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SHORT PAPER

Finite-Time Interval Observers' Design for Switched Systems

Xiang Ma[1] · Jun Huang[1] · Liang Chen[1]

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Abstract

In this study, a finite-time interval observers' design method is developed for switched systems suffering from disturbance. First, the interval observer frames for the systems are constructed. Then, sufficient conditions are derived to guarantee that the upper and lower error systems are both positive and finite-time bound. Unlike the current studies, all the conditions proposed in this paper are formulated in the form of linear programming. Finally, two numerical examples are provided to show the efficiency of designed observers.

Keywords Interval observers · Finite-time boundedness · Switched systems · Linear programming

List of Symbols

R^n	n-dimensional Euclidean space
$R^{n\times m}$	The set of $n\times m$ real matrices
$x > (\geq)0$	Its components are positive (nonnegative), i.e., $x_i > (\geq)0$
$A > (\geq)0$	Its components are positive (nonnegative), i.e., $A_{ij} > (\geq)0$
E^+	$\max\{E, 0\}$
E^-	$E^+ - E$
$\|\|x\|\|_1$	The 1-norm of the vector x
$\overline{\lambda}(v)$	The maximum value of the elements of the vector v
$\underline{\lambda}(v)$	The minimum value of the elements of the vector v
$\mathbf{1}_n$	The vector whose entries equal to 1

✉ Jun Huang
cauchyhot@163.com

Xiang Ma
maxiang2017@126.com

Liang Chen
chenl@suda.edu.cn

[1] School of Mechanical and Electrical Engineering, Soochow University, Suzhou 215131, China

1 Introduction

By developing theories and applications, state estimation has been widely applied in nonlinear systems [3,17]. Because uncertainty exists in real systems, it is difficult to estimate the states exactly. Thus, it is often very useful to estimate the upper and lower bounds of states for uncertain systems [1,12]. Interval observer has become a hot topic in the field of control theory [9,20–22]. Rami et al. [22] studied the design of interval observers for nonlinear systems containing uncertainty in the state equation by linear programming (LP) approach. Under the construction composition, Moisan and Bernard [20] designed robust interval observers to estimate the states of uncertain chaotic systems. Raissi et al. [21] studied the interval observers' design problem for a class of nonlinear continuous-time systems by the linear time-invariant change of coordinates. Under the time-varying transformation, Efimov et al. [9] proposed the interval observers' design method for nonlinear time-varying systems. Besides, with respect to some specified systems, such as linear parameter varying systems [25], singular systems [33], and impulsive systems [4,7], the problems of interval observers' design were also solved.

As a type of important hybrid systems, switched systems have drawn considerable attention in the past decades [18,23]. The switched systems comprise finite numbers of subsystems, which are activated by a switching law. To determine the stability of switched systems, common Lyapunov functions method [16], multiple Lyapunov functions method [24], and average dwell time (ADT) method [31] were proposed. Because of the uncertainty of switching law, the interval observer design of switched systems is much more complex than the systems without switching modes. Recently, the problem of interval observers for switched systems was studied [10,13,14]. In [14], based on the framework of interval observers, feedback laws were designed to stabilize switched systems whose nonlinear functions satisfy the Lipschitz condition. Ethabet et al. [10] focused on the interval observer design for continuous-time linear switched systems with disturbance, which is unknown but bounded. Guo and Zhu [13] considered discrete-time switched systems and used the time-varying coordinate transformation method to design the interval observers. Interestingly, the observers designed in Refs. [10,13,14] are asymptotical interval observers, and sufficient conditions are given by the form of linear matrix inequalities (LMIs).

Recently, a large number of studies were conducted on finite-time stability and stabilization [2,27–30]. Amato et al. [2] solved the finite-time control problem for linear systems with parametric uncertainties and exogenous disturbances. In [27], the problem of finite-time stabilization for uncertain singular Markovian jump systems was addressed. Furthermore, Zhang et al. [28] studied the problem of finite-time H_∞ control of discrete-time Markovian jump nonlinear systems with time delays. Based on [27,28], the nonfragile and robust finite-time H_∞ controllers were designed for a class of uncertain Markovian jump nonlinear systems via matrix decomposition approach in [29]. Combining stochastic analysis technique with matrix decomposition method, Zhang et al. [30] studied the robust and resilient finite-time H_∞ controller design method for uncertain discrete-time Markovian jump nonlinear systems. Based on the above-mentioned studies [2,27–30], we will apply the finite-time stability theory to design the interval observer.

Motivated by the above discussion, we will study a design method of finite-time interval observers for switched systems. Unlike the asymptotical interval observers proposed in [10,13,14], we design a type of finite-time interval observers for switched systems based on the finite-time stability theory. In addition, the observer gains are determined by multiple linear copositive Lyapunov function (MLCLF), and sufficient conditions are derived using LP technique. The rest of the paper is organized as follows: The problem statement and some necessary preliminary are presented in Sect. 2. The main results, consisting of sufficient conditions that make the error systems bounded in finite time, are stated in Sect. 3. Two numerical examples are shown in Sect. 4 to illustrate the effectiveness of the designed interval observers.

2 Problem Formulation and Preliminaries

Consider the following switched system described by

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}w(t), \\ y(t) = C_{\sigma(t)}x(t), \\ x^-(0) \le x(0) \le x^+(0), \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, and $y(t) \in R^q$ is the output of the system. $w(t) \in R^r$ is the disturbance and satisfies $w^- \le w(t) \le w^+$, and w^-, w^+ are known vectors. $\sigma(t)$ is a continuous mapping and takes values in a finite set $S = \{1, 2, \ldots, N\}$. For $\sigma(t) = i \in S$, $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$, $C_i \in R^{q\times n}$, and $E_i \in R^{n\times r}$ are determined matrices. $x(0)$ is the initial state, and its upper bound $x^+(0)$ and lower bound $x^-(0)$ are both known.
First, some general definitions of interval observer are introduced.

Definition 1 [22] An interval observer for (1) is a pair of upper and lower recovered states $\{\hat{x}^+(t), \hat{x}^-(t)\}$, which satisfy for any $t > 0$

$$\hat{x}^-(t) \le x(t) \le \hat{x}^+(t),$$

under the initial condition

$$\hat{x}^-(0) \le x(0) \le \hat{x}^+(0).$$

Definition 2 [22] An interval observer for (1) is said to be an asymptotical interval observer for any $t > 0$

$$\begin{cases} \lim_{t\to\infty} ||\hat{x}^+(t) - x(t)||_1 = \alpha, \\ \lim_{t\to\infty} ||x(t) - \hat{x}^-(t)||_1 = \beta, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ are constants.

Remark 1 In Definition 2, $\lim_{t\to\infty}||\hat{x}^{+}(t)-x(t)||=\alpha$ and $\lim_{t\to\infty}||x(t)-\hat{x}^{-}(t)||=\beta$ indicate that $\lim_{t\to\infty}||\hat{x}^{+}(t)-\hat{x}^{-}(t)||$ is bounded. Thus, Definition 2 is equivalent to Definition 2.2 in [22], even though we use the 1-norm here.

