Functional Interval Observer for Discrete-Time Switched Descriptor Systems

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***Abstract*—This article investigates functional interval observer based estimation methods for discrete-time switched descriptor systems with disturbances. A functional observer is presented using the H_∞ formalism, and then a zonotope-based approach is applied to estimate its boundaries with a reduction of observer design constraints. To avoid the wrapping effect brought by the order reduction operations in the zonotope approach, reachability analysis is combined with the H_∞ technique to design a switched functional interval observer with a better performance. Moreover, the relationship between the existing method, zonotope approach, and reachability analysis method is discussed in detail. The aforementioned methods are applied to an example for comparisons and to show their performance.**

***Index Terms*—Functional interval observers (FIOs), reachability analysis, switched descriptor systems (SDSs), zonotopes.**

I. Introduction

Functional observers (FOs), unlike general observers, estimate the linear function of state because of actual requirements [1]. Compared with classical full-order observers, both the complexity and order of FOs are reduced. Since Luenberger proposed the concept of FO in 1966 [2], abundant achievements have been made in the design methods of FOs [3]–[6]. Considering the uncertainties in the process of design as well as the requirements of interval estimation for practical systems [7], Gu *et al.* combined FOs with interval observers (IOs) for the first time and presented a functional IO (FIO) design method for linear systems [8]. Che *et al.* in [9] further improved the work in [8] and designed an optimal FIO with more relaxed design conditions. A distributed FIO was constructed for interconnected systems in [10], which is a recent study on distributed FIOs.

Switched descriptor systems (SDSs) can represent a kind of dynamic systems with algebraic constraints. Up to now, many valuable achievements regarding the design of observers for SDSs have been made [11]–[15]. In the works of Koenig *et al.* and Lin *et al.*, a linear matrix inequality (LMI) technique was used in the observer design of discrete-time switched descriptor linear and nonlinear systems with unknown inputs, respectively [12], [13]. However, it should be pointed out that Koenig *et al.* [12] and Lin *et al.* [13] did not consider the effect of average dwell time (ADT) switching. In fact, ADT is an important performance index in the synthesis problem of switched systems. In respond to this problem, Hou *et al.* designed an observer for a class of SDSs with unknown input under an appropriate ADT switching [14]. Zhang *et al.* further improved the result of Hou *et al.* [14] and presented a systematical reduced-order observer design method for SDSs [15]. In view of the great practical value of FOs, there are some interesting results about FOs for SDSs [16], [17]. Nevertheless, as far as the authors' knowledge, switched FIOs (SFIOs) design for SDSs has not been fully considered in the previous literature.

Actually, the main problem in the design of IOs for switched systems is that the error systems are not easy to satisfy the stability and positivity conditions. In [18], under the assumption that the coefficient matrices of the error systems are Metzler, an IO was presented for a class of switched systems. The works in [19] and [20] applied coordinate transformation to improve the design conditions of IOs. Although the coordinate transformation method ensures the cooperativity (positivity) condition, it may cause a more conservative accuracy of IOs [21]. Recently, the optimization issue of IOs for switched systems has also been addressed in [22]–[24].

On the other hand, the combination of H_∞ techniques and set membership estimation (SME) is known as an effective design method for interval estimation. It is well known that the H_∞ approach is always employed to reduce the effect of disturbances [25], [26]. Meanwhile, SME techniques are useful to improve the estimation precision. In SME, the state vectors are assumed to belong to a feasible set, which can be characterized by proper geometrical shapes. Geometric bodies commonly include ellipsoids [27], ordinary polyhedrons [28], zonotopes [29], [30], etc. In view of less conservatism and computational burden, the zonotope method has been widely concerned and applied to state estimation [31]–[33]. However, its implementation relies on the order reduction operations, and the original zonotope is outer-approximated by a new zonotope with fixed dimensions. Such a procedure may lead to the additional conservatism because of the wrapping effect [34]. Nevertheless, interval estimation can be realized by calculating the approximation of the reachable set in a bounded time interval as it was proposed in the recent work [35]. This approach not only avoids the order reduction operations in the zonotope method, but also improves the estimation accuracy.

Thus, this article investigates the SFIO design problem for discrete-time SDSs (DSDSs). First, an optimal H_∞ FO with an ADT condition is constructed and the functional interval estimation of DSDSs is given by the zonotopic SME method. Then, to avoid the conservatism brought by the order reduction operations, the H_∞ FO is combined with reachability analysis to improve the accuracy of the designed SFIO. The contributions of this article lie in the following aspects.

1) For the first time, the SFIO design method is presented for DSDSs so that the application scope of IOs is expanded. The construction

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complexity of the proposed observer is reduced, since the observation object of SFIOs is more convenient than that of traditional IOs.

2) Compared with Ethabet *et al.* [20], the cooperativity conditions of the error dynamics are not required in this article, which relaxes the design constraints.
3) The conservatism caused by order reduction operations in the zonotope method is reduced by the reachability analysis technique, and the estimation accuracy is further improved.

The remainder of this article starts with some preliminaries in Section II. In Section III, an H_∞ FO is designed and the interval estimation of the linear function is realized by the zonotope-based and reachability analysis approaches, respectively. In Section IV, the relationship between the existing method and the proposed approaches is investigated from a theoretical point of view. An example is given in Section V to illustrate the performance of the aforementioned approaches. Finally, Section VI concludes this article.

Notation: The symbols $>$, $\geq$, $<$, and $\leq$ should be understood elementwise for vectors and matrices. If $A \in \mathbb{R}^{p\times q}$, $A^+ = \max\{0, A\}$, and $A^- = A^+ - A$. For a given matrix $P \in \mathbb{R}^{n\times n}$ with $P = P^T$, $P \prec 0 (\succ 0)$ means that P is negative (positive) definite. $\overline{\lambda}(P)$ and $\underline{\lambda}(P)$ represent the largest and the smallest real part of eigenvalues of P, respectively. $\oplus$ is the Minkowski sum, and $\odot$ represents a linear mapping.

II. Preliminaries

Consider the following DSDS:

$$\begin{cases} Ex(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) + D_{\sigma(k)}\omega(k) \\ \quad\quad y(k) = C_{\sigma(k)}x(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $y(k) \in \mathbb{R}^q$ are the state, control input, and output, respectively. $\omega(k) \in \mathbb{R}^p$ is an unknown but bounded disturbance satisfying $\underline{\omega}(k) \leq \omega(k) \leq \overline{\omega}(k)$, where $\underline{\omega}(k)$ and $\overline{\omega}(k)$ are the known boundaries of $\omega(k)$. $\sigma(k)$ is a discrete mapping taking values in a finite set $\mathbb{S} = \{1, 2, \ldots, N\}$. For any $\sigma(k) = i \in \mathbb{S}$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{q\times n}$, $D_i \in \mathbb{R}^{n\times p}$, and $E \in \mathbb{R}^{n\times n}$ are constant matrices, where $\text{rank}(E) = s \leq n$. To simplify symbols, k is omitted whenever necessary. We denote

$$v = \Gamma_i x, i \in \mathbb{S} \tag{2}$$

where $\Gamma_i \in \mathbb{R}^{\mu\times n}$ is a known matrix. This article aims at deriving a pair of bounds $\{\underline{v}, \overline{v}\}$ such that $\underline{v} \leq v \leq \overline{v} \ \ \forall k > 0$, and the initial condition satisfies $\underline{v}(0) \leq v(0) \leq \overline{v}(0)$.

