Physics Letters B 798 (2019) 134994

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Short-distance constraints for the HLbL contribution to the muon anomalous magnetic moment

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ARTICLE INFO

Article history:
Received 12 August 2019
Received in revised form 26 September 2019
Accepted 1 October 2019
Available online 3 October 2019
Editor: B. Grinstein

ABSTRACT

We derive short-distance constraints for the hadronic light-by-light contribution (HLbL) to the anomalous magnetic moment of the muon in the kinematic region where the three virtual momenta are all large. We include the external soft photon via an external field leading to a well-defined Operator Product Expansion. We establish that the perturbative quark loop gives the leading contribution in a well defined expansion. We compute the first nonzero power correction. It is related to the magnetic susceptibility of the QCD vacuum. The results can be used as model-independent short-distance constraints for the very many different approaches to the HLbL contribution. Numerically the power correction is found to be small.

1. Introduction

The anomalous magnetic moment of the muon is one of the most powerful low-energy probes of the Standard Model (SM). Its experimental value via $a_\mu = (g_\mu - 2)/2$, [1,2],

$$a_\mu^{\text{exp}} = 116\,592\,091(63) \times 10^{-11}\,, \tag{1}$$

is expected to be significantly improved [3,4]. The present theoretical prediction is [2]

$$a_\mu^{\text{SM}} = 116\,591\,823(43) \times 10^{-11}\,. \tag{2}$$

The tension between (1) and (2) might be a sign of physics beyond the SM. Both the theoretical prediction and the measured value thus need improvement. Reviews of the theory are [5,6].

A major contributor to the theoretical error is the hadronic light-by-light contribution (HLbL or a_μ^{HLbL}) depicted in Fig. 1. It involves the evaluation of the 4-point correlation function of electromagnetic quark currents

$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = -i\int d^4x\, d^4y\, d^4z\, e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)} \times \langle 0|T\left\{J^\mu(x)J^\nu(y)J^\lambda(z)J^\sigma(0)\right\}|0\rangle\,, \tag{3}$$

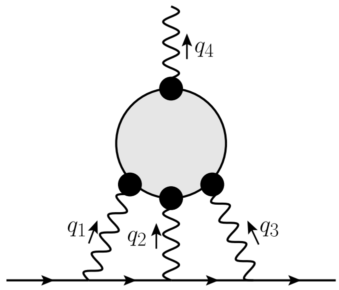

Fig. 1. The HLbL contribution to the $g-2$. The bottom line is the muon. The blob is filled with hadrons.

the HLbL tensor. The currents are $J^\mu(x) = \bar{q}Q_q\gamma^\mu q$ with the quark fields $q = (u,d,s)$ and charge matrix $Q_q = \text{diag}(2/3, -1/3, -1/3)$. The contribution from the heavy quarks, c, b, and t, can be evaluated fully perturbatively [7]. The evaluation of a_μ^{HLbL} involves an integration with the loop momenta, q_1, q_2, and q_3, running over all possible values and the fourth, $q_4 = q_1 + q_2 + q_3$, in the static limit, *i.e.* $q_4 \to 0$. This class of diagrams thus contains a complex interplay of strong interactions at different scales. In the below we work in the Euclidean domain and use $Q_i^2 = -q_i^2$.

The first full calculations of HLbL were done in the 1990s [8, 9] using mainly models. A model independent approach using dispersive theory allows for a much more precise determination [10, 11] for individual intermediate states but the short-distance part contains very many. Perturbative short-distance constraints have been used in constraining individual contributions starting in [8,

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https://doi.org/10.1016/j.physletb.2019.134994