The following definition about the finite-time interval observer is given for the first time.

Definition 3 An interval observer for (1) is said to be a finite-time interval observer if $T > 0$ so that

$$||\hat{x}^{+}(0)-x(0)||_1 \le \alpha_1 \Rightarrow ||\hat{x}^{+}(t)-x(t)||_1 \le \alpha_2, \quad \forall\, t \in [0, T], \tag{2}$$
$$||x(0)-\hat{x}^{-}(0)||_1 \le \beta_1 \Rightarrow ||x(t)-\hat{x}^{-}(t)||_1 \le \beta_2, \quad \forall\, t \in [0, T], \tag{3}$$

where $\alpha_1, \alpha_2, \beta_1$, and β_2 are positive constants, and $\alpha_1 < \alpha_2, \beta_1 < \beta_2$.

Remark 2 The concept of finite-time interval observers is introduced based on the finite-time stability theory. Unlike the finite-time stability and stabilization problem such as [2,27–30], the upper and lower error systems in this paper should be finite-time bounded (FTB) and positive. Definition 3 indicates that $||\hat{x}^{+}(t)-\hat{x}^{-}(t)||_1$ is bounded in the time interval $[0, T]$. It focuses on the transient performance of interval observer, and the error can be retained in a bounded region when the time belongs to a finite interval. However, the asymptotical interval observer indicates that the error converges to a bounded value when the time tends to infinity. From application perspective, the finite-time interval observer is necessary. Similar definitions can be found in the finite-time filters [6,19].

Because the theory of positive system plays an important role in the design of interval observer, some results of positive system are reviewed. More details can be found in [11]. Consider the following system,

$$\begin{cases} \dot{x}(t) = Mx(t) + f(t), \\ x(0) = x_0 \ge 0, \end{cases} \tag{4}$$

where $x(t) \in R^n$, $f(t) \in R^n \ge 0$, $M \in R^{n\times n}$ is a constant matrix.

Definition 4 System (4) is said to be positive if the corresponding trajectory $x(t) \ge 0$ for any $t \ge 0$.

Definition 5 Matrix M is said to be a Metzler matrix if all its off-diagonal entries are negative.

The following lemma gives the condition that can guarantee the positivity of system (4).

Lemma 1 *Consider system* (4). *If matrix M is a Metzler matrix, then system* (4) *is positive.*

In fact, we need the results of positive switched system; thus, we extend system (4) to the following switched system, i.e.,

$$\begin{cases} \dot{x}(t) = M_{\sigma(t)}x(t) + f_{\sigma(t)}(t), \\ x(0) = x_0 \geq 0, \end{cases} \tag{5}$$

where $x(t) \in R^n$ is the state, $\sigma(t)$ is a switched signal which takes values in the finite set S. For any $\sigma(t) = i \in S$, $M_i \in R^{n\times n}$ is the given matrix, and $f_i(t) \in R^n \geq 0$.

Definition 6 System (5) is said to be positive if the corresponding trajectory $x(t) \geq 0$ for any $t \geq 0$.

Definition 7 Matrix $M_{\sigma(t)}$ is said to be a Metzler matrix if all the off-diagonal entries of M_i are negative for any $\sigma(t) = i \in S$.

Lemma 2 *System (5) is positive if the matrix $M_{\sigma(t)}$ is a Metzler matrix.*

Lemma 3 [15] *$M_{\sigma(t)}$ is a Metzler matrix if and only if a constant γ exists so that $M_{\sigma(t)} + \gamma I \geq 0$.*

According to switched system (1), we design the finite-time interval observer as follows:

$$\begin{cases} \dot{\hat{x}}^+(t) = A_{\sigma(t)}\hat{x}^+(t) + B_{\sigma(t)}u(t) + \epsilon^+_{\sigma(t)} + L_{\sigma(t)}(y - C_{\sigma(t)}\hat{x}^+(t)), \\ \dot{\hat{x}}^-(t) = A_{\sigma(t)}\hat{x}^-(t) + B_{\sigma(t)}u(t) + \epsilon^-_{\sigma(t)} + L_{\sigma(t)}(y - C_{\sigma(t)}\hat{x}^-(t)), \\ \hat{x}^+(0) = x^+(0), \\ \hat{x}^-(0) = x^-(0), \end{cases} \tag{6}$$

where $\epsilon^+_{\sigma(t)} = E^+_{\sigma(t)}w^+ - E^-_{\sigma(t)}w^-$ and $\epsilon^-_{\sigma(t)} = E^+_{\sigma(t)}w^- - E^-_{\sigma(t)}w^+$. Subtracting (6) from (1), we can obtain the error system

$$\begin{cases} \dot{e}^+(t) = (A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)})e^+(t) + \epsilon^+_{\sigma(t)} - E_{\sigma(t)}w(t), \\ \dot{e}^-(t) = (A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)})e^-(t) + E_{\sigma(t)}w(t) - \epsilon^-_{\sigma(t)}, \\ e^+(0) \geq 0, \quad e^-(0) \geq 0, \end{cases} \tag{7}$$

where $e^+(t) = \hat{x}^+(t) - x(t)$, and $e^-(t) = x(t) - \hat{x}^-(t)$. Based on the error system (7), we introduce the definition of finite-time boundedness.

Definition 8 [8,26] Let c_1, c_2, c_3, c_4, T, h be positive constants with $c_1 < c_2, c_3 < c_4$. If for $\forall\ w(t) : \int_0^{\mathrm{T}} \|w(t)\|_1 \mathrm{d}t \leq h$

$$||e^+(t_0)||_1 \leq c_1 \Rightarrow ||e^+(t)||_1 \leq c_2, \quad \forall\ t \in [0, T], \tag{8}$$

$$||e^-(t_0)||_1 \leq c_3 \Rightarrow ||e^-(t)||_1 \leq c_4, \quad \forall\ t \in [0, T], \tag{9}$$

then, system (7) is FTB with respect to $(c_1, c_2, c_3, c_4, T, h)$.

If the error system is FTB, then observer (6) is the finite-time interval observer from Definition 3. When we analyze the performance of switched systems, we need the definition of ADT.

Definition 9 [32] Consider the time interval $[t_1, t_2)$ where $t_1 \geq 0$. Let the switching number of $\sigma(t)$ on $[t_1, t_2)$ be $N_\sigma(t_1, t_2)$. If the following inequality holds

$$N_\sigma(t_1, t_2) \leq N_0 + (t_2 - t_1)/\tau^*,$$

where $N_0 \geq 0$ and $\tau^* > 0$, then τ^* is an ADT of the switching signal $\sigma(t)$. For simplicity, we let $N_0 = 0$ in the sequel.

To develop the main result of the paper, the following lemma is necessary.