Without loss of generality, it is assumed that for any $i \in \mathbb{S}$, rank $\begin{bmatrix} E \\ C_i \end{bmatrix} = n$, and rank $\begin{bmatrix} hE - A_i \\ C_i \end{bmatrix} = n \ \ \forall h \in \mathbb{C}, |h| \geq 1$, h is finite. Since rank $\begin{bmatrix} E \\ C_i \end{bmatrix} = n$, there exist a set of solutions of matrices $\Pi_i \in \mathbb{R}^{n\times n}$ and $\Sigma_i \in \mathbb{R}^{n\times q}$ such that

$$\Pi_i E + \Sigma_i C_i = I_n, i \in \mathbb{S}. \tag{3}$$

By using (3), system (1) is equivalent to

$$x(k+1) = \Pi_i A_i x(k) + \Pi_i B_i u(k) + \Pi_i D_i \omega(k) + \Sigma_i y(k+1). \tag{4}$$

Lemma 1 (see[36]): If rank $\begin{bmatrix} E \\ C_i \end{bmatrix} = n$, then the general solution of (3) is

$$\begin{bmatrix} \Pi_i & \Sigma_i \end{bmatrix} = \begin{bmatrix} E \\ C_i \end{bmatrix}^\dagger + S_i \left(I_{n+q} - \begin{bmatrix} E \\ C_i \end{bmatrix} \begin{bmatrix} E \\ C_i \end{bmatrix}^\dagger \right) \tag{5}$$

where $S_i \in \mathbb{R}^{n\times(n+q)}$ is an arbitrarily given matrix, and $\begin{bmatrix} E \\ C_i \end{bmatrix}^\dagger \in \mathbb{R}^{n\times(n+q)}$ is the pseudoinverse matrix of $\begin{bmatrix} E \\ C_i \end{bmatrix}$.

Remark 1: The rank condition rank $\begin{bmatrix} E \\ C_i \end{bmatrix} = n$ is one of the observability conditions of standard descriptor systems [11], [15]. In practice, such a necessary condition may be restrictive. However, even if this condition is satisfied, the design of FIOs for DSDSs with disturbances remains challenging. This fact motivates this work.

Remark 2: If E is in the standard form, i.e., $E = \begin{bmatrix} I_s & 0 \\ 0 & 0_{n-s} \end{bmatrix}$, the matrix S_i is chosen so that Π_i is full rank. To obtain good performances of the designed observer, the matrix composed of the first n columns of S_i in Lemma 1 is then suggested to be chosen as the identity matrix. If E is not in the standard form, according to Dai [11], there are two invertible matrices Q_1 and Q_2 such that $\overline{E} = Q_1 E Q_2 = \begin{bmatrix} I_s & 0 \\ 0 & 0_{n-s} \end{bmatrix}$. By taking the coordinate transformation $x(k) = Q_2 \widetilde{x}(k)$ and left multiplying system (1) by the matrix Q_1, we can obtain the corresponding system

$$\begin{cases} \overline{E}\widetilde{x}(k+1) = \overline{A}_i \widetilde{x}(k) + \overline{B}_i u(k) + \overline{D}_i \omega(k) \\ \quad\quad y(k) = \overline{C}_i \widetilde{x}(k) \end{cases}$$

where $\overline{E} = Q_1 E Q_2$, $\overline{A}_i = Q_1 A_i Q_2$, $\overline{B}_i = Q_1 B_i$, $\overline{D}_i = Q_1 D_i$, and $\overline{C}_i = C_i Q_2$. Then, we can use the same principle to choose the matrix S_i for this system.

Lemma 2 (see[37]): Suppose that v satisfies $\underline{v} \leq v \leq \overline{v}$, then for any given constant $\Psi \in \mathbb{R}^{n\times n}$, $\Psi^+ \underline{v} - \Psi^- \overline{v} \leq \Psi v \leq \Psi^+ \overline{v} - \Psi^- \underline{v}$.

Theorem 1: The following dynamical system

$$\begin{cases} \overline{x}(k+1) = \Pi_i A_i \overline{x}(k) + \Pi_i B_i u(k) + L_i^n (y(k) - C_i \overline{x}(k)) \\ \quad\quad + \Sigma_i y(k+1) + (\Pi_i D_i)^+ \overline{\omega}(k) - (\Pi_i D_i)^- \underline{\omega}(k) \\ \underline{x}(k+1) = \Pi_i A_i \underline{x}(k) + \Pi_i B_i u(k) + L_i^n (y(k) - C_i \underline{x}(k)) \\ \quad\quad + \Sigma_i y(k+1) + (\Pi_i D_i)^+ \underline{\omega}(k) - (\Pi_i D_i)^- \overline{\omega}(k) \end{cases} \tag{6}$$

is an IO for the switched system (4) if the inequalities

$$\begin{bmatrix} -P & * \\ P\Pi_i A_i - N_i C_i & -\frac{1}{\alpha_i} P \end{bmatrix} \prec 0 \tag{7}$$

$$\Pi_i A_i - L_i^n C_i \geq 0 \tag{8}$$

are satisfied where $P = P^T \in \mathbb{R}^{n\times n} \succ 0$, $L_i^n = P^{-1} N_i$, $\alpha_i = 1 + \frac{1}{\delta_i}$, and δ_i are positive scalars.

Proof: The proof of this theorem can be easily obtained based on the work in [20]. For brevity, it is omitted in this article. ■

Remark 3: Based on Lemma 2, an SFIO for the linear function (2) is constructed

$$\begin{cases} \overline{v} = \Gamma_i^+ \overline{x} - \Gamma_i^- \underline{x} \\ \underline{v} = \Gamma_i^+ \underline{x} - \Gamma_i^- \overline{x}. \end{cases} \tag{9}$$

In what follows, an H_∞ FO is designed as

$$\begin{cases} \hat{x}(k+1) = & \Pi_i A_i \hat{x}(k) + \Pi_i B_i u(k) + L_i^o (y(k) - C_i \hat{x}(k)) \\ & + \Sigma_i y(k+1) \\ \hat{v}(k) = & \Gamma_i \hat{x}(k) \end{cases} \tag{10}$$

where $\hat{x} \in \mathbb{R}^n$ and $\hat{v} \in \mathbb{R}^\mu$ are the estimations of x and v, and $L_i^o \in \mathbb{R}^{n\times q}$ is the observer gain. The state estimation error is obtained as

$$\begin{aligned} e(k+1) &= x(k+1) - \hat{x}(k+1) \\ &= (\Pi_i A_i - L_i^o C_i)e + \Pi_i D_i \omega \\ &= \tilde{A}_i e + \Pi_i D_i \omega \end{aligned} \tag{11}$$

where $\tilde{A}_i = \Pi_i A_i - L_i^o C_i$, and the estimation error is formulated as

$$\begin{aligned} \epsilon(k+1) &= v(k+1) - \hat{v}(k+1) \\ &= \Gamma_i \tilde{A}_i e + \Gamma_i \Pi_i D_i \omega. \end{aligned} \tag{12}$$

Some definitions and properties of the zonotope are reviewed now.

Definition 1: A zonotope $\mathscr{S} \in \mathbb{R}^\mu$ is denoted as

$$\mathscr{S} = \Xi \oplus G\mathbb{B}^\iota = \{\Xi + Gz, z \in \mathbb{B}^\iota\}$$

where $\Xi \in \mathbb{R}^\mu$ is its center vector, $G \in \mathbb{R}^{\mu\times\iota}$ is its generator matrix, and $\mathbb{B}^\iota = [-1,1]^\iota$ is a hypercube with ι unitary intervals. For simplicity, the zonotope $\mathscr{S}$ is abbreviated as $\langle \Xi, G\rangle$.

Lemma 3 (see[30]): For a zonotope $\mathscr{S} = \langle \Xi, G\rangle$, we have

$$1)\ \langle \Xi_1, G_1\rangle \oplus \langle \Xi_2, G_2\rangle = \langle \Xi_1 + \Xi_2, [G_1\ G_2]\rangle$$

$$2)\ \Phi \odot \langle \Xi, G\rangle = \langle \Phi\Xi, \Phi G\rangle$$

$$3)\ \langle \Xi, G\rangle \subseteq \langle \Xi, \bar{G}\rangle$$

where $\Xi, \Xi_1, \Xi_2 \in \mathbb{R}^\mu$, $G, G_1, G_2 \in \mathbb{R}^{\mu\times\iota}$, and $\Phi \in \mathbb{R}^{l\times\mu}$. The matrix $\bar{G} \in \mathbb{R}^{\mu\times\mu}$ is diagonal, where $\bar{G}_{r,r} = \sum_{s=1}^{\iota} |G_{r,s}|, r = 1,\ldots,\mu$ stands for the rth row, and $s = 1,\ldots,\iota$ represents the sth column.

