi B - \Xi)u \\ &= \Gamma e + (\Phi A - \Gamma\Phi - \Sigma C)x + (\Phi B - \Xi)u. \end{aligned} \tag{40}$$

Substituting (34) and (35) into the last line of (40) yields

$$e(k+1) = \Gamma e(k). \tag{41}$$

Hence, one can get

$$\Delta V(k) = V(k+1) - V(k) = e^T\Gamma^T P\Gamma e - e^T Pe = e^T\left(\Gamma^T P\Gamma - P\right)e. \tag{42}$$

The inequality (38) implies that $\Gamma^T P\Gamma - \xi P + I \prec 0$. Thus, it is deduced from (42) that

$$\Delta V(k) < (\xi - 1)e^T Pe - e^T e < (\xi - 1)e^T Pe, \tag{43}$$

which means

$$V(k+1) < \xi V(k). \tag{44}$$

By repeating (44), it is easy to find that

$$V(k) < \xi^k V(0). \tag{45}$$

Since $V(k) \geq \underline{\lambda}(P)||e(k)||^2$ and $V(0) \leq \overline{\lambda}(P)||e(0)||^2$, we have

$$\begin{aligned} &\underline{\lambda}(P) \parallel e(k)\parallel^2 < \xi^k\overline{\lambda}(P) \parallel e(0)\parallel^2 \\ &\parallel e(k) \parallel < \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}\xi^k} \parallel e(0) \parallel . \end{aligned} \tag{46}$$

Given $0 < \xi < 1$, then

$$\lim_{k\to+\infty} ||e(k)|| = \lim_{k\to+\infty} \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}\xi^k}||e(0)|| = 0. \tag{47}$$

Thus, the error (31) converges to zero asymptotically when $f(k) = 0$.

(ii) $f(k) \neq 0$, the error (31) is derived by

$$\begin{aligned} e(k+1) &= \Phi x(k+1) - \hat{g}(k+1) \\ &= \Phi Ax + \Phi Bu + \Phi Df - \Gamma\hat{g} - \Sigma y - \Xi u \\ &= \Phi Ax + \Gamma\Phi x - \Gamma\Phi x - \Gamma\hat{g} - \Sigma Cx - \Xi u + \Phi Bu + \Phi Df \\ &= \Gamma\left(\Phi x - \hat{g}\right) + (\Phi A - \Gamma\Phi - \Sigma C)x + (\Phi B - \Xi)u + \Phi Df \\ &= \Gamma e + (\Phi A - \Gamma\Phi - \Sigma C)x + (\Phi B - \Xi)u + \Phi Df, \end{aligned} \tag{48}$$

where $\Gamma \in R^{\mu\times\mu}$ is the observer gain defined in observer (30) and $\Phi \in R^{\mu\times n}$ is defined in the linear function (2). Substituting (34) and (35) into the last line of (48) yields

$$e(k+1) = \Gamma e + \Phi Df. \tag{49}$$

Let

$$J = \sum_{k=0}^{\infty} e^T e - \gamma \sum_{k=0}^{\infty} f^T f = \sum_{k=0}^{\infty} \left(e^T e - \gamma f^T f\right). \tag{50}$$

Under zero initial condition, one can obtain

$$\sum_{k=0}^{\infty}\Delta V(k)=\lim_{k\to+\infty}[V(k+1)-V(k)+V(k)-V(k-1)+\cdots+V(1)-V(0)]=\lim_{k\to+\infty}V(k+1)\geq 0, \tag{51}$$

and thus

$$J\leq \textstyle\sum_{k=0}^{\infty}[e^Te-\gamma f^Tf+\Delta V(k)], \tag{52}$$

where the sum of $\Delta V(k)$ is defined in Eq. (51). Denote $S(k)=e^Te-\gamma f^Tf+\Delta V(k)$, by the expression of error (49), then we obtain

$$\begin{aligned} S(k)&=e^T(k)e(k)-\gamma f^T(k)f(k)+V(k+1)-V(k)\\ &=e^T(k+1)Pe(k+1)-e^T(k)Pe(k)+e^T(k)e(k)-\gamma f^T(k)f(k)\\ &=e^T\big(\Gamma^TP\Gamma-P+I\big)e+2f^TD^T\Phi^TP\Gamma e+f^T\big(D^T\Phi^TP\Phi D-\gamma I\big)f. \end{aligned} \tag{53}$$

