Lattice QCD - Fall 2017 ETH Zürich

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Exercise Nr. 8

Discussion on November 21st, 14:45

1) SU(2) Parameterization

The Lie group SU(2) as a manifold is diffeomorphic to the sphere S^3, i.e. there exists a smooth mapping between both parameter spaces. Show that the usual parameterization of the matrix $U \in$ SU(2) in terms of complex numbers

$$U = \begin{pmatrix} c & d \\ -d^* & c^* \end{pmatrix}$$

can be re-expressed in terms of the pauli matrices

$$U = a_0\sigma_0 + i(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)$$

where σ_0 is the identity, such that a_μ forms a S^3 vector.

Show in this parameterization, that the Haar measure

$$dU \equiv \prod_{i=0}^{3} da_i \delta\left(\sum_i a_i^2 - 1\right)$$

is left invariant. Show also how the S^3 vector is related to the exponential representation in terms of the angles θ_i

$$U = \exp\left(-i\sum_{i=1}^{3}\theta_i\frac{\sigma_i}{2}\right)$$

For this purpose it is useful to introduce a new parameterization of the angles:

$$\theta_1 = \theta\sin\delta\cos\phi, \qquad \theta_2 = \theta\sin\delta\sin\phi, \qquad \theta_3 = \theta\cos\delta, \qquad 0 \le \theta,\phi \le 2\pi, \quad 0 \le \delta \le \pi$$

Express the Haar measure in these new parameters.

2) SU(3) Parameterization

Explain why $U \in$ SU(3) can be parameterized by complex 3-vectors X, Y, Z which are constrained by the conditions

$$Z = X \times Y, \qquad X^* \cdot Y = 0, \qquad X \cdot X^* = Y \cdot Y^* = 1.$$

Note that U in the exponential parameterization

$$U = \exp\left(-i\sum_{a=1}^{8}\theta_a T_a\right)$$

has 8 real parameters θ_a. Express the Haar measure for SU(3) in terms of these 3-vectors.

Since each matrix U is diagonalizable, the determinant and the trace of U can be expressed in terms of the Eigenvalues:

$$\det U = \lambda_1 \lambda_2 \lambda_3, \qquad \mathrm{tr} U = \lambda_1 + \lambda_2 + \lambda_3.$$

With these identities, plot the distribution of $\mathrm{tr}\, U$, i.e. fill the plane spanned by its real and imaginary parts $\mathrm{Re}\,\mathrm{tr}\, U$ and $\mathrm{Im}\,\mathrm{tr}\, U$ for all allowed combinations of Eigenvalues obtained from the representation in terms of X, Y, Z. Note that for an element $U \in \mathrm{SU}(N)$

$$-N \leq \mathrm{tr} U \leq N.$$

How can the apparent Z(3) symmetry be explained?