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Interval observer design method for asynchronous switched systems

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Abstract: When designing interval observers for switched systems, for simplicity, it is always assumed that the switching signal of interval observers is synchronised with one of subsystems. However, in fact, they are asynchronous in most cases. In this study, the interval observer design problem for discrete-time switched systems with asynchronous switching law is investigated. At first, a robust observer is designed based on the H_∞ observer theory. Then, in order to reduce the conservatism of design conditions and improve the performance of the interval observer, the zonotope method is applied to estimate the bounds of system state. Finally, two examples are provided to highlight the efficiency of the proposed method.

Nomenclature

$x \le y$	components of x are less than the corresponding components of y
$\oplus$	Minkowski sum
$\odot$	linear mapping
$A \prec 0$	A is a negative definite matrix
$\Vert x \Vert$	Euclidean norm of the vector x, i.e. $\sqrt{x^{\mathrm{T}}x}$
I	identity matrix with appropriate dimensions
$\underline{\phi}(P)$	minimum value among all eigenvalues of matrix P
$\overline{\overline{\phi}}(P)$	maximum value among all eigenvalues of matrix P

1 Introduction

Switched systems are an important kind of hybrid systems, which have become a hot topic in the past decades. A switched system consists of a series of subsystems along with a switching rule determining the active mode. There are many practical applications of switched systems in the real world, for instance, motor motion control, traffic systems, network congestion control, flight control systems, etc. In correspondence with the stability analysis of switched systems, many valid approaches have been proposed. The common Lyapunov function method is considered as a basic tool for analysing Lyapunov stability [1], while the multiple Lyapunov function method is an advanced method which needs to find a proper switching signal so as to make switched systems stable [2]. In [3], average dwell time (ADT) method was employed to investigate the stability problem when the switching signal was constrained. Besides, other methods can be found in [4–10].

State estimation has received considerable attention in control theory community [11–13]. After Luenberger designed an asymptotic observer for linear time-invariant systems in 1966 [14], numerous works on designing Luenberger-like observers for certain systems sprung up. Since there always exist external disturbances and unmeasured noises, estimating a system state exactly is really a hard task. As an alternative way, some researchers have focused on estimating the bounds of the state [15, 16], which relates to the interval observer design. In recent years, the design methods of interval observers for switched systems have been studied intensively [17–22]. In [17], when the error systems satisfied the condition that $A_i - L_iC_i$ are both Metzler and Hurwitz, the interval observers were directly designed for continuous-time switched systems. However, in most cases, it is impossible to obtain the suitable gains L_i satisfying that $A_i - L_iC_i$ are Metzler (or non-negative). The authors in [18] used a transformation matrix P to overcome the mentioned drawback. While in [19], the framework of interval observers design was established via the time-varying transformation of coordinates. Furthermore, the different time-invariant coordinate transformation P_i and time-varying coordinate transformation $P_i(t)$ were introduced for corresponding subsystems in [20, 21], respectively, which reduces the conservatism of the existing conditions of interval observers. On the basis of [18–21], the work [22] employed bounded-real lemma to get a more compact interval so that an optimal interval observer can be designed. However, as pointed out in [23], it is unable to determine the appropriate transformation matrices and observer gains simultaneously such that the cooperativity, as well as, the disturbance performance is guaranteed. Besides, most of the current works only deal with the problem of interval observers for synchronous switched systems, and the interval observers for asynchronous switched systems have been rarely reported.

Recently, as another effective approach to estimate system state in the presence of bounded noises and perturbations, the method of set-membership estimation has been developed. There are several geometrical forms used to describe the set of state, such as ellipsoid [24], polytope [25], parallelotope [26] and zonotope [27]. Methods based on polytope and parallelotope increase the computational complexity although it can improve the accuracy of estimation. Meanwhile, approaches based on ellipsoid decrease the complexity with a loss of accuracy. In order to get a balance between the computational cost and the estimation accuracy, zonotope was proposed in [28, 29]. Moreover, the zonotope method can also reduce the wrapping effect efficiently [30]. In the last years, the studies on the state estimation via zonotope method have attracted increasing attention [31–34]. In [31, 32], the P-radius and F-radius of the zonotopic set were, respectively minimised to estimate the state of the corresponding system. The work [33] presented a novel zonotope method and constructed a framework of zonotopic Kalman observer design for descriptor systems with unknown inputs. In [34], two set-membership estimation algorithms, i.e. cone and zonotope intersection and polyhedron and zonotope intersection were developed to solve the identification problem of time-varying parameters for linear static models with uncertainties.

The discussion above is the major motivation of our study. This paper investigates the interval observer design method for discrete-time switched systems under asynchronous switching. There are two main challenges.

- Due to the different modes between the switched system and interval observer, it is necessary to establish an augmented system including the original system and error system, which increases the complexity of synthesis. The process of transforming the sufficient conditions into a linear matrix inequality (LMI) also becomes more complicated. Besides, we extend the zonotope method to the augmented system. Different from the synchronous case, the iteration of zonotope is more computationally intensive.
- As stated in [35, 36], ellipsoids were first used to bound the set of states. Because of the shape restriction, polytopes suitable for more bounding tasks were proposed. However, employing polytopes or simplified parallelotopes increases computational complexity greatly. Therefore, we select zonotopes that offer a good trade-off between the precision of estimation and computational complexity. The difficulty is how to construct a proper zonotope and apply it to estimate the bound of the error system with a switching signal. The proposed method needs to be realised by an advanced iterative algorithm, which brings a great challenge to this work.

The contribution can be summarised as three aspects.

- The framework of interval observer for asynchronous switched systems is constructed by asynchronisation technique, which expands the scope of application.
- An H_∞ observer for asynchronous switched systems is designed on the basis of H_∞ observer theory, which makes the sufficient conditions less restrictive.
- The detailed process to estimate the boundaries of the system state by using the zonotope method is presented, and the interval estimation is more accurate.

The rest of this paper has the following structure. The problem formulation and some important definitions, as well as, lemmas are introduced in Section 2 briefly. Section 3 mainly presents the sufficient conditions of designing H_∞ observers and the process of estimating bounds described by zonotope. Two simulation examples are shown in Section 4. Finally, a conclusion is given in Section 5.

2 Problem formulation and preliminary

Consider a discrete-time switched system

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + E_{\sigma(k)}w(k), \\ y(k) = C_{\sigma(k)}x(k) + F_{\sigma(k)}v(k), \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state, $w(k) \in R^r$ is the unknown but bounded process perturbation, $-w \le w(k) \le w$ (w is a constant vector), $y(k) \in R^q$ is the output, $v(k) \in R^s$ is the observation noise satisfying $-v \le v(k) \le v$ (v is a constant vector), $\sigma(\cdot): Z^+ \to S: \{1, 2, \ldots, N\}$ is a switching signal that possesses finite modes. $A_{\sigma(k)}$, $E_{\sigma(k)}$, $C_{\sigma(k)}$ and $F_{\sigma(k)}$ are given matrices with appropriate dimensions.

To begin with, we review some concepts of interval observer, H_∞ observer and switching system [37, 38].