According to inequality (38), the following inequality is satisfied

$$N^T\begin{bmatrix}\Gamma^TP\Gamma-\xi P+I & *\\ D^T\Phi^TP\Gamma & D^T\Phi^TP\Phi D-\gamma I\end{bmatrix}N<0, \tag{54}$$

where $N=[e(k)^T \quad f(k)^T]^T$. Based on the derivation of (54), we have

$$e^T(\Gamma^TP\Gamma-\xi P+I)e+2f^TD^T\Phi^TP\Gamma e+f^T(D^T\Phi^TP\Phi D-\gamma I)f<0. \tag{55}$$

Further simplifying inequality (55) results in

$$e^T(\Gamma^TP\Gamma-P+I)e+2f^TD^T\Phi^TP\Gamma e+f^T(D^T\Phi^TP\Phi D-\gamma I)f<(\xi-1)e^TPe. \tag{56}$$

By (53), (56) and the fact that $0<\xi<1$, we have

$$S(k)<(\xi-1)e^TPe<0. \tag{57}$$

In the light of (52) and (57), the following inequality holds

$$J\leq\sum_{k=0}^{\infty}S(k)<0, \tag{58}$$

which means that

$$\textstyle\sum_{k=0}^{\infty}(e^Te-\gamma f^Tf)<0, \tag{59}$$

that is,

$$\textstyle\sum_{k=0}^{\infty}e^Te<\gamma\sum_{k=0}^{\infty}f^Tf. \tag{60}$$

In view of (i), (ii), and Definition 3, the conclusion can be drawn that (30) is the H_∞ functional observer of system (1). □

Remark 8. It is known that the H_∞ performance depends on the disturbance attenuation level γ. The smaller the γ is, the better the performance of the designed observer is. Therefore, the optimal problem of H_∞ functional observer can be formulated as follows:

$$\min\ \gamma \tag{61}$$

s.t.

$$\begin{bmatrix}-\xi P+I & * & *\\ D^T\Phi^TR & D^T\Phi^TP\Phi D-\gamma I & *\\ R & 0 & -P\end{bmatrix}\prec 0, \tag{62}$$

$$\Phi A=\Gamma\Phi+\Sigma C, \tag{63}$$

$$\Xi=\Phi B. \tag{64}$$

3.2. Design of the FIO by zonotope

Below, the zonotope method is used to recover the boundaries of system (1) based on the observer (30). Some definitions and lemmas are given firstly.

Definition 4. An s-order zonotope $\mathscr{S} \in R^{\mu}$ is an affine transformation of hypercube $B^s = [-1, 1]^s$, that is,

$$\mathscr{S} = g \oplus MB^s = \{g + Mz, z \in B^s\}, \tag{65}$$

where $g \in R^{\mu}$ is the center of $\mathscr{S}$, and $M \in R^{\mu \times s}$ is the generating matrix of $\mathscr{S}$. To simplify symbols, the zonotope $\mathscr{S}$ is described by $\langle g, M \rangle$.

Remark 9. It is supposed that the initial values of the linear state vectors and the disturbances of system (1) are unknown but bounded, and they can be regarded as the following zonotopes:

$$\begin{aligned} &\nu = \Phi x \in \chi = \langle g_0, M \rangle, \\ &f \in \Omega = \langle 0, W \rangle, \end{aligned} \tag{66}$$

where g_0 is a known vector, and M, W are given matrices.

Lemma 8. *[40] Given a zonotope $\mathscr{S} = \langle g, M \rangle$, the following properties hold:*

$$(1)\ \langle g_1, M_1 \rangle \oplus \langle g_2, M_2 \rangle = \langle g_1 + g_2, [M_1\ M_2] \rangle,$$
$$(2)\ P \odot \langle g, M \rangle = \langle Pg, PM \rangle,$$
$$(3)\ \langle g, M \rangle \quad \subseteq \langle g, \overline{M} \rangle,$$

where g, g_1, $g_2 \in R^{\mu}$, M, M_1, $M_2 \in R^{\mu \times s}$, $P \in R^{l \times \mu}$, and $\bar{M} \in R^{\mu \times \mu}$ is a diagonal matrix with

$$\bar{M}_{i,i} = \sum_{j=1}^{s} |M_{i,j}|, \quad i = 1, 2, \ldots, \mu,$$

i.e.,

$$\bar{M} = \begin{bmatrix} \sum_{j=1}^{s} |M_{1,j}| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{s} |M_{\mu,j}| \end{bmatrix}.$$

With time going by, the number of zonotope orders continues to increase, which will eventually lead to the impossibility of computation. Therefore, the order of the zonotope needs to be reduced. A simple method of order reduction was mentioned in a prior study [49].