Lemma 4 [5] *Let $w(t)$ be a vector satisfying $w^- \leq w(t) \leq w^+$, and E be a constant matrix. Then*

$$E^+ w^- - E^- w^+ \leq Ew(t) \leq E^+ w^+ - E^- w^-. \tag{10}$$

3 Main Result

In this section, we first provide sufficient conditions for the existence of finite-time interval observer.

Theorem 1 *Let $\eta > 0$, $\rho > 1$ and γ be given constants. If there exist constants λ_1, λ_2, λ_3, λ_4 and vectors $v_i \in R^n > 0$, $v_j \in R^n > 0$, $z_i \in R^q$, $\forall\, i, j \in S$, $i \neq j$ so that*

$$E_i^{\mathrm{T}} v_i + \lambda_1 \boldsymbol{I}_n \geq 0, \tag{11}$$
$$E_i^{\mathrm{T}} v_i - \lambda_3 \boldsymbol{I}_n \leq 0, \tag{12}$$
$$(\epsilon_i^+)^{\mathrm{T}} v_i - \lambda_2 \leq 0, \tag{13}$$
$$(\epsilon_i^-)^{\mathrm{T}} v_i + \lambda_4 \geq 0, \tag{14}$$
$$(A_i^{\mathrm{T}} - \eta I) v_i + C_i^{\mathrm{T}} z_i \leq 0, \tag{15}$$
$$\xi_i^{\mathrm{T}} v_i (\xi_i^{\mathrm{T}} v_i A_i + \xi_i z_i^{\mathrm{T}} C_i + \gamma I) \geq 0, \tag{16}$$
$$v_i \leq \rho v_j, \tag{17}$$

where $\xi_i \in R^n \neq 0$ is a prescribed vector, and the observer gain is designed by

$$L_i = -\frac{\xi_i z_i^{\mathrm{T}}}{\xi_i^{\mathrm{T}} v_i}, \tag{18}$$

and ADT satisfies

$$\tau^* \geq \max\left\{ \frac{T \ln \rho}{\ln \mu_1 - \ln \zeta_1 - \eta T}, \frac{T \ln \rho}{\ln \mu_2 - \ln \zeta_2 - \eta T} \right\}, \tag{19}$$

where $\mu_1 = \underline{\lambda}(v_i)c_2$, $\zeta_1 = \bar{\lambda}(v_{\sigma(0)})c_1 + |\lambda_1|h + |\lambda_2|T$, $\mu_2 = \underline{\lambda}(v_i)c_4$, $\zeta_2 = \bar{\lambda}(v_{\sigma(0)})c_3 + |\lambda_3|h + |\lambda_4|T$, c_1, c_2, c_3, c_4 *are positive constants determined by* $\mu_1 > \zeta_1 \exp\{\eta T\}$ *and* $\mu_2 > \zeta_2 \exp\{\eta T\}$. *Then the error system* (7) *is positive and FTB, i.e.,* (6) *is a finite-time interval observer.*

Proof From Lemma 2 and Definition 8, we will prove the positivity as well as finite-time boundedness of error system (7). Next, the proof is divided into two steps:

(i) We first show the positivity of upper error system, i.e,

$$\begin{cases} \dot{e}^+(t) = (A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)})e^+(t) + \epsilon^+_{\sigma(t)} - E_{\sigma(t)}w(t), \\ e^+(0) \geq 0. \end{cases} \tag{20}$$

When $\sigma(t) = i \in S$, in view of (18),

$$A_i - L_i C_i = A_i + \frac{\xi_i z_i^{\mathrm{T}}}{\xi_i^{\mathrm{T}} v_i} C_i. \tag{21}$$

Because $\xi_i^{\mathrm{T}} v_i$ is a scalar, it is deduced from (16) that

$$A_i + \frac{\xi_i z_i^{\mathrm{T}}}{\xi_i^{\mathrm{T}} v_i} C_i + \gamma I \geq 0. \tag{22}$$

Using Lemma 3, (22) indicates that $A_i + \frac{\xi_i z_i^{\mathrm{T}}}{\xi_i^{\mathrm{T}} v_i} C_i$ is a Metzler matrix, i.e., $A_i - L_i C_i$ is a Metzler matrix. Using Lemmas 2 and 4, system (20) is positive. Similarly, the lower error system

$$\begin{cases} \dot{e}^-(t) = (A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)})e^+(t) - \epsilon^-_{\sigma(t)} + E_{\sigma(t)}w(t), \\ e^-(0) \geq 0. \end{cases} \tag{23}$$

is also positive.

(ii) Then, we prove the finite-time boundedness of system (7). First, we consider the upper error system (20). Without loss of generality, it is supposed that the switching sequence is $\{t_i, i = 1, 2, \ldots\}$, and $0 < t_1 < t_2 < \cdots$. Let $\sigma(t_k) = i \in S$, we select the following MLCLF candidate

$$V_i(t) = (e^+(t))^{\mathrm{T}} v_i, \quad i \in S. \tag{24}$$

When $t \in [t_k, t_{k+1})$, taking the derivative of $V_i(t)$ results in

$$\begin{aligned} \dot{V}_i(t) = (e^+(t))^{\mathrm{T}} A_i^{\mathrm{T}} v_i - (e^+(t))^{\mathrm{T}} C_i^{\mathrm{T}} L_i^{\mathrm{T}} v_i \\ + (\epsilon_i^+)^{\mathrm{T}} v_i - (w(t))^{\mathrm{T}} E_i^{\mathrm{T}} v_i. \end{aligned} \tag{25}$$

Substituting (18) into (25),

$$\dot{V}_i(t) = (e^+(t))^{\mathrm{T}}(A_i^{\mathrm{T}} v_i + C_i^{\mathrm{T}} z_i) + (\epsilon_i^+)^{\mathrm{T}} v_i - (w(t))^{\mathrm{T}} E_i^{\mathrm{T}} v_i. \tag{26}$$

Using (11), (13), and (15), we obtain

$$\begin{aligned}\dot{V}_i(t) &\leq \eta(e^+(t))^{\mathrm{T}} v_i + \lambda_2 + \lambda_1 (w(t))^{\mathrm{T}} \mathbf{1}_n \\ &= \eta V_i(t) + \lambda_2 + \lambda_1 ||w(t)||_1.\end{aligned} \tag{27}$$

From t_k to t, integrating both sides of (27), we obtain

$$\begin{aligned}V_i(t) \leq& \exp\{\eta(t-t_k)\} V_i(t_k) + \lambda_1 \int_{t_k}^t \exp\{\eta(t-s)\} ||w(s)||_1 \mathrm{d}s \\ &+ \lambda_2 \int_{t_k}^t \exp\{\eta(t-s)\} \mathrm{d}s.\end{aligned} \tag{28}$$

In view of (17), the following holds

$$\begin{aligned}V_i(t) \leq& \rho \exp\{\eta(t-t_k)\} V_j(t_k) + \lambda_1 \int_{t_k}^t \exp\{\eta(t-s)\} ||w(s)||_1 \mathrm{d}s \\ &+ \lambda_2 \int_{t_k}^t \exp\{\eta(t-s)\} \mathrm{d}s.\end{aligned} \tag{29}$$

