CHAPTER 20
Volatility Smiles and Volatility Surfaces

Practice Questions

20.1
(a) A smile similar to that in Figure 20.7 is observed.
(b) An upward sloping volatility smile is observed.

20.2
Jumps tend to make both tails of the stock price distribution heavier than those of the lognormal distribution. This creates a volatility smile similar to that in Figure 20.1. The volatility smile is likely to be more pronounced for the three-month option.

20.3
The put has a price that is too low relative to the call's price. The correct trading strategy is to buy the put, buy the stock, and sell the call.

20.4
The heavier left tail should lead to high prices, and therefore high implied volatilities, for out-of-the-money (low-strike-price) puts. Similarly, the less heavy right tail should lead to low prices, and therefore low volatilities for out-of-the-money (high-strike-price) calls. A volatility smile where volatility is a decreasing function of strike price results.

20.5
With the notation in the text

$$c_{\text{bs}} + Ke^{-rT} = p_{\text{bs}} + Se^{-qT}$$

$$c_{\text{mkt}} + Ke^{-rT} = p_{\text{mkt}} + Se^{-qT}$$

It follows that

$$c_{\text{bs}} - c_{\text{mkt}} = p_{\text{bs}} - p_{\text{mkt}}$$

In this case, $c_{\text{mkt}} = 3.00$; $c_{\text{bs}} = 3.50$; and $p_{\text{bs}} = 1.00$. It follows that p_{mkt} should be 0.50.

20.6
The crashophobia argument is an attempt to explain the pronounced volatility skew in equity markets since 1987. (This was the year equity markets shocked everyone by crashing more than 20% in one day). The argument is that traders are concerned about another crash and as a result increase the price of out-of-the-money puts. This creates the volatility skew.

20.7
The probability distribution of the stock price in one month is not lognormal. Possibly, it consists of two lognormal distributions superimposed upon each other and is bimodal. Black–Scholes is clearly inappropriate, because it assumes that the stock price at any future time is lognormal.

20.8
When the asset price is positively correlated with volatility, the volatility tends to increase as

the asset price increases, producing less heavy left tails and heavier right tails. Implied volatility then increases with the strike price.

20.9
There are a number of problems in testing an option pricing model empirically. These include the problem of obtaining synchronous data on stock prices and option prices, the problem of estimating the dividends that will be paid on the stock during the option's life, the problem of distinguishing between situations where the market is inefficient and situations where the option pricing model is incorrect, and the problems of estimating stock price volatility.

20.10
In this case, the probability distribution of the exchange rate has a thin left tail and a thin right tail relative to the lognormal distribution. We are in the opposite situation to that described for foreign currencies in Section 20.2. Both out-of-the-money and in-the-money calls and puts can be expected to have lower implied volatilities than at-the-money calls and puts. The pattern of implied volatilities is likely to be similar to Figure 20.7.

20.11
A deep-out-of-the-money option has a low value. Decreases in its volatility reduce its value. However, this reduction is small because the value can never go below zero. Increases in its volatility, on the other hand, can lead to significant percentage increases in the value of the option. The option does, therefore, have some of the same attributes as an option on volatility.

20.12
Put–call parity implies that European put and call options have the same implied volatility. If a call option has an implied volatility of 30% and a put option has an implied volatility of 33%, the call is priced too low relative to the put. The correct trading strategy is to buy the call, sell the put and short the stock. This does not depend on the lognormal assumption underlying Black–Scholes–Merton. Put–call parity is true for any set of assumptions.