Definition 1: For any $k > 0$, if a bundle of states $\{\hat{x}^+(k), \hat{x}^-(k)\}$ satisfies

$$\hat{x}^-(k) \le x(k) \le \hat{x}^+(k),$$

under the initial condition

$$\hat{x}^-(0) \le x(0) \le \hat{x}^+(0),$$

then the bundle of states $\{\hat{x}^+(k), \hat{x}^-(k)\}$ is an interval observer.

Here, with respect to system (1), a robust observer is designed, and it has the following form:

$$\hat{x}(k+1) = A_{\hat{\sigma}(k)}\hat{x}(k) + L_{\hat{\sigma}(k)}(y - C_{\hat{\sigma}(k)}\hat{x}(k)), \tag{2}$$

where the matrix $L_{\hat{\sigma}(k)} \in R^{n \times q}$ is the observer gain. $\hat{\sigma}(k)$ is also a switching signal possessing finite modes, but it is not synchronised with $\sigma(k)$. Denote the error $e(k) = x(k) - \hat{x}(k)$, then

$$\begin{aligned} e(k+1) = {} & (A_{\hat{\sigma}(k)} - L_{\hat{\sigma}(k)}C_{\hat{\sigma}(k)})e(k) + (A_{\sigma(k)} - A_{\hat{\sigma}(k)} \\ & + L_{\hat{\sigma}(k)}C_{\hat{\sigma}(k)} - L_{\hat{\sigma}(k)}C_{\sigma(k)})x(k) \\ & + E_{\sigma(k)}w(k) - L_{\hat{\sigma}(k)}F_{\sigma(k)}v(k). \end{aligned} \tag{3}$$

For simplicity, $\hat{\sigma}(k)$ and $\sigma(k)$ are taken as $\hat{i}$ and i, respectively. Define $z(k) = [x^{\mathrm{T}}(k)\, e^{\mathrm{T}}(k)]^{\mathrm{T}}$ and $\bar{w}(k) = [w^{\mathrm{T}}(k)\, v^{\mathrm{T}}(k)]^{\mathrm{T}}$, (3) can be written as

case I $\hat{i} \ne i$:

$$z(k+1) = \tilde{A}_i z(k) + \tilde{B}_i \bar{w}(k), \tag{4}$$

where

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ A_i - A_{\hat{i}} + L_{\hat{i}}C_{\hat{i}} - L_{\hat{i}}C_i & A_{\hat{i}} - L_{\hat{i}}C_{\hat{i}} \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} E_i & 0 \\ E_i & -L_{\hat{i}}F_i \end{bmatrix}.$$

or

case II $\hat{i} = i$:

$$z(k+1) = \tilde{A}_i^* z(k) + \tilde{B}_i^* \bar{w}(k), \tag{5}$$

where

$$\tilde{A}_i^* = \begin{bmatrix} A_i & 0 \\ 0 & A_i - L_iC_i \end{bmatrix}, \quad \tilde{B}_i^* = \begin{bmatrix} E_i & 0 \\ E_i & -L_iF_i \end{bmatrix}.$$

In this paper, the asynchronous case is our primary research content. Thus, we focus on the cascade system (4).

Definition 2: Observer (2) is called the H_∞ observer of system (1), if

(i) $\bar{w}(k) = 0$, the cascade system (4) is asymptotically stable.
(ii) $\bar{w}(k) \ne 0$, when the initial state is equal to zero, the defined state $z(k)$ in (4) satisfies

$$\sum_{k=0}^{\infty} z^{\mathrm{T}}(k)z(k) \le \gamma \sum_{k=0}^{\infty} \bar{w}^{\mathrm{T}}(k)\bar{w}(k),$$

where $\gamma > 0$ is the disturbance attenuation level.

Definition 3: With regard to any $k_{n+1} \ge k_n \ge 0$, if the following inequality holds

$$N_\sigma(k_n, k_{n+1}) \le N_0 + (k_{n+1} - k_n)/\tau^*,$$

where N_0 and τ^* are non-negative and positive constants, respectively, and $N_\sigma(k_n, k_{n+1})$ represents the switching number from k_n to k_{n+1}, then τ^* is called ADT. In what follows, we let $N_0 = 0$.

As an important set-membership estimation method, zonotope is a powerful tool to study the bounds of state in linear systems. It is essential to introduce some definitions and properties of zonotope herein [32, 39].

Definition 4: Given vector $r \in R^n$ and matrix $Y \in R^{n \times s}$, the set

$$Z = r \oplus YB^s = r + Yb, \quad b \in B^s$$

is called a zonotope, where $B = [-1, 1]$ is a unitary interval and B^s is a s-dimensional unitary box.

For simplicity, a zonotope Z is denoted by $Z = \langle r, Y \rangle$.

Lemma 1: Consider the zonotope $Z = \langle r, Y \rangle$ defined above, then

$$\begin{aligned} &(i)\ \langle r_1, Y_1 \rangle \oplus \langle r_2, Y_2 \rangle = \langle r_1 + r_2, [Y_1\ \ Y_2] \rangle, \\ &(ii)\ L \odot \langle r, Y \rangle = \langle Lr, LY \rangle, \\ &(iii)\ \langle r, Y \rangle \subseteq \langle r, \bar{Y} \rangle, \end{aligned}$$

where $\bar{Y} \in R^{n \times n}$ is a diagonal matrix with $\bar{Y}_{i,i} = \sum_{j=1}^{s} |Y_{i,j}|$, $i = 1, \dots, n$.

Remark 1: $\bar{Y}$ has the following form

$$\bar{Y} = \begin{bmatrix} \sum_{j=1}^{s} |Y_{1,j}| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{s} |Y_{n,j}| \end{bmatrix}.$$

If $s > n$, then $\langle r, Y \rangle \subseteq \langle r, \bar{Y} \rangle$ is used for reducing the order of zonotope.

Lemma 2: Given a zonotope $Z = \langle r, Y \rangle$, where $r \in R^n$, $Y \in R^{n \times s}$, then the following holds

$$Z \subseteq \langle r, \hat{Y} \rangle, \quad n \le q \le s,$$

where $\hat{Y} = [\check{Y}_{q-n}\ \ H] \in R^{n \times q}$. $\check{Y}$ can be generated by sorting the Euclidean norm of each column of Y in descending order, and $\check{Y}_{q-n}$ is obtained from the first $q - n$ columns of matrix $\check{Y}$. $H \in R^{n \times n}$ is a diagonal matrix, whose diagonal elements $H_{i,i} = \sum_{j=q-n+1}^{s} |\check{Y}_{i,j}|$, $i = 1, \dots, n$.

Remark 2: $\bar{Y}$, $\hat{Y}$ and $\check{Y}$ are three different matrices derived from Y. Some details about computation of $\bar{Y}$, $\hat{Y}$, $\check{Y}$ are shown in the following example.

$$\begin{aligned} &\text{If } Y = \begin{bmatrix} 2.3 & -0.8 & 1.7 & 1.2 \\ -1.3 & -2.4 & 1.6 & -1.8 \end{bmatrix}, \\ &\text{then } \check{Y} = \begin{bmatrix} 2.3 & -0.8 & 1.7 & 1.2 \\ -1.3 & -2.4 & 1.6 & -1.8 \end{bmatrix}, \quad \check{Y}_2 = \begin{bmatrix} 2.3 & -0.8 \\ -1.3 & -2.4 \end{bmatrix}, \\ &H = \begin{bmatrix} 2.9 & 0 \\ 0 & 3.4 \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} 2.3 & -0.8 & 2.9 & 0 \\ -1.3 & -2.4 & 0 & 3.4 \end{bmatrix}, \\ &\bar{Y} = \begin{bmatrix} 6 & 0 \\ 0 & 7.1 \end{bmatrix}. \end{aligned}$$

It is noted that $\hat{Y} = \bar{Y}$ when $q = n$.