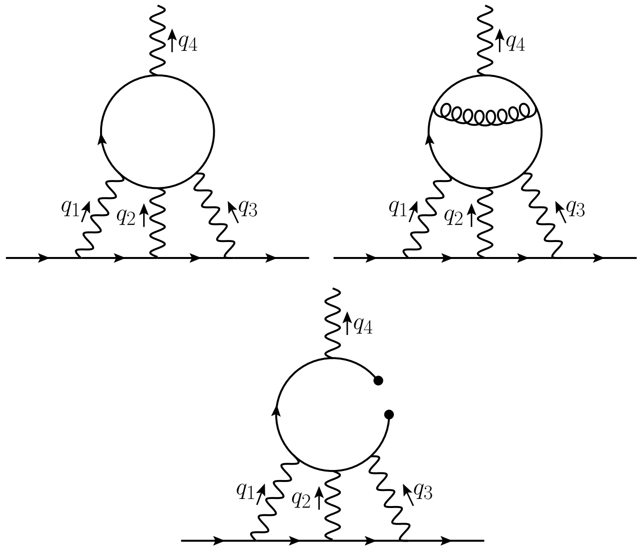

Fig. 2. Three examples of short-distance contributions to the HLbL when all q_i are large. (a) Pure quark loop, (b) gluonic corrections, (c) contribution from a vev.

12] as well as some matching with the quark loop [8]. The part with $Q_1^2 \approx Q_2^2 \gg Q_3^2$ was treated in [13].

Our best theoretical understanding of $\Pi^{\mu\nu\lambda\sigma}$ lies in the kinematic regions where the four Euclidean momenta are large, $Q_1 \sim Q_2 \sim Q_3 \sim Q_4 \gg \Lambda_{\rm QCD}$, where $\Lambda_{\rm QCD}$ is the hadronic scale. This allows for a perturbative description in terms of quarks and gluons. In this regime, one may construct a well-defined Operator Product Expansion (OPE), where the leading contribution corresponds to a simple quark loop with $\alpha_s = 0$. Nonperturbative corrections arising from nonzero expectation values of operators involving quarks and gluons [14], are suppressed by powers of $(\Lambda_{\rm QCD}/Q_i)^D$, starting at $D = 4$. Some of the different contributions are sketched in Fig. 2. While the calculation of the different terms of that expansion may be interesting for constraining some of the models, it does not correspond to any of the kinematic regions associated with the $g-2$ integral, *i.e.* $q_4 \to 0$, and we will not discuss this region further.

When considering that the photon associated to the external field should be set as soft, the OPE mentioned above is no longer valid, even though the three loop momenta, Q_1, Q_2 and Q_3, are large. This can *e.g.* be seen at the perturbative level when gluonic corrections are considered. Setting $\mu \sim Q_{i\neq 4}$, so that $\alpha_s(\mu)$ remains small, one would obtain corrections scaling as $\alpha_s^n(Q_i)\ln^m \frac{Q_4}{Q_i}$, which break the expansion. The invalidity of the simple OPE for this region becomes even more evident when trying to compute power corrections such as the one in Fig. 2c. Since no loop momentum flows through the loop, one of the quark propagators depends only on the soft momentum q_4 and is thus manifestly divergent in the static limit.

An analogous problem arises when trying to estimate the nucleon magnetic moment through the use of baryonic sum rules and it was successfully solved by formulating an alternative OPE in the presence of an external electromagnetic background field [15]. Note the anomalous magnetic moment is defined classically in an external magnetic field.[1] In this formalism, the soft emission (or, equivalently, response to the constant external field) can be produced not only by hard quark lines, but also by low-energy degrees of freedom via vacuum expectation values of operators. Note that not only operators with vacuum quantum numbers acquire non-zero values, but also those with the same quantum numbers as the external electromagnetic field $F_{\mu\nu}$, *e.g.* $\langle \overline{q}\sigma^{\mu\nu}q\rangle$.

In this letter we show how this formalism can be used to provide a model-independent and accurate description of the region where the three incoming loop momenta are large.

2. Some generalities about the HLbL tensor

We use the notation of [10,11] to facilitate using our results together with theirs. This section summarizes what we need from there. The HLbL tensor satisfies the Ward identities

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\}\,\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = 0\,. \tag{4}$$

Note that this implies that

$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = -\,q_{4\rho}\,\frac{\partial\,\Pi^{\mu\nu\lambda\rho}}{\partial q_{4\sigma}}(q_1,q_2,q_3)\,. \tag{5}$$

The dependence on q_4 is via $q_4 = q_1 + q_2 + q_3$. Equation (5) allows to compute $a_\mu^{\rm HLbL}$ directly from the derivative [17]. In [11], the HLbL tensor is decomposed in a basis with 54 Lorentz scalar functions $\hat{\Pi}_i$ free of kinematic singularities as

$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = \sum_{i=1}^{54} \hat{T}_i^{\mu\nu\lambda\sigma}\,\hat{\Pi}_i(q_1,q_2,q_3)\,. \tag{6}$$

