

math notes

0.1 矩阵的三则运算性质

$$A+B=B+A \quad (A+B)+C=A+(B+C) \quad (AB)C=A(BC) \quad (k+l)A=kA+lA \quad k(A+B)=kA+kB \\ A(B+C)=AB+AC$$

0.2 矩阵转置的性质

$$(A^T)^T=A \quad (kA)^T=kA^T \quad (AB)^T=B^TA^T \quad (A^{-1})^T=((A)^T)^{-1} \quad (A^T)^m=(A^m)^T$$

0.3 伴随矩阵的性质

$$(A^T)^*= (A^*)^T \quad (kA)^*=k^{n-1}A^* \quad (AB)^*=B^*A^*$$

0.4 逆矩阵的性质

$$(A^{-1})^{-1}=A \quad (kA)^{-1}=\frac{1}{k}A \quad (AB)^{-1}=B^{-1}A^{-1} \quad (A^T)^{-1}=(A^{-1})^T \quad (A^n)^{-1}=(A^{-1})^n \\ A_{m \times m}, B_{n \times n}: \quad \begin{pmatrix} A & O \\ O & B \end{pmatrix} = \begin{pmatrix} A^{-1} & O \\ O & B^{-1} \end{pmatrix}$$

0.5 矩阵秩的性质

$$r(A)=r(A^T)=r(A^TA)=r(AA^T) \quad \text{设 } A, B \text{ 都是同型矩阵, 则 } r(A \pm B) \leq r(A) + r(B) \quad \text{设 } A, B \text{ 分别为 } m \times n, n \times s \text{ 矩阵, 且 } AB=O, \text{ 则 } r(A)+r(B) \leq n$$

0.6 证明向量组 $\alpha_1 \dots \alpha_n$ 线性相关的充分必要条件是向量组中至少有一个向量被其余向量线性表示

1. " \Rightarrow ": 存在不全为 0 的 k_1, k_2, \dots, k_n 使得 $k_1\alpha_1 + \dots + k_n\alpha_n = 0$ 设 $k_1 \neq 0, \alpha_1 = -\frac{k_2}{k_1}\alpha_2 - \dots - \frac{k_n}{k_1}\alpha_n$ 2. " \Leftarrow "
 $l_1\alpha_1 + \dots + l_{k-1}\alpha_{k-1} + l_{k+1}\alpha_{k+1} + \dots + l_n\alpha_n$
 $\Rightarrow l_1\alpha_1 + \dots + l_{k-1}\alpha_{k-1} + (-1)\alpha_k + l_{k+1}\alpha_{k+1} + \dots + l_n\alpha_n = 0$
所以 $\alpha_1 \dots \alpha_n$ 线性相关

0.7 证明 α, β 线性相关 $\Leftrightarrow \alpha, \beta$ 成比例

证明:" \Rightarrow " \exists 不全为 0 的 k_1, k_2 , 使得 $k_1\alpha + k_2\beta = 0$ 设 $k_2 \neq 0 \Rightarrow \beta = -\frac{k_1}{k_2}\alpha \Rightarrow \alpha, \beta$ 对应成比例
" \Leftarrow " 设 $\beta = l\alpha \Rightarrow l\alpha + (-1)\beta = 0 \Rightarrow \alpha, \beta$ 线性相关

0.8 向量组相关性与线性表示的性质

2. 设 $\alpha_1 \dots \alpha_n$ 线性无关

(1) 若 $\alpha_1 \dots \alpha_n, \beta$ 线性相关, 则向量 β 可以由 $\alpha_1 \dots \alpha_n$ 唯一线性表示
" \Rightarrow " \exists 不全为 0 的 $k_1 \dots k_n, k_0$ 使 $k_1\alpha_1 + \dots + k_n\alpha_n + k_0\beta = 0$; 若 $k_0 = 0 \Rightarrow k_1\alpha_1 + \dots + k_n\alpha_n = 0, k_0 \neq 0 \Rightarrow \beta = -\frac{k_1}{k_0}\alpha_1 - \dots - \frac{k_n}{k_0}\alpha_n$