The lemma below is known as the Schur complement [40], which is used in the further study.

Lemma 3: Consider a given symmetrical matrix

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^{\mathrm{T}} & G_{22} \end{bmatrix},$$

the three conditions below are equivalent:

$$\begin{aligned} &(i)\ G \prec 0, \\ &(ii)\ G_{11} \prec 0,\ G_{22} - G_{12}^{\mathrm{T}} G_{11}^{-1} G_{12} \prec 0, \\ &(iii)\ G_{22} \prec 0,\ G_{11} - G_{12} G_{22}^{-1} G_{12}^{T} \prec 0. \end{aligned}$$

3 Main result

In this section, the interval observer design includes two parts in comparison to other general methods. First, we design the H_∞ observer and give sufficient conditions.

Theorem 1: Suppose that $0 < \xi < 1$ and $\mu > 1$ are pre-specified constants, if there exist scalar $\gamma > 0$ and matrices $\tilde{P}_i \in R^{2n \times 2n} \succ 0$, $\tilde{P}_j \in R^{2n \times 2n} \succ 0$, $\forall i, j, i \ne j$ such that

$$\begin{bmatrix} \Lambda & * \\ (\tilde{B}_i)^{\mathrm{T}} \tilde{P}_i \tilde{A}_i & (\tilde{B}_i)^{\mathrm{T}} \tilde{P}_i \tilde{B}_i - \gamma I \end{bmatrix} \prec 0, \tag{6}$$

$$\tilde{P}_i \prec \mu \tilde{P}_j, \tag{7}$$

where $\Lambda = (\tilde{A}_i)^{\mathrm{T}} \tilde{P}_i \tilde{A}_i - \xi \tilde{P}_i + I$,

$$\tilde{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & Q_i \end{bmatrix}, \quad \tilde{P}_j = \begin{bmatrix} P_j & 0 \\ 0 & Q_j \end{bmatrix},$$

$P_i \succ 0$, $P_j \succ 0$, $Q_i \succ 0$, $Q_j \succ 0$, and ADT satisfying

$$\frac{1}{\tau^*} + \frac{\ln \xi}{\ln \mu} \le 0, \tag{8}$$

then (2) is the H_∞ observer of (1).

Proof: Let $\sigma(k_l) = i$, we select a Lyapunov function

$$V_i(k) = z^{\mathrm{T}}(k) \tilde{P}_i z(k). \tag{9}$$

From Definition 2, two cases that $\bar{w}(k) = 0$ and $\bar{w}(k) \ne 0$ will be argued.

(i) $\bar{w}(k) = 0$. When $k \in [k_l, k_{l+1})$, $V_i(k)$ can be taken the forward difference based on system (4)

$$\begin{aligned} \Delta V_i(k) &= V_i(k+1) - V_i(k) \\ &= z^{\mathrm{T}}(k)[(\tilde{A}_i)^{\mathrm{T}} \tilde{P}_i \tilde{A}_i - \tilde{P}_i] z(k). \end{aligned} \tag{10}$$

By Lemma 3, (6) means that $\Lambda \prec 0$. Then,

$$\begin{aligned} \Delta V_i(k) &\le (\xi - 1) z^{\mathrm{T}}(k) \tilde{P}_i z(k) \\ &\le (\xi - 1) V_i(k), \end{aligned} \tag{11}$$

i.e.

$$V_i(k+1) \le \xi V_i(k). \tag{12}$$

Considering the interval $[k_l, k)$, we obtain

$$V_i(k) \le \xi^{k - k_l} V_i(k_l). \tag{13}$$

Suppose that $\sigma(k_{l-1}) = j$. From (7), we have

$$V_i(k) \le \mu \xi^{k - k_l} V_j(k_l). \tag{14}$$

Repeating (13) and (14), one can get

$$\begin{aligned} V_i(k) &\le \mu \xi^{k - k_l} V_j(k_l) \\ &\le \mu \xi^{k - k_{l-1}} V_j(k_{l-1}) \\ &\le \mu^2 \xi^{k - k_{l-1}} V_{\sigma(k_{l-2})}(k_{l-1}) \\ &\le \mu^2 \xi^{k - k_{l-2}} V_{\sigma(k_{l-2})}(k_{l-2}) \\ &\le \cdots \\ &\le \mu^{N_\sigma(0,k)} \xi^k V_{\sigma(0)}(0). \end{aligned} \tag{15}$$

From Definition 3, we obtain $N_\sigma(0,k) \le k/\tau^*$. Then, it follows that

$$\begin{aligned} V_i(k) &\le \mu^{\frac{k}{\tau^*}} \xi^k V_{\sigma(0)}(0) \\ &\le \mu^{\frac{k}{\tau^*}} \mu^{\frac{k\ln\xi}{\ln\mu}} V_{\sigma(0)}(0) \\ &\le \mu^{k(\frac{1}{\tau^*}+\frac{\ln\xi}{\ln\mu})} V_{\sigma(0)}(0)\,. \end{aligned} \tag{16}$$

Due to $V_i(k) \ge \underline{\phi}(\tilde{P}_i) \parallel z(k) \parallel^2$ and $V_{\sigma(0)}(0) \le \overline{\phi}(\tilde{P}_i) \parallel z(0) \parallel^2$, it holds that

$$\begin{aligned} \underline{\phi}(\tilde{P}_i) \parallel z(k) \parallel^2 &\le \mu^{k(\frac{1}{\tau^*}+\frac{\ln\xi}{\ln\mu})} \overline{\phi}(\tilde{P}_i) \parallel z(0) \parallel^2 , \\ \parallel z(k) \parallel &\le \sqrt{\frac{\mu^{k(\frac{1}{\tau^*}+\frac{\ln\xi}{\ln\mu})} \overline{\phi}(\tilde{P}_i)}{\underline{\phi}(\tilde{P}_i)}} \parallel z(0) \parallel \ . \end{aligned} \tag{17}$$

By (8), we obtain

$$\lim_{k\to\infty} \parallel z(k) \parallel = \lim_{k\to\infty} \sqrt{\frac{\mu^{k(\frac{1}{\tau^*}+\frac{\ln\xi}{\ln\mu})} \overline{\phi}(\tilde{P}_i)}{\underline{\phi}(\tilde{P}_i)}} \parallel z(0) \parallel \ = 0. \tag{18}$$

Thus, the system (4) is asymptotically stable.
(ii) When $\bar{w}(k) \ne 0$, define that

$$\begin{aligned} J &= \sum_{k=0}^{\infty} z^{\mathrm{T}}(k) z(k) - \gamma \sum_{k=0}^{\infty} \bar{w}^{\mathrm{T}}(k) \bar{w}(k) \\ &= \sum_{k=0}^{\infty} [z^{\mathrm{T}}(k) z(k) - \gamma \bar{w}^{\mathrm{T}}(k) \bar{w}(k)]\,. \end{aligned} \tag{19}$$