Lemma 9. *[50] For a given zonotope $\mathscr{S} = g \oplus MB^s \subset R^{\mu}$, it can be included by the following zonotope:*

$$\mathscr{S} = \langle g, M \rangle \subseteq \left\langle g, \hat{M} \right\rangle = g \oplus \hat{M}B^q \ (\mu \le q \le s), \tag{67}$$

where $\hat{M} = [M_z \quad Q]$, M_z is comprised of the first $q - \mu$ columns of $\widetilde{M}$, Q is a diagonal matrix whose diagonal elements are

$$Q_{i,i} = \sum_{j=q-\mu+1}^{s} |\widetilde{M}_{i,j}|, \quad i = 1, 2, \ldots, \mu, \tag{68}$$

and matrix $\widetilde{M}$ is generated by sorting the Euclidean norm of each column of M in descending order.

Remark 10. An example is given to further explain the reduced-order algorithm. It is assumed that $\mu = 2$, $q = 3$, and $s = 4$. If

$$M = \begin{bmatrix} 1 & -0.8 & 1.2 & 1.7 \\ -1 & -1.7 & 0.9 & -1.2 \end{bmatrix},$$

then

$$\tilde{M} = \begin{bmatrix} 1.7 & 1 & -0.8 & 1.2 \\ -1.2 & -1 & -1.7 & 0.9 \end{bmatrix}, \quad M_z = \begin{bmatrix} 1.7 \\ -1.2 \end{bmatrix},$$

$$Q = \begin{bmatrix} 3 & 0 \\ 0 & 3.6 \end{bmatrix}, \quad \hat{M} = \begin{bmatrix} 1.7 & 3 & 0 \\ -1.2 & 0 & 3.6 \end{bmatrix}.$$

When $\mu = q = 2$,

$$\hat{M} = \bar{M} = \begin{bmatrix} 4.7 & 0 \\ 0 & 4.8 \end{bmatrix}.$$

Theorem 3. *Suppose that $f(k) \in \Omega = \langle 0, W \rangle$, then the linear function $v(k) = \Phi x(k) \in \hat{\mathscr{X}}(k)$ can be obtained and $\hat{\mathscr{X}}(k)$ is determined by*

$$\hat{\mathscr{X}}(k) = \langle \hat{g}(k), M(k) \rangle, \tag{69}$$

where $\hat{g}(k)$ is generated by the H_∞ functional observer (30), and $M(k)$ satisfies

$$M(k+1) = [\Gamma \hat{M}(k) \quad \Phi DW], \tag{70}$$

where $\hat{M}(k)$ is defined in Lemma 9.

Proof. According to $e(k)$ defined by (31), one can obtain $v(k) = \hat{g}(k) + e(k)$. We assume that

$$e(k) \in \hat{\zeta}(k) = \langle 0, M(k) \rangle. \tag{71}$$

Then,

$$v(k) \in \hat{\mathscr{X}}(k) = \langle \hat{g}(k), M(k) \rangle. \tag{72}$$

It follows from the error (49) that

$$e(k+1) \in \zeta(k+1) = \Gamma \odot \langle 0, M(k) \rangle \oplus \Phi D \odot \langle 0, W \rangle = \langle 0, [\Gamma M(k) \quad \Phi DW] \rangle. \tag{73}$$

By Lemma 9, we obtain

$$e(k) \in \langle 0, M(k) \rangle \subseteq \langle 0, \hat{M}(k) \rangle, \tag{74}$$

and

$$e(k+1) \in \zeta(k+1) \subseteq \hat{\zeta}(k+1) = \Gamma \odot \langle 0, \hat{M}(k) \rangle \oplus \Phi D \odot \langle 0, W \rangle = \langle 0, \left[\Gamma \hat{M}(k) \quad \Phi DW\right] \rangle. \tag{75}$$