The $\hat{T}_i^{\mu\nu\lambda\sigma}$ satisfy Ward identities equivalent to (4) and thus, in the static limit $q_4 \to 0$,

$$\frac{\partial\,\Pi^{\mu\nu\lambda\rho}(q_1,q_2,q_3)}{\partial q_{4\sigma}} = \sum_{i=1}^{54} \frac{\partial\,\hat{T}_i^{\mu\nu\lambda\rho}(q_1,q_2,q_3)}{\partial q_{4\sigma}}\,\hat{\Pi}_i(q_1,q_2,q_3)\,. \tag{7}$$

However, in this limit only 19 terms survive [11,13] and using the symmetry $(q_1,\mu) \leftrightarrow (q_2,\nu)$ one obtains

$$a_\mu^{\rm HLbL} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau\,\sqrt{1-\tau^2}\,Q_1^3 Q_2^3 \times \sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\,\overline{\Pi}_i(Q_1,Q_2,\tau)\,. \tag{8}$$

The integration variable τ is defined via $Q_3^2 = Q_1^2 + Q_2^2 + 2\tau Q_1 Q_2$. Expressions for the T_i can be found in [11], and the $\overline{\Pi}_i$ are related to the $\hat{\Pi}_i$ according to

$$\begin{aligned}
&\overline{\Pi}_1 = \hat{\Pi}_1\,,\ \overline{\Pi}_2 = C_{23}\left[\hat{\Pi}_1\right],\ \overline{\Pi}_3 = \hat{\Pi}_4\,,\ \overline{\Pi}_4 = C_{23}\left[\hat{\Pi}_4\right],\\
&\overline{\Pi}_5 = \hat{\Pi}_7\,,\ \overline{\Pi}_6 = C_{12}\left[C_{13}\left[\hat{\Pi}_7\right]\right],\ \overline{\Pi}_7 = C_{23}\left[\hat{\Pi}_7\right],\\
&\overline{\Pi}_8 = C_{13}\left[\hat{\Pi}_{17}\right],\ \overline{\Pi}_9 = \hat{\Pi}_{17}\,,\ \overline{\Pi}_{10} = \hat{\Pi}_{39}\,,\\
&\overline{\Pi}_{11} = -C_{23}\left[\hat{\Pi}_{54}\right],\ \overline{\Pi}_{12} = \hat{\Pi}_{54}\,,
\end{aligned} \tag{9}$$

where C_{ij} permutes the momenta according to $q_i \leftrightarrow q_j$ for $i,j \in \{1,2,3\}$. As can be seen, only the six functions $\hat{\Pi}_i$ for $i \in \{1,4,7,17,39,54\}$ are needed.

[1] We realized during the course of this work that a similar method has been used for another contribution to a_μ in [16].

3. The HLBL tensor in an external field

The HLBL tensor in (3) can be obtained from

$$\int d^4x d^4y\, e^{-i(q_1\cdot x+q_2\cdot y)} \langle 0|T\left\{J^\mu(x)J^\nu(y)J^\lambda(0)\right\}|\gamma(-q_4)\rangle$$
$$\equiv -\Pi^{\mu\nu\lambda}(q_1,q_2,q_3) = -\epsilon_\sigma(-q_4)\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3)\,, \tag{10}$$

where we have captured the fourth photon vertex via the matrix element with a possibly off-shell photon and defined $q_3 = q_4 - q_1 - q_2$.

In the static limit, $q_4 \to 0$, one can factor out the soft photon part according to

$$\Pi^{\mu\nu\lambda}(q_1,q_2,q_3) \equiv \Pi_F^{\mu\nu\lambda\rho\sigma}(q_1,q_2)\langle 0|F_{\rho\sigma}|\gamma(-q_4)\rangle$$
$$= i\,q_{4\rho}\epsilon_\sigma(-q_4)\Pi_F^{\mu\nu\lambda[\rho\sigma]}(q_1,q_2)\,, \tag{11}$$

where $[\rho\sigma]$ indicates antisymmetrization. Combining (11) with (5) and (10) one obtains

$$\lim_{q_4\to 0}\frac{\partial\,\Pi^{\mu\nu\lambda\rho}}{\partial q_{4\sigma}}(q_1,q_2,q_3) = i\Pi_F^{\mu\nu\lambda[\rho\sigma]}(q_1,q_2)\,. \tag{12}$$

The momentum conservation in the static limit reads $q_1+q_2+q_3 = 0$. From the above equivalence, (12), together with (7), it is possible to obtain the required $\hat{\Pi}_i$ to calculate a_μ^{HLbL}.