Remark 4: For a given vector $x_0 \in \mathbb{R}^n$ and matrices $M_0 \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, the bounded initial state and the disturbances can be enclosed by the following zonotopes:

$$\begin{aligned} & x \in \chi = \langle x_0, M_0\rangle \\ & v = \Gamma_i x \in \Gamma_i \odot \chi = \Gamma_i \langle x_0, M_0\rangle \\ & \omega \in \Omega = \langle 0, W\rangle. \end{aligned}$$

In the estimation process of the zonotope method, the increasing dimension of the zonotopes may result in a growing computation burden. A solution was presented in [29] as shown below.

Lemma 4 (see[29]): A zonotope $\mathscr{S} = \Xi \oplus G\mathbb{B}^\iota \subset \mathbb{R}^\mu$ can be included in

$$\mathscr{S} = \langle \Xi, G\rangle \subseteq \left\langle \Xi, \hat{G}\right\rangle = \Xi \oplus \hat{G}\mathbb{B}^q (\mu \le q \le \iota)$$

where $\hat{G} = [G_z \quad Q]$, and G_z is made up of the first $q-\mu$ columns of $\widetilde{G}$. The matrix Q is in diagonal form, where

$$Q_{r,r} = \sum_{s=q-\mu+1}^{\iota} |\widetilde{G}_{r,s}|, r = 1,2,\ldots,\mu$$

and $\widetilde{G}$ is formed by sorting the Euclidean norm of the columns of G in a decreasing order.

The following basics are essential to reachability analysis.

Definition 2: An interval vector $I \subset \mathbb{R}^\mu$ is defined as $I = \{x : x \in \mathbb{R}^\mu, \alpha_r \le x_r \le \beta_r, r = 1,\ldots,\mu\}$. For simplicity, the interval vector I is simplified as $I = [\alpha, \beta]$, where $\alpha = [\alpha_1,\ldots,\alpha_\mu]^T$, and $\beta = [\beta_1,\ldots,\beta_\mu]^T$.

Property 1: Given interval vectors $[\alpha,\beta] \subset \mathbb{R}^\mu$, $[\eta,\rho] \subset \mathbb{R}^\mu$, then, $[\alpha,\beta] \oplus [\eta,\rho] = [\alpha+\eta, \beta+\rho]$. For a set of interval vectors, $\overset{m}{\underset{i=1}{\oplus}} I_i = I_1 \oplus I_2 \cdots \oplus I_m$.

Definition 3: Given a set $Z \subset \mathbb{R}^\mu$, its interval hull is defined as

$$\text{Hull}(Z) = [\alpha, \beta]$$

where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_\mu]^T$, and $\beta = [\beta_1, \beta_2, \ldots, \beta_\mu]^T$. For any vector $z_r \in Z$, $\alpha_r \le z_r \le \beta_r, r = 1,\ldots,\mu$.

Property 2: For the sets $Z_i \subset \mathbb{R}^\mu, i = 1,\ldots,m$, $\text{Hull}(\overset{m}{\underset{i=1}{\oplus}} Z_i) = \overset{m}{\underset{i=1}{\oplus}} \text{Hull}(Z_i)$.

Lemma 5 (see[35]): Given an interval vector $I = [\alpha,\beta] \subset \mathbb{R}^\mu$ and a matrix $\Psi \in \mathbb{R}^{m\times\mu}$, then $\text{Hull}(\Psi I) = [\Psi^+\alpha - \Psi^-\beta, \Psi^+\beta - \Psi^-\alpha]$.

Lemma 6 (see[30]): Given a zonotope $\mathscr{S} = \Xi \oplus G\mathbb{B}^\iota \subset \mathbb{R}^\mu$. Its interval hull is given by

$$\text{Hull}(\mathscr{S}) = [\alpha, \beta]$$

where $\alpha_r = \Xi_r - \sum_{s=1}^{\iota} |G_{r,s}|$ and $\beta_r = \Xi_r + \sum_{s=1}^{\iota} |G_{r,s}|$.

III. Main Result

A. Zonotope-Based SFIO Design

To reduce the design constraints and obtain better estimation accuracy, the H_∞ technique is combined with the zonotopic SME method to design an SFIO for system (1).

Theorem 2: Given two positive scalars $0 < \xi < 1$ and $\mu > 1$, if there exist a positive constant γ, matrices $P_i = P_i^T, P_j = P_j^T \in \mathbb{R}^{n\times n} \succ 0$, and $R_i \in \mathbb{R}^{n\times q}$ such that for all $(\sigma(k_n) = i, \sigma(k_n - 1) = j) \in \mathbb{S}$, $i \ne j$

$$\begin{bmatrix} -\xi P_i + \Gamma_i^T \Gamma_i & * & * \\ \Theta_i & D_i^T \Pi_i^T P_i \Pi_i D_i - \gamma I & * \\ P_i \Pi_i A_i - R_i C_i & 0 & -P_i \end{bmatrix} \prec 0 \tag{13}$$

$$P_i \prec \mu P_j \tag{14}$$

with an ADT satisfying

$$\frac{1}{\tau^*} + \frac{\ln\xi}{\ln\mu} \le 0 \tag{15}$$

where $\Theta_i = D_i^T \Pi_i^T P_i \Pi_i A_i - D_i^T \Pi_i^T R_i C_i$, then (10) is an H_∞ FO for system (1) with $L_i^o = (P_i)^{-1} R_i$. Furthermore, the optimal H_∞ FO is formulated by

$$\min \gamma \text{ s.t. (13)–(15)}. \tag{16}$$

Proof: Based on the definition of the H_∞ performance (refer for instance to Xiang *et al.* [38]), the proof is divided into following two steps.

(i) $\omega = 0$, consider the following Lyapunov candidate $V_i = e^T P_i e, i \in \mathbb{S}$. For $k \in [k_n, k_{n+1})$, and take the forward difference of $V_i(k)$

$$\Delta V_i(k) = V_i(k+1) - V_i(k) = e^T (\tilde{A}_i{}^T P_i \tilde{A}_i - P_i) e. \tag{17}$$

For each mode, the error dynamic equation (12) is asymptotically stable when $\omega = 0$ via

$$\Delta V_i(k) < (\xi - 1)e^T P_i e - e^T \Gamma_i^T \Gamma_i e < (\xi - 1)e^T P_i e$$

which means

$$V_i(k+1) < \xi V_i(k) \tag{18}$$

and the sufficient condition for the inequality is given by $\tilde{A}_i{}^T P_i \tilde{A}_i - \xi P_i + \Gamma_i^T \Gamma_i \prec 0$. For the interval $[k_n, k)$, repeating (18), it is easy to get that

$$V_i(k) < \xi^{k-k_n} V_i(k_n). \tag{19}$$

Since $\sigma(k_n - 1) = j$, combining with the inequality (14), we have

$$V_i(k) < \mu \xi^{k-k_n+1} V_j(k_n - 1). \tag{20}$$

Repeating (19) and (20), the following inequality is obtained:

$$V_i(k) < \mu^{N_{\sigma(0,k)}} \xi^k V_{\sigma(0)}(0) \tag{21}$$

where $N_{\sigma(0,k)}$ is the switching number of the signal σ on $[0, k)$. By the definition of ADT (see [38, Definition 1]), we have $N_{\sigma(0,k)} \leq \frac{k}{\tau^*}$. Then, from (21), it yields

$$\begin{aligned} V_i(k) &< \mu^{\frac{k}{\tau^*}} \xi^k V_{\sigma(0)}(0) < \mu^{\frac{k}{\tau^*}} \mu^{\frac{k\ln\xi}{\ln\mu}} V_{\sigma(0)}(0) \\ &< \mu^{k(\frac{1}{\tau^*} + \frac{\ln\xi}{\ln\mu})} V_{\sigma(0)}(0). \end{aligned}$$

Since $\tilde{A}_i{}^T P_i \tilde{A}_i - \xi P_i + \Gamma_i^T \Gamma_i \prec 0$, we have $-\xi P_i + \Gamma_i^T \Gamma_i \prec 0$. According to $0 < \xi < 1$, it follows that $\Gamma_i^T \Gamma_i \prec P_i$. Therefore

$$\epsilon^T \epsilon = e^T \Gamma_i^T \Gamma_i e < V_i(k) < \mu^{k(\frac{1}{\tau^*} + \frac{\ln\xi}{\ln\mu})} V_{\sigma(0)}(0).$$

Since $V_{\sigma(0)}(0) \leq \overline{\lambda}(P_{\sigma(0)})||e(0)||^2$ and $\mu > 1$, then, from (15), it is deduced that

$$\lim_{k\to+\infty} ||\epsilon(k)|| = \lim_{k\to+\infty} \sqrt{\overline{\lambda}(P_{\sigma(0)}) \mu^{k(\frac{1}{\tau^*} + \frac{\ln\xi}{\ln\mu})}} ||e(0)|| = 0.$$

Thus, the error (12) is asymptotically stable when $\omega = 0$.