Denote $\hat{\zeta}(k+1) = \langle 0, M(k+1) \rangle$, and then

$$M(k+1) = [\Gamma \hat{M}(k) \quad \Phi DW]. \tag{76}$$

□

Remark 11. According to Theorem 3, $M(k)$ can be calculated by following the iterative relation:

$$M(0) \Rightarrow \hat{M}(0) \Rightarrow M(1) \Rightarrow \hat{M}(1) \Rightarrow \cdots \Rightarrow M(k). \tag{77}$$

By Lemma 8, $\langle \hat{g}, M \rangle \subseteq \langle \hat{g}, \bar{M} \rangle$, and thus the interval observer structure of the system is described as

$$\begin{cases} \hat{v}_i^+ = \hat{g}_i + \sum_{j=1}^{s} \left| M_{i,j} \right|, \\ \hat{v}_i^- = \hat{g}_i - \sum_{j=1}^{s} \left| M_{i,j} \right|, \end{cases} \tag{78}$$

where $i = 1, \ldots, \mu$ is the i-th row, and $j = 1, \ldots, s$ is the j-th column.

Thus, the following algorithm can be used for interval estimation by the zonotope method.
Similar to Corollary 1, system (28) is also considered.

Corollary 2. *The H_∞ functional observer of system (28) is described as*

$$\tilde{g}(k+1) = \Gamma \tilde{g}(k) + \Sigma y(k), \tag{79}$$

and let $0 < \xi < 1$ be a given constant. If there exists a positive constant γ and matrices $P \in R^{\mu \times \mu} \succ 0$, $R \in R^{\mu \times \mu}$ such that

$$\begin{bmatrix} -\xi P + I & * & * \\ D^T \Phi^T R & D^T \Phi^T P \Phi D - \gamma I & * \\ R & 0 & -P \end{bmatrix} \prec 0,$$

$$\Phi A = \Gamma \Phi + \Sigma C,$$

then system (79) is the H_∞ functional observer of system (28), and the observer gain Γ is determined by $\Gamma = P^{-1} R$.

Proof. The proof process is the same as Theorem 2, and thus it is omitted here. □

Algorithm 2 Algorithm for interval estimation by zonotope method.

Input: $\nu(0)$, $\hat{g}(0)$, $M(0)$, W, ξ, Φ
Output: $\hat{\nu}^{+}$, $\hat{\nu}^{-}$
Given the values of ξ and matrix Φ;
do
Remark 8;
output: Γ, Σ, Ξ;
Given initial values:
$\nu(0)=\Phi x_0$, $\hat{g}(0)=\hat{g}_0$, $M(0)=M_0$, $f(k)\in\langle 0,W\rangle$;
for $k\geq 0$ **do**
  $\hat{g}(k+1)=\Gamma\hat{g}(k)+\Sigma y(k)+\Xi u(k)$
  $\nu(k)=\hat{g}(k)+e(k)$
  $e(k)\in\hat{\zeta}(k)=\langle 0,M(k)\rangle$
  $\nu(k)=\Phi x(k)\in\hat{\mathscr{X}}=\langle\hat{g}(k),M(k)\rangle$
  $e(k+1)\in\langle 0,\Gamma M(k)\rangle\oplus\Phi D\odot\langle 0,W\rangle=\langle 0,\ [\Gamma M(k)\quad\Phi DW]\rangle$
  $e(k)\in\langle 0,M(k)\rangle\subseteq\langle 0,\hat{M}(k)\rangle$
  $e(k+1)\in\hat{\zeta}(k+1)=\langle 0,M(k+1)\rangle=\langle 0,[\Gamma\hat{M}(k)\quad\Phi DW]\rangle$
  $M(k+1)=[\Gamma\hat{M}(k)\quad\Phi DW]$
  $\hat{\nu}_i^{+}=\hat{g}_i+\sum_{j=1}^{s}|M_{i,j}|$
  $\hat{\nu}_i^{-}=\hat{g}_i-\sum_{j=1}^{s}|M_{i,j}|$;
end for

