Lattice QCD - Fall 2017 ETH Zürich

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Exercise Nr. 6

Discussion on October 31st, 14:45

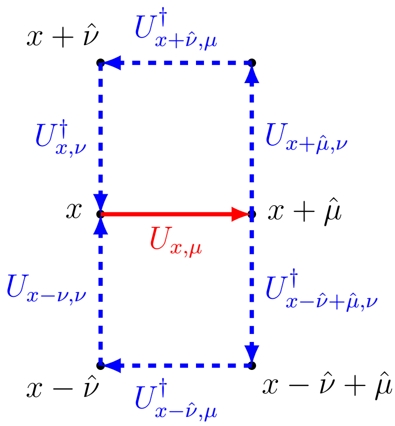
1) U(1) Lattice Gauge Theory - a photon gas

In a pure lattice gauge theory the only variables are the gauge fields, $U_{x,\mu}$ living on the links and belonging to the gauge group, $\mathcal{G}$. The links transform under gauge transformations like,

$$U_{x,\mu} \to U'_{x,\mu} = \Lambda(x) U_{x,\mu} \Lambda^\dagger(x+\hat{\mu}),\ \Lambda \in \mathcal{G}. \tag{1}$$

Gague invariance dictates that the action can only contain closed loops and in the lecture you have seen that the simplest action reproducing the continuum action is the Wilson action:

$$S = \beta \sum_p \left(1 - \mathrm{Re}\,\mathrm{Tr}\,U_p / \,\mathrm{Tr}[I]\right), \tag{2}$$



where the sum is over all plaquettes and

$$U_p = U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu} \tag{3}$$

is the ordered product of all links around one plaquette. For all practical purposes the constant can be dropped. For $U(1)$ we have

$$U = \exp(i\theta),\ \theta \in [0, 2\pi)\,(\,\text{alternatively: } \theta \in [-\pi, \pi)\,). \tag{4}$$

Starting from the Wilson action (2), implement a Metropolis algorithm for simulating a $4d$ $U(1)$ pure gauge theory. Note that when updating a link you need to consider the sum of all incomplete plaquettes (the dashed "staples" in the picture) surrounding the given link. Use the fact that the sum of $U(1)$ elements are proportinal to a $U(1)$ element to arrive at the local action,

$$S_{x,\mu} = -\beta\alpha \cos(\theta_{x,\mu} + \theta_s), \tag{5}$$

where $\alpha e^{i\theta_s}$, $\alpha \in \mathbb{R}$ is the sum over the staples. From here you can propose to shift $\theta_{x,\mu}$. Don't forget that θ is defined up to shifts of 2π.

2) Overrelaxation

To speed up the exploration of configuration space we can apply overrelaxation. This means that we reflect a given link with respect to the minimum of the Bolzman distribution. How this is realized can be seen by looking at the local action (5). The action is minimized for $\theta_{x,\mu} = -\theta_s$. Reflecting $\theta_{x,\mu}$ with respect to this value yields $\theta^{\text{new}}_{x,\mu} = -2\theta_s - \theta^{\text{old}}_{x,\mu}$ and $S^{\text{new}}_{x,\mu} = -\beta\alpha\cos(-\theta^{\text{old}}_{x,\mu} - \theta_s) = S^{\text{old}}_{x,\mu}$. Since the action is unchanged such a move is always accepted. Implement this in your program and alternate between Metropolis and overrelaxation updates as you generate new configurations.

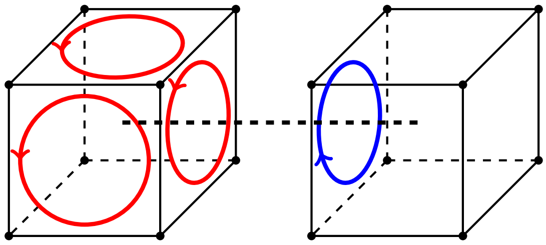


Figure 1: A Dirac string in 3 dimensions, starting at a monopole and ending at an anti-monopole.

3) Monopoles

The electromagnetic flux through a plaqeutte is given by the plaquette angle, $\theta_{\mu,\nu}(x) \in (-4\pi, 4\pi)$. This can be decomposed into a physical flux, $\tilde{\theta}_{\mu,\nu}(x)$, and a Dirac flux, $m_{\mu,\nu}(x)$, via

$$\theta_{\mu,\nu}(x) = \tilde{\theta}_{\mu,\nu}(x) + m_{\mu,\nu}(x)2\pi,\ \tilde{\theta}_{\mu,\nu}(x) \in [-\pi, \pi). \tag{6}$$

The Dirac flux is equal to the number of Dirac strings passing through the plaquette. If the total flux through an elementary spatial cube is non-zero, there must be a source or a sink located in that cube, i.e. a monopole or an anti-monopole. We can therefore define the monopole-content of an elementary 3D cube as:

$$M(x) \ = \ \frac{1}{2}\epsilon_{\sigma\mu\nu4}(m_{\mu,\nu}(x+\hat{\sigma}) - m_{\mu,\nu}(x)). \tag{7}$$

As we are on a Euclidean lattice and at the moment considering only the zero temperature case (i.e. the lattice is hypercubic), there is no prefered time-direction. It therefore makes sense to define more generally,

$$M_\rho(x) \ = \ \frac{1}{2}\epsilon_{\sigma\mu\nu\rho}(m_{\mu,\nu}(x+\hat{\sigma}) - m_{\mu,\nu}(x)), \tag{8}$$

which measures the monopole content of a 3D cube that is spatial w.r.t. the time-direction $\hat{\rho}$.

Measure in your simulation the monopole density,

$$Q_{\text{mon}} = \frac{1}{4V}\sum_{x,\rho}|M_\rho(x)| \tag{9}$$

and verify that there are two phases separated by a phase transition at $\beta_c \approx 1$. In the high β phase the monopoles are heavy and long Dirac strings are exponentially suppressed whereas the monopole mass vanishes at the phase transition and they condence in the low β phase.

4*) Continuum Limit of the Wilson Action

In the lecture you have seen that the Wilson Action has the correct continuum limit in the case of $U(1)$. Now you will show it also for $SU(N)$. Let,

$$U_{\mu,x} = \exp(iagA_\mu(x)) \in SU(N),\ A_\mu(x) = A^a_\mu(x)T^a \tag{10}$$

$$\mathrm{Tr}(T^a) = 0,\ \mathrm{Tr}\left(T^aT^b\right) = \frac{1}{2}\delta^{ab},\ T^aT^b - T^bT^a \equiv \left[T^a, T^b\right] = if^{abc}T^c. \tag{11}$$

With these relations, show that the Wilson Action, Eq. (2), reduces to the continuum action,

$$S_E = \int \mathrm{d}^4x\,\frac{1}{2}\,\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right),\ F_{\mu\nu} = \left(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + gf^{abc}A^a_\mu(x)A^b_\nu(x)\right)T^a, \tag{12}$$

when $a \to 0$. In doing so, find the expression for β in terms of g and a.

There are $N^2 - 1$ generators, T^a of $SU(N)$. In the above all repeated indices are summed over.