(ii) $\omega \neq 0$, consider $J = \sum_{k=0}^{\infty} (\epsilon^T \epsilon - \gamma \omega^T \omega)$. Since $V_{\sigma(0)}(0) = 0$, and one can obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \Delta V_i(k) &= \lim_{k\to+\infty} [V_i(k+1) - V_i(k) + V_i(k) \cdots V_{\sigma(0)}(0)] \\ &= \lim_{k\to+\infty} V_i(k+1) \geq 0 \end{aligned}$$

and thus

$$J \leq \sum_{k=0}^{\infty} [\epsilon^T \epsilon - \gamma \omega^T \omega + \Delta V_i]. \tag{22}$$

Denote that $\mathscr{Q}(k) = \epsilon^T \epsilon - \gamma \omega^T \omega + \Delta V_i$, from the error equation (12), we have

$$\begin{aligned} \mathscr{Q}(k) = e^T [\tilde{A}_i^T P_i \tilde{A}_i - P_i + \Gamma_i^T \Gamma_i] e + 2\omega^T D_i^T \Pi_i^T P_i \\ \tilde{A}_i e + \omega^T (D_i^T \Pi_i^T P_i \Pi_i D_i - \gamma I)\omega. \end{aligned} \tag{23}$$

The error dynamic equation (12) satisfies the H_∞ performance when $\omega \neq 0$ via

$$\begin{aligned} &e^T [\tilde{A}_i{}^T P_i \tilde{A}_i - P_i + \Gamma_i^T \Gamma_i] e + 2\omega^T D_i^T \Pi_i^T P_i \tilde{A}_i e \\ &+ \omega^T (D_i^T \Pi_i^T P_i \Pi_i D_i - \gamma I)\omega < (\xi - 1) e^T P_i e \end{aligned}$$

and a sufficient condition is given by the matrix inequality.

$$\begin{bmatrix} \tilde{A}_i{}^T P_i \tilde{A}_i - \xi P_i + \Gamma_i^T \Gamma_i & * \\ D_i^T \Pi_i^T P_i \tilde{A}_i & D_i^T \Pi_i^T P_i \Pi_i D_i - \gamma I \end{bmatrix} \prec 0. \tag{24}$$

Using (23) and the fact that $0 < \xi < 1$, one can get

$$\mathscr{Q}(k) < (\xi - 1) e^T P_i e < 0. \tag{25}$$

By (22) and (25), it is obvious that

$$J \leq \sum_{k=0}^{\infty} \mathscr{Q}(k) < 0$$

that is

$$\sum_{k=0}^{\infty} \epsilon^T \epsilon < \gamma \sum_{k=0}^{\infty} \omega^T \omega.$$

In the light of aforementioned discussion, it can be concluded that (10) is an H_∞ FO of system (1). Since (24) is not a standard LMI, the solution of (24) cannot be found by the LMI toolbox in MATLAB. Thus, it is essential to transform (24) into the following LMI form:

$$\begin{bmatrix} -\xi P_i + \Gamma_i^T \Gamma_i & * & * \\ \Theta_i & D_i^T \Pi_i^T P_i \Pi_i D_i - \gamma I & * \\ P_i \Pi_i A_i - R_i C_i & 0 & -P_i \end{bmatrix} \prec 0.$$

The result can be easily obtained by the Schur complement lemma and the proof is omitted here. Moreover, if the optimal problem (16) is solvable, then the H_∞ FO is optimal. ■

Given the H_∞ observer (10), a zonotopic SME method is now extended to the design of SFIO.

Theorem 3: Suppose $\omega \in \Omega = \langle 0, W \rangle$ and assume that the conditions of Theorem 2 are satisfied. The linear function $v = \Gamma_i x$ belongs to a zonotope given by

$$\hat{\mathscr{V}}(k) = \langle \hat{v}(k), \Gamma_i M(k) \rangle$$

where $M(k)$ satisfies

$$M(k+1) = [\tilde{A}_i \hat{M}(k) \quad \Pi_i D_i W]$$

and $\hat{M}(k)$ is given in Lemma 4.

Proof: From (12), it can be concluded that $v = \hat{v} + \Gamma_i e$. According to Remark 4, the error (11) belongs to the zonotope $\hat{\zeta}^z = \langle 0, M \rangle$. Then, we have $\epsilon \in \Gamma_i \odot \hat{\zeta}^z = \Gamma_i \odot \langle 0, M \rangle = \langle 0, \Gamma_i M \rangle$ and $v \in \hat{\mathscr{V}} = \langle \hat{v}, \Gamma_i M \rangle$. Consider the error (11), then it can be enclosed by the following zonotope:

$$\begin{aligned} e(k+1) \in \zeta^z(k+1) &= \tilde{A}_i \odot \langle 0, M(k) \rangle \oplus \Pi_i D_i \odot \langle 0, W \rangle \\ &= \langle 0, [\tilde{A}_i M(k) \; \Pi_i D_i W] \rangle. \end{aligned}$$

By using Lemma 4, we have $e \in \langle 0, M \rangle \subseteq \langle 0, \hat{M} \rangle$, and

$$\begin{aligned} e(k+1) \in \zeta^z(k+1) &\subseteq \hat{\zeta}^z(k+1) \\ &= \tilde{A}_i \odot \langle 0, \hat{M}(k) \rangle \oplus \Pi_i D_i \odot \langle 0, W \rangle \\ &= \langle 0, [\tilde{A}_i \hat{M}(k) \quad \Pi_i D_i W] \rangle. \end{aligned}$$

Define $\hat{\zeta}^z(k+1) = \langle 0, M(k+1) \rangle$, we have

$$M(k+1) = [\tilde{A}_i \hat{M}(k) \quad \Pi_i D_i W]$$

which completes the proof. ■

Algorithm 1: Algorithm for SFIO by the Zonotopic SME Method.

Input: $\hat{x}(0), M(0), W, \Gamma_i$

Output: $\underline{v}^z, \overline{v}^z$

Given initial values:

$\hat{x}(0) = x_0,\ e(0) \in \hat{\zeta}^z(0) = \langle 0, M(0)\rangle,\ \omega \in \Omega = \langle 0, W\rangle$;

for $k \geq 0$ **do**

$\hat{x}(k+1) = \Pi_i A_i \hat{x}(k) + \Pi_i B_i u(k) + L_i^o(y(k) - C_i\hat{x}(k)) + \Sigma_i y(k+1)$

$\hat{v} = \Gamma_i \hat{x}$

$v \in \mathscr{V} = \langle \hat{v}, \Gamma_i M\rangle$

$e(k+1) \in \tilde{A}_i \odot \langle 0, M(k)\rangle \oplus \Pi_i D_i \odot \langle 0, W\rangle = \langle 0, [\tilde{A}_i M(k) \quad \Pi_i D_i W]\rangle$

$e \in \langle 0, M\rangle \subseteq \langle 0, \hat{M}\rangle$

$e(k+1) \in \hat{\zeta}^z(k+1) = \langle 0, M(k+1)\rangle = \langle 0, [\tilde{A}_i \hat{M}(k) \quad \Pi_i D_i W]\rangle$

$M(k+1) = [\tilde{A}_i \hat{M}(k) \quad \Pi_i D_i W]$

$\langle \hat{v}, H\rangle = \hat{\mathscr{V}} = \langle \hat{v}, \Gamma_i M\rangle$

$\overline{v}_r^z = \hat{v}_r + \sum_{s=1}^{\iota} |H_{r,s}|, \underline{v}_r^z = \hat{v}_r - \sum_{s=1}^{\iota} |H_{r,s}|$;

end for

Algorithm 2: Algorithm for SFIO by Reachability Analysis.

