CHAPTER 18
Futures Options and Black's Model

Practice Questions

18.1

In this case, $u = 1.12$ and $d = 0.92$. The probability of an up movement in a risk-neutral world is

$$\frac{1-0.92}{1.12-0.92} = 0.4$$

From risk-neutral valuation, the value of the call is

$$e^{-0.06\times 0.5}(0.4\times 6 + 0.6\times 0) = 2.33$$

18.2

The American futures call option is worth more than the corresponding American option on the underlying asset when the futures price is greater than the spot price prior to the maturity of the futures contract. This is the case when the risk-free rate is greater than the income on the asset plus the convenience yield.

18.3

In this case, $F_0 = 19$, $K = 20$, $r = 0.12$, $\sigma = 0.20$, and $T = 0.4167$. The value of the European futures put option is

$$20N(-d_2)e^{-0.12\times 0.4167} - 19N(-d_1)e^{-0.12\times 0.4167}$$

where

$$d_1 = \frac{\ln(19/20) + (0.04/2)0.4167}{0.2\sqrt{0.4167}} = -0.3327$$

$$d_2 = d_1 - 0.2\sqrt{0.4167} = -0.4618$$

This is

$$e^{-0.12\times 0.4167}[20N(0.4618) - 19N(0.3327)]$$

$$= e^{-0.12\times 0.4167}(20\times 0.6778 - 19\times 0.6303)$$

$$= 1.50$$

or \$1.50.

18.4

An amount (1,400 – 1,380) × 100 = \$2,000 is added to your margin account and you acquire a short futures position obligating you to sell 100 ounces of gold in October. This position is marked to market in the usual way until you choose to close it out.

18.5

In this case, an amount (1.35−1.30) × 40,000 = \$2,000 is subtracted from your margin account and you acquire a short position in a live cattle futures contract to sell 40,000 pounds of cattle in April. This position is marked to market in the usual way until you choose to close it out.

18.6

Lower bound if option is European is

$$(F_0 - K)e^{-rT} = (47-40)e^{-0.1\times 2/12} = 6.88$$

Lower bound if option is American is

$$F_0 - K = 7$$

18.7

Lower bound if option is European is

$$(K - F_0)e^{-rT} = (50-47)e^{-0.1\times 4/12} = 2.90$$

Lower bound if option is American is

$$K - F_0 = 3$$

18.8

In this case, $u = e^{0.3\times\sqrt{1/4}} = 1.1618$; $d = 1/u = 0.8607$; and

$$p = \frac{1-0.8607}{1.1618-0.8607} = 0.4626$$

In the tree shown in Figure S18.1, the middle number at each node is the price of the European option and the lower number is the price of the American option. The tree shows that the value of the European option is 4.3155 and the value of the American option is 4.4026. The American option should sometimes be exercised early.

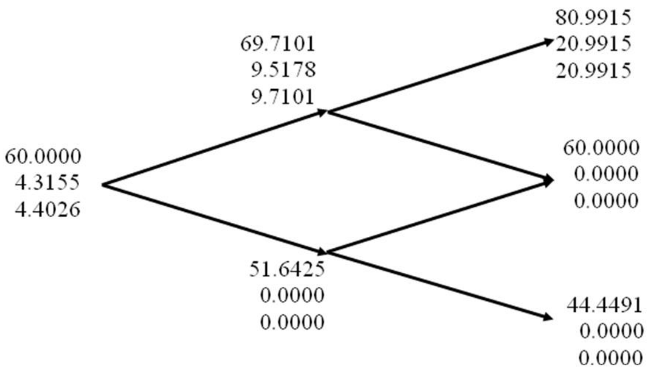

Figure S18.1: *Tree to evaluate European and American call options in Problem 18.8*

18.9

The parameters u, d and p are the same as in Problem 18.8. The tree in Figure S18.2 shows that the prices of the European and American put options are the same as those calculated for call options in Problem 18.8. This illustrates a symmetry that exists for at-the-money futures options. The American option should sometimes be exercised early. Because $K = F_0$ and $c = p$, the European put–call parity result holds.

$$c + Ke^{-rT} = p + F_0e^{-rT}$$

Also, because $C = P$, $F_0e^{-rT} < K$, and $Ke^{-rT} < F_0$ the result in equation (18.2) holds. (The first expression in equation (18.2) is negative; the middle expression is zero, and the last expression is positive.)