By repeating (28) and (29), we have

$$\begin{aligned}V_i(t) \leq\ & \rho \exp\{\eta(t-t_k)\} V_{\sigma(t_{k-1})}(t_k) + \lambda_1 \int_{t_k}^t \exp\{\eta(t-s)\} ||w(s)||_1 \mathrm{d}s \\ & + \lambda_2 \int_{t_k}^t \exp\{\eta(t-s)\} \mathrm{d}s \\ \leq\ & \cdots \\ \leq\ & \rho^k \exp\{\eta t\} V_{\sigma(0)}(0) + \lambda_1 \int_0^t \exp\{\eta(t-s) + N_\sigma(s,t) \ln \rho\} ||w(s)||_1 \mathrm{d}s \\ & + \lambda_2 \int_0^t \exp\{\eta(t-s) + N_\sigma(s,t) \ln \rho\} \mathrm{d}s.\end{aligned} \tag{30}$$

From Definition 9, $k = N_\sigma \leq N_0 + \frac{t}{\tau^*} = \frac{t}{\tau^*}$. Since $\rho \geq 1$, $t \leq T$ and $\int_0^{\mathrm{T}} \|w(t)\|_1 \mathrm{d}t \leq h$, inequality (30) implies that

$$\begin{aligned}V_i(t) &\leq \rho^k \exp\{\eta t\} V_{\sigma(0)}(0) + \rho^k \exp\{\eta t\}(|\lambda_1| h + |\lambda_2| T) \\ &\leq \rho^{\frac{T}{\tau^*}} \exp\{\eta T\}(V_{\sigma(0)}(0) + |\lambda_1| h + |\lambda_2| T).\end{aligned} \tag{31}$$

From $\underline{\lambda}(v_i)\mathbf{1}_n < v_i < \bar{\lambda}(v_i)\mathbf{1}_n$, it can be deduced that

$$\begin{cases} V_i(t) = (e^+(t))^{\mathrm{T}} v_i \geq \underline{\lambda}(v_i)(e^+(t))^{\mathrm{T}}\mathbf{1}_n = \underline{\lambda}(v_i)||e^+(t)||_1, \\ V_{\sigma(0)}(0) = (e^+(0))^{\mathrm{T}} v_{\sigma(0)} \leq \bar{\lambda}(v_{\sigma(0)})(e^+(0))^{\mathrm{T}}\mathbf{1}_n = \bar{\lambda}(v_{\sigma(0)})||e^+(0)||_1. \end{cases} \tag{32}$$

Substituting (32) into (31) yields

$$||e^+(t)||_1 \leq \frac{1}{\underline{\lambda}(v_i)}\rho^{\frac{T}{\tau^*}}\exp\{\eta T\}(\bar{\lambda}(v_{\sigma(0)})||e^+(0)||_1 + |\lambda_1|h + |\lambda_2|T). \tag{33}$$

Combining (19) and $\mu > \zeta\exp\{\eta T\}$, we obtain

$$||e^+(t)||_1 \leq \frac{1}{\underline{\lambda}(v_i)}\frac{\mu_1}{\zeta_1}(\bar{\lambda}(v_{\sigma(0)})||e^+(0)||_1 + |\lambda_1|h + |\lambda_2|T). \tag{34}$$

When $||e^+(0)||_1 \leq c_1$, it follows

$$||e^+(t)||_1 \leq \frac{1}{\underline{\lambda}(v_i)}\frac{\mu_1}{\zeta_1}(\bar{\lambda}(v_{\sigma(0)})c_1 + |\lambda_1|h + |\lambda_2|T) = c_2. \tag{35}$$

From Definition 8, the upper error system (20) is FTB.
Next, we consider the lower error system. Using the similar treatment as that used in the upper one, we select the following MLCLF candidate

$$\bar{V}_i(t) = (e^-(t))^{\mathrm{T}} v_i, \quad i \in S. \tag{36}$$

When $t \in [t_k, t_{k+1})$, we obtain the derivative of $\bar{V}_i(t)$

$$\begin{aligned} \dot{\bar{V}}_i(t) = &(e^-(t))^{\mathrm{T}} A_i^{\mathrm{T}} v_i - (e^-(t))^{\mathrm{T}} C_i^{\mathrm{T}} L_i^{\mathrm{T}} v_i \\ &- (\epsilon_i^-)^{\mathrm{T}} v_i + (w(t))^{\mathrm{T}} E_i^{\mathrm{T}} v_i. \end{aligned} \tag{37}$$

By substituting (18) into (37),

$$\dot{\bar{V}}_i(t) = (e^-(t))^{\mathrm{T}}(A_i^{\mathrm{T}} v_i + C_i^{\mathrm{T}} z_i) - (\epsilon_i^-)^{\mathrm{T}} v_i + (w(t))^{\mathrm{T}} E_i^{\mathrm{T}} v_i. \tag{38}$$

In view of (12), (14), and (15), it is deduced from (38) that

$$\begin{aligned} \dot{\bar{V}}_i(t) &\leq \eta(e^-(t))^{\mathrm{T}} v_i + \lambda_4 + \lambda_3(w(t))^{\mathrm{T}}\mathbf{1}_n \\ &= \eta V_i(t) + \lambda_4 + \lambda_3||w(t)||_1. \end{aligned} \tag{39}$$

From Definition 9, $k = N_\sigma \leq N_0 + \frac{t}{\tau^*} = \frac{t}{\tau^*}$, we can also conclude that

$$\begin{aligned} \bar{V}_i(t) &\leq \rho^k \exp\{\eta t\} V_{\sigma(0)}(0) + \rho^k \exp\{\eta t\}(|\lambda_3|h + |\lambda_4|T) \\ &\leq \rho^{\frac{T}{\tau^*}}\exp\{\eta T\}(V_{\sigma(0)}(0) + |\lambda_3|h + |\lambda_4|T). \end{aligned} \tag{40}$$

Similar to the upper error system, when $||e^-(0)||_1 \leq c_3$, we have

$$||e^-(t)||_1 \leq \frac{1}{\underline{\lambda}(v_i)} \frac{\mu_2}{\zeta_2} (\bar{\lambda}(v_{\sigma(0)})c_1 + |\lambda_3|h + |\lambda_4|T) = c_4. \tag{41}$$

Hence, from Definition 8, the lower error system (23) is FTB. By combining (i) with (ii), we complete the proof. □

Remark 3 MLCLF is a powerful tool to determine the stability of positive systems. However, the quadratic Lyapunov function is usually applied to more general systems. By selecting the MLCLFs as (25) and (36), sufficient conditions (11)–(17) are derived to guarantee the positivity as well as the finite-time boundedness of upper and lower errors.

Remark 4 Compared with the quadratic Lyapunov function, MLCLF is used to derive sufficient conditions in the form of LP instead of LMI. LP is more tractable than LMI, especially when the optimization problem is considered.