Input: $\hat{x}(0), M(0), W, \Gamma_i$

Output: $\underline{v}^h, \overline{v}^h$

Given initial values:

$\hat{x}(0) = x_0, e(0) \in \hat{\zeta}^z(0) = \langle 0, M(0)\rangle, \omega \in \Omega = \langle 0, W\rangle$;

for $k \geq 0$ **do**

$\hat{x}(k+1) = \Pi_i A_i \hat{x}(k) + \Pi_i B_i u(k) + L_i^o(y(k) - C_i\hat{x}(k)) + \Sigma_i y(k+1)$

$\hat{v} = \Gamma_i \hat{x}$

$\epsilon(k) = \Gamma_i[\tilde{A}_i^k e(0) + \sum_{l=0}^{k-1} \tilde{A}_i^l \Pi_i D_i w(k-1-l)]$

$\hat{\zeta}^h(0) = \langle 0, \Gamma_i M(0)\rangle$

$\epsilon(0) \in \hat{\zeta}^h(0) \subseteq [\underline{\epsilon}^h(0), \overline{\epsilon}^h(0)] = Hull(\langle 0, \Gamma_i M(0)\rangle)$

$\epsilon(k) \in \hat{\zeta}^h(k) = \Gamma_i \tilde{A}_i^k \hat{\zeta}^z(0) \oplus \bigoplus_{l=0}^{k-1} \Gamma_i \tilde{A}_i^l \Pi_i D_i \Omega$

$[\underline{\epsilon}^h, \overline{\epsilon}^h] = Hull(\hat{\zeta}^h) = Hull(\Gamma_i \tilde{A}_i^k \hat{\zeta}^z(0)) \oplus \bigoplus_{l=0}^{k-1} Hull(\Gamma_i \tilde{A}_i^l \Pi_i D_i \Omega)$

$\overline{v}^h = \hat{v} + \overline{\epsilon}^h, \underline{v}^h = \hat{v} + \underline{\epsilon}^h$;

end for

Remark 5: Define $\langle \hat{v}, H\rangle = \hat{\mathscr{V}} = \langle \hat{v}, \Gamma_i M\rangle$. Based on Lemma 3, $\langle \hat{v}, H\rangle \subseteq \langle \hat{v}, \bar{H}\rangle$, the SFIO is described by

$$\begin{cases} \overline{v}_r^z = \hat{v}_r + \sum_{s=1}^{\iota} |H_{r,s}| \\ \underline{v}_r^z = \hat{v}_r - \sum_{s=1}^{\iota} |H_{r,s}| \end{cases} \tag{26}$$

where $r = 1, \ldots, \mu$ is the rth row, and $s = 1, \ldots, \iota$ is the sth column.

Thus, Theorem 3 can be implemented by Algorithm 1.

B. SFIO Design by Reachability Analysis

To avoid the conservatism brought by the order reduction operations in Algorithm 1, reachability analysis is applied to the design of SFIOs for DSDSs. The H_∞ FO (10) is still considered. Given the error (12), if there exists an interval vector $[\underline{\epsilon}^h, \overline{\epsilon}^h]$ satisfying $\underline{\epsilon}^h \leq \epsilon \leq \overline{\epsilon}^h$, then we obtain

$$\hat{v} + \underline{\epsilon}^h \leq v \leq \hat{v} + \overline{\epsilon}^h.$$

The SFIO structure obtained by the reachability analysis method has the following form:

$$\begin{cases} \overline{v}^h = \hat{v} + \overline{\epsilon}^h \\ \underline{v}^h = \hat{v} + \underline{\epsilon}^h. \end{cases} \tag{27}$$

Remark 6: For the form of observer (26), the zonotope $\langle \hat{v}, H\rangle$ is converted to the interval vector $[\underline{v}^z, \overline{v}^z]$, that is

$$[\underline{v}^z, \overline{v}^z] = Hull\left(\langle \hat{v}, H\rangle\right)$$

where $\overline{v}_r{}^z = \hat{v}_r + \sum_{s=1}^{\iota} |H_{r,s}|$ and $\underline{v}_r{}^z = \hat{v}_r - \sum_{s=1}^{\iota} |H_{r,s}|$. Thus, the estimation (26) can be rewritten as

$$\begin{cases} \overline{v}^z = \hat{v} + \overline{\epsilon}^z \\ \underline{v}^z = \hat{v} + \underline{\epsilon}^z \end{cases} \tag{28}$$

where $\underline{\epsilon}^z \leq \epsilon \leq \overline{\epsilon}^z$. Define $\hat{\zeta}^{\mathscr{Z}} = \Gamma_i \hat{\zeta}^z$, and $[\underline{\epsilon}^z, \overline{\epsilon}^z] = Hull(\hat{\zeta}^{\mathscr{Z}}) = Hull(\langle 0, H\rangle)$.

Theorem 4: Consider system (1) with the observer (10) and assume that the conditions of Theorem 2 are satisfied. The linear function of the state variable v belongs to the interval vector $[\underline{v}^h, \overline{v}^h]$ given by (27), where $[\underline{\epsilon}^h, \overline{\epsilon}^h]$ is determined by

$$[\underline{\epsilon}^h, \overline{\epsilon}^h] = \text{Hull}\left(\Gamma_i \tilde{A}_i^k \langle 0, M(0)\rangle\right) \oplus \bigoplus_{l=0}^{k-1} \text{Hull}\left(\Gamma_i \tilde{A}_i^l \Pi_i D_i \langle 0, W\rangle\right)$$

with the initial condition $[\underline{\epsilon}^h(0), \overline{\epsilon}^h(0)] = \text{Hull}(\langle 0, \Gamma_i M(0)\rangle)$.

Proof: The error (12) is equivalent to

$$\epsilon(k) = \Gamma_i \left[\tilde{A}_i^k e(0) + \sum_{l=0}^{k-1} \tilde{A}_i^l \Pi_i D_i w(k-1-l)\right]. \tag{29}$$

From Remark 4, the initial value of the error (11) and the disturbances of system (1) can be described by

$$e(0) \in \hat{\zeta}^z(0) = \langle 0, M(0)\rangle \tag{30}$$

$$\omega(k) \in \Omega = \langle 0, W\rangle. \tag{31}$$

For $\epsilon(0) = \hat{v}(0) - v(0) = \Gamma_i e(0)$, we have

$$\epsilon(0) \in \hat{\zeta}^h(0) = \Gamma_i \hat{\zeta}^z(0) = \langle 0, \Gamma_i M(0)\rangle.$$

Thus, the interval hull of $\hat{\zeta}^h(0)$ can be obtained by

$$\hat{\zeta}^h(0) \subseteq [\underline{\epsilon}^h(0), \overline{\epsilon}^h(0)] = \text{Hull}\left(\langle 0, \Gamma_i M(0)\rangle\right).$$

By substituting (30) and (31) into (29), the error (12) is enclosed by the following reachable set:

$$\epsilon(k) \in \hat{\zeta}^h(k) = \Gamma_i \tilde{A}_i^k \hat{\zeta}^z(0) \oplus \bigoplus_{l=0}^{k-1} \Gamma_i \tilde{A}_i^l \Pi_i D_i \langle 0, W\rangle. \tag{32}$$

From (32) and Property 2, the interval hull of $\hat{\zeta}^h$ can be obtained by

$$\text{Hull}\left(\hat{\zeta}^h\right) = \text{Hull}\left(\Gamma_i \tilde{A}_i^k \hat{\zeta}^z(0)\right) \oplus \bigoplus_{l=0}^{k-1} \text{Hull}\left(\Gamma_i \tilde{A}_i^l \Pi_i D_i \Omega\right). \tag{33}$$

Since $\epsilon \in \hat{\zeta}^h \subseteq \text{Hull}(\hat{\zeta}^h)$, we get $[\underline{\epsilon}^h, \overline{\epsilon}^h] = \text{Hull}(\hat{\zeta}^h)$. ■

Theorem 4 can be implemented by Algorithm 2.

IV. Comparisons Analysis

In this section, the reachability analysis method is compared with the zonotope technique and with the approach given in [20] based on monotone systems theory.