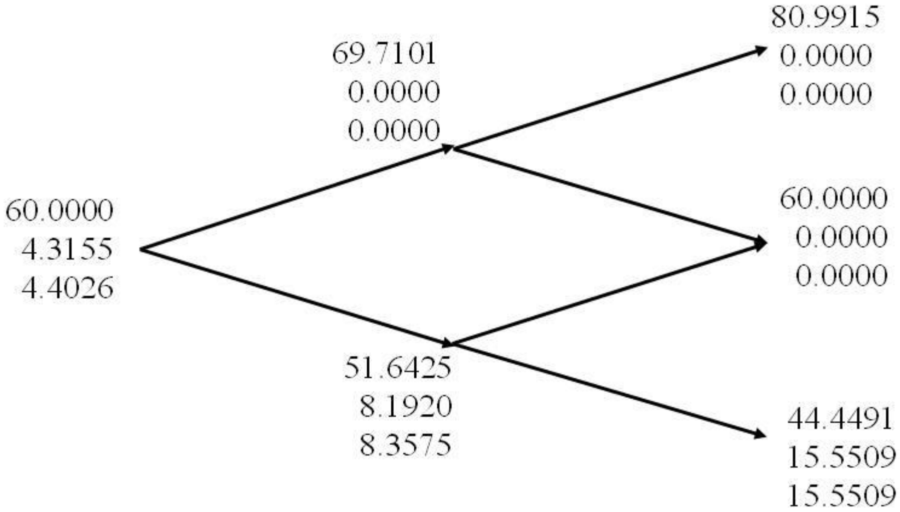

Figure S18.2: *Tree to evaluate European and American put options in Problem 18.9*

18.10

In this case, $F_0 = 25$, $K = 26$, $\sigma = 0.3$, $r = 0.1$, $T = 0.75$

$$d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma\sqrt{T}} = -0.0211$$

$$d_2 = \frac{\ln(F_0 / K) - \sigma^2 T / 2}{\sigma\sqrt{T}} = -0.2809$$

$$c = e^{-0.075}[25N(-0.0211) - 26N(-0.2809)]$$

$$= e^{-0.075}[25 \times 0.4916 - 26 \times 0.3894] = 2.01$$

18.11

In this case, $F_0 = 70$, $K = 65$, $\sigma = 0.2$, $r = 0.06$, $T = 0.4167$

$$d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma\sqrt{T}} = 0.6386$$

$$d_2 = \frac{\ln(F_0 / K) - \sigma^2 T / 2}{\sigma\sqrt{T}} = 0.5095$$

$$p = e^{-0.025}[65N(-0.5095) - 70N(-0.6386)]$$

$$= e^{-0.025}[65 \times 0.3052 - 70 \times 0.2615] = 1.495$$

18.12

In this case,

$$c + Ke^{-rT} = 2 + 34e^{-0.1\times 1} = 32.76$$

$$p + F_0e^{-rT} = 2 + 35e^{-0.1\times 1} = 33.67$$

Put–call parity shows that we should buy one call, short one put and short a futures contract. This costs nothing up front. In one year, either we exercise the call or the put is exercised against us. In either case, we buy the asset for 34 and close out the futures position. The gain on the short futures position is $35 - 34 = 1$.

18.13

The put price is

$$e^{-rT}[KN(-d_2) - F_0N(-d_1)]$$

Because $N(-x) = 1 - N(x)$ for all x the put price can also be written

$$e^{-rT}[K - KN(d_2) - F_0 + F_0N(d_1)]$$

Because $F_0 = K$ this is the same as the call price:

$$e^{-rT}[F_0N(d_1) - KN(d_2)]$$

This result also follows from put–call parity showing that it is not model dependent.