Let $z(0) = 0$, one can get

$$\begin{aligned} \sum_{k=0}^{\infty} \Delta V_i(k) &= \lim_{k\to\infty} [V_i(k) - V_{\sigma(k-1)}(k-1) + V_{\sigma(k-1)}(k-1) \\ &\quad - V_{\sigma(k-2)}(k-2) + \cdots + V_{\sigma(1)}(1) - V_{\sigma(0)}(0)] \\ &= \lim_{k\to\infty} V_i(k) \\ &\ge 0. \end{aligned} \tag{20}$$

Thus,

$$J \le \sum_{k=0}^{\infty} [z^{\mathrm{T}}(k) z(k) - \gamma \bar{w}^{\mathrm{T}}(k) \bar{w}(k) + \Delta V_i(k)]\,. \tag{21}$$

Denote that $S(k) = z^{\mathrm{T}}(k) z(k) - \gamma \bar{w}^{\mathrm{T}}(k) \bar{w}(k) + \Delta V_i(k)$. By (6), together with $0 < \xi < 1$, we obtain

$$\begin{aligned} S(k) &= z^{\mathrm{T}}(k)[(\tilde{A}_i)^{\mathrm{T}} \tilde{P}_i \tilde{A}_i - \tilde{P}_i + I] z(k) \\ &\quad + 2\bar{w}^{\mathrm{T}}(k)(\tilde{B}_i)^{\mathrm{T}} \tilde{P}_i \tilde{A}_i z(k) + \bar{w}^{\mathrm{T}}(k)[(\tilde{B}_i)^{\mathrm{T}} \tilde{P}_i \tilde{B}_i - \gamma I] \bar{w}(k) \\ &\le (\xi - 1) z^{\mathrm{T}}(k) \tilde{P}_i z(k) \\ &\le 0. \end{aligned} \tag{22}$$

From (21) and (22), it deduces that

$$J \le \sum_{k=0}^{\infty} S(k) \le 0. \tag{23}$$

Substituting (19) into (23) yields

$$\sum_{k=0}^{\infty} z^{\mathrm{T}}(k) z(k) - \gamma \sum_{k=0}^{\infty} \bar{w}^{\mathrm{T}}(k) \bar{w}(k) \le 0, \tag{24}$$

i.e.

$$\sum_{k=0}^{\infty} z^{\mathrm{T}}(k) z(k) \le \gamma \sum_{k=0}^{\infty} \bar{w}^{\mathrm{T}}(k) \bar{w}(k)\,. \tag{25}$$

Combining (i) with (ii), it can be proved that (2) is the H_∞ observer of (1).

□

Remark 3: Compared with the synchronised case, the asynchronised case considered here is more general. Nevertheless, the error system (3) is complicated, thus the amount of computation of observer gains L_i also increases. It is obvious that (6) is not a standard LMI. It is essential to transform (6) into an LMI so that (6) can be solved by Matlab. Moreover, to design an optimal H_∞ observer, we have to make the disturbance attenuation level γ as small as possible.

Theorem 2: Suppose that $0 < \xi < 1$ and $\mu > 1$ are pre-specified constants, if there exist scalar $\gamma > 0$ and matrices $\tilde{P}_i \in R^{2n\times 2n} \succ 0$, $\tilde{P}_j \in R^{2n\times 2n} \succ 0$, $M_i \in R^{n\times q}$, $\forall\, i, j, i \ne j$ such that

$$\begin{cases} \min\ \gamma \\ \text{subject to:} \\ \begin{bmatrix} \Phi & * & * & * & * \\ 0 & -\xi Q_i + I & * & * & * \\ E_j^{\mathrm{T}} P_i A_j & 0 & E_j^{\mathrm{T}} P_i E_j - \gamma I & * & * \\ 0 & 0 & 0 & -\gamma I & * \\ \Omega^T & \Psi^T & Q_i E_j & -M_i F_j & -Q_i \end{bmatrix} \prec 0, \\ \tilde{P}_i \prec \mu \tilde{P}_j, \end{cases} \tag{26}$$

where $\Phi = A_j^{\mathrm{T}} P_i A_j - \xi P_i + I$, $\Omega = C_i^{\mathrm{T}} M_i^{\mathrm{T}} - C_j^{\mathrm{T}} M_i^{\mathrm{T}} - A_i^{\mathrm{T}} Q_i^{\mathrm{T}} + A_j^{\mathrm{T}} Q_i^{\mathrm{T}}$, $\Psi = -C_i^{\mathrm{T}} M_i^{\mathrm{T}} + A_i^{\mathrm{T}} Q_i^{\mathrm{T}}$,

$$\tilde{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & Q_i \end{bmatrix}, \quad \tilde{P}_j = \begin{bmatrix} P_j & 0 \\ 0 & Q_j \end{bmatrix},$$

$P_i \succ 0$, $P_j \succ 0$, $Q_i \succ 0$, $Q_j \succ 0$. The observer gain is determined by $L_i = (Q_i)^{-1} M_i$, and ADT satisfies (8), then (2) is the H_∞ observer with the optimal performance of disturbance attenuation.

Proof: Substituting

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ A_i - A_{\hat{i}} + L_{\hat{i}} C_{\hat{i}} - L_{\hat{i}} C_i & A_{\hat{i}} - L_{\hat{i}} C_{\hat{i}} \end{bmatrix},$$
$$\tilde{B}_i = \begin{bmatrix} E_i & 0 \\ E_i & -L_{\hat{i}} F_i \end{bmatrix}, \quad \tilde{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & Q_i \end{bmatrix},$$

and $M_i = Q_i L_i$ into (6), we have

$$\begin{bmatrix} \Theta_{1,1} & * & * & * \\ \Theta_{2,1} & \Theta_{2,2} & * & * \\ \Theta_{3,1} & E_j^{\mathrm{T}} Q_i A_i - E_j^{\mathrm{T}} M_i C_i & \Theta_{3,3} & * \\ \Theta_{4,1} & -F_j^{\mathrm{T}} M_i^{\mathrm{T}} A_i + F_j^{\mathrm{T}} M_i^{\mathrm{T}} L_i C_i & -F_j^{\mathrm{T}} M_i^{\mathrm{T}} E_j & \Theta_{4,4} \end{bmatrix} \prec 0, \tag{27}$$

