CHAPTER 5
Determination of Forward and Futures Prices

Practice Questions

5.1
The forward price of an asset today is the price at which you would agree to buy or sell the asset at a future time. The value of a forward contract is zero when you first enter into it. As time passes, the underlying asset price changes and the value of the contract may become positive or negative.

5.2
The forward price is $30e^{0.05\times0.5} = 30.76$.

5.3
The futures price is $350e^{(0.04-0.03)\times0.3333} = 351.17$.

5.4
Gold is an investment asset. If the futures price is too high, investors will find it profitable to increase their holdings of gold and short futures contracts. If the futures price is too low, they will find it profitable to decrease their holdings of gold and go long in the futures market. Copper is a consumption asset. If the futures price is too high, a strategy of buy copper and short futures works. However, because investors do not in general hold the asset, the strategy of sell copper and buy futures is not available to them. There is therefore an upper bound, but no lower bound, to the futures price.

5.5
A foreign currency provides a known interest rate, but the interest is received in the foreign currency. The value in the domestic currency of the income provided by the foreign currency is therefore known as a percentage of the value of the foreign currency. This means that the income has the properties of a known yield.

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The futures price of a stock index is always less than the expected future value of the index. This follows from Section 5.14 and the fact that the index has positive systematic risk. For an alternative argument, let μ be the expected return required by investors on the index so that $E(S_T) = S_0e^{(\mu-q)T}$. Because $\mu > r$ and $F_0 = S_0e^{(r-q)T}$, it follows that $E(S_T) > F_0$.

5.7

a) The forward price, F_0, is given by equation (5.1) as:

$$F_0 = 40e^{0.05\times1} = 42.05$$

or $42.05. The initial value of the forward contract is zero.

b) The delivery price K in the contract is $42.05. The value of the contract, f, after six months is given by equation (5.5) as:

$$f = 45 - 42.05e^{-0.05\times 0.5} = 3.99$$

i.e., it is \$3.99. The forward price is:

$$45e^{0.05\times 0.5} = 46.14$$

or \$46.14.

5.8

Using equation (5.3), the six month futures price is

$$150e^{(0.07-0.032)\times 0.5} = 152.88$$

or \$152.88.

5.9

The futures contract lasts for five months. The dividend yield is 2% for three of the months and 5% for two of the months. The average dividend yield is therefore

$$\frac{1}{5}(3\times 2 + 2\times 5) = 3.2\%$$

The futures price is therefore

$$1300e^{(0.04-0.032)\times 0.4167} = 1304.34$$

or \$1304.34.

5.10

The theoretical futures price is

$$400e^{(0.06-0.04)\times 4/12} = 402.68$$

The actual futures price is 405. This shows that the index futures price is too high relative to the index. The correct arbitrage strategy is the following:

1. Sell futures contracts.
2. Buy the shares underlying the index.

5.11

The settlement prices for the futures contracts are to

Jun: 0.93070

Sept: 0.93200

The September price is 0.14% above the June price. This suggests that the short-term interest rate in Japan was less than the short-term interest rate in the U.S. by about 0.14% per three months or about 0.56% per year.

5.12

The theoretical futures price is

$$1.0500e^{(0.02-0.01)\times 2/12} = 1.0518$$

The actual futures price is too low. This suggests that a Swiss arbitrageur should sell Swiss francs for US dollars and buy Swiss francs back in the futures market.

5.13

The present value of the storage costs for nine months are

$$0.06 + 0.06e^{-0.05\times 0.25} + 0.06e^{-0.05\times 0.5} = 0.178$$

or \$0.178. The futures price is from equation (5.11) given by F_0 where

$F_0 = (25.000 + 0.178)e^{0.05\times0.75} = 26.14$
i.e., it is \$26.14 per ounce.

5.14

If

$$F_2 > F_1 e^{r(t_2-t_1)}$$

an investor could make a riskless profit by:

1. Taking a long position in a futures contract which matures at time t_1.
2. Taking a short position in a futures contract which matures at time t_2.