The short-distance quantity, $\Pi_F^{\mu\nu\lambda[\rho\sigma]}(q_1,q_2)$ does not depend on the soft-photon momentum and can be calculated directly using the methods of OPE in an external electromagnetic field of [15]. The arguments for our case, all three Q_i^2 large, are the same as those used there for the 2-point function at large p^2. By construction this procedure is free from infrared divergent propagators. The coupling to an external field can arise in two different ways, either via a soft insertion on a hard quark line or from the vacuum expectation values induced by the external electromagnetic field.

In order to simplify calculations we work in the radial gauge for the external electromagnetic field. This implies to first order, *i.e.* in the static limit,

$$A_\sigma(z) = \frac{1}{2}z^\rho F_{\rho\sigma}(0) + \dots\,, \tag{13}$$

allowing to calculate immediately in the $q_4 = 0$ limit. This gauge is particularly convenient for the soft QCD parts as well, since it allows to easily expand non-local terms such as $\langle\bar{q}(x)q(0)\rangle$ into gauge invariant local ones. This stems from the equivalence between partial derivatives and covariant derivatives in expansions of fields such as for instance $q(x) = q(0) + x^\mu D_\mu q(0) + \dots$. A pedagogical introduction is in [18].

We first look at the contributions with a soft insertion on a hard line. The lowest order topology is illustrated in Fig. 3a. It consists of a quark loop with three hard insertions and one soft. This soft insertion enters through the NLO term in a propagator modified by the external field [19]:

$$iS(x,y) = iS_0(x-y) + ie_q\int d^4u\, iS_0(x-u)\not{A}(u)iS_0(u-y)\,, \tag{14}$$

where $S_0(x-y)$ is the usual free Dirac propagator. The static limit is automatically set when using (13) for $A_\sigma(u)$.

Notice how an insertion of the second piece in (14) is equivalent in the parton limit ($\alpha_s = 0$) to an insertion of an electromagnetic vertex, $\mathcal{L}_{EM} = e_q\bar{q}\not{A}q$. Then, the calculation leads to the same result as the usual quark loop obtained from the calculation with Fig. 2a, including the dependence on the quark mass. We have calculated using both methods as well as compared with quark loop expressions from [20]. The agreement is exact, both numerical and

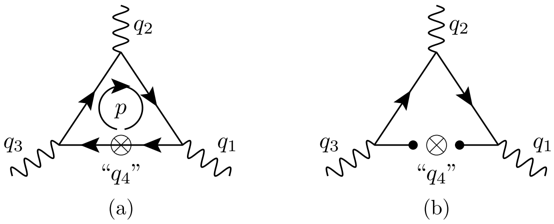

Fig. 3. The two leading terms in the external field OPE: (a) The quark loop with loop momentum p, and (b) the condensate $\langle\bar{q}\sigma_{\alpha\beta}q\rangle$. The presence of the external field is here represented by a crossed vertex. Note that there is no divergent propagator here as the momentum q_4 never enters the diagram explicitly.

analytical. In future work we intend to calculate the gluonic corrections to this. This part shows that the quark loop at short distances is indeed the first term in a systematic expansion. We do not quote the analytical expressions since they are rather lengthy.

We now turn to the power corrections. The lowest dimensional contribution comes from

$$\langle\bar{q}\,\sigma_{\alpha\beta}\,q\rangle \equiv e_q F_{\alpha\beta}X_q\,, \tag{15}$$

where e_q is the one of the light quark charges in the matrix Q_q, and the X_q are so-called tensor coefficients related to the magnetic susceptibility that are known from lattice QCD [21]. Regarding the suppression of this condensate as compared to the leading term, the only scale to compensate dimensions is Λ_{QCD}. From naive dimensional analysis, this contribution is thus suppressed by at least a factor of $\frac{\Lambda_{\mathrm{QCD}}}{Q_{\mathrm{hard}}}$. The contribution is schematically drawn in Fig. 3b. The resulting Dirac structure is of the form, ij indicate Dirac indices:

$$(\gamma_\mu iS(x-y)\gamma_\nu iS(y)\gamma_\lambda)_{ij}\,\langle q_0\bar{q}_x\rangle_{ji}\,, \tag{16}$$

where $\langle q_0\bar{q}_x\rangle_{ji} \equiv \langle 0|q_j(0)\bar{q}_i(x)|0\rangle$ is expanded as described above to leave the result as a function of local operators:

$$\langle q_0\bar{q}_x\rangle_{ji} = \langle q_0\bar{q}_0\rangle_{ji} + x^\mu\langle 0|q_j(\bar{q}\overleftarrow{D}_\mu^\dagger)_i|0\rangle\; + \mathcal{O}(\Lambda_{\mathrm{QCD}}^{5-2}). \tag{17}$$

If one decomposes both terms in the Clifford basis $\Gamma \equiv \{\mathcal{I},\gamma^5,\gamma^\mu,\gamma^\mu\gamma^5,\sigma^{\mu\nu}\}$, one finds that the magnetic susceptibility gives a contribution proportional to $\sigma^{\mu\nu}$ for the first term and $\gamma^\mu\gamma^5$ (suppressed by the quark mass through the Dirac equation of motion) for the second. Our result agrees with the one of [22]. In our calculation, both contributions enter at the same level, since for the former one also needs an extra mass insertion coming from one of the free propagators in order to recover a non-zero Dirac trace. The Ward identities from (4) are only recovered when both pieces are added together. The needed extra insertion of a quark mass leads to a suppression compared to the quark loop of two powers of the hard scale. The analytical result for the leading power suppressed contributions are

$$\hat{\Pi}_1 = m_q X_q e_q^4\,\frac{-4\left(Q_1^2+Q_2^2-Q_3^2\right)}{Q_1^2Q_2^2Q_3^4}\,,\qquad \hat{\Pi}_7 = 0\,,$$
$$\hat{\Pi}_4 = m_q X_q e_q^4\,\frac{8}{Q_1^2Q_2^2Q_3^2}\,,\qquad \hat{\Pi}_{39} = 0\,,$$
$$\hat{\Pi}_{17} = m_q X_q e_q^4\,\frac{8}{Q_1^2Q_2^2Q_3^4}\,,$$
$$\hat{\Pi}_{54} = m_q X_q e_q^4\,\frac{-4\left(Q_1^2-Q_2^2\right)}{Q_1^4Q_2^4Q_3^2}\,. \tag{18}$$

Work is in progress to calculate the power corrections that are not suppressed by quark masses but these will be suppressed by more

Table 1
The total contributions to a_μ^{HLbL} from both the quark loop and the next term in the OPE. The condensate contributions have been divided into two parts, one for the up and down quarks and the other for the strange quark.

Q_{min}	Quark loop	$m_u X_u + m_d X_d$	$m_s X_s$
1 GeV	17.3×10^{-11}	5.40×10^{-13}	8.29×10^{-13}
2 GeV	4.35×10^{-11}	3.40×10^{-14}	5.22×10^{-14}

powers of the hard scale. Preliminary results indicate that the contributions not suppressed by quark masses occur first suppressed by four powers of the hard scale.

4. Numerical results

In this section we present numerical results obtained from the external field OPE. For the numerical integration of a_μ^{HLbL} in (8), we use the Cuba library [23], both employing a Monte Carlo algorithm (`Vegas`) as well as a deterministic algorithm (`Cuhre`) for a cross-check.

First of all we consider the quark loop. In order to compare with [7], we use constituent quark masses of $m_{u,d,s} = 240$ MeV and $N_c = 3$. This yields $a_\mu^{\text{HLbL}} = 80.30 \times 10^{-11}$, which is in excellent agreement with the result quoted in [7]. This was of course expected given that our leading result analytically agrees with the quark loop.