where

$$\Theta_{1,1} = A_j^{\mathrm{T}}P_iA_j + A_j^{\mathrm{T}}Q_iA_j - A_j^{\mathrm{T}}Q_iA_i + A_j^{\mathrm{T}}M_iC_i - A_j^{\mathrm{T}}M_iC_j$$
$$-A_i^{\mathrm{T}}Q_iA_j + A_i^{\mathrm{T}}Q_iA_i - A_i^{\mathrm{T}}M_iC_i + A_i^{\mathrm{T}}M_iC_j + C_i^{\mathrm{T}}M_i^{\mathrm{T}}A_j$$
$$-C_i^{\mathrm{T}}M_i^{\mathrm{T}}A_i + C_i^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_i - C_i^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_j - C_j^{\mathrm{T}}M_i^{\mathrm{T}}A_j$$
$$+C_j^{\mathrm{T}}M_i^{\mathrm{T}}A_i - C_j^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_i + C_j^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_j - \xi P_i + I,$$
$$\Theta_{2,1} = A_i^{\mathrm{T}}Q_iA_j - A_i^{\mathrm{T}}Q_iA_i + A_i^{\mathrm{T}}M_iC_i - A_i^{\mathrm{T}}M_iC_j - C_i^{\mathrm{T}}M_i^{\mathrm{T}}A_j$$
$$+C_i^{\mathrm{T}}M_i^{\mathrm{T}}A_i - C_i^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_i + C_i^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_j,$$
$$\Theta_{2,2} = A_i^{\mathrm{T}}Q_iA_i - A_i^{\mathrm{T}}M_iC_i - C_i^{\mathrm{T}}M_i^{\mathrm{T}}A_i + C_i^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_i - \xi Q_i + I,$$
$$\Theta_{3,1} = E_j^{\mathrm{T}}P_iA_j + E_j^{\mathrm{T}}Q_iA_j - E_j^{\mathrm{T}}Q_iA_i + E_j^{\mathrm{T}}M_iC_i - E_j^{\mathrm{T}}M_iC_j,$$
$$\Theta_{3,3} = E_j^{\mathrm{T}}P_iE_j + E_j^{\mathrm{T}}Q_iE_j - \gamma I,$$
$$\Theta_{4,1} = -F_j^{\mathrm{T}}M_i^{\mathrm{T}}A_j + F_j^{\mathrm{T}}M_i^{\mathrm{T}}A_i - F_j^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_i + F_j^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_j,$$
$$\Theta_{4,4} = F_j^{\mathrm{T}}M_i^{\mathrm{T}}L_iF_j - \gamma I\,.$$

In (27), there are some bilinear items which should be converted to linear terms. According to Lemma 3, we select

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^{\mathrm{T}} & G_{22} \end{bmatrix},$$

where

$$G_{11} = \begin{bmatrix} \Phi & * & * & * \\ 0 & -\xi Q_i + I & * & * \\ E_j^{\mathrm{T}}P_iA_j & 0 & E_j^{\mathrm{T}}P_iE_j - \gamma I & * \\ 0 & 0 & 0 & -\gamma I \end{bmatrix},$$

$$G_{12} = \begin{bmatrix} (M_iC_i - M_iC_j - Q_iA_i + Q_iA_j)^T \\ (-M_iC_i + Q_iA_i)^T \\ (Q_iE_j)^T \\ (-M_iF_j)^T \end{bmatrix},$$

$$G_{22} = -Q_i\,.$$

Then, we can verify that

$$G_{22} = -Q_i \prec 0,$$

and

$$G_{11} - G_{12}G_{22}^{-1}G_{12}^T = \begin{bmatrix} \Theta_{1,1} & * & * & * \\ \Theta_{2,1} & \Theta_{2,2} & * & * \\ \Theta_{3,1} & E_j^{\mathrm{T}}Q_iA_i - E_j^{\mathrm{T}}M_iC_i & \Theta_{3,3} & * \\ \Theta_{4,1} & -F_j^{\mathrm{T}}M_i^{\mathrm{T}}A_i + F_j^{\mathrm{T}}M_i^{\mathrm{T}}L_iC_i & -F_j^{\mathrm{T}}M_i^{\mathrm{T}}E_j & \Theta_{4,4} \end{bmatrix} \prec 0.$$

By Lemma 3, we have

$$G = \begin{bmatrix} \Phi & * & * & * & * \\ 0 & -\xi Q_i + I & * & * & * \\ E_j^{\mathrm{T}}P_iA_j & 0 & E_j^{\mathrm{T}}P_iE_j - \gamma I & * & * \\ 0 & 0 & 0 & -\gamma I & * \\ \Omega^T & \Psi^T & Q_iE_j & -M_iF_j & -Q_i \end{bmatrix} \prec 0.$$

Combining with (7) and (8), the sufficient conditions of Theorem 3.2 are equivalent to those of Theorem 1. Thus, the proof is completed. □

Remark 4: Unlike [17, 41], the condition that $A_i - L_iC_i$ are non-negative does not need to hold in this paper. Meanwhile, the present method also avoids the hybrid behaviour in [19, 20]. Consequently, it is worthwhile to point out that the feasible solutions of sufficient conditions in Theorem 2 can be obtained by Matlab.

Remark 5: Actually, ADT is an important performance index of switched systems. According to Definition 2.3, ADT represents the average time that the system stays in a mode, and determines the times that the system switches during a period of time. From the aspect of design, ADT is expected to be small. Since $\tau^* \geq -(\ln\mu/\ln\xi)$, we want to make $-(\ln\mu/\ln\xi)$ as small as possible, and the designed interval observer is subsequently suitable for more general kind of switched systems. So, it is important to find proper μ and ξ to minimise the ADT.

Based on the designed H_∞ observer, we then give the interval estimation of system state by zonotope method. Denote that $w = [w_1\, w_2 \cdots w_r]^{\mathrm{T}}$ and $v = [v_1\, v_2 \cdots v_s]^{\mathrm{T}}$. In view of $-w \leq w(k) \leq w$ and $-v \leq v(k) \leq v$, $\bar{w}(k)$ is described by the zonotope $\langle 0, \bar{W}\rangle$, where $\bar{W} \in R^{(r+s)\times(r+s)}$ has the following form

$$\bar{W} = \begin{bmatrix} w_1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots & 0 \\ \vdots & \cdots & w_r & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & v_1 & \cdots & \vdots \\ 0 & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & v_s \end{bmatrix}.$$

The state estimation of system (1) is stated by the following theorem.

Theorem 3: If there exists the H_∞ observer (2) for (1), then the state is described by

$$x(k) \in \langle \hat{x}(k), Y_i(k)\rangle, \tag{28}$$

where $Y_i(k) \in R^{n\times n}$. $\tilde{Y}_i(k)$ has a recursive form:

$$\tilde{Y}_i(k+1) = [\tilde{A}_i\hat{\tilde{Y}}_i(k) \quad \tilde{B}_i\bar{W}], \tag{29}$$

where

$$\tilde{Y}_i(k) = \begin{bmatrix} Y_i(k) & 0 \\ 0 & Y_i(k) \end{bmatrix},$$

and $\hat{\tilde{Y}}_i(k)$ is defined in Lemma 2.