Given (79) and (28), the following error is governed by

$$\begin{aligned}e(k+1)&=\Phi x(k+1)-\tilde{g}(k+1)\\&=\Phi Ax+\Phi Df-\Gamma\tilde{g}-\Sigma y\\&=\Phi Ax+\Gamma\Phi x-\Gamma\Phi x-\Gamma\tilde{g}-\Sigma Cx+\Phi Df\\&=\Gamma(\Phi x-\tilde{g})+(\Phi A-\Gamma\Phi-\Sigma C)x+\Phi Df\\&=\Gamma e+(\Phi A-\Gamma\Phi-\Sigma C)x+\Phi Df\\&=\Gamma e+\Phi Df.\end{aligned}$$

Corollary 3. *According to Theorem 3, the linear function $\nu(k)=\Phi x(k)\in\widetilde{\mathscr{X}}(k)$ can be obtained and $\widetilde{\mathscr{X}}(k)$ is determined by*

$$\widetilde{\mathscr{X}}(k)=\langle\tilde{g}(k),M(k)\rangle,$$

where $\tilde{g}(k)$ is generated by the H_∞ functional observer (79), and $M(k)$ satisfies

$$M(k+1)=[\Gamma\hat{M}(k)\quad\Phi DW],$$

where $\hat{M}(k)$ is defined in Lemma 9. The proof is omitted here.

Remark 12. Similarly, by Lemma 8, $\langle\tilde{g},M\rangle\subseteq\langle\tilde{g},\bar{M}\rangle$, and thus the interval observer structure of system (28) is described as

$$\begin{cases}\tilde{\nu}_i^{+}=\tilde{g}_i+\sum\limits_{j=1}^{s}\left|M_{i,j}\right|,\\ \tilde{\nu}_i^{-}=\tilde{g}_i-\sum\limits_{j=1}^{s}\left|M_{i,j}\right|,\end{cases}$$

where $i=1,\ldots,\mu$ is the i-th row, and $j=1,\ldots,s$ is the j-th column.

4. Examples

4.1. Numerical example

Consider system (1) with

$$A=\begin{bmatrix}0.3276&0&0\\0&0.026&-0.1582\\-0.0916&-0.0437&0.5017\end{bmatrix},\quad B=\begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$C=\begin{bmatrix}1&0.5&1\end{bmatrix},\quad D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Let

$$f^- = \begin{bmatrix} -0.2 \\ -0.2 \\ -0.4 \end{bmatrix} \le f(k) = \begin{bmatrix} 0.2\sin(k) \\ 0.2\cos(k) \\ 0.4\cos(k) \end{bmatrix} \le f^+ = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.4 \end{bmatrix}, \quad u(k) = 100\sin(k).$$

Define the linear function (2) with

$$\Phi = \begin{bmatrix} 0.0031 & 0.0011 & 0.1 \\ 0.1 & 0.168 & 0.1 \end{bmatrix}.$$

Firstly, a Luenberger-like FIO is designed for this system. Choose Γ as follows:

$$\Gamma = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.7 \end{bmatrix},$$

which is both a non-negative and a Schur matrix. The initial values are given by

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{g}^+(0) = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}, \quad \bar{g}^-(0) = \begin{bmatrix} -0.5 \\ -0.25 \end{bmatrix}, \quad v(0) = \Phi x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By using Lemma 5 and Theorem 1, select $\Sigma = \begin{bmatrix} -0.01 \\ 0.0336 \end{bmatrix}$, and then $\Pi = \begin{bmatrix} 0.0152 & 0.0047 & 0.0310 \\ -0.0514 & -0.0147 & -0.1577 \end{bmatrix}$. By solving the conditions (13) and (15), the following can be obtained:

$$\Xi = \begin{bmatrix} 0.031 \\ -0.1577 \end{bmatrix}, \quad \Psi = \begin{bmatrix} -7.5709 & -2.0369 \\ -93.2803 & -13.8648 \end{bmatrix}, \quad L = \begin{bmatrix} 0.0135 \\ 0.805 \end{bmatrix}.$$

Thus,

$$(\Pi D)^+ = \begin{bmatrix} 0.0152 & 0.0047 & 0.0310 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\Pi D)^- = \begin{bmatrix} 0 & 0 & 0 \\ 0.0514 & 0.0147 & 0.1577 \end{bmatrix},$$

$$\Psi^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \Psi^- = \begin{bmatrix} 7.5709 & 2.0369 \\ 93.2803 & 13.8648 \end{bmatrix},$$

$$\bar{v}^+(0) = \begin{bmatrix} 4.297 \\ 50.1064 \end{bmatrix}, \quad \bar{v}^-(0) = \begin{bmatrix} -4.297 \\ -50.1064 \end{bmatrix}.$$

