Lattice QCD - Fall 2017 ETH Zürich

Lecturer:	Philippe de Forcrand	Office: K22.2	forcrand@phys.ethz.ch
Tutor:	Liam Keegan	Office: K31.3	keeganl@phys.ethz.ch

Exercise Nr. 12

Discussion on December 19th, 14:45

1) Finite Size Scaling

Derive from the scaling hypothesis for the singular part of the free energy density

$$b^{-d} f_s(b^{y_t} t, b^{y_h} h, b^d V^{-1}) = f_s(t, h, V^{-1}) \quad \forall b \in \mathbb{R}, \qquad t = \frac{T - T_c}{T_c}, \quad h = \frac{H}{k_b T_c}$$

the scaling relations for the magnetization $M = \frac{\partial}{\partial H} f_s$ and susceptibility $\chi = \frac{\partial^2}{\partial H^2} f_s$:

$$M(t < 0, h = 0, V^{-1} = 0) \sim |t|^\beta, \quad M(t = 0, h, V^{-1} = 0) \sim h^{1/\delta}, \quad \chi(t = 0, h, V^{-1} = 0) \sim h^{1/\delta - 1}$$

and the central relation for finite size scaling (FSS) of a second order phase transition:

$$T_{c,L} = T_c(1 + \tilde{t} L^{-1/\nu}), \tag{1}$$

and relate the critical exponents β, γ, δ and ν to the homogeneity coefficients y_t, y_h. Here, $T_{c,L}$ is a pseudo-critical temperature which depends on the lattice volume $V = L^d$, and the FSS relation can be used to extrapolate the critical temperature T_c from a set of $\{T_{c,L_i}\}_i$. The pseudo-critical temperature is defined by the peak maximum of the susceptibility, where H is the symmetry breaking external field (e.g. an external magnetic field). As an intermediate step, show that the peak maxima scale in the following way with the volume:

$$\chi_{c,L} \sim L^{\gamma/\nu}$$

Hint: in general, to obtain a scaling relation for an observable derived from the free energy, set b to the specific value which scales the scaling variables of interest to 1, and set those scaling variables to zero when required by the context (e.g. the peak maxima are characterized by $t = 0$).

2) The Binder Cumulant

One observable with very nice scaling properties is the Binder Cumulant

$$B_4 = \frac{\langle (M - \langle M \rangle)^4 \rangle}{\langle (M - \langle M \rangle)^2 \rangle^2}, \tag{2}$$

where $\langle (M - \langle M \rangle)^k \rangle$ is the expectation value of the k^{th}-central moment of the field M. Use the scaling hypothesis to show that

$$B_4(t, h = 0, L) = \Phi\left(t L^{1/\nu}\right), \tag{3}$$

where $\Phi(x)$ is some unknown scaling function. Note that B_4 is independent of L at criticality ($t = 0$) and that its argument depends only on one critical exponent. Hence, by measuring B_4 at a few different volumes we can determine the critical coupling by finding where B_4 is the same for all volumes and the critical exponent ν by plotting B_4 vs. $tL^{1/\nu}$ and finding ν for which the curves collapse into one.

3) Determination of the Universality class of the SU(2) Deconfinement Transition

Measure with your SU(2) program at fixed $N_\tau = 4$ the Polyakov loop susceptibility

$$\chi_P = \langle P^2 \rangle - \langle |P| \rangle^2$$

(high statistics needed, please give errorbars) and the Binder cumulant eq.(3). From the location of the maxima of the susceptibility (the pseudo-critical points) for different system sizes ($L = 8, 12, 16, 20$), determine the critical value β_c by fitting (1) to the data. Note that to a first approximation, $t \sim \beta - \beta_c$. Also plot B_4 vs $L^{1/\nu} t$ and look for a data collapse. Show that the critical exponents are consistent with those of the 3d Ising model: $\gamma = 1.2396$, $\nu = 0.6304$