20.13
Suppose that p is the probability of a favorable ruling. The expected price of the company's stock tomorrow is

$$75p + 50(1-p) = 50 + 25p$$

This must be the price of the stock today. (We ignore the expected return to an investor over one day.) Hence,

$$50 + 25p = 60$$

or $p = 0.4$.
If the ruling is favorable, the volatility, σ, will be 25%. Other option parameters are $S_0 = 75$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 26.502.
If the ruling is unfavorable, the volatility, σ will be 40% Other option parameters are $S_0 = 50$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 6.310.
The value today of a European call option with a strike price today is the weighted average of 26.502 and 6.310 or,

$$0.4 \times 26.502 + 0.6 \times 6.310 = 14.387$$

DerivaGem can be used to calculate the implied volatility when the option has this price. The parameter values are $S_0 = 60$, $K = 50$, $T = 0.5$, $r = 0.06$ and $c = 14.387$. The implied volatility is 47.76%.
These calculations can be repeated for other strike prices. The results are shown in the table below. The pattern of implied volatilities is shown in Figure S20.1.

Strike Price	*Call Price: Favorable Outcome*	*Call Price: Unfavorable Outcome*	*Weighted Price*	*Implied Volatility (%)*
30	45.887	21.001	30.955	46.67
40	36.182	12.437	21.935	47.78
50	26.502	6.310	14.387	47.76
60	17.171	2.826	8.564	46.05
70	9.334	1.161	4.430	43.22
80	4.159	0.451	1.934	40.36

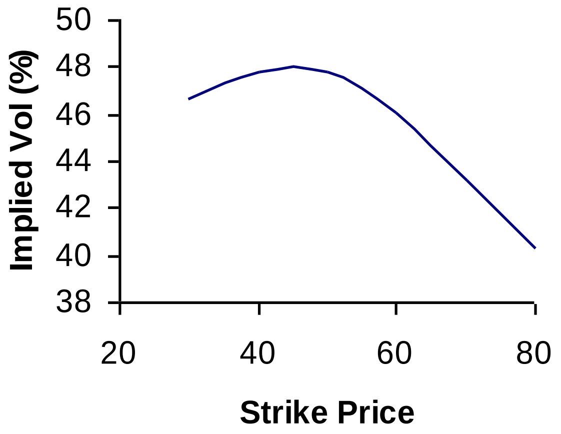

Figure S20.1: *Implied Volatilities in Problem 20.13*

20.14
As pointed out in Chapters 5 and 17, an exchange rate behaves like a stock that provides a dividend yield equal to the foreign risk-free rate. Whereas the growth rate in a non-dividend-paying stock in a risk-neutral world is r, the growth rate in the exchange rate in a risk-neutral world is $r - r_f$. Exchange rates have low systematic risks and so we can reasonably assume that this is also the growth rate in the real world. In this case, the foreign risk-free rate equals the domestic risk-free rate ($r = r_f$). The expected growth rate in the exchange rate is therefore zero. If S_T is the exchange rate at time *T*, its probability distribution is given by equation (14.3) with $\mu = 0$:

$$\ln S_T \sim \varphi\left(\ln S_0 - \sigma^2 T/2, \sigma^2 T\right)$$

where S_0 is the exchange rate at time zero and σ is the volatility of the exchange rate. In this case, $S_0 = 0.8000$ and $\sigma = 0.12$, and $T = 0.25$ so that

$$\ln S_T \sim \varphi\left(\ln 0.8 - 0.12^2 \times 0.25/2, 0.12^2 \times 0.25\right)$$

or

$$\ln S_T \sim \varphi\left(-0.2249, 0.06^2\right)$$

a) ln 0.70 = –0.3567. The probability that $S_T < 0.70$ is the same as the probability that $\ln S_T < -0.3567$. It is

$$N\left(\frac{-0.3567+0.2249}{0.06}\right)=N(-2.1955)$$

This is 1.41%.

b) ln 0.75 = –0.2877. The probability that $S_T < 0.75$ is the same as the probability that $\ln S_T < -0.2877$. It is

$$N\left(\frac{-0.2877+0.2249}{0.06}\right)=N(-1.0456)$$

This is 14.79%. The probability that the exchange rate is between 0.70 and 0.75 is therefore $14.79-1.41=13.38\%$.

c) ln 0.80 = –0.2231. The probability that $S_T < 0.80$ is the same as the probability that $\ln S_T < -0.2231$. It is

$$N\left(\frac{-0.2231+0.2249}{0.06}\right)=N(0.0300)$$

This is 51.20%. The probability that the exchange rate is between 0.75 and 0.80 is therefore $51.20-14.79=36.41\%$.

