some points

$$\kappa = \frac{1}{2(m_0 a + 4)}$$

$$\downarrow$$

$$2\kappa \mathcal{M}[U_\mu]_{x,y} = \sigma_{x,y} - \kappa \Sigma_{\mu=1}^{4}[(1-\gamma_\mu)U_\mu(x)\sigma_{x+\mu,y} + (1+\gamma_\mu)U_\mu^\dagger(x-\mu)\sigma_{x-\mu,y}]$$

$$\downarrow \dagger(\gamma_\mu \ is \ Hermitic)$$

$$\sigma_{x,y} - \kappa \Sigma_{\mu=1}^{4}[(1-\gamma_\mu)U_\mu^\dagger(x)\sigma_{y+\mu,x} + (1+\gamma_\mu)U_\mu(x-\mu)\sigma_{y-\mu,x}]$$

$$\downarrow reform \ eqution$$

$$\sigma_{x,y} - \kappa \Sigma_{\mu=1}^{4}[(1-\gamma_\mu)U_\mu^\dagger(x-\mu)\sigma_{x-\mu,y} + (1+\gamma_\mu)U_\mu(x)\sigma_{x+\mu,y}]$$

$$\downarrow reform \ eqution$$

$$\sigma_{x,y} - \kappa \Sigma_{\mu=1}^{4}[(1+\gamma_\mu)U_\mu(x)\sigma_{x+\mu,y} + (1-\gamma_\mu)U_\mu^\dagger(x-\mu)\sigma_{x-\mu,y}]$$

Fig. 1: 24850c8ae14ac5bfc3ad2a9a56528a0a.png

24850c8ae14ac5bfc3ad2a9a56528a0a.png

even-odd

all in all

> x - 1 (backward x)
>
> | parity\eo | 0 | 1 |
> |---|---|---|
> | 0 | b_x | x |
> | 1 | x | b_x |
>
> x + 1 (forward x) \
>
> | parity\eo | 0 | 1 |
> |---|---|---|
> | 0 | x | f_x |
> | 1 | f_x | x |

Fig. 2: b6914007e64b53434f905acbea83e392.png

b6914007e64b53434f905acbea83e392.png

$$A_{ee}x_e - \kappa D_{eo}x_o = b_e$$
$$(A_{oo} - \kappa^2 D_{oe}A_{ee}^{-1}D_{eo})x_o = \kappa D_{oe}A_{ee}^{-1}b_e + b_o$$

> origin one : $\mathcal{M}x = b$
>
> $$\begin{pmatrix} 1-\kappa T_{ee} & -\kappa D_{eo} \\ -\kappa D_{oe} & 1-\kappa T_{oo} \end{pmatrix} \begin{pmatrix} x_e \\ x_o \end{pmatrix} = \begin{pmatrix} b_e \\ b_o \end{pmatrix}$$
>
> $A = 1 + T$
>
> in this case, $T = 0$, so $A = 1$
>
> so,
>
> $$\begin{pmatrix} 1 & -\kappa D_{eo} \\ -\kappa D_{oe} & 1 \end{pmatrix} \begin{pmatrix} x_e \\ x_o \end{pmatrix} = \begin{pmatrix} b_e \\ b_o \end{pmatrix}$$
>
> $x_e - \kappa D_{eo}x_o = b_e$
>
> $x_o - \kappa D_{oe}x_e = b_o$
>
> so,
>
> $x_e - \kappa D_{eo}x_o = b_e$
>
> $x_o - \kappa D_{oe}(\kappa D_{eo}x_o + b_e) = b_o$
>
> then
>
> $x_e - \kappa D_{eo}x_o = b_e$
>
> $x_o - \kappa^2 D_{oe}(D_{eo}x_o) = \kappa D_{oe}b_e + b_o$
>
> so,
>
> just solve $x_o - \kappa D_{oe}(\kappa D_{eo}x_o) = \kappa D_{oe}b_e + b_o$
>
> then will easily get x_e by $x_e - \kappa D_{eo}x_o = b_e$
>
> so,
>
> give Dslash :
>
> $tmp = D_{eo}src_o$
>
> $dest_o = src_o - \kappa D_{oe}(\kappa D_{eo}src_o) = src_o - \kappa^2 D_{oe}tmp$
>
> give b:
>
> $b_e = anw_e - \kappa D_{eo}anw_o$
>
> $b_o = anw_o - \kappa D_{oe}anw_e$
>
> $b'_o = b_o + \kappa D_{oe}b_e$
>
> so,
>
> Dslash(x_o)=b'_o
>
> then get x_o by BistabCg,
>
> finally get x_e by $b_e + \kappa D_{eo}x_o$
>
> done!