Next, the FIO designed by the zonotope method and the initial values are

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad v(0) = \Phi x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \hat{g}(0) = 0,$$

$$M(0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}.$$

Select the constant $\xi = 0.63$. Solving (61)-(64), we can obtain

$$P = \begin{bmatrix} 3.8562 & 0 \\ 0 & 4.1759 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.6 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad \Xi = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} -0.01 \\ 0.0336 \end{bmatrix},$$

$$\gamma = 2.8771, \quad \hat{v}^+(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \hat{v}^+(0) = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

To show the advatage of the zonotope method, a simulation of the two proposed FIOs was conducted. Fig. 1 depicts the evolution of original state v_1, the estimated bounds $\bar{v}_1^+$, $\bar{v}_1^-$ (by monotone system theory), and the estimated bounds $\hat{v}_1^+$, $\hat{v}_1^-$ (by the zonotope method). Fig. 2 presents the evolution of original state v_2, the estimated bounds $\bar{v}_2^+$, $\bar{v}_2^-$, and the estimated bounds $\hat{v}_2^+$, $\hat{v}_2^-$. As shown, the boundaries estimated by the zonotope method are more precise than those estimated by monotone system theory. Furthermore, from the design procedure, the observer gain Γ in the first FIO needs to be a Schur and non-negative matrix, while Γ in the second one is only a Schur matrix. Thus, it can be concluded that the performance of the FIO by the zonotope method is better.

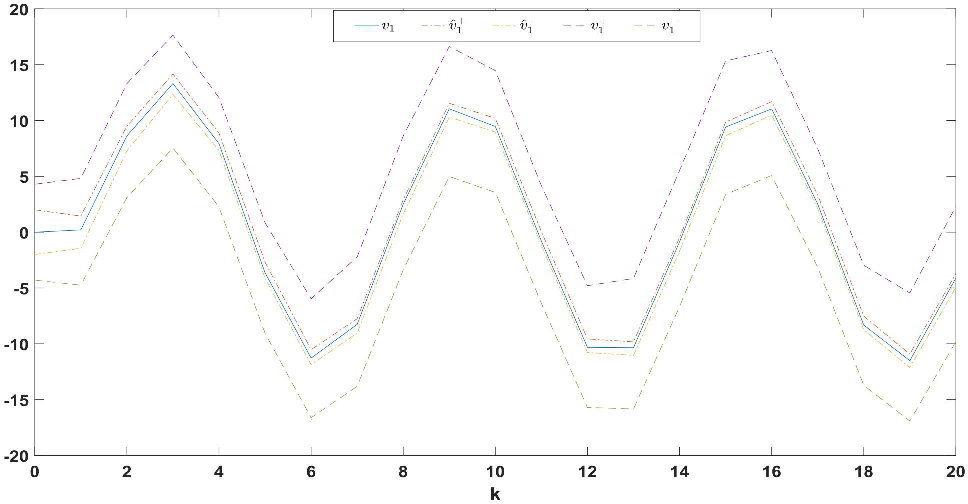

Fig. 1. Evolution of $v_1(k)$, $\bar{v}_1^+$, $\bar{v}_1^-$, $\hat{v}_1^+$ and $\hat{v}_1^-$.

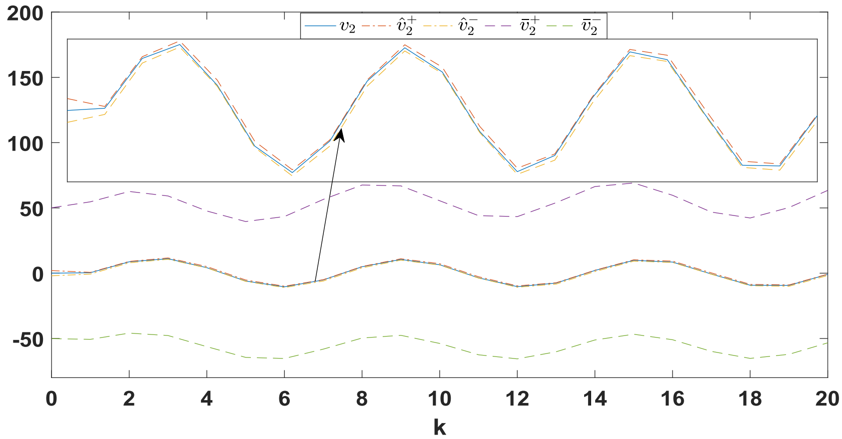

Fig. 2. Evolution of $v_2(k)$, $\bar{v}_2^+$, $\bar{v}_2^-$, $\hat{v}_2^+$ and $\hat{v}_2^-$.