A. Comparison Between the Reachability Analysis and Monotone Systems Theory

Theorem 5: For the observers (9) and (27), with the same observer gains $L_i^n = L_i^o = L_i$, the initial conditions $\underline{v}(0) = \underline{v}^h(0)$ and $\overline{v}(0) = \overline{v}^h(0)$, the following inequalities hold:

$$\begin{cases} \underline{v} \le \underline{v}^h \\ \overline{v} \ge \overline{v}^h \end{cases}$$

where $\underline{v}$, $\overline{v}$, $\underline{v}^h$, and $\overline{v}^h$ are generated by the observers (9) and (27).

Proof: Consider $\overline{x} = \hat{x} + \overline{e}$ and $\underline{x} = \hat{x} + \underline{e}$, and we have

$$\begin{cases} \overline{e}(k) = \hat{A}_i^k \overline{e}(0) + \sum_{l=0}^{k-1} \hat{A}_i^l d^+ \\ \underline{e}(k) = \hat{A}_i^k \underline{e}(0) + \sum_{l=0}^{k-1} \hat{A}_i^l d^- \end{cases}$$

where $\hat{A}_i = \Pi_i A_i - L_i C_i$, $d^+ = (\Pi_i D_i)^+ \overline{\omega} - (\Pi_i D_i)^- \underline{\omega}$ and $d^- = (\Pi_i D_i)^+ \underline{\omega} - (\Pi_i D_i)^- \overline{\omega}$. Since $L_i^n = L_i^o = L_i$, then $\tilde{A}_i = \hat{A}_i$. According to Lemma 5 and $\omega \in \Omega = \langle 0, W \rangle$, one can get $\mathrm{Hull}(\Pi_i D_i \Omega) = [d^-, d^+]$. Since $\overline{v}(0) = \overline{v}^h(0)$ and $\underline{v}(0) = \underline{v}^h(0)$, we define $[\underline{e}(0), \overline{e}(0)] = \mathrm{Hull}(\hat{\zeta}^z(0))$. In conclusion, the following interval vector can be obtained:

$$[\underline{e}, \overline{e}] = \widetilde{\zeta}(k) = \hat{A}_i^k \mathrm{Hull}(\hat{\zeta}^z(0)) \oplus \bigoplus_{l=0}^{k-1} \hat{A}_i^l \mathrm{Hull}(\Pi_i D_i \Omega).$$

Similarly, we define

$$\begin{cases} \overline{v} = \hat{v} + \overline{\epsilon} \\ \underline{v} = \hat{v} + \underline{\epsilon}. \end{cases} \tag{34}$$

Under the derivation of (34), it yields

$$\begin{cases} \overline{\epsilon} = \Gamma_i^+ \overline{e} - \Gamma_i^- \underline{e} \\ \underline{\epsilon} = \Gamma_i^+ \underline{e} - \Gamma_i^- \overline{e}. \end{cases} \tag{35}$$

By Lemma 5 and (35), we have

$$\begin{aligned} [\underline{\epsilon}, \overline{\epsilon}] = \mathrm{Hull}(\Gamma_i \widetilde{\zeta}) = \mathrm{Hull}\Big(& \Gamma_i \hat{A}_i^k \mathrm{Hull}(\hat{\zeta}^z(0)) \\ & \oplus \bigoplus_{l=0}^{k-1} \Gamma_i \hat{A}_i^l \mathrm{Hull}(\Pi_i D_i \Omega) \Big). \end{aligned} \tag{36}$$

Comparing (36) with (33), then $[\underline{\epsilon}^h, \overline{\epsilon}^h] = \mathrm{Hull}(\hat{\zeta}^h) \subseteq \mathrm{Hull}(\Gamma_i \widetilde{\zeta})$, i.e., $\underline{\epsilon} \le \underline{\epsilon}^h$ and $\overline{\epsilon} \ge \overline{\epsilon}^h$. By (27) and (34), we can conclude that $\underline{v} \le \underline{v}^h$ and $\overline{v} \ge \overline{v}^h$. ∎

B. Comparison Between the Reachability Analysis and the Zonotope Method

Theorem 6: For the observers (26) and (27), with the same initial conditions $\underline{v}^h(0) = \underline{v}^z(0)$ and $\overline{v}^h(0) = \overline{v}^z(0)$, the following inequalities hold:

$$\begin{cases} \overline{v}^z \ge \overline{v}^h \\ \underline{v}^z \le \underline{v}^h \end{cases}$$

where $\overline{v}^h$, $\underline{v}^h$, $\overline{v}^z$, and $\underline{v}^z$ are generated by (26) and (27).

Proof: From Remark 6, the observer (26) is equivalent to the observer (28). Therefore, the observer (28) is compared with the observer (27) in the sequel. By Algorithm 1, it follows from Theorem 3 that

$$\begin{aligned} \hat{\zeta}^z(k+1) = \langle 0, M(k+1) \rangle = \tilde{A}_i \odot \langle 0, \hat{M}(k) \rangle \\ \oplus \Pi_i D_i \odot \langle 0, W \rangle. \end{aligned} \tag{37}$$

Since $\langle 0, M(k) \rangle \subseteq \langle 0, \hat{M}(k) \rangle$, we obtain

$$\hat{\zeta}^z(k+1) \supseteq \tilde{A}_i \odot \langle 0, M(k) \rangle \oplus \Pi_i D_i \odot \langle 0, W \rangle. \tag{38}$$

Repeating (37) and (38), it follows that

$$\hat{\zeta}^z(k) \supseteq \tilde{A}_i^k \hat{\zeta}^z(0) \oplus \bigoplus_{l=0}^{k-1} \tilde{A}_i^l \Pi_i D_i \langle 0, W \rangle$$

and

$$\mathrm{Hull}\left(\Gamma_i \hat{\zeta}^z\right) \supseteq \mathrm{Hull}\left(\Gamma_i \tilde{A}_i^k \hat{\zeta}^z(0)\right) \oplus \bigoplus_{l=0}^{k-1} \mathrm{Hull}\left(\Gamma_i \tilde{A}_i^l \Pi_i D_i \Omega\right). \tag{39}$$

Based on (33) and (39), we have

$$\mathrm{Hull}(\Gamma_i \hat{\zeta}^z) = \mathrm{Hull}(\hat{\zeta}^{\mathscr{Z}}) \supseteq \mathrm{Hull}(\hat{\zeta}^h). \tag{40}$$

According to Remark 6 and Theorem 4, $\mathrm{Hull}(\hat{\zeta}^{\mathscr{Z}}) = [\underline{\epsilon}^z, \overline{\epsilon}^z]$ and $\mathrm{Hull}(\hat{\zeta}^h) = [\underline{\epsilon}^h, \overline{\epsilon}^h]$, and (40) implies $\overline{\epsilon}^z \ge \overline{\epsilon}^h$ and $\underline{\epsilon}^z \le \underline{\epsilon}^h$. By (27) and (28), we can obtain $\overline{v}^z \ge \overline{v}^h$ and $\underline{v}^z \le \underline{v}^h$. ∎

Remark 7: From a theoretical point of view, Theorems 5 and 6 show that the bounds estimated by the proposed method are better than those by the method in [20] and the zonotope method. For the purpose of quantitative statement, we will employ a simulation example to illustrate it. Refer to Table I in Section V.

TABLE I
NUMERICAL COMPARISONS

	Method in [20]	Algorithm 1	Algorithm 2
$\overline{v} - \underline{v}$	0.1743	0.0988	2×10^{-4}
Design conditions	$\Pi_i A_i - L_i^n C_i \ge 0$	–	–
Algorithm steps	–	12	11

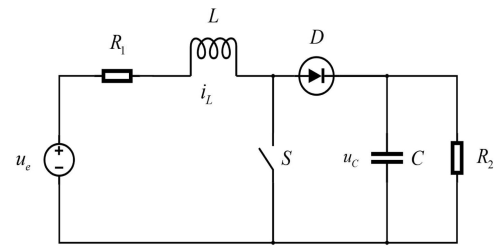

Fig. 1. Diagram of circuit system.

V. Simulation

Consider the circuit system depicted in Fig. 1.