The conditions in Theorem 3.1 cannot be solved by LP toolbox in MATLAB because the nonlinear term $\xi_i^{\mathrm{T}} v_i$ is present in (16). It is necessary to derive the equivalent forms. Then we have the following theorem.

Theorem 2 *For given constants $\eta > 0$, $\rho > 1$ and γ. If there exist constants λ_1, λ_2, λ_3, λ_4 and vectors $v_i \in R^n > 0$, $v_j \in R^n > 0$, $z_i \in R^q$, $\forall\, i, j \in S$, $i \neq j$ such that*

$$\begin{cases} E_i^{\mathrm{T}} v_i + \lambda_1 \boldsymbol{1}_n \geq 0, \\ E_i^{\mathrm{T}} v_i - \lambda_3 \boldsymbol{1}_n \leq 0, \\ (\epsilon_i^+)^{\mathrm{T}} v_i - \lambda_2 \leq 0, \\ (\epsilon_i^-)^{\mathrm{T}} v_i + \lambda_4 \geq 0, \\ (A_i^{\mathrm{T}} - \eta I) v_i + C_i^{\mathrm{T}} z_i \leq 0, \\ \xi_i^{\mathrm{T}} v_i \geq 0, \\ \xi_i^{\mathrm{T}} v_i A_i + \xi_i z_i^{\mathrm{T}} C_i + \gamma I \geq 0, \\ v_i \leq \rho v_j, \end{cases} \tag{42}$$

or

$$\begin{cases} E_i^{\mathrm{T}} v_i + \lambda_1 \boldsymbol{1}_n \geq 0, \\ E_i^{\mathrm{T}} v_i - \lambda_3 \boldsymbol{1}_n \leq 0, \\ (\epsilon_i^+)^{\mathrm{T}} v_i - \lambda_2 \leq 0, \\ (\epsilon_i^-)^{\mathrm{T}} v_i + \lambda_4 \geq 0, \\ (A_i^{\mathrm{T}} - \eta I) v_i + C_i^{\mathrm{T}} z_i \leq 0, \\ \xi_i^{\mathrm{T}} v_i \leq 0, \\ \xi_i^{\mathrm{T}} v_i A_i + \xi_i z_i^{\mathrm{T}} C_i + \gamma I \leq 0, \\ v_i \leq \rho v_j, \end{cases} \tag{43}$$

where $\xi_i \in R^n \neq 0$ is a prescribed vector, and the observer gain is designed by (18) *and ADT satisfying* (19). *Then, the error system* (7) *is positive as well as FTB, i.e.,* (6) *is a finite-time interval observer for system* (1).

Proof In the nonlinear condition (16), we just need to argue whether the scalar $\xi_i^{\mathrm{T}} v_i$ is positive or negative. If $\xi_i^{\mathrm{T}} v_i \geq 0$, then $\xi_i^{\mathrm{T}} v_i A_i + \xi_i z_i^{\mathrm{T}} C_i + \gamma I \geq 0$. If $\xi_i^{\mathrm{T}} v_i \leq 0$, then $\xi_i^{\mathrm{T}} v_i A_i + \xi_i z_i^{\mathrm{T}} C_i + \gamma I \leq 0$. Thus, sufficient conditions (42) and (43) are equivalent to (11)–(17). The proof is completed. □

Remark 5 The designed observer (6) reflects the transient characteristic. From the proof of Theorem 1, we can see that the error system is kept bounded in a finite-time interval $[0, T]$, but it may be unstable. Thus, the method used in the previous studies cannot be applied in this paper.

Remark 6 Unlike previous studies [10,13,14], sufficient conditions are derived in the form of LP. From computation perspective, LP can be more tractable than LMI by MATLAB. In fact, if conditions (11), (13), (15)–(17) hold, the upper interval observer is valid. Conditions (12), (14), (15)–(17) guarantee the existence of lower interval observer.

Remark 7 To design the interval observer, the following steps are followed:

Step 1 Solve (42) or (43) by MATLAB;
Step 2 Compute (18) to obtain the observer gain;
Step 3 Determine μ_1, ζ_1, μ_2, and ζ_2;
Step 4 Estimate c_2 and c_4 with $\mu_1 > \zeta_1 \exp\{\eta T\}$ and $\mu_2 > \zeta_2 \exp\{\eta T\}$.

The designed interval observer can only converge to the original system in a bounded neighborhood. From practice perspective, we want to make the estimated interval as small as possible.

Theorem 3 *If the optimal gain L_i can be calculated from the optimization LP problem as follows*

$$\begin{cases} \min \quad c_2 \\ \text{subject to :} \\ E_i^{\mathrm{T}} v_i + \lambda_1 \boldsymbol{1}_n \geq 0, \\ E_i^{\mathrm{T}} v_i - \lambda_3 \boldsymbol{1}_n \leq 0, \\ (\epsilon_i^{+})^{\mathrm{T}} v_i - \lambda_2 \leq 0, \\ (\epsilon_i^{-})^{\mathrm{T}} v_i + \lambda_4 \geq 0, \\ (A_i^{\mathrm{T}} - \eta I) v_i + C_i^{\mathrm{T}} z_i \leq 0, \\ \xi_i^{\mathrm{T}} v_i \geq 0, \\ \xi_i^{\mathrm{T}} v_i A_i + \xi_i z_i^{\mathrm{T}} C_i + \gamma I \geq 0, \\ v_i \leq \rho v_j, \end{cases} \tag{44}$$

or

$$\begin{cases} \min \quad c_2 \\ \text{subject to :} \\ E_i^{\mathrm{T}} v_i + \lambda_1 \boldsymbol{1}_n \geq 0, \\ E_i^{\mathrm{T}} v_i - \lambda_3 \boldsymbol{1}_n \leq 0, \\ (\epsilon_i^+)^{\mathrm{T}} v_i - \lambda_2 \leq 0, \\ (\epsilon_i^-)^{\mathrm{T}} v_i + \lambda_4 \geq 0, \\ (A_i^{\mathrm{T}} - \eta I) v_i + C_i^{\mathrm{T}} z_i \leq 0, \\ \xi_i^{\mathrm{T}} v_i \leq 0, \\ \xi_i^{\mathrm{T}} v_i A_i + \xi_i z_i^{\mathrm{T}} C_i + \gamma I \leq 0, \\ v_i \leq \rho v_j, \end{cases} \tag{45}$$

then the interval observer (6) *is an optimal interval observer.*

The proof of Theorem 3 can be obtained from that in Theorem 2, and it is omitted here.

Remark 8 Note that the conditions $\mu_1 > \zeta_1 \exp\{\eta T\}$ and $\mu_2 > \zeta_2 \exp\{\eta T\}$, where $\mu_1 = \underline{\lambda}(v_i)c_2$, $\zeta_1 = \bar{\lambda}(v_{\sigma(0)})c_1 + |\lambda_1|h + |\lambda_2|T$, $\mu_2 = \underline{\lambda}(v_i)c_4$, $\zeta_2 = \bar{\lambda}(v_{\sigma(0)})c_3 + |\lambda_3|h + |\lambda_4|T$. c_2 is dependent on λ_1, λ_2, c_1, whereas c_4 is dependent on λ_3, λ_4, c_3. Generally, if c_1, c_3, and v_i are given, to make c_2 and c_4 as small as possible, we should select proper parameters λ_1, λ_2, λ_3, and λ_4.