18.14

From equation (18.2), $C - P$ must lie between

$$30e^{-0.05\times 3/12} - 28 = 1.63$$

and

$$30 - 28e^{-0.05\times 3/12} = 2.35$$

Because $C = 4$ we must have $1.63 < 4 - P < 2.35$ or

$$1.65 < P < 2.37$$

18.15

In this case, we consider:

Portfolio A: A European call option on futures plus an amount K invested at the risk-free interest rate.

Portfolio B: An American put option on futures plus an amount F_0e^{-rT} invested at the risk-free interest rate plus a long futures contract maturing at time *T*.

Following the arguments in Chapter 5, we will treat all futures contracts as forward contracts. Portfolio A is worth $c + K$ while portfolio B is worth $P + F_0e^{-rT}$. If the put option is exercised at time $\tau\,(0 \le \tau < T)$, portfolio B is worth

$$K - F_\tau + F_0e^{-r(T-\tau)} + F_\tau - F_0$$

$$= K + F_0e^{-r(T-\tau)} - F_0 < K$$

at time τ where F_τ is the futures price at time τ. Portfolio A is worth

$$c + Ke^{r\tau} \ge K$$

Hence, Portfolio A more than Portfolio B. If both portfolios are held to maturity (time *T*), Portfolio A is worth

$$\max(F_T - K, 0) + Ke^{rT}$$

$$= \max(F_T, K) + K(e^{rT} - 1)$$

Portfolio B is worth

$$\max(K - F_T, 0) + F_0 + F_T - F_0 = \max(F_T, K)$$

Hence, portfolio A is worth more than portfolio B.
Because portfolio A is worth more than portfolio B in all circumstances:

$$P + F_0 e^{-r(T-t)} < c + K$$

Because $c \leq C$ it follows that

$$P + F_0 e^{-rT} < C + K$$

or

$$F_0 e^{-rT} - K < C - P$$

This proves the first part of the inequality.

For the second part of the inequality consider:
Portfolio C: An American call futures option plus an amount Ke^{-rT} invested at the risk-free interest rate.
Portfolio D: A European put futures option plus an amount F_0 invested at the risk-free interest rate plus a long futures contract.
Portfolio C is worth $C + Ke^{-rT}$ while portfolio D is worth $p + F_0$. If the call option is exercised at time $\tau (0 \leq \tau < T)$ portfolio C becomes:

$$F_\tau - K + Ke^{-r(T-\tau)} < F_\tau$$

while portfolio D is worth

$$\begin{aligned} &p + F_0 e^{r\tau} + F_\tau - F_0 \\ &= p + F_0 (e^{r\tau} - 1) + F_\tau \geq F_\tau \end{aligned}$$

Hence, portfolio D is worth more than portfolio C. If both portfolios are held to maturity (time T), portfolio C is worth $\max(F_T, K)$ while portfolio D is worth

$$\begin{aligned} &\max(K - F_T, 0) + F_0 e^{rT} + F_T - F_0 \\ &= \max(K, F_T) + F_0 (e^{rT} - 1) \\ &> \max(K, F_T) \end{aligned}$$

Hence portfolio D is worth more than portfolio C.
Because portfolio D is worth more than portfolio C in all circumstances

$$C + Ke^{-rT} < p + F_0$$

Because $p \leq P$ it follows that

$$C + Ke^{-rT} < P + F_0$$

or

$$C - P < F_0 - Ke^{-rT}$$

This proves the second part of the inequality. The result:

$$F_0 e^{-rT} - K < C - P < F_0 - Ke^{-rT}$$

has therefore been proved.

18.16
This has the same value as a three-month call option on silver futures where the futures contract expires in three months. It can therefore be valued using equation (18.7) with $F_0 = 12$, $K = 13$, $r = 0.04$, $\sigma = 0.25$ and $T = 0.25$. The value is 0.244.