When the first futures contract matures, the asset is purchased for F_1 using funds borrowed at rate r. It is then held until time t_2 at which point it is exchanged for F_2 under the second contract. The costs of the funds borrowed and accumulated interest at time t_2 is $F_1 e^{r(t_2-t_1)}$. A positive profit of

$$F_2 - F_1 e^{r(t_2-t_1)}$$

is then realized at time t_2. This type of arbitrage opportunity cannot exist for long. Hence:

$$F_2 \leq F_1 e^{r(t_2-t_1)}$$

5.15

In total, the gain or loss under a futures contract is equal to the gain or loss under the corresponding forward contract. However, the timing of the cash flows is different. When the time value of money is taken into account, a futures contract may prove to be more valuable or less valuable than a forward contract. Of course, the company does not know in advance which will work out better. The long forward contract provides a perfect hedge. The long futures contract provides a slightly imperfect hedge.

a) In this case, the forward contract would lead to a slightly better outcome. The company will make a loss on its hedge. If the hedge is with a forward contract, the whole of the loss will be realized at the end. If it is with a futures contract, the loss will be realized day by day throughout the contract. On a present value basis the former is preferable.
b) In this case, the futures contract would lead to a slightly better outcome. The company will make a gain on the hedge. If the hedge is with a forward contract, the gain will be realized at the end. If it is with a futures contract, the gain will be realized day by day throughout the life of the contract. On a present value basis, the latter is preferable.
c) In this case, the futures contract would lead to a slightly better outcome. This is because it would involve positive cash flows early and negative cash flows later.
d) In this case, the forward contract would lead to a slightly better outcome. This is because, in the case of the futures contract, the early cash flows would be negative and the later cash flow would be positive.

5.16

From the discussion in Section 5.14 of the text, the forward exchange rate is an unbiased predictor of the future exchange rate when the exchange rate has no systematic risk. To have no systematic risk, the exchange rate must be uncorrelated with the return on the market.

5.17

Suppose that F_0 is the futures price at time zero for a contract maturing at time T and F_1 is the futures price for the same contract at time t_1. It follows that

$$F_0 = S_0 e^{(r-q)T}$$

$$F_1 = S_1 e^{(r-q)(T-t_1)}$$

where S_0 and S_1 are the spot price at times zero and t_1, r is the risk-free rate, and q is the dividend yield. These equations imply that

$$\frac{F_1}{F_0} = \frac{S_1}{S_0} e^{-(r-q)t_1}$$

Define the excess return of the portfolio underlying the index over the risk-free rate as x. The total return is $r+x$ and the return realized in the form of capital gains is $r+x-q$. It follows that $S_1 = S_0 e^{(r+x-q)t_1}$ and the equation for F_1/F_0 reduces to

$$\frac{F_1}{F_0} = e^{xt_1}$$

which is the required result.

5.18

To understand the meaning of the expected future price of a commodity, suppose that there are N different possible prices at a particular future time: P_1, P_2, …, P_N. Define q_i as the (subjective) probability the price being P_i (with $q_1 + q_2 + ... + q_N = 1$). The expected future price is

$$\sum_{i=1}^{N} q_i P_i$$

Different people may have different expected future prices for the commodity. The expected future price in the market can be thought of as an average of the opinions of different market participants. Of course, in practice the actual price of the commodity at the future time may prove to be higher or lower than the expected price.

Keynes and Hicks argue that speculators on average make money from commodity futures trading and hedgers on average lose money from commodity futures trading. If speculators tend to have short positions in crude oil futures, the Keynes and Hicks argument implies that futures prices overstate expected future spot prices. If crude oil futures prices decline at 2% per year, the Keynes and Hicks argument therefore implies an even faster decline for the expected price of crude oil in this case.

5.19

When the geometric average of the price relatives is used, the changes in the value of the index do not correspond to changes in the value of a portfolio that is traded. Equation (5.8) is therefore no longer correct. The changes in the value of the portfolio are monitored by an index calculated from the arithmetic average of the prices of the stocks in the portfolio. Since the geometric average of a set of numbers is always less than the arithmetic average, equation (5.8) overstates the futures price. It is rumored that at one time (prior to 1988), equation (5.8) did hold for the Value Line Index. A major Wall Street firm was the first to recognize that this represented a trading opportunity. It made a financial killing by buying the stocks underlying the index and shorting the futures.

5.20

(a) The relationship between the futures price F_t and the spot price S_t at time t is

$$F_t = S_t e^{(r-r_f)(T-t)}$$

Suppose that the hedge ratio is h. The price obtained with hedging is

$$h(F_0 - F_t) + S_t$$

where F_0 is the initial futures price. This is

$$hF_0 + S_t - hS_t e^{(r-r_f)(T-t)}$$

If $h = e^{(r_f-r)(T-t)}$, this reduces to hF_0 and a zero variance hedge is obtained.

(b) When t is one day, h is approximately $e^{(r_f-r)T} = S_0/F_0$. The appropriate hedge ratio is therefore S_0/F_0.