证明唯一性: 令 $\beta = l_1\alpha_1 + \dots + l_n\alpha_n$; $\beta = t_1\alpha_1 + \dots + t_n\alpha_n \Rightarrow (l_1 - t_1)\alpha_1 + \dots + (l_n - t_n)\alpha_n = 0$ 因为 $\alpha_1 \dots \alpha_n$ 线性无关, 所以 $l_1 = t_1, \dots, l_n = t_n$, 证毕

(2) $\alpha_1 \dots \alpha_n, \beta$ 线性无关 \Leftrightarrow 向量 β 不可以由 $\alpha_1 \dots \alpha_n$ 线性表示;

3. 全组线性无关 \Rightarrow 部分组线性无关

4. 部分组相关 \Rightarrow 全组线性相关

5. $\alpha_1 \dots \alpha_n$ 为 n 个 n 维向量 $\alpha_1 \dots \alpha_n$ 线性无关的充分必要条件是 $|\alpha_1, \dots, \alpha_n| \neq 0$

用结论 $A = (\alpha_1 \dots \alpha_n)$, $\alpha_1 \dots \alpha_n$ 线性无关 $\Leftrightarrow \alpha_1 \dots \alpha_n$ 的秩 $= n \Leftrightarrow r(A) = n \Leftrightarrow |A| \neq 0$

6. $\alpha_1 \dots \alpha_n$ 线性相关 $\Leftrightarrow |\alpha_1 \dots \alpha_n| = 0$ 证: 令 $A = (\alpha_1 \dots \alpha_n)$. $\alpha_1 \dots \alpha_n$ 线性相关 $\Leftrightarrow \alpha_1 \dots \alpha_n$ 的秩 $< n \Leftrightarrow r(A) = n \Leftrightarrow |A| = 0$

7. 设 $\alpha_1 \dots \alpha_n$ 为 n 个 m 维向量, 若 $m < n$, 则向量组 $\alpha_1 \dots \alpha_n$ 一定线性相关

qqquad 口诀向量组左右长上下短一点线性相关

向量的维数代表了方程的个数; ; 方程数少了, 有自由变量, 一定有非零解, 则一定线性相关

证明: 令 $A_{m \times n} = (\alpha_1 \dots \alpha_n) \Rightarrow r(A) \leq m < n \Rightarrow \alpha_1 \dots \alpha_n$ 线性相关 $\Leftrightarrow \alpha_1 \dots \alpha_n$ 的秩 $< n \Leftrightarrow r(A) < n \Rightarrow r(A) \leq m < n$. 所以 $\alpha_1 \dots \alpha_n$ 线性相关

0.9 汤家凤行列式强化提高

$$1. D = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 7 & 7 & 3 & 3 \\ 3 & 2 & 4 & 5 & 2 \\ 3 & 3 & 3 & 2 & 2 \\ 4 & 6 & 5 & 2 & 3 \end{vmatrix}, \text{ 则 } A_{31} + A_{32} + A_{33} =$$

$$A_{31} + A_{32} + A_{33} = 1 * A_{31} + 1 * A_{32} + 1 * A_{33} + 0 * A_{34} + 0 * A_{35} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 7 & 7 & 3 & 3 \\ 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 2 & 2 \\ 4 & 6 & 5 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 3 & 3 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 4 & 6 & 5 & 2 & 3 \end{vmatrix} = 0$$