18.17
The rate received will be less than 6.5% when LIBOR is less than 7%. The corporation requires a three-month call option on a Eurodollar futures option with a strike price of 93. If

three-month LIBOR is greater than 7% at the option maturity, the Eurodollar futures quote at option maturity will be less than 93 and there will be no payoff from the option. If the three-month LIBOR is less than 7%, one Eurodollar futures options provide a payoff of $25 per 0.01%. Each 0.01% of interest costs the corporation $125 (= 5,000,000 × 0.0001 × 0.25). A total of 125/25 = 5 contracts are therefore required.

18.18

In this case, $u = 1.125$ and $d = 0.875$. The risk-neutral probability of an up move is

$$(1-.875)/(1.125-0.875) = 0.5$$

The value of the option is

$$e^{-0.07\times 0.25}[0.5\times 3+0.5\times 0] = 1.474$$

18.19

Put–call parity for European options gives

$$6.5 + 78e^{-0.03\times 0.5} = c + 80e^{-0.03\times 0.5}$$

so that c = 4.53.

The relation for American options gives

$$78\, e^{-0.03\times 0.5} - 80 < C - 6.5 < 78 - 80\, e^{-0.03\times 0.5}$$

so that

$$-3.16 < C - 6.5 < -0.81$$

so that C lies between 3.34 and 5.69.

18.20

u =1.331, d = 0.8825, and p = 0.4688. As the tree in Figure S18.3 shows the value of the option is 4.59.

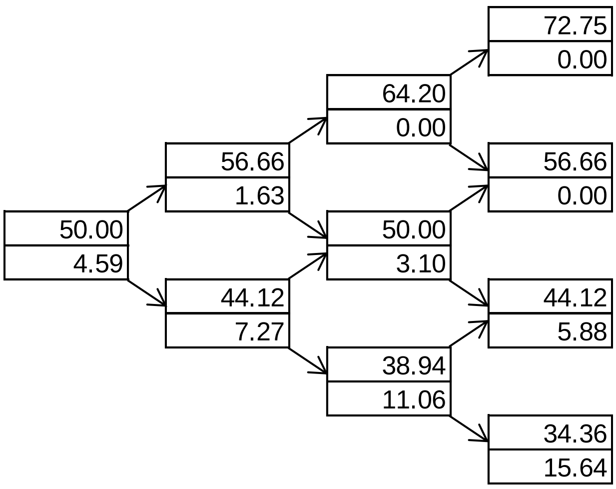

Figure S18.3: *Tree for Problem 18.20*

18.21

There are 135 days to maturity (assuming this is not a leap year). Using DerivaGem with $F_0 = 278.25$, $r = 1.1\%$, T = 135/365, and 500 time steps gives the implied volatilities shown in the table below.

Strike Price	*Call Price*	*Put Price*	*Call Implied Vol*	*Put Implied Vol*
260	26.75	8.50	24.69	24.59
270	21.25	13.50	25.40	26.14
280	17.25	19.00	26.85	26.86
290	14.00	25.625	28.11	27.98
300	11.375	32.625	29.24	28.57

We do not expect put–call parity to hold exactly for American options and so there is no reason why the implied volatility of a call should be exactly the same as the implied volatility of a put. Nevertheless it is reassuring that they are close.
There is a tendency for high strike price options to have a higher implied volatility. As explained in Chapter 20, this is an indication that the probability distribution for corn futures prices in the future has a heavier right tail and less heavy left tail than the lognormal distribution.

18.22
In this case, $F_0 = 525$, $K = 525$, $r = 0.06$, $T = 0.4167$. We wish to find the value of σ for which $p = 20$ where

$$p = Ke^{-rT}N(-d_2) - F_0e^{-rT}N(-d_1)$$

This must be done by trial and error. When σ = 0.2, *p* = 26.35. When σ = 0.15, *p* = 19.77 When $\sigma = 0.155$, *p* = 20.43. When $\sigma = 0.152$, *p* = 20.03. These calculations show that the implied volatility is approximately 15.2% per annum.

18.23
The price of the option is the same as the price of a European put option on the forward price of the index where the forward contract has a maturity of six months. It is given by equation (18.8) with $F_0 = 1400$, $K = 1450$, $r = 0.05$, $\sigma = 0.15$, and $T = 0.5$. It is 86.35.