Proof: Since $x(k) = \hat{x}(k) + e(k)$, $x(k) \in \langle \hat{x}(k), Y_i(k)\rangle$ is equivalent to $e(k) \in \langle 0, Y_i(k)\rangle$. By Lemma 2, we obtain

$$x(k) \in \langle \hat{x}(k), \hat{Y}_i(k)\rangle, \tag{30}$$

$$e(k) \in \langle 0, \hat{Y}_i(k)\rangle\,. \tag{31}$$

Let $\hat{z}(k) = [\hat{x}^{\mathrm{T}}(k)\, 0]^{\mathrm{T}}$, it holds that

$$z(k) \in \langle \hat{z}(k), \hat{\tilde{Y}}_i(k)\rangle\,. \tag{32}$$

According to the system (4), we have

$$z(k+1) \in \tilde{A}_i \odot \langle \hat{z}(k), \hat{\tilde{Y}}_i(k)\rangle \oplus \tilde{B}_i \odot \langle 0, \bar{W}\rangle\,. \tag{33}$$

In view of Lemma 1, one can get that

$$\begin{aligned} z(k+1) &\in \langle \tilde{A}_i\hat{z}(k), \tilde{A}_i\hat{\tilde{Y}}_i(k)\rangle \oplus \langle 0, \tilde{B}_i\bar{W}\rangle \\ &= \langle \tilde{A}_i\hat{z}(k), [\tilde{A}_i\hat{\tilde{Y}}_i(k) \quad \tilde{B}_i\bar{W}]\rangle\,. \end{aligned} \tag{34}$$

$$\begin{array}{l}
\text{Input } A_i, C_i, E_i, F_i, \mu, \xi; \\
\text{Output } P_i, Q_i, M_i; \\
L_i = (P_i)^{-1} M_i; \\
\hat{x}(k+1) = A_{\hat{\sigma}(k)} \hat{x}(k) + L_{\hat{\sigma}(k)}(y - C_{\hat{\sigma}(k)} \hat{x}(k)); \\
z(k+1) = \tilde{A}_i z(k) + \tilde{B}_i \bar{w}(k); \\
x(k) \in \langle \hat{x}(k), Y_i(k) \rangle, \bar{w}(k) = \langle 0, \bar{W} \rangle; \\
e(k) \in \langle 0, Y_i(k) \rangle, z(k) \in \langle \hat{z}(k), \tilde{Y}_i(k) \rangle; \\
\text{If } k \geq 0 \\
\qquad z(k+1) \in \tilde{A}_i \odot \langle \hat{z}(k), \hat{\tilde{Y}}_i(k) \rangle \oplus \tilde{B}_i \odot \langle 0, \bar{W} \rangle; \\
\qquad \tilde{Y}_i(k+1) = [\tilde{A}_i \hat{\tilde{Y}}_i(k) \quad \tilde{B}_i \bar{W}]; \\
\text{end}
\end{array}$$

Fig. 1 *Algorithm 1: Steps to design an interval observer*

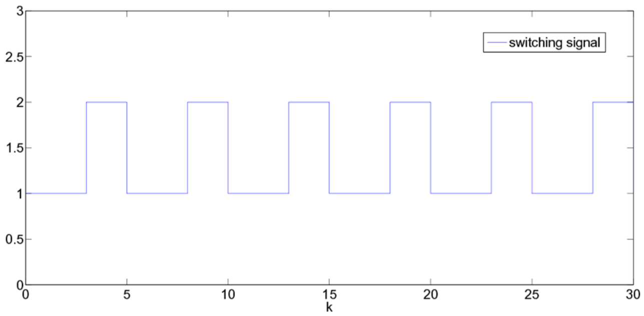

Fig. 2 *Switching signal of switched system*

Since $z(k+1) \in \langle \hat{z}(k+1), \tilde{Y}_i(k+1) \rangle$, (34) implies that

$$\tilde{Y}_i(k+1) = [\tilde{A}_i \hat{\tilde{Y}}_i(k) \quad \tilde{B}_i \bar{W}] .$$

Hence, the proof is completed. □

Remark 6: The zonotope method is used to obtain a tighter set which can be computed recursively by Theorem 3. The detailed procedure to compute $\tilde{Y}_i(k)$ is suggested as follows:

$$\tilde{Y}_{\sigma(0)}(0) \to \hat{\tilde{Y}}_{\sigma(0)}(0) \to \tilde{Y}_{\sigma(1)}(1) \to \cdots \to \hat{\tilde{Y}}_{\sigma(k-1)}(k-1) \to \tilde{Y}_i(k)$$

Due to $z(k) = [x^{\mathrm{T}}(k)\, e^{\mathrm{T}}(k)]^{\mathrm{T}}$, we only employ the last n rows of $\tilde{Y}_i(k)$.

From Theorem 3, an interval observer can be designed by

$$\begin{cases} (x^m)^+(k) = \hat{x}^m(k) + \displaystyle\sum_{j=1}^{2q} \left| (\tilde{Y}_i)^{m+n,j}(k) \right|, \\ (x^m)^-(k) = \hat{x}^m(k) - \displaystyle\sum_{j=1}^{2q} \left| (\tilde{Y}_i)^{m+n,j}(k) \right|, \end{cases} \tag{35}$$

where $m = 1, \ldots, n$ is the mth row, $j = 1, \ldots, 2q$ is the jth column and $i \in S$ is the ith mode.

The following steps are given to design the interval observer (see Fig. 1).

Remark 7: This paper focuses on the interval observer design for asynchronous switched systems. There are few studies devoted to the associated problem. The direct LMI method in [17] requires that $A_i - L_i C_i$, $i = 1, 2, \ldots, n$ are all Metzler (or non-negative), which is known as a strong conservative condition. While in [18–22], the method of coordinate transformation is utilised to make $P_i^{-1}(A_i - L_i C_i) P_i$, $i = 1, 2, \ldots, n$ be Metzler (or non-negative) by the matrices P_i, $i = 1, 2, \ldots, n$, which may also lead to some conservatism because the matrices P_i, $i = 1, 2, \ldots, n$ are not easy to determine. Compared with these methods, this paper just designs a robust H_∞ observer, and $A_i - L_i C_i$, $i = 1, 2, \ldots, n$ are not required to be Metzler (or non-negative). Therefore, the zonotope method can reduce the conservatism of design conditions. Furthermore, as stated in [23], it is difficult to determine the appropriate transformation matrices and observer gains simultaneously such that both the cooperativity and the disturbance performance are guaranteed. Similar conclusions can also be found in the recent works [42, 43], the reachable domain of estimation error by the method of coordinate transformation is larger than that by zonotope method. Thus, the interval observer designed by zonotope method has a more accurate estimation bound and the zonotope method improves the performance of the interval observer.

4 Numerical example

To verify the performance of designed interval observer, two examples are presented for illustration in this section.

Example 1: Consider system (1) with

$$A_1 = \begin{bmatrix} 0.3 & -0.22 \\ 0.19 & -0.23 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.05 \\ 0.07 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.15 & 0.24 \\ 0.08 & -0.32 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.11 \\ 0.08 \end{bmatrix},$$

$$C_1 = [0.25 \quad 0.37], \quad F_1 = 0.1,$$
$$C_2 = [0.44 \quad 0.5], \quad F_2 = 0.17.$$

Given $\mu = 2.3$, $\xi = 0.7$. By Theorem 2, we obtain

$$P_1 = \begin{bmatrix} 3.3113 & -7.014 \\ -7.014 & 6.3508 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.4591 & -4.8638 \\ -4.8638 & 2.7319 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 4.3268 & -2.9611 \\ -2.9611 & 7.4475 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 5.1453 & -2.0828 \\ -2.0828 & 5.9873 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -0.0469 \\ 0.0316 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.0814 \\ -0.1529 \end{bmatrix},$$

$$\gamma = 9, \quad \tau^* \geq 2.3352.$$

Thus,

$$L_1 = \begin{bmatrix} -0.0109 \\ -0.0001 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.0064 \\ -0.0233 \end{bmatrix}.$$