4 Numerical Simulations

In this section, we will provide two numerical examples to verify the effectiveness of proposed approach.

Example 1 Consider system (1) with

$$A_1 = \begin{bmatrix} -1.7 & 2.1 \\ 1.8 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 2 \\ 1.2 & 2.3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1.5 & 2.2 \\ 1.7 & -1.9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.1 & 1.9 \\ 1.3 & 2.1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1.4 & 1.6 \\ 2.5 & 1.9 \end{bmatrix}, \quad E_1 = \begin{bmatrix} -1 & 1.3 \\ -0.7 & 2.2 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 1.6 & 1.5 \\ 2.4 & 1.8 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1.7 & 0.3 \\ -1 & -1.2 \end{bmatrix}.$$

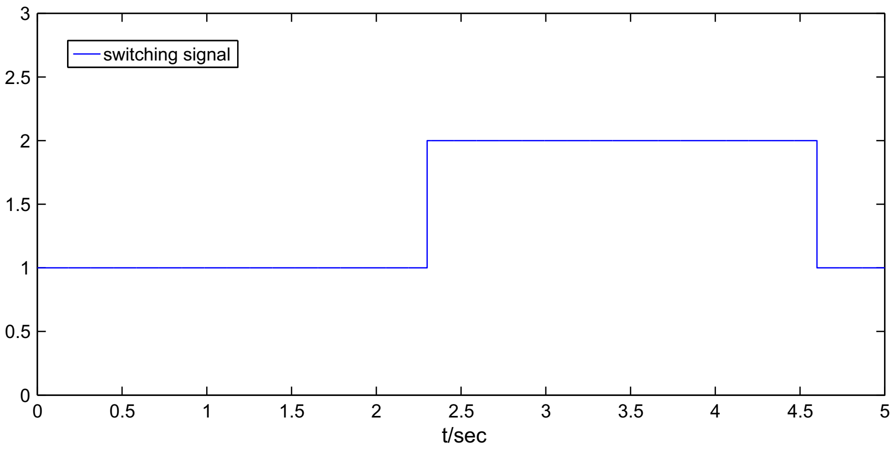

Fig. 1 Simulation of switching signal $\sigma(t)$

Given $\xi^{(1)} = [1; 2]$, $\xi^{(2)} = [2; 1]$, $\eta = 0.01$, $\rho = 1.35$, $\gamma = 1$, $T = 5$. Using Theorem 3, we obtain

$$\begin{aligned} v^{(1)} &= \begin{bmatrix} 0.1044 \\ 0.0851 \end{bmatrix}, \quad z^{(1)} = \begin{bmatrix} -0.1674 \\ 0.0441 \end{bmatrix}, \\ v^{(2)} &= \begin{bmatrix} 0.0913 \\ 0.1143 \end{bmatrix}, \quad z^{(2)} = \begin{bmatrix} 0.2636 \\ -0.2421 \end{bmatrix}, \\ \lambda_1 &= 0.164, \quad \lambda_2 = 0.0323, \\ \lambda_3 &= 0.041, \quad \lambda_4 = 0.0487. \end{aligned}$$

Thus, we obtain the observer gain as follows:

$$L_1 = \begin{bmatrix} 0.6096 & -0.1606 \\ 1.2192 & -0.3212 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -1.7757 & 1.6309 \\ -0.8878 & 0.8154 \end{bmatrix}.$$

The inputs, disturbance, and initial conditions of system (1) are selected as follows:

$$\begin{aligned} u &= \begin{bmatrix} \sin t \\ \sin^2 2t \end{bmatrix}, \quad w = \begin{bmatrix} 0.1 \sin^2 t \\ 0.1 \sin t \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \\ x_0^+ &= \begin{bmatrix} 0.15 \\ 0.25 \end{bmatrix}, \quad x_0^- = \begin{bmatrix} 0.05 \\ 0.15 \end{bmatrix}. \end{aligned}$$

Using Theorem 3, we obtain $\tau^* \geq 2.25$, $c_2 \leq 6.5582$ and $c_4 \leq 6.6231$. Figure 1 shows the time response of switching signal $\sigma(t)$. The simulation results of interval observer are shown in Figs. 2 and 3. The designed observer can converge to a bounded neighborhood of the original system whose trajectory is divergent. To be more clear, the upper error and lower error are also shown in Figs. 4 and 5. The errors are bounded in the finite-time interval [0, 5]. Thus, the finite-time interval observer is effective.

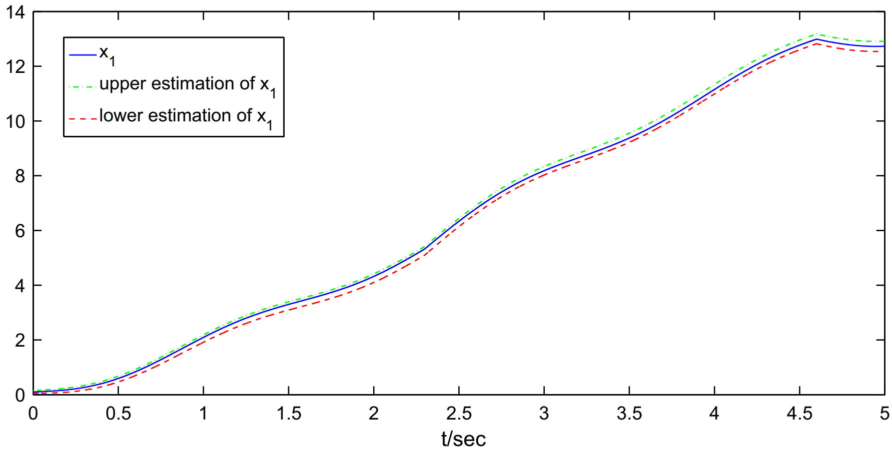

Fig. 2 Evolution of the real state $x_1(t)$ and the estimations $\hat{x}_1^+(t)$, $\hat{x}_1^-(t)$