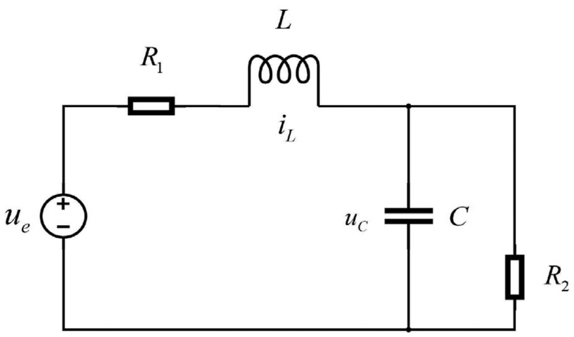

Fig. 3. The diagram of circuit system.

4.2. Application example

Consider the following circuit system, whose diagram is shown in Fig. 3. The mathematic model is described by

$$\begin{cases} \dfrac{di_L(t)}{dt} = -\dfrac{R_1}{L}i_L(t) - \dfrac{1}{L}u_c(t) + \dfrac{1}{L}u_e(t), \\ \dfrac{du_c(t)}{dt} = \dfrac{1}{C}i_L(t) - \dfrac{1}{R_2C}u_c(t), \end{cases} \tag{80}$$

where i_L is the inductance current and u_c is the capacitance voltage. It is defined that $x(t)=[i_L(t),u_c(t)]^T$ and $y(t)=u_c$, and thus the state-space model of system (80) is

$$\begin{cases} \dfrac{dx(t)}{dt}=Ax(t)+Bu_e(t), \\ y(t)=Cx(t), \end{cases} \tag{81}$$

where

$$A=\begin{bmatrix} -\dfrac{R_1}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{R_2C} \end{bmatrix},\quad B=\begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix},\quad C=\begin{bmatrix} 0 & 1 \end{bmatrix}.$$

According to the definition of the derivative, one can obtain

$$\frac{dx(t)}{dt}=\lim_{\delta h\to 0}\frac{x(t+\delta h)-x(t)}{\delta h},$$

where δh is the sampling time. When δh is taken as a very small value,

$$\frac{dx(t)}{dt}=Ax(t)+Bu_e(t)\approx\frac{x(t+\delta h)-x(t)}{\delta h},$$
$$x(t+\delta h)\approx(A\delta h+I)x(t)+B\delta hu_e(t).$$

Let $t=k\delta h$, and then

$$\begin{cases} x((k+1)\delta h)=(A\delta h+I)x(k\delta h)+B\delta hu_e(k\delta h), \\ y(k\delta h)=Cx(k\delta h). \end{cases} \tag{82}$$

For simulation, $\delta h=1$, and (82) becomes

$$\begin{cases} x(k+1)=(A+I)x(k)+Bu_e(k), \\ y(k)=Cx(k). \end{cases} \tag{83}$$

Standard parameters of the circuit system can be chosen as: $L=20H,\quad C=1\times10^3F,\quad R_1=20.7\Omega$, and $R_2=1\times10^{-3}\Omega$, and then

$$A+I=\begin{bmatrix} -0.035 & -0.05 \\ 0.001 & 0 \end{bmatrix},\quad B=\begin{bmatrix} 0.05 \\ 0 \end{bmatrix},\quad C=[0\quad 1].$$

In practice, the circuit system is often affected by disturbances, such as electromagnetic disturbance or temperature. In this paper, the non-linear term $Df(k)$ is used to represent the disturbances in circuit system, where $D=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $f(k)=\begin{bmatrix} 0.2\sin(k) \\ 0.4\cos(k) \end{bmatrix}$ and the bounds of f are $f^-=\begin{bmatrix} -0.2 \\ -0.4 \end{bmatrix}$, $f^+=\begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$. Let $u_e(k)=10^4\sin(k)$, and $\Phi=\begin{bmatrix} -1.53\times10^{-3} & 0.0992 \end{bmatrix}$.