Let

$$w(k) = \sin 2k, \quad v(k) = 2\cos k, \quad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$$x^+(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad x^-(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\bar{W} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \tilde{Y}_{\sigma(0)}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The signals $\sigma(t)$ and $\hat{\sigma}(t)$ which determine the switching between two subsystems are plotted in Figs. 2 and 3, respectively. $\sigma(t)$ is a periodic signal with the period of 5 s. The switched system is in the mode 1 for the first 3 s, and then stays at the mode 2 for the subsequent 2 s in the first period. $\hat{\sigma}(t)$ is the opposite of $\sigma(t)$, the interval observer is in the mode 2 for the first 3 s, and then stays at the mode 1 for the next 2 s in the first period. By a direct computation, the ADT τ^* is about 2.5 s. Figs. 4 and 5 present the component-wise results of state vector and interval estimation. It can be seen from Fig. 4 that the boundaries x_1^+ and x_1^- of interval observer are strictly above and below the state x_1 of the original

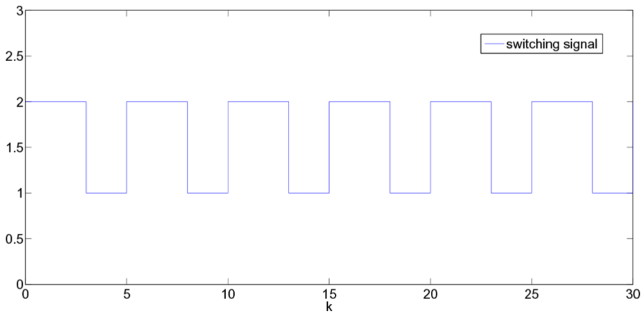

Fig. 3 *Switching signal of interval observer*

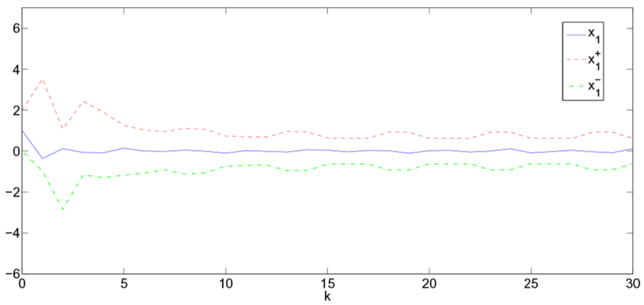

Fig. 4 *State x_1 and estimation bounds x_1^+, x_1^-*

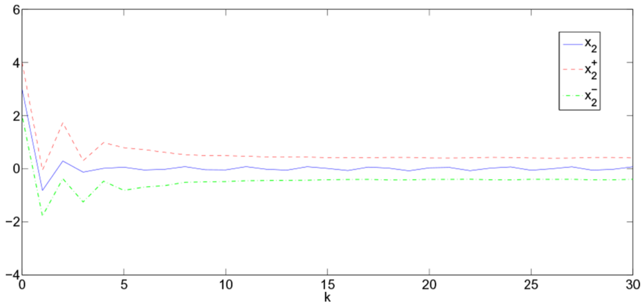

Fig. 5 *State x_2 and estimation bounds x_2^+, x_2^-*

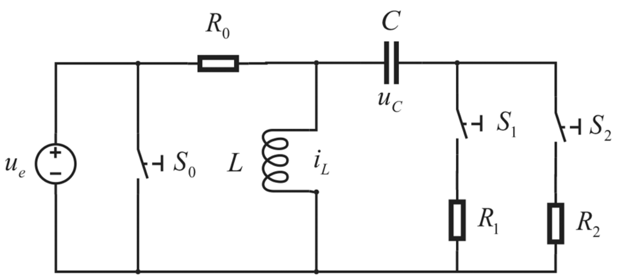

Fig. 6 *A simple circuit diagram*

system, respectively, and the steady-state upper and lower estimation errors of x_1 are both <1. The same relationship between x_2^+, x_2^- and x_2 can be obtained from Fig. 5. Therefore, the designed interval observer is valid.

Example 2: Consider a circuit system as shown in Fig. 6. When the capacitor is discharged, we have

$$\begin{cases} \dot{u}_{\rm C}(t) = -\dfrac{1}{(R_0 + R_1 S(t) + R_2(1 - S(t)))C} u_{\rm C}(t) \\ \qquad - \dfrac{R_0}{(R_0 + R_1 S(t) + R_2(1 - S(t)))C} i_{\rm L}(t), \\ \dot{i}_{\rm L}(t) = \dfrac{R_0}{L(R_0 + R_1 S(t) + R_2(1 - S(t)))} u_{\rm C}(t) \\ \qquad - \dfrac{R_0(R_1 S(t) + R_2(1 - S(t)))}{L(R_0 + R_1 S(t) + R_2(1 - S(t)))} i_{\rm L}(t), \end{cases} \tag{36}$$

where $u_{\rm C}(t)$ represents the capacitor voltage, $i_{\rm L}(t)$ is the current through inductor. S_0, S_1, S_2 are switches, each of which equals to 1 (switch off) or 0 (switch off). $S(t) = S_1(t)$, and $S_1(t) + S_2(t) = 1$.

Define that $x(t) = [u_{\rm C}(t), i_{\rm L}(t)]^{\rm T}$ and $y(t)$ is the voltage of R_1 or R_2, i.e.

$$\begin{aligned} y(t) = &-\frac{R_1 S(t) + R_2(1 - S(t))}{R_0 + R_1 S(t) + R_2(1 - S(t))} u_{\rm C}(t) \\ &-\frac{R_0(R_1 S(t) + R_2(1 - S(t)))}{R_0 + R_1 S(t) + R_2(1 - S(t))} i_{\rm L}(t), \end{aligned}$$

then the state space description of the system (36) is

$$\begin{cases} \dot{x}(t) = \bar{A}_{\sigma(t)} x(t), \\ y(t) = \bar{C}_{\sigma(t)} x(t), \end{cases} \tag{37}$$

where

$$\bar{A}_1 = \begin{bmatrix} -\dfrac{1}{(R_0 + R_1)C} & -\dfrac{R_0}{(R_0 + R_1)C} \\ \dfrac{R_0}{L(R_0 + R_1)} & \dfrac{R_0 R_1}{L(R_0 + R_1)} \end{bmatrix},$$

$$\bar{C}_1 = \begin{bmatrix} -\dfrac{R_1}{R_0 + R_1} & -\dfrac{R_0 R_1}{R_0 + R_1} \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} -\dfrac{1}{(R_0 + R_2)C} & -\dfrac{R_0}{(R_0 + R_2)C} \\ \dfrac{R_0}{L(R_0 + R_2)} & \dfrac{R_0 R_2}{L(R_0 + R_2)} \end{bmatrix},$$

$$\bar{C}_2 = \begin{bmatrix} -\dfrac{R_2}{R_0 + R_2} & -\dfrac{R_0 R_2}{R_0 + R_2} \end{bmatrix}.$$