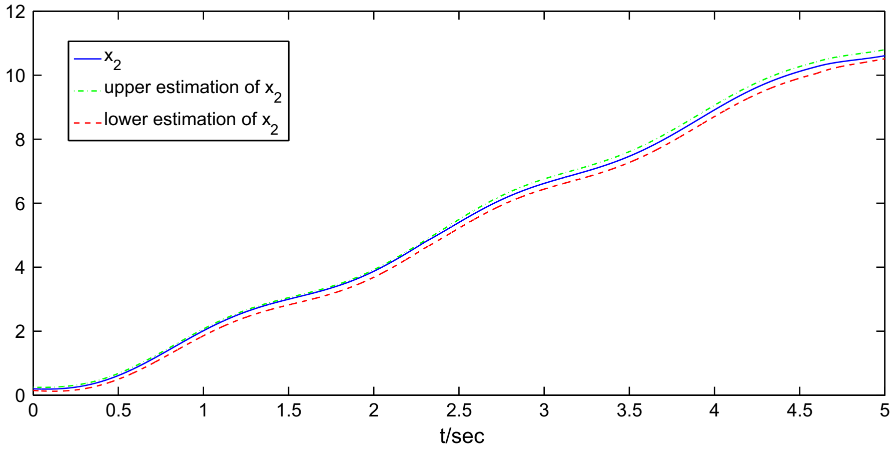

Fig. 3 Evolution of the real state $x_2(t)$ and the estimations $\hat{x}_2^+(t)$, $\hat{x}_2^-(t)$

Example 2 Consider the system (1) with

$$
\begin{aligned}
A_1 &= \begin{bmatrix} -1.3 & 2 \\ 1.8 & -1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.2 & 1.4 \\ 2.1 & 1.6 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -1.2 & 1.6 \\ 1.7 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.1 & 1.9 \\ 1.5 & 2 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 1.5 & 1.6 \\ 2.3 & 1.7 \end{bmatrix}, \quad E_1 = \begin{bmatrix} -1.1 & 1.9 \\ 1 & -0.9 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 1.4 & 1.3 \\ 2.1 & 1.8 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1.7 & -1.5 \\ -1.3 & 1.2 \end{bmatrix}.
\end{aligned}
$$

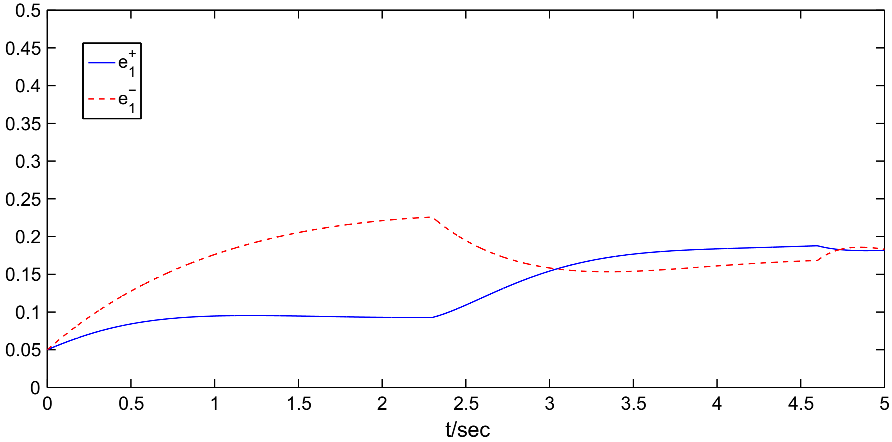

Fig. 4 Evolution of the errors e_1^+, e_1^-

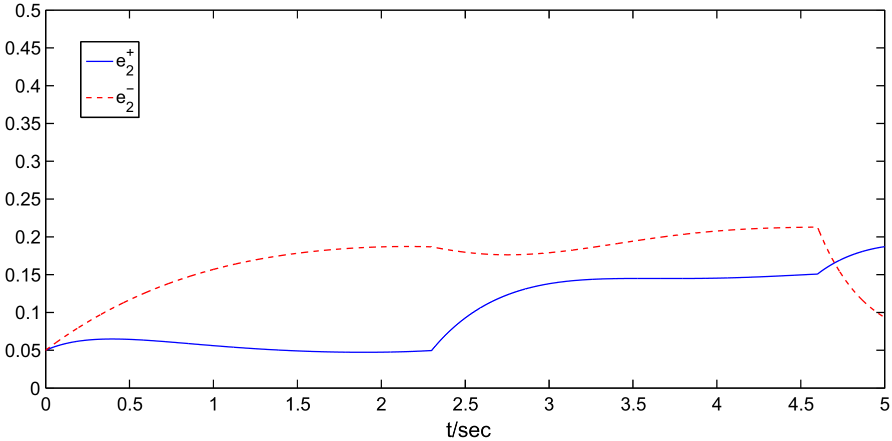

Fig. 5 Evolution of the errors e_2^+, e_2^-

Given $\xi^{(1)} = [1; 2]$, $\xi^{(2)} = [2; 1]$, $\eta = 0.2$, $\rho = 1.3$, $\gamma = 1$, $T = 5$. By solving the conditions in Theorem 3, we obtain

$$
\begin{aligned}
&v^{(1)} = \begin{bmatrix} 0.1685 \\ 0.1812 \end{bmatrix}, \quad z^{(1)} = \begin{bmatrix} 0.1055 \\ -0.1209 \end{bmatrix}, \\
&v^{(2)} = \begin{bmatrix} 0.1896 \\ 0.2316 \end{bmatrix}, \quad z^{(2)} = \begin{bmatrix} 0.2092 \\ -0.2026 \end{bmatrix}, \\
&\lambda_1 = 0.0065, \quad \lambda_2 = 0.0902, \\
&\lambda_3 = 0.1571, \quad \lambda_4 = 0.0908.
\end{aligned}
$$