Firstly, a Luenberger-like FIO is designed. Γ is chosen as 0.7, and the initial values are

$$x(0)=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\quad \nu(0)=\Phi x(0)=0,\quad \hat{g}^+(0)=0.3,\quad \hat{g}^-(0)=-0.3.$$

By using Lemma 5 and Theorem 1, for a given $\Sigma=0.1$, then $\Pi=\begin{bmatrix} -1.94\times10^{-4} & -0.143 \end{bmatrix}$. Solving the conditions (13) and (15) results in

$$\Xi=-9.7\times10^{-6},\quad \Psi=7.89,\quad L=1.23.$$

Thus,

$$(\Pi D)^+=\begin{bmatrix} 0 & 0 \end{bmatrix},\quad (\Pi D)^-=\begin{bmatrix} 1.94\times10^{-4} & 0.143 \end{bmatrix},$$

$$\Psi^+=7.98,\quad \Psi^-=0,\quad \bar{\nu}^+(0)=2.367,\quad \bar{\nu}^-(0)=-2.367.$$

Next, the FIO is designed by the zonotope method. Choose that $M(0)=1$, $W=\begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$. Let $\xi=0.54$, and by solving (61)-(64), we can obtain

$$P=96.523,\quad \Gamma=-0.7,\quad \Xi=-7.63\times10^{-5},\quad \Sigma=0.01,$$

$$\gamma=81.82,\quad \hat{\nu}^+(0)=1,\quad \hat{\nu}^-(0)=-1.$$

The simulation results of the FIOs by two methods are shown in Fig. 4. ν is the state of the original system, $\bar{\nu}^+$, $\bar{\nu}^-$ are the estimated boundaries by monotone system theory, and $\hat{\nu}_1^+$, $\hat{\nu}_1^-$ are the estimated boundaries by the zonotope method. As shown, the FIO designed by the zonotope method is better than that designed by using monotone system theory.

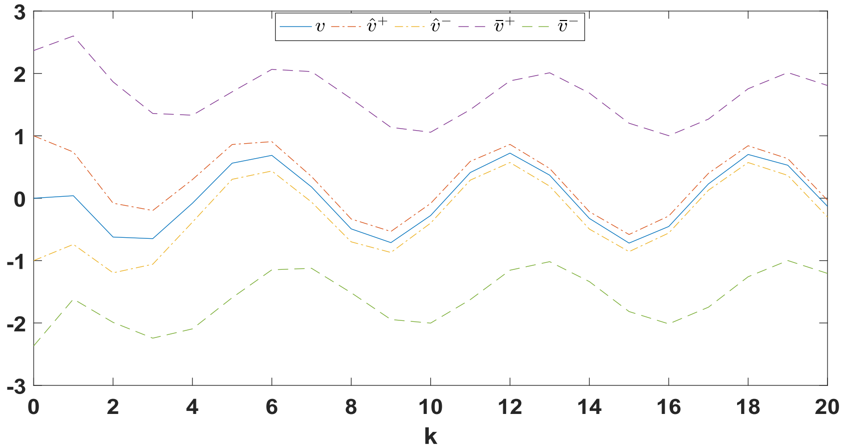

Fig. 4. Evolution of v, $\bar{v}^+$, $\bar{v}^-$, $\hat{v}^+$, and $\hat{v}^-$.

5. Conclusion

The FIO design problem for linear discrete-time systems with disturbances is studied in this paper. Two methods are proposed. In the first method, a Luenberger-like FIO is designed by using monotone system theory and the observer gains are determined by solving the Sylvester equations. In the second method, to improve the design conditions as well as the performance of the FIO, an H_∞ observer is constructed based on the functional observer, and the boundaries of the linear function are recovered by the zonotope method. Finally, two examples are simulated and the correctness of two methods is evaluated. It is concluded that the FIO designed by the zonotope method can obtain a more accurate boundary and preform better. However, at present, the zonotope method can only be applied in linear systems, and it is less involved in nonlinear systems. As a further investigation, the FIO design method for nonlinear systems will be studied.

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