d) ln 0.85 = –0.1625. The probability that $S_T < 0.85$ is the same as the probability that $\ln S_T < -0.1625$. It is

$$N\left(\frac{-0.1625+0.2249}{0.06}\right)=N(1.0404)$$

This is 85.09%. The probability that the exchange rate is between 0.80 and 0.85 is therefore $85.09-51.20=33.89\%$

e) ln 0.90 = –0.1054. The probability that $S_T < 0.90$ is the same as the probability that $\ln S_T < -0.1054$. It is

$$N\left(\frac{-0.1054+0.2249}{0.06}\right)=N(1.9931)$$

This is 97.69%. The probability that the exchange rate is between 0.85 and 0.90 is therefore $97.69-85.09=12.60\%$.

f) The probability that the exchange rate is greater than 0.90 is $100-97.69=2.31\%$.

The volatility smile encountered for foreign exchange options is shown in Figure 20.1 of the text and implies the probability distribution in Figure 20.2. Figure 20.2 suggests that we would expect the probabilities in (a), (c), (d), and (f) to be too low and the probabilities in (b) and (e) to be too high.

20.15

The difference between the two implied volatilities is consistent with Figure 20.3 in the text. For equities, the volatility smile is downward sloping. A high strike price option has a lower implied volatility than a low strike price option. The reason is that traders consider the probability of a large downward movement in the stock price is higher than that predicted by the lognormal probability distribution. The implied distribution assumed by traders is shown in Figure 20.4.

To use DerivaGem to calculate the price of the first option, proceed as follows. Select Equity as the Underlying Type in the first worksheet. Select Black–Scholes European as the Option Type. Input the stock price as 40, volatility as 35%, risk-free rate as 5%, time to exercise as 0.5 year, and exercise price as 30. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 11.155. Change the volatility to 28% and the strike price to 50. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 0.725.

Put–call parity is

$$c + Ke^{-rT} = p + S_0$$

so that

$$p = c + Ke^{-rT} - S_0$$

For the first option, $c = 11.155$, $S_0 = 40$, $r = 0.054$, $K = 30$, and $T = 0.5$ so that

$$p = 11.155 + 30e^{-0.05\times0.5} - 40 = 0.414$$

For the second option, $c = 0.725$, $S_0 = 40$, $r = 0.06$, $K = 50$, and $T = 0.5$ so that

$$p = 0.725 + 50e^{-0.05\times0.5} - 40 = 9.490$$

To use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 30. Input the price as 0.414 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 34.99%.

Similarly, to use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 50. Input the price as 9.490 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 27.99%.

These results are what we would expect. DerivaGem gives the implied volatility of a put with strike price 30 to be almost exactly the same as the implied volatility of a call with a strike price of 30. Similarly, it gives the implied volatility of a put with strike price 50 to be almost exactly the same as the implied volatility of a call with a strike price of 50.

20.16

When plain vanilla call and put options are being priced, traders do use the Black–Scholes–Merton model as an interpolation tool. They calculate implied volatilities for the options whose prices they can observe in the market. By interpolating between strike prices and between times to maturity, they estimate implied volatilities for other options. These implied volatilities are then substituted into Black–Scholes–Merton to calculate prices for these options. In practice, much of the work in producing a table such as Table 20.2 in the over-the-counter market is done by interdealer brokers. These brokers often act as intermediaries between participants in the over-the-counter market and usually have more information on the trades taking place than any individual financial institution. The brokers provide a table such as Table 20.2 to their clients as a service.

20.17
The implied volatility is 13.45%. We get the same answer by (a) interpolating between strike prices of 1.00 and 1.05 and then between maturities six months and one year, and (b) interpolating between maturities of six months and one year and then between strike prices of 1.00 and 1.05.