2. 已知 $|E - A| = |E - 2A| = |E - 3A| = 0$ 求 $|B^{-1} + 2E|$

解: 因为 $|E - A| = |E - 2A| = |E - 3A| = 0$ 所以 A 的特征值为 $\frac{1}{3}, \frac{1}{2}, 1$

$$(u+v)^{(n)} = u^{(n)} + v^{(n)} \quad (uv)^{(n)} = C_n^0 u^{(n)} + C_n^1 u^{(n-1)} v' + \dots + C_n^n u v^{(n)} \quad (\sin x)^{(n)} = \sin(x + \frac{n\pi}{2}) \quad (\cos x)^{(n)} = \cos(x + \frac{n\pi}{2})$$

$$\frac{1}{(ax+b)^{(n)}} = \frac{(-1)^n n! a^n}{(ax+b)^{(n+1)}}$$

设 $y=f(x)$ 可导且 $f'(x) \neq 0, x = \varphi(y)$ 为反函数, 则 $x = \varphi(y)$ 可导, 且 $\varphi'(y) = \frac{1}{f'(x)}$

设 $y=f(x)$ 二阶可导且 $f'(x) \neq 0, x = \varphi(y)$ 为反函数, 则 $x = \varphi(y)$ 二阶可导, 且 $\varphi''(y) = -\frac{f''(x)}{f'^3(x)}$

$x \rightarrow 0$ 常用的等价无穷小 $x \sim \sin x \sim \tan x \sim \arcsin x \sim \arctan x \sim \ln(1+x) \sim e^x - 1, \quad 1 - \cos x \sim \frac{x^2}{2}, 1 - \cos^a x \sim \frac{a}{2} x^2$

$(1+x)^a - 1 \sim ax, \quad a^x - 1 \sim x \ln a$

$x \rightarrow 0$ 常用的麦克劳林公式 $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n) \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + o(x^{2n+1})$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n}{(2n)!} x^{2n} + o(x^{2n}) \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots + x^n + o(x^n)$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots + \frac{(-1)^{n-1}}{n} x^n + o(x^n)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n-1}}{n} x^n + o(x^n) \quad (1+x)^a = 1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \dots + \frac{a(a-1)\dots(a+1-n)}{n!} x^n$

$f'(x) \in C[a, b], (a, b) \quad f'_+(a) > 0, f(a) = f(b) : (1) \exists \xi \in (a, b), f'(\xi) > 0, f'(\eta) < 0. \quad (2) \exists \xi \in (a, b), f''(\xi) < 0.$

$: (1) f'_+(a) > 0 \Rightarrow c \in (a, b), f(c) > f(a); \exists \xi \in (a, c), \eta \in (c, b) \quad f'(\xi) = \frac{f(c)-f(a)}{c-a} > 0, f'(\eta) = \frac{f(b)-f(c)}{b-c} < 0.$

2. $f(x) \in C[0, 1], (0, 1) \quad , f(0) = 0, f(1) = 1, : (1) \exists c \in (0, 1), f(c) = \frac{2}{3}; (2) \exists \xi, \eta \in (0, 1) \frac{2}{f'(\xi)} + \frac{1}{f'(\eta)} = 3$

$: (1) \quad , \quad , \varphi(x) = f(x) - \frac{2}{3} \cdot \varphi(0) = -\frac{2}{3} < 0, \varphi(1) = 13 > 0. \exists c \in (0, 1), \varphi(c) = 0 \Rightarrow f(c) = \frac{2}{3}.$

(2) $0, 1, c \in (0, 1), \exists \xi \in (0, c), \eta \in (c, 1), f'(\xi) = \frac{f(c)-f(0)}{c} = \frac{2}{3c}, f'(\eta) = \frac{f(1)-f(c)}{1-c} = \frac{1}{3(1-c)},$

3. $\xi, \quad ,$

P58 1 $f(x) \in C[a, b], (a, b) \quad , f(a) = f(b) = 1, \exists \xi, \eta \in (a, b), e^{\eta-\xi} [f'(\eta) + f(\eta)] = 1. : \varphi(x) = e^x f(x), \exists \in (a, b), \frac{\varphi(b)-\varphi(a)}{b-a} = \varphi'(\eta) \Rightarrow$

$\frac{e^b - e^a}{b-a} = e^\eta [f'(\eta) + f(\eta)]. \exists \xi \in (a, b), \frac{e^b - e^a}{b-1} = e^\xi.$