We also numerically evaluate the contribution to a_μ^{HLbL} from the regime where our OPE is valid. In order to allow for future cross-checks, we first calculate it for two lower cut-offs $Q_{min} = 1$, 2 GeV such that $Q_{i=1,2,3} \geq Q_{min}$. The condensates X_q have been estimated in [21] on the lattice, and the values are[2]

$$X_u = 40.7 \pm 1.3\,\text{MeV}\,, \quad X_d = 39.4 \pm 1.4\,\text{MeV}\,,$$
$$X_s = 53.0 \pm 7.2\,\text{MeV}\,. \tag{19}$$

The quark masses we use are $m_u = m_d = 5$ MeV and $m_s = 100$ MeV.[3] The results are presented in Table 1. For an order of magnitude comparison also the quark loop with zero quark masses is included there with the same region of integration. As can be seen, the contributions from the condensates are strongly suppressed as compared to the quark loop. This is expected given the smallness of $m_q X_q$. Finally, in addition to the above comparison we also look at a_μ^{HLbL} for a range of Q_{min} in Fig. 4. The running of the $\overline{\text{MS}}$ quark masses is implemented using the package `CRunDec` [24]. In addition to the condensate contribution and massless quark loop, also the mass correction to the massless quark loop is plotted. As can be seen, both the condensate contribution and the mass correction scale the same way in Q_{min}. This Q_{min} dependence goes as $1/Q_{min}^4$, while the massless quark loop scales perfectly as $1/Q_{min}^2$. Note that higher-dimensional contributions contain condensates not suppressed by the small quark mass values. They are expected to dominate the power corrections when the cut-off Q_{min} is small enough.

5. Conclusions and outlook

Due to the long-standing deviation between the experimental value and the Standard Model prediction of the muon magnetic moment, there is at present much work going into reducing the errors on both quantities. One of the two main uncertainties in the Standard Model value comes from the HLbL contribution, a_μ^{HLbL}.

[2] The sign differs from [21] due to differences in conventions.

[3] Given the numerical smallness of the result more precise values are not needed.

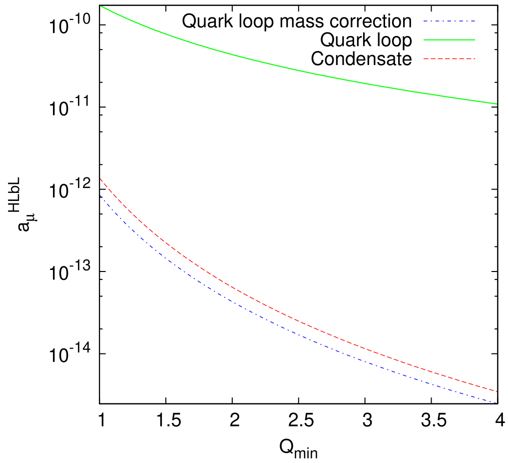

Fig. 4. The Q_{min} dependence of a_μ^{HLbL}.

The loop integral in a_μ^{HLbL} is particularly complicated due to the various regions of virtual or internal photon momenta. In this letter we have focused on the region where the three (Euclidean) virtual photon momenta are large. We have shown how the standard OPE in the vacuum of the associated four-point correlation function breaks down beyond the leading order in the static limit in which $g - 2$ is defined. Instead, an OPE in the presence of an electromagnetic background field has been used. The photon associated to the soft momentum $q_4 \to 0$ can be emitted from both high-energy degrees of freedom, *i.e.* quarks, or from long distance ones parametrized by induced vacuum expectation values of QCD operators.

The leading order contribution arises from the radiation of a hard line and is analytically identical to the purely perturbative quark loop. This proves the expectation that the perturbative quark loop is the first term in a systematic expansion in this region. The first power correction in our OPE contains a condensate related to the magnetic susceptibility of the QCD vacuum. Our numerical study has shown that its contribution to a_μ^{HLbL} is suppressed, as compared to the quark loop, by roughly three orders of magnitude, as a consequence of the small values of the quark masses and the condensate itself. The leading contribution scales as suppressed by two powers of the hard scale while the first power correction is suppressed by four powers of the hard scale.

The higher order power corrections are not suppressed by the small quark masses. Together with the purely perturbative α_s correction, they should be enough to give a first reliable estimate of the onset of the asymptotic domain. Both calculations are underway and are expected to be presented in a forthcoming publication.

Acknowledgements

We thank Martin Hoferichter and Peter Stoffer for useful discussions and the sharing of their analytic quark loop expressions. This work is supported in part by the Swedish Research Council grants contract numbers 2015-04089 and 2016-05996, and by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 668679.

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