(c) When a futures contract is used for hedging, the price movements in each day should in theory be hedged separately. This is because the daily settlement means that a futures contract is closed out and rewritten at the end of each day. From (b) the correct hedge ratio at any given time is, therefore, S/F where S is the spot price and F is the futures price. Suppose there is an exposure to N units of the foreign currency and M units of the foreign currency underlie one futures contract. With a hedge ratio of 1, we should trade N/M contracts. With a hedge ratio of S/F, we should trade

$$\frac{SN}{FM}$$

contracts. In other words, we should calculate the number of contracts that should be traded as the dollar value of our exposure divided by the dollar value of one futures contract. (This is not the same as the dollar value of our exposure divided by the dollar value of the assets underlying one futures contract.) Since a futures contract is settled daily, we should in theory rebalance our hedge daily so that the outstanding number of futures contracts is always $(SN)/(FM)$. This is known as tailing the hedge. (See Chapter 3.)

5.21

a) The risk-free rate, b) the excess of the risk-free rate over the dividend yield, c) the risk-free rate plus the storage cost, d) the excess of the domestic risk-free rate over the foreign risk-free rate.

5.22

The theoretical forward exchange rate is $1.0404e^{(0.0025-0)\times 0.25} = 1.041$.

If the actual forward exchange rate is 1.03, an arbitrageur can a) borrow X Swiss francs, b) convert the Swiss francs to $1.0404X$ dollars and invest the dollars for three months at 0.25%, and c) buy X Swiss francs at 1.03 in the three-month forward market. In three months, the arbitrageur has $1.0404Xe^{0.0025\times 0.25} = 1.041X$ dollars. A total of $1.3X$ dollars are used to buy the Swiss francs under the terms of the forward contract and a gain of $0.011X$ is made.

If the actual forward exchange rate is 1.05, an arbitrageur can a) borrow X dollars, b) convert the dollars to $X/1.0404$ Swiss francs and invest the Swiss francs for three months at zero interest rate, and c) enter into a forward contract to sell $X/1.0404$ Swiss francs in three months. In three months, the arbitrageur has $X/1.0404$ Swiss francs. The forward contract converts these to $(1.05X)/1.0404=1.0092X$ dollars. A total of $Xe^{0.0025\times 0.25} =1.0006X$ is needed to repay the dollar loan. A profit of 0.0086*X* dollars is therefore made.

5.23

The futures price for the three-month contract is $1200e^{(0.03\text{-}0.012)\times 0.25}=1205.41$. The futures price for the six-month contract is $1200e^{(0.035\text{-}0.01)\times 0.5}=1215.09$.

5.24

If the six-month euro interest rate is r_f then

$$1.1950 = 1.2000e^{(0.01-r_f)\times 0.5}$$

so that

$$0.01 - r_f = 2\ln\left(\frac{1.1950}{1.2000}\right) = -0.00835$$

and $r_f = 0.01835$. The six-month euro interest rate is 1.835%.

5.25.

The present value of the storage costs per barrel is $3e^{-0.05\times 1} = 2.854$. An upper bound to the one-year futures price is $(50 + 2.854)e^{0.05\times 1} = 55.56$.

5.26

It is likely that the bank will price the product on assumption that the company chooses the delivery date least favorable to the bank. If the foreign interest rate is higher than the domestic interest rate, then:

1. The earliest delivery date will be assumed when the company has a long position.
2. The latest delivery date will be assumed when the company has a short position.

If the foreign interest rate is lower than the domestic interest rate, then:

1. The latest delivery date will be assumed when the company has a long position.
2. The earliest delivery date will be assumed when the company has a short position.

If the company chooses a delivery which, from a purely financial viewpoint, is suboptimal the bank makes a gain.

5.27

The value of the contract to the bank at time T_1 is $S_1 - K_1$. The bank will choose K_2 so that the new (rolled forward) contract has a value of $S_1 - K_1$. This means that

$$S_1 e^{-r_f(T_2-T_1)} - K_2 e^{-r(T_2-T_1)} = S_1 - K_1$$

where r and r_f and the domestic and foreign risk-free rate observed at time T_1 and applicable to the period between time T_1 and T_2. This means that

$$K_2 = S_1 e^{(r-r_f)(T_2-T_1)} - (S_1 - K_1)e^{r(T_2-T_1)}$$

This equation shows that there are two components to K_2. The first is the forward price at time T_1. The second is an adjustment to the forward price equal to the bank's gain on the first part of the contract compounded forward at the domestic risk-free rate.