20.18
In liquidation, the company's stock price must be at least 300,000/100,000 = \$3. The company's stock price should therefore always be at least \$3. This means that the stock price distribution that has a thinner left tail and fatter right tail than the lognormal distribution. An upward sloping volatility smile can be expected.

20.19 (Excel file)
The following table shows the percentage of daily returns greater than 1, 2, 3, 4, 5, and 6 standard deviations for each currency. The pattern is similar to that in Table 20.1.

	>1sd	*>2sd*	*>3sd*	*>4sd*	*>5sd*	*>6sd*
EUR	22.62	5.21	1.70	0.50	0.20	0.10
CAD	23.12	5.01	1.60	0.50	0.20	0.10
GBP	22.62	4.70	1.30	0.80	0.50	0.10
JPY	25.23	4.80	1.50	0.40	0.30	0.10
Normal	31.73	4.55	0.27	0.01	0.00	0.00

20.20 (Excel file)
The percentage of times up and down movements happen are shown in the table below.

	>3sd down	*>3sd up*
S&P 500	1.10	0.90
NASDAQ	0.80	0.90
FTSE	1.30	0.90
Nikkei	1.00	0.60
Average	1.05	0.83

As might be expected from the shape of the volatility smile, large down movements occur more often than large up movements. However, the results are not significant at the 95% level. (The standard error of the Average >3sd down percentage is 0.185% and the standard error of the Average >3sd up percentage is 0.161%. The standard deviation of the difference between the two is 0.245%).

20.21
Define c_1 and p_1 as the values of the call and the put when the volatility is σ_1. Define c_2 and p_2 as the values of the call and the put when the volatility is σ_2. From put–call parity

$$p_1 + S_0 e^{-qT} = c_1 + Ke^{-rT}$$

$$p_2 + S_0 e^{-qT} = c_2 + Ke^{-rT}$$

If follows that

$$p_1 - p_2 = c_1 - c_2$$

20.22
Define:

$$g(S_T) = g_1 \text{ for } 0.7 \le S_T < 0.8$$

$$g(S_T) = g_2 \text{ for } 0.8 \le S_T < 0.9$$

$$g(S_T) = g_3 \text{ for } 0.9 \le S_T < 1.0$$

$$g(S_T) = g_4 \text{ for } 1.0 \le S_T < 1.1$$

$$g(S_T) = g_5 \text{ for } 1.1 \le S_T < 1.2$$

$$g(S_T) = g_6 \text{ for } 1.2 \le S_T < 1.3$$

The value of g_1 can be calculated by interpolating to get the implied volatility for a six-month option with a strike price of 0.75 as 12.5%. This means that options with strike prices of 0.7, 0.75, and 0.8 have implied volatilities of 13%, 12.5% and 12%, respectively. From DerivaGem, their prices are \$0.2963, \$0.2469, and \$0.1976, respectively. Using equation (20A.1) with $K = 0.75$ and $\delta = 0.05$, we get

$$g_1 = \frac{e^{0.025\times 0.5}(0.2963 + 0.1976 - 2\times 0.2469)}{0.05^2} = 0.0315$$

Similar calculations show that $g_2 = 0.7241$, $g_3 = 4.0788$, $g_4 = 3.6766$, $g_5 = 0.7285$, and $g_6 = 0.0898$. The total probability between 0.7 and 1.3 is the sum of these numbers multiplied by 0.1 or 0.9329. If the volatility had been flat at 11.5%, the values of g_1, g_2, g_3, g_4, g_5, and g_6 would have been 0.0239, 0.9328, 4.2248, 3.7590, 0.9613, and 0.0938. The total probability between 0.7 and 1.3 is in this case 0.9996. This shows that the volatility smile gives rise to heavy tails for the distribution.

20.23
Interpolation gives the volatility for a six-month option with a strike price of 98 as 12.82%. Interpolation also gives the volatility for a 12-month option with a strike price of 98 as 13.7%. A final interpolation gives the volatility of an 11-month option with a strike price of 98 as 13.55%. The same answer is obtained if the sequence in which the interpolations are done is reversed.