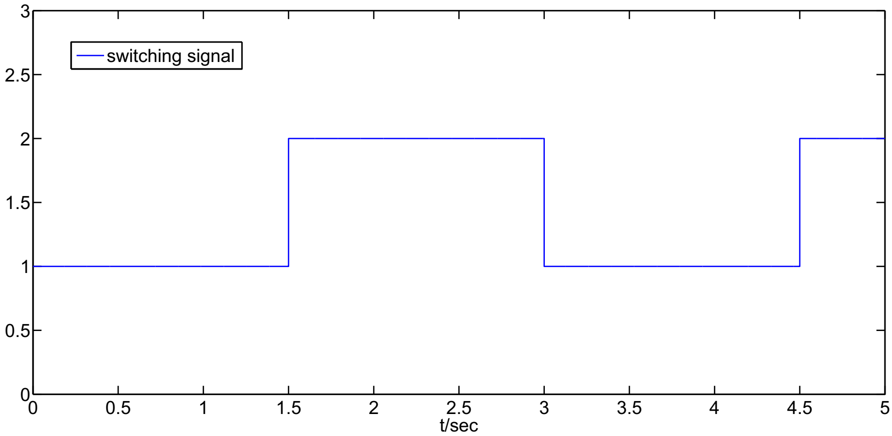

Fig. 6 Simulation of switching signal $\sigma(t)$

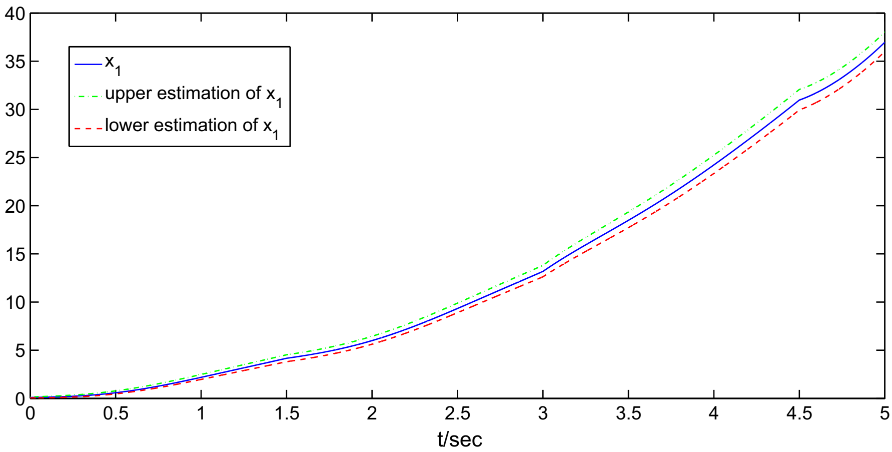

Fig. 7 Evolution of the real state $x_1(t)$ and the estimations $\hat{x}_1^+(t)$, $\hat{x}_1^-(t)$

Thus,

$$L_1 = \begin{bmatrix} -0.1987 & 0.2277 \\ -0.3974 & 0.4555 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.6850 & 0.6634 \\ -0.3425 & 0.3317 \end{bmatrix}.$$

The inputs, disturbance, and initial conditions of system (1) are given as follows:

$$u = \begin{bmatrix} \sin t \\ \sin^2 2t \end{bmatrix}, \quad w = \begin{bmatrix} 0.1 \sin 2t \\ 0.1 \sin^2 t \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$
$$x_0^+ = \begin{bmatrix} 0.15 \\ 0.25 \end{bmatrix}, \quad x_0^- = \begin{bmatrix} 0.05 \\ 0.15 \end{bmatrix}.$$

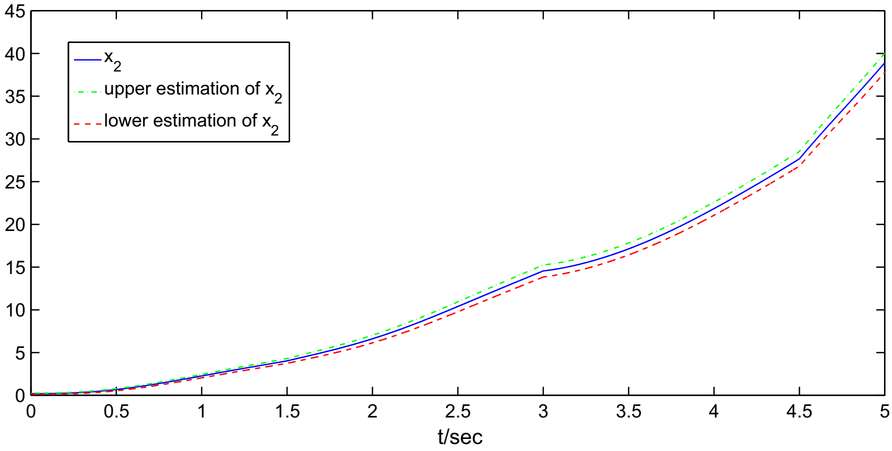

Fig. 8 Evolution of the real state $x_2(t)$ and the estimations $\hat{x}_2^+(t)$, $\hat{x}_2^-(t)$

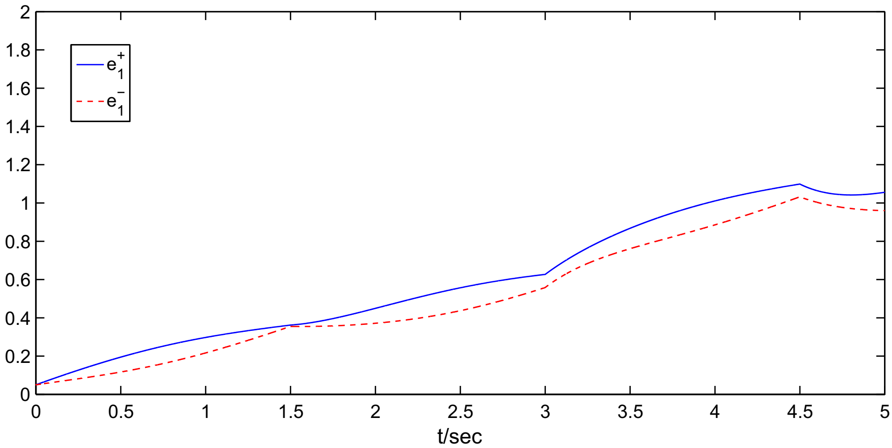

Fig. 9 Evolution of the errors e_1^+, e_1^-

Using Theorem 3, we get $\tau^* \geq 1.5$, $c_2 \leq 17.9602$ and $c_4 \leq 21.8319$. Figure 6 shows the time response of switching signal $\sigma(t)$. The simulation results of interval observer are shown in Figs. 7 and 8. The upper and lower errors are shown in Figs. 9 and 10. The upper and lower errors are in the bounded interval in 5 s.

5 Conclusions

In this study, the finite-time interval observers' design problem for switched systems is investigated, and the state equation suffers from disturbance. A finite-time interval observer is designed, and sufficient conditions are also given as LP forms. To the best of our knowledge, this is the first study on the design of finite-time interval observers for

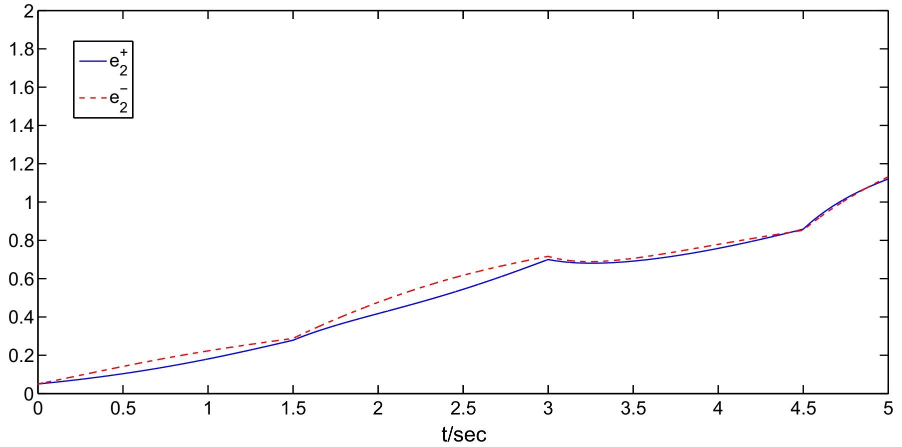

Fig. 10 Evolution of the errors e_2^+, e_2^-

switched systems with uncertainty using LP approach. From computation perspective, the LP conditions are more tractable than LMI conditions. In the future, we will develop a finite-time interval observers' design for nonlinear switched systems.

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