The dynamics of the plant are described by

$$\begin{cases} \dfrac{di_L(t)}{dt} = -\dfrac{R_1}{L} i_L(t) - (1 - S(t)) \dfrac{1}{L} u_c(t) + \dfrac{1}{L} u_e(t) \\ \dfrac{du_c(t)}{dt} = (1 - S(t)) \dfrac{1}{C} i_L(t) - \dfrac{1}{R_2 C} u_c(t) \end{cases} \tag{41}$$

where $S(t) = \begin{cases} 1, & \text{Switchon} \\ 0, & \text{Switchoff} \end{cases}$ is the switch, i_L is the current flowing over the inductance, and u_c is the voltage of capacitance. Here, the state is $g(t) = [i_L(t), u_c(t)]^T$ and the output is $y(t) = u_c(t)$. $D_i \omega(k)$

is the disturbance in the circuit system, where $D_1 = D_2 = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$ and $\omega(k) = 0.05\cos(k)$ with $\underline{\omega} = -0.05$, $\overline{\omega} = 0.05$. In addition, $f(k) = \begin{cases} 0, & k \leq 3 \\ 2\sin(0.2k), & k > 3 \end{cases}$ is supposed to be a sensor fault in the output. The parameters are chosen as: $L = 0.04\text{H}, C = 1\text{F}, R_1 = 1\Omega$, and $R_2 = 0.02\Omega$. For simulation, the sampling period is chosen as 0.01 s and the following discrete system can be obtained by the explicit Euler method:

$$\begin{cases} g(k+1) = A_i g(k) + B_i u_e(k) + D_i \omega(k) \\ y(k) = C_i x(k) + f(k) \end{cases}$$

where

$$A_1 = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.5 \end{bmatrix}, A_2 = \begin{bmatrix} 0.75 & -0.25 \\ 0.01 & 0.5 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}, C_1 = C_2 = [0 \quad 1].$$

In fault estimation, the fault may be included in an augmented state $x = \begin{bmatrix} g^T, f^T \end{bmatrix}^T$. The augmented system has the following form:

$$\begin{cases} Ex(k+1) = \breve{A}_i x(k) + \breve{B}_i u_e(k) + \breve{D}_i \omega(k) \\ y(k) = \breve{C}_i x(k) \end{cases}$$

where

$$\breve{A}_1 = \begin{bmatrix} 0.75 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \breve{A}_2 = \begin{bmatrix} 0.75 & -0.25 & 0 \\ 0.01 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\breve{B}_1 = \breve{B}_2 = \begin{bmatrix} 0.25 \\ 0 \\ 0 \end{bmatrix}, \breve{D}_1 = \breve{D}_2 = \begin{bmatrix} 0.4 \\ 0.2 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\breve{C}_1 = \breve{C}_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}.$$

It is worth to note that the discretization error is not taken into account in this article. According to Remark 2, matrices S_1 and S_2 are selected as $S_1 = S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. By a direct computation of (5), we have

$$\Pi_1 = \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \Sigma_1 = \Sigma_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $u_e(k) = \sin(k)$. To estimate the output with the fault, the parameters of (2) are chosen as $\Gamma_1 = \Gamma_2 = \breve{C}_1 = \breve{C}_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$. The same initial conditions are given, namely, $M(0) = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$, $[\underline{\epsilon}^z(0), \overline{\epsilon}^z(0)] = [\underline{\epsilon}^h(0), \overline{\epsilon}^h(0)] = \text{Hull}(\langle 0, \Gamma_2 M(0) \rangle)$ and $\hat{v}(0) = 0$. Given $\delta_1 = \delta_2 = 19$, the following observer gains are obtained by solving LMIs (7)–(8):

$$L_1^n = \begin{bmatrix} -0.2057 \\ -0.2090 \\ -0.5023 \end{bmatrix}, L_2^n = \begin{bmatrix} -0.4066 \\ -0.2139 \\ -0.5027 \end{bmatrix}.$$

Let $\mu = 1.01$ and $\xi = 0.835$, by Theorem 2, then

$$L_1^o = \begin{bmatrix} -0.2 \\ 0.02 \\ -0.6923 \end{bmatrix}, L_2^o = \begin{bmatrix} -0.4066 \\ 0.0139 \\ -0.7027 \end{bmatrix}$$

$$\gamma = 45.6726, \tau^* \geq 0.18.$$

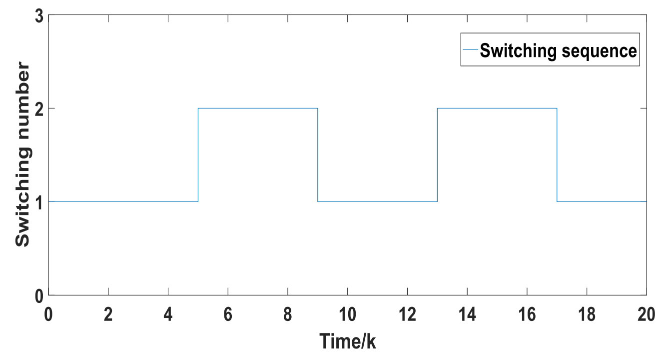

Fig. 2. Time response of the switching sequence.

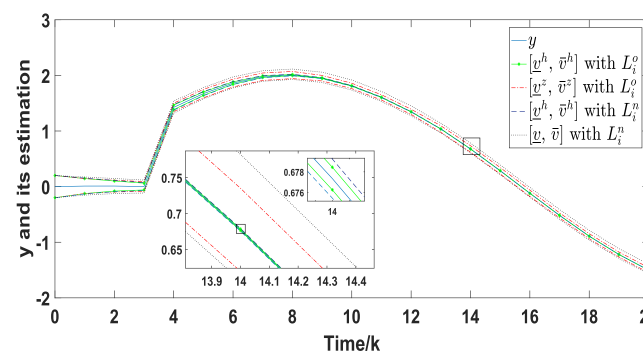

Fig. 3. Evolution of y, $\underline{v}^h$, $\overline{v}^h$, $\underline{v}^z$, $\overline{v}^z$, $\underline{v}$, and $\overline{v}$.

In Table I, the conservatism and computational complexity of the three methods are further illustrated. The interval estimation widths obtained by the three methods show that the SFIO designed in Algorithm 2 has a better estimation accuracy than that designed in Algorithm 1 and the method in [20] under the same conditions, which is consistent with the theoretical analysis in Section IV. In addition, the method in [20] is easier to implement but has strict design conditions. Due to the existence of order reduction operations, Algorithm 1 has more calculation steps and heavier computational burden than the method proposed in [20]. Algorithm 2 achieves a good tradeoff between the estimation accuracy and calculational burden. It not only avoids the computational burden of the order reduction operations in Algorithm 1, but also has a better estimation accuracy.

The switching sequence is shown in Fig. 2 and the simulation results are illustrated in Fig. 3, which is the same as that described earlier.

Remark 8: The values of μ and ξ are fixed a priori. The fact that $\mu > 1$ and $0 < \xi < 1$ makes Theorem 2 reasonable, and the conditions (13)–(16) can be solved by the LMI toolbox in MATLAB. The value $\gamma = 45.6726$ is the minimum value obtained using the toolbox.

Remark 9: In practice, the output of the system is inevitably disturbed by measurement noise, such as radio frequency interference in the circuit system. The treatment method is similar to that of process noise. Due to the layout limitation, it is omitted here.

VI. Conclusion

In this article, novel estimation methods are presented to design SFIO for DSDSs. First, an optimal H_∞ FO is constructed, and the functional interval estimation is realized by the zonotopic SME method with a reduction of the observer design constraints. Second, to further improve the estimation accuracy of the SFIO, an H_∞ robust observer technique is combined with reachability analysis. Different from previous works, the proposed methods can reduce the design constraints as well as the additional conservatism. Furthermore, compared with the monotone systems theory and zonotope, the reachability analysis technique can effectively improve the estimation accuracy. Finally, an application example is performed in simulation and the effectiveness of the aforementioned methods is illustrated. As a further investigation, the SFIO for DSDSs with asynchronous switching law will be studied.

References

[1] K. Emami, T. Fernando, B. Nener, H. Trinh, and Y. Zhang, "A functional observer based fault detection technique for dynamical systems," *J. Franklin Inst.*, vol. 352, no. 5, pp. 2113–2128, 2015.

