Lattice Gauge Theory - Fall 2017 ETH Zürich

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Exercise Nr. 3

Discussion on October 17, 11:45

1) Scalar Propagator

Show that the Fourier transform of a free scalar massive propagator

$$G(p) = 1/(p^2 + m^2)$$

in d spatial dimensions is given by

$$G(r) \sim \frac{e^{-mr}}{r^{(d-1)/2}}.$$

in the infinite distance limit $|r| \to \infty$.

2) Lattice Discretization for the Scalar Field

Calculate the Fourier transfrom of the lattice two-point correlation function for the free real scalar field

$$G(n) \equiv \langle \phi_n \phi_0 \rangle,$$

based on the dimensionless quantities $\phi_n = a\phi(na)$, $M = ma$. This can be done by making use of Gaussian integrals in momentum space. For this purpose, prove that the discretized Euclidean action,

$$S_E = \frac{1}{2} \sum_{n,m \in \Lambda} \phi_n K_{nm} \phi_m,$$

with

$$K_{nm} = -\sum_{\mu=1}^{4} (\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{nm}) \; + \; M^2 \delta_{nm}$$

can be written as

$$S_E = \frac{1}{2} \sum_p \tilde{K}(p) \tilde{\phi}_p \tilde{\phi}_p^*, \qquad \tilde{K}(p) = 4 \sum_{\mu=1}^{4} \sin^2(p_\mu/2) + M^2 \; = \; G^{-1}(p).$$

By going through these steps, show where the cut-off dependence of the 2-point function for the free field is hidden.

3) Time Correlation Function

Redo exercise (3.2) of the last sheet for various values of $\mu^2 \in \{-1, -2, -3\}$ and determine the first two energy levels. Recall that the two-point function can be used to extract the excited states:

$$\lim_{T\to\infty} \Gamma^{(2)} = \langle 0|x(0)x(\tau)|0\rangle - \langle 0|x|0\rangle^2 = \sum_{n\neq 0} e^{-1/\hbar (E_n - E_0)\tau} |\langle 0|x|n\rangle|.$$

The energy gap can hence be determined by

$$\frac{1}{\hbar}(E_1 - E_0) = \lim_{T\to\infty} \left(\frac{-1}{\Delta\tau} \log[\Gamma^2(\tau + \Delta\tau)/\Gamma^2(\tau)] \right).$$

George Green

(14 July 1793 - 31 May 1841) was a British mathematician and physicist

[...] His father (also named George) was a baker who had built and owned a brick windmill used to grind grain. The younger Green only had about one year of formal schooling as a child, between the ages of 8 and 9. In his youth, George Green was described as having a frail constitution and a dislike for doing work in his father's bakery. He had no choice in the matter, however, and as was common for the time he likely began working daily to earn his living at the age of five.

Roughly 25-50% of children in Nottingham received any schooling in this period. The majority of schools were Sunday schools, run by the Church, and children would typically attend for one or two years only. Recognizing the young Green's above average intellect, and being in a strong financial situation due to his successful bakery, his father enrolled him in March 1801 at Robert Goodacre's Academy. [...] He stayed for only four terms (one year), and it was speculated by his contemporaries that he probably exhausted all they had to teach him.

[...] In 1807, George Green senior bought a plot of land in Sneinton, a small town about a mile away from Nottingham. On this plot of land he built a "brick wind corn mill", the wind-mill now famously referred to as Green's Windmill. It was technologically impressive for its time, but required nearly twenty-four hour maintenance, which was to become George Green's burden for the next twenty years. [...] In 1828, Green published "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism", which is the essay he is most famous for today. When Green published his Essay, it was sold on a subscription basis to 51 people, most of whom were friends and probably could not understand it. The wealthy landowner and mathematician Edward Bromhead bought a copy and encouraged Green to do further work in mathematics. Not believing the offer was sincere, Green did not contact Bromhead for two years.

By 1829, the time when Green's father died, the senior Green had become one of the gentry due to his considerable accumulated wealth and land owned, roughly half of which he left to his son and the other half to his daughter. The young Green, now thirty-six years old, consequently was able to use this wealth to abandon his miller duties and pursue mathematical studies. [...] In 1832, aged nearly forty, Green was admitted as an undergraduate at Gonville and Caius College, Cambridge. He was particularly insecure about his lack of knowledge of Greek or Latin, which was a prerequisite, but it turned out not to be as hard for him to learn as he had expected (and the expected mastery was not as high as he expected). In the mathematics examinations, he won the first-year mathematical prize. He graduated BA in 1838 as a 4th Wrangler (the 4th highest scoring student in his graduating class).

Following his graduation, Green was elected a fellow of the Cambridge Philosophical Society. Even without his stellar academic standing, the Society had already read and made note of his Essay and three other publications, and so Green was warmly welcomed. The next two years provided an unparalleled opportunity for Green to read, write and discuss his scientific ideas. In this short time he published an additional six publications with applications to hydrodynamics, sound and optics.

In his final years at Cambridge, Green became rather ill, and in 1840 he returned to Sneinton, only to die a year later. [...] Green's work on the motion of waves in a canal anticipates the WKB approximation of quantum mechanics, while his research on light-waves and the properties of the ether produced what is now known as the Cauchy-Green tensor. Green's theorem and functions were important tools in classical mechanics, and were revised by Schwinger's 1948 work on electrodynamics that led to his 1965 Nobel prize (shared with Feynman and Tomonaga). Green's functions later also proved useful in analyzing superconductivity. [...]

`[from http://en.wikipedia.org/wiki/George_Green]`