Since the above circuit system is a continuous-time system, it is necessary to discretise it as:

$$\begin{cases} x((k+1)h) = (\bar{A}_i h + I)x(kh), \\ y(kh) = \bar{C}_i x(kh), \end{cases} \tag{38}$$

where h is the sampling time, and we let $h = 1$ here. Substituting the standard parameters $R_0 = 1\,\Omega$, $R_1 = 2\,\Omega$, $R_2 = 3\,\Omega$, $L = 0.8\,$H, and $C = 0.5\,$F into the matrices above, we have the discrete-time system (1), where

$$A_1 = \begin{bmatrix} -0.6667 & -0.6667 \\ 0.4167 & -0.8333 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.5 & -0.5 \\ 0.3125 & -0.9375 \end{bmatrix},$$

$$C_1 = [-0.6667 \quad -0.6667], \quad C_2 = [-0.75 \quad -0.75].$$

To highlight the effectiveness of the proposed method, we add the disturbance terms to the circuit system. We choose that

$$E_1 = \begin{bmatrix} 0.02 \\ 0.09 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0.05 \\ 0.07 \end{bmatrix}, \; F_1 = 0.03, \quad F_2 = 0.06,$$

$$w(k) = \cos^2 2k, \quad v(k) = \sin k.$$

Given $\mu = 1.55$, $\xi = 0.33$. By Theorem 2, we obtain the following feasible solutions

$$P_1 = \begin{bmatrix} -0.2755 & -0.0531 \\ -0.0531 & 0.0574 \end{bmatrix}, \quad P_2 = \begin{bmatrix} -0.2781 & -0.0367 \\ -0.0367 & 0.026 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.1853 & -0.0147 \\ -0.0147 & 0.2824 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.1864 & -0.0156 \\ -0.0156 & 0.2833 \end{bmatrix},$$

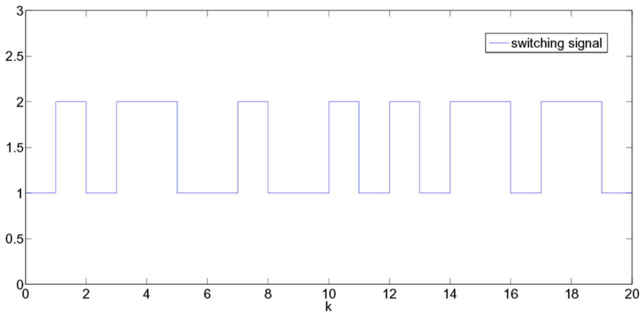

Fig. 7 *Switching signal of circuit system*

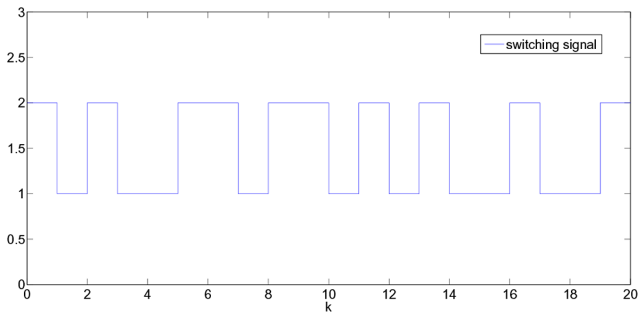

Fig. 8 *Switching signal of interval observer*

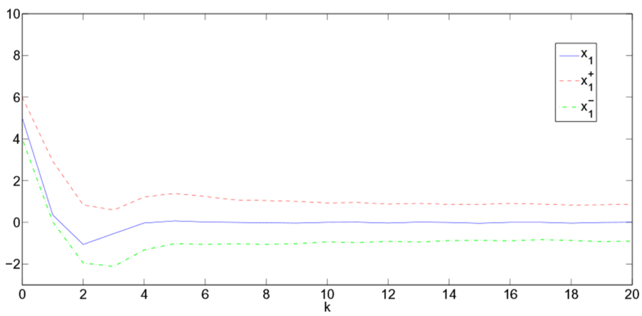

Fig. 9 *Voltage and its estimation bounds*

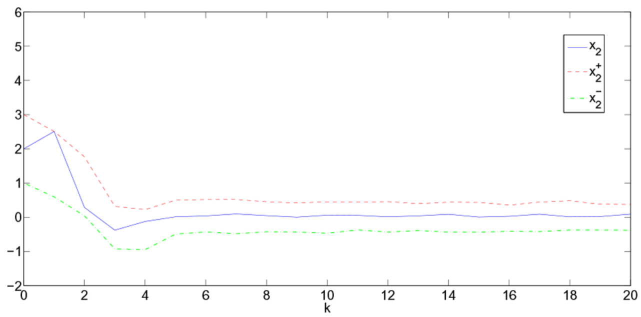

Fig. 10 *Current and its estimation bounds*

$$M_1 = \begin{bmatrix} 0.0428 \\ -0.119 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -0.0035 \\ -0.0651 \end{bmatrix},$$

$$\gamma = 5, \quad \tau^* \geq 0.3953.$$

Thus, the observer gain

$$L_1 = \begin{bmatrix} 0.1983 \\ -0.4111 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.0385 \\ -0.2318 \end{bmatrix}.$$

In addition, we use the initial conditions that

$$x(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad x^+(0) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad x^-(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix},$$

$$\bar{W} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{Y}_{\sigma(0)}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Figs. 7 and 8 present the time response of switching signals $\sigma(t)$ and $\hat{\sigma}(t)$, respectively. $\sigma(t)$ and $\hat{\sigma}(t)$ both switch randomly between two modes. The simulation time is chosen as 20 s. It can be seen from Figs. 7 and 8 that $\sigma(t)$ and $\hat{\sigma}(t)$ are different completely. By a direct computation, the ADT τ^* is 1.33 s. The evolution of voltage, current and their related estimation are depicted in Figs. 9 and 10, where x_1 is the voltage and x_2 stands for the current. As we can see from Fig. 9, in the given simulation time, the upper estimation x_1^+ and lower estimation x_1^- stay in both sides of x_1 strictly, and the steady-state estimation errors of the voltage are maintained within 1 V. In Fig. 10, the upper estimation x_2^+ and lower estimation x_2^- are above and below x_2, and the steady-state estimation errors of the current are kept in 0.4 A. Thus, the proposed interval observer is also effective.

5 Conclusion

In this paper, a technique to study the interval estimation problem for a class of asynchronous switched systems is proposed. The design of interval observer is mainly divided into two steps. First, a robust observer is constructed via H_∞ observer theory. The sufficient conditions are derived by the form of LMIs, which do not require the cooperativity of error system. Then, based on the designed H_∞ observer, the system state is described by zonotope. The iterative algorithm is employed to compute the bounds of system state. Finally, two illustrative examples are given to demonstrate the effectiveness of the designed method. In the future work, we will consider the interval observer design problem for continuous-time non-linear asynchronous switched systems. As we know, the zonotope method is an iterative algorithm which is usually used for discrete-time systems. How to extend the zonotope method to continuous-time systems directly is a difficult problem. Besides, interval observer design for non-linear asynchronous switched systems is rarely reported, and the zonotope method is also not easy to be applied to non-linear switched systems. Therefore, it will be full of challenge to design interval observers for continuous-time non-linear asynchronous switched systems.

6 Acknowledgments

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