[2] D. Luenberger, "Observers for multivariable systems," *IEEE Trans. Autom. Control*, vol. AC-11, no. 2, pp. 190–197, Apr. 1966.

[3] H. Trinh and T. Fernando, *Functional Observers for Dynamical Systems*. Berlin, Germany: Springer, 2011.

[4] T. Fernando, H. Trinh, and L. Jennings, "Functional observability and the design of minimum order linear functional observers," *IEEE Trans. Autom. Control*, vol. 55, no. 5, pp. 1268–1273, May 2010.

[5] L. Jennings, T. Fernando, and H. Trinh, "Existence conditions for functional observability from an eigenspace perspective," *IEEE Trans. Autom. Control*, vol. 56, no. 12, pp. 2957–2961, Dec. 2011.

[6] S. Islam, C.-C. Lim, and P. Shi, "Functional observer based controller for stabilizing Takagi-Sugeno fuzzy systems with time-delays," *J. Franklin Inst.*, vol. 355, no. 8, pp. 3619–3640, 2018.

[7] J. Gouzé, A. Rapaport, and M. Hadj-Sadok, "Interval observers for uncertain biological systems," *Ecological Model.*, vol. 133, no. 1/2, pp. 45–56, 2000.

[8] D. Gu, L. Liu, and G. Duan, "Functional interval observer for the linear systems with disturbances," *IET Control Theory Appl.*, vol. 12, no. 18, pp. 2562–2568, 2018.

[9] H. Che, J. Huang, X. Zhao, X. Ma, and N. Xu, "Functional interval observer for discrete-time systems with disturbances," *Appl. Math. Comput.*, vol. 383, 2020, Art. no. 125352.

[10] D. Huong, V. Huynh, and H. Trinh, "Distributed functional interval observers for nonlinear interconnected systems with time-delays and additive disturbances," *IEEE Syst. J.*, vol. 15, no. 1, pp. 411–422, Mar. 2021.

[11] L. Dai, "Observers for discrete singular systems," *IEEE Trans. Autom. Control*, vol. AC-33, no. 2, pp. 187–191, Feb. 1988.

[12] D. Koenig, B. Marx, and D. Jacquet, "Unknown input observers for switched nonlinear discrete time descriptor systems," *IEEE Trans. Autom. Control*, vol. 53, no. 1, pp. 373–379, Feb. 2008.

[13] J. Lin and Z. Gao, "Observers design for switched discrete-time singular time-delay systems with unknown inputs," *Nonlinear Anal.-Hybrid Syst.*, vol. 18, pp. 85–99, 2015.

[14] Y. Hou, F. Zhu, X. Zhao, and S. Guo, "Observer design and unknown input reconstruction for a class of switched descriptor systems," *IEEE Trans. Syst. Man Cybern. Syst.*, vol. 48, no. 8, pp. 1411–1419, Aug. 2017.

[15] J. Zhang, X. Zhao, F. Zhu, and H. R. Karimi, "Reduced-order observer design for switched descriptor systems with unknown inputs," *IEEE Trans. Autom. Control*, vol. 65, no. 1, pp. 287–294, Jan. 2020.

[16] J. Lin, Y. Shi, Z. Gao, and J. Ding, "Functional observer for switched discrete-time singular systems with time delays and unknown inputs," *IET Control Theory Appl.*, vol. 9, no. 14, pp. 2146–2156, 2015.

[17] D. Koenig, B. Marx, and S. Varrier, "Filtering and fault estimation of descriptor switched systems," *Automatica*, vol. 63, pp. 116–121, 2016.

[18] H. Ethabet, T. Raïssi, M. Amairi, C. Combastel, and M. Aoun, "Interval observer design for continuous-time switched systems under known switching and unknown inputs," *Int. J. Control*, vol. 93, no. 5, pp. 1088–1101, 2020.

[19] S. Guo and F. Zhu, "Interval observer design for discrete-time switched system," *IFAC-Papers Online*, vol. 50, no. 1, pp. 5073–5078, 2017.

[20] H. Ethabet, D. Rabehi, D. Efimov, and T. Raïssi, "Interval estimation for continuous-time switched linear systems," *Automatica*, vol. 90, pp. 230–238, 2018.

[21] E. Chambon, L. Burlion, and P. Apkarian, "Overview of linear time-invariant interval observer design: Towards a non-smooth optimisation-based approach," *IET Control Theory Appl.*, vol. 10, no. 11, pp. 1258–1268, 2016.

[22] Z. Wang, C.-C. Lim, and Y. Shen, "Interval observer design for uncertain discrete-time linear systems," *Syst. Control Lett.*, vol. 116, pp. 41–46, 2018.

[23] T. Dinh, G. Marouani, T. Raïssi, Z. Wang, and H. Messaoud, "Optimal interval observers for discrete-time linear switched systems," *Int. J. Control*, vol. 93, no. 11, pp. 2613–2621, 2020.

[24] H. Ito, "Interval observer of minimal error dynamics," *Automatica*, vol. 113, 2020, Art. no. 108794.

[25] D. Koenig and B. Marx, "H_∞-filtering and state feedback control for discrete-time switched descriptor systems," *IET Control Theory Appl.*, vol. 3, no. 6, pp. 661–670, 2009.

[26] Z. Wang, P. Shi, and C.-C. Lim, "$H-/H_\infty$ fault detection observer in finite frequency domain for linear parameter-varying descriptor systems," *Automatica*, vol. 86, pp. 38–45, 2017.

[27] Z. Bo, Q. Kun, X. Ma, and X. Dai, "A new nonlinear set membership filter based on guaranteed bounding ellipsoid algorithm," *Acta Automatica Sinica*, vol. 39, no. 2, pp. 146–154, 2013.

[28] E. Walter and H. Piet-lahanier, "Exact recursive polyhedral description of the feasible parameter set for bounded-error models," *IEEE Trans. Autom. Control*, vol. 34, no. 8, pp. 911–915, Aug. 1989.

[29] T. Alamo, J. Bravo, and E. Camacho, "Guaranteed state estimation by zonotopes," *Automatica*, vol. 41, no. 6, pp. 1035–1043, 2005.

[30] C. Combastel, "Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence," *Automatica*, vol. 55, pp. 265–273, 2015.

[31] Y. Wang, V. Puig, and G. Cembrano, "Set-membership approach and Kalman observer based on zonotopes for discrete-time descriptor systems," *Automatica*, vol. 93, pp. 435–443, 2018.

[32] Y. Wang, Z. Wang, V. Puig, and G. Cembrano, "Zonotopic set-membership state estimation for discrete-time descriptor LPV systems," *IEEE Trans. Autom. Control*, vol. 64, no. 5, pp. 2092–2099, May 2018.

[33] J. Huang, X. Ma, H. Che, and Z. Han, "Further result on interval observer design for discrete-time switched systems and application to circuit systems," *IEEE Trans. Circuits Syst. II-Exp. Briefs*, vol. 67, no. 11, pp. 2542–2546, Nov. 2020.

[34] W. Kühn, "Rigorously computed orbits of dynamical systems without the wrapping effect," *Computing*, vol. 61, no. 1, pp. 47–67, 1998.

[35] W. Tang, Z. Wang, Y. Wang, T. Raïssi, and Y. Shen, "Interval estimation methods for discrete-time linear time-invariant systems," *IEEE Trans. Autom. Control*, vol. 64, no. 11, pp. 4717–4724, Nov. 2019.

[36] C. R. Rao and S. K. Mitra, "Generalized inverse of a matrix and its applications," in *Proc. 6th Berkeley Symp. Math. Statist. Probability*, Vol. 1: *Theory Statist.*, 1972, pp. 601–620.

[37] D. Efimov, L. Fridman, T. Raïssi, A. Zolghadri, and R. Seydou, "Interval estimation for LPV systems applying high order sliding mode techniques," *Automatica*, vol. 48, no. 9, pp. 2365–2371, 2012.

[38] W. Xiang, J. Xiao, and M. Iqbal, "Robust observer design for nonlinear uncertain switched systems under asynchronous switching," *Nonlinear Anal.-Hybrid Syst.*, vol. 6, no. 1, pp. 754–773, 2012.