so $Dslash : (1-\kappa^2 D_{oe} D_{eo})$
then $Dslash^\dagger : (1-\kappa^2 D_{oe} D_{eo})^\dagger = (1-\kappa^2 D_{eo}^\dagger D_{oe}^\dagger)$
let $Dslash^\dagger Dslash\ x_o = Dslash^\dagger\ b'_o$

Fig. 3: f01486a17a32914c803c6e92667e2753.png

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clover dslash

本文的研究工作在 u, d 夸克不很轻的条件下进行，我们实际上选用了四叶草作用量进行后续的数值计算，暂不考虑其对手征对称性的影响。结合 (1-49)，最终得到了格点上的 QCD 作用量：

$$S_{\mathrm{LQCD}} = \frac{2}{g^2}\sum_x \sum_{\mu<\nu}\left[\frac{5}{3u_0^4}P_{\mu\nu}(x) - \frac{1}{12u_0^6}R_{\mu\nu}(x) - \frac{1}{12u_0^6}R_{\nu\mu}(x)\right] + a^4\bar{\psi}\mathcal{M}\psi .$$

$$\begin{aligned} 2\kappa a\mathcal{M}_{xy} = & \left[1 + \sum_{\mu<\nu}\frac{a^2\kappa}{u_0^4}\sigma_{\mu\nu}\hat{F}_{\mu\nu}(x)\right]\delta_{x,y} \\ & - \frac{\kappa}{u_0}\sum_{\mu=1}^{4}\left[(1-\gamma_\mu)U_\mu(x)\delta_{x+\mu,y} + (1+\gamma_\mu)U_\mu^\dagger(x-\mu)\delta_{x-\mu,y}\right] . \end{aligned} \tag{1-60}$$

Fig. 4: ee0e5623a05ece2d5df411c3b521aeef.png

ee0e5623a05ece2d5df411c3b521aeef.png

其中 c_{SW} 参数即为 Sheikholeslami-Wohlert 系数，其值在微扰树图阶为 1； $\sigma_{\mu\nu} = -\frac{i}{2}[\gamma_\mu, \gamma_\nu]$； $\hat{F}_{\mu\nu}$ 为格点上的场强张量形式。c_{SW} 也可以使用平均场方法[32] 或者非微扰地[34] 确定。

$$\begin{aligned} \hat{F}_{\mu\nu}(x) = \frac{1}{a^2}\frac{1}{8i}\Big[& U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) \\ & + U_\nu(x)U_\mu^\dagger(x+\nu-\mu)U_\nu^\dagger(x-\mu)U_\mu(x-\mu) \\ & + U_\mu^\dagger(x-\mu)U_\nu^\dagger(x-\mu-\nu)U_\mu(x-\mu-\nu)U_\nu(x-\nu) \\ & + U_\nu^\dagger(x-\nu)U_\mu(x-\nu)U_\nu(x-\nu+\mu)U_\mu^\dagger(x) - h.c.\Big] . \end{aligned} \tag{1-55}$$

Fig. 5: b85b86a0502bc2e32d8fc8a2e7229f11.png

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$$A = 1 - \kappa T : \{A_{ee} = 1 - \kappa T_{ee}; A_{oo} = 1 - \kappa T_{oo}\}$$

for the coeffi for T:

$$(-a^2/\mu_0^4) * (-i/) * (1/a^2 * 1/(8 * i)) * \{\gamma_\mu * \gamma_\nu * u......\} = (1/\mu_0^4) * (1/8) * \{\gamma_\mu * \gamma_\nu * u......\}$$

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \gamma_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
\sum_{\mu<\nu} \sigma_{\mu\nu} F_{\mu\nu} \delta_{x,y}[x=y] &= 2\gamma_\mu\gamma_\nu \frac{1}{4} \sum_p \frac{1}{2}[U_p(x) - U_p^\dagger(x)] \\
&= \frac{1}{4}\gamma_\mu\gamma_\nu[\\
& u(x,\mu)u(x+\mu,\nu)u^\dagger(x+\nu,\mu)u^\dagger(x,\nu) \\
& + u(x,\nu)u^\dagger(x-\mu+\nu,\mu)u^\dagger(x-\mu,\nu)u(x-\mu,\mu) \\
& + u^\dagger(x-\mu,\mu)u^\dagger(x-\mu-\nu,\nu)u(x-\mu-\nu,\mu)u(x-\nu,\nu) \\
& + u^\dagger(x-\nu,\nu)u(x-\nu,\mu)u(x-\nu+\mu,\nu)u^\dagger(x,\mu) - BEFORE^\dagger] \\
&= \frac{1}{4}\gamma_0\gamma_1[U(x,y,z,t;x)U(x+1,y,z,t;y)U^\dagger(x,y+1,z,t;x)U^\dagger(x,y,z,t;y) \\
& +U(x,y,z,t;y)U^\dagger(x-1,y+1,z,t;x)U^\dagger(x-1,y,z,t;y)U(x-1,y,z,t;x) \\
& +U^\dagger(x-1,y,z,t;x)U^\dagger(x-1,y-1,z,t;y)U(x-1,y-1,z,t;x)U(x,y-1,z,t;y) \\
& +U^\dagger(x,y-1,z,t;y)U(x,y-1,z,t;x)U(x+1,y-1,z,t;y)U^\dagger(x,y,z,t;x) - BEFORE^\dagger] \\
& + \frac{1}{4}\gamma_0\gamma_2[U(x,y,z,t;x)U(x+1,y,z,t;y)U^\dagger(x,y,z+1,t;x)U^\dagger(x,y,z,t;z) \\
& +U(x,y,z,t;y)U^\dagger(x-1,y,z+1,t;x)U^\dagger(x-1,y,z,t;z)U(x-1,y,z,t;x) \\
& +U^\dagger(x-1,y,z,t;x)U^\dagger(x-1,y,z-1,t;z)U(x-1,y,z-1,t;x)U(x,y,z-1,t;z) \\
& +U^\dagger(x,y,z-1,t;z)U(x,y,z-1,t;x)U(x+1,y,z-1,t;z)U^\dagger(x,y,z,t;x) - BEFORE^\dagger] \\
& + \frac{1}{4}\gamma_0\gamma_3[U(x,y,z,t;x)U(x+1,y,z,t;t)U^\dagger(x,y,z,t+1;x)U^\dagger(x,y,z,t;y) \\
& +U(x,y,z,t;t)U^\dagger(x-1,y,z,t+1;x)U^\dagger(x-1,y,z,t;t)U(x-1,y,z,t;x) \\
& +U^\dagger(x-1,y,z,t;x)U^\dagger(x-1,y,z,t-1;t)U(x-1,y,z,t-1;x)U(x,y,z,t-1;t) \\
& +U^\dagger(x,y,z,t-1;t)U(x,y,z,t-1;x)U(x+1,y,z,t-1;t)U^\dagger(x,y,z,t;x) - BEFORE^\dagger] \\
& + \frac{1}{4}\gamma_1\gamma_2[U(x,y,z,t;y)U(x,y+1,z,t;z)U^\dagger(x,y,z+1,t;y)U^\dagger(x,y,z,t;z) \\
& +U(x,y,z,t;z)U^\dagger(x,y-1,z+1,t;y)U^\dagger(x,y-1,z,t;z)U(x,y-1,z,t;y) \\
& +U^\dagger(x,y-1,z,t;y)U^\dagger(x,y-1,z-1,t;z)U(x,y-1,z-1,t;y)U(x,y,z-1,t;z) \\
& +U^\dagger(x,y,z-1,t;z)U(x,y,z-1,t;y)U(x,y+1,z-1,t;z)U^\dagger(x,y,z,t;y) - BEFORE^\dagger] \\
& + \frac{1}{4}\gamma_1\gamma_3[U(x,y,z,t;y)U(x,y+1,z,t;t)U^\dagger(x,y,z,t+1;y)U^\dagger(x,y,z,t;t) \\
& +U(x,y,z,t;t)U^\dagger(x,y-1,z,t+1;y)U^\dagger(x,y-1,z,t;t)U(x,y-1,z,t;y) \\
& +U^\dagger(x,y-1,z,t;y)U^\dagger(x,y-1,z,t-1;t)U(x,y-1,z,t-1;y)U(x,y,z,t-1;t) \\
& +U^\dagger(x,y,z,t-1;t)U(x,y,z,t-1;y)U(x,y+1,z,t-1;t)U^\dagger(x,y,z,t;y) - BEFORE^\dagger] \\
& + \frac{1}{4}\gamma_2\gamma_3[U(x,y,z,t;z)U(x,y,z+1,t;t)U^\dagger(x,y,z,t+1;z)U^\dagger(x,y,z,t;t) \\
& +U(x,y,z,t;t)U^\dagger(x,y,z-1,t+1;z)U^\dagger(x,y,z-1,t;t)U(x,y,z-1,t;z) \\
& +U^\dagger(x,y,z-1,t;z)U^\dagger(x,y,z-1,t-1;t)U(x,y,z-1,t-1;z)U(x,y,z,t-1;t) \\
& +U^\dagger(x,y,z,t-1;t)U(x,y,z,t-1;z)U(x,y,z+1,t-1;t)U^\dagger(x,y,z,t;z) - BEFORE^\dagger]
\end{aligned}$$

Fig. 6: 578c344355637a18ca77561ba0e18e51.png

578c344355637a18ca77561ba0e18e51.png

成奇偶两部分，

$$
\begin{aligned}
e &= \{(x,y,z,t), x+y+z+t \equiv 0(\bmod 2)\}\,, \\
o &= \{(x,y,z,t), x+y+z+t \equiv 1(\bmod 2)\}\,.
\end{aligned} \tag{1-160}
$$

自然的，可以将费米子矩阵按照行列指标的奇偶性拆成四个部分，

$$
2\kappa a_t \mathcal{M}_{xy} = \begin{pmatrix} M_{ee} & M_{eo} \\ M_{oe} & M_{oo} \end{pmatrix} = \begin{pmatrix} 1-\kappa T_{ee} & -\kappa D_{eo} \\ -\kappa D_{oe} & 1-\kappa T_{oo} \end{pmatrix}, \tag{1-161}
$$

$$
T_{xy} = \left[c_{\mathrm{SW}}^{t} \sum_{i} \sigma_{i4} \hat{F}_{i4} + \frac{c_{\mathrm{SW}}^{s}}{\xi_0} \sum_{i<j} \sigma_{ij} \hat{F}_{ij} \right] \delta_{x,y}, \tag{1-162}
$$

$$
\begin{aligned}
D_{xy} = &\left[(1-\gamma_4)U_4(x)\delta_{x+\hat{4},y} + (1+\gamma_4)U_4(x-\hat{4})\delta_{x-\hat{4},y}\right] \\
&+ \frac{\nu}{\xi_0} \sum_{i} \left[(1-\gamma_i)U_i(x)\delta_{x+i,y} + (1+\gamma_i)U_i(x-i)\delta_{x-i,y}\right].
\end{aligned} \tag{1-163}
$$

那么 D 就是 Wilon 作用量的非对角部分，T 就是四叶草部分。在实践中，我们称呼这里的 D 矩阵为 Dslash。现在对上述分离成四部分的矩阵进行 LDU 分解，定义 $A = 1 + T$，

$$
\begin{aligned}
&\begin{pmatrix} A_{ee} & -\kappa D_{eo} \\ -\kappa D_{oe} & A_{oo} \end{pmatrix} = \\
&\quad \begin{pmatrix} 1 & 0 \\ -\kappa D_{oe} A_{ee}^{-1} & 1 \end{pmatrix} \begin{pmatrix} A_{ee} & 0 \\ 0 & A_{oo} - \kappa^2 D_{oe} A_{ee}^{-1} D_{eo} \end{pmatrix} \begin{pmatrix} 1 & -\kappa A_{ee}^{-1} D_{eo} \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{1-164}
$$

左边下三角矩阵的逆可以轻松得到，

$$
\begin{pmatrix} 1 & 0 \\ -\kappa D_{oe} A_{ee}^{-1} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \kappa D_{oe} A_{ee}^{-1} & 1 \end{pmatrix}. \tag{1-165}
$$

考虑将 x 和 b 也分成奇偶两部分，初始的线性方程组 $\mathcal{M}x = b$ 变为

$$
\begin{cases}
A_{ee} x_e - \kappa D_{eo} x_o = b_e, \\
\left(A_{oo} - \kappa^2 D_{oe} A_{ee}^{-1} D_{eo}\right) x_o = \kappa D_{oe} A_{ee}^{-1} b_e + b_o.
\end{cases} \tag{1-166}
$$

由于 A 关于时空指标是对角的，它的逆矩阵就是对每个对角部分求逆。而这些对角元是一个颜色指标上复矩阵（四叶草是小方格的线性组合）和 Dirac 指标上

Fig. 7: 26facbb878f5726ce648784aaecd648b.png

26facbb878f5726ce648784aaecd648b.png

奇偶预处理中的 LDU 分解并不是唯一的。上文中介绍的是奇奇非对称的形式。偶偶非对称形式会使式 (1-164) 分解中间项对角变为 $A_{ee} - \kappa^2 D_{eo} A_{oo}^{-1} D_{oe}$ 和 A_{oo}，自然左右两侧的三角矩阵也需要做相应的改变。另外还有奇奇对称形式和偶偶对称形式，以前者为例，分解结果为

$$\begin{pmatrix} A_{ee} & 0 \\ -\kappa D_{oe} & A_{oo} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - \kappa^2 A_{oo}^{-1} D_{oe} A_{ee}^{-1} D_{eo} \end{pmatrix} \begin{pmatrix} 1 & -\kappa A_{ee}^{-1} D_{eo} \\ 0 & 1 \end{pmatrix}. \tag{1-171}$$

这时中间对角项有一个变成了单位矩阵。

Fig. 8: f271fdc93b6303ffba82c28d13ce8575.png

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