# Session 10

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## 1 Vandermonde matrix

The interpolation problem is, given a smooth function $f$, to find an order $n$ polynomial $p_n(x) = \sum_{j=0}^{n} a_j x^j$ equal to $f$ on $\{x_0, x_1, \cdots, x_n\}$, which can be converted to a linear system:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

Here the matrix on the left is called the Vandermonde matrix, which is also very important in quantum mechanics. We will proof the existence and uniqueness of $p_n(x)$ by studying the Vandermonde matrix.

### 1.1

$$\det\left(\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} 1 & -x_0 & & & \\ & 1 & -x_0 & & \\ & & \ddots & \ddots & \\ & & & 1 & -x_0 \end{bmatrix}\right)$$

What is the result of matrix product on the right?

### 1.2

Prove that for a Vandermonde matrix $V$, $\det(V) = \prod_{0 \le i < j \le n}(x_j - x_i)$, by conduction.

### 1.3

Conclude that when $x_i$ are distinct, the Vandermonde matrix is invertible.

## 2 Interpolation basics

- True or False? For the nodes $x_0 = 0, x_1 = 1, x_2 = 2$, the Lagrange interpolation polynomial $L_0(x)$ is $-x^2 + 1$.
- True or False? We compute the Hermite interpolant with 3 distinct nodes of a function $f$ that is a polynomial of degree 4. Then this Hermite interpolant is identical to $f$. (In short: Hermite interpolation with 3 nodes is exact for polynomials of degree 4.)
- True or False? Hermite interpolation with 4 distinct nodes is exact for polynomials of degree 6.
- True or false: Let $p_n$ be the Lagrange interpolant to a function $f$ with $n+1$ interpolation points, and $e_n(x) = |p_n(x) - f(x)|$. The interpolation error $\|e_n\|_\infty$ *always* gets arbitrarily small for large $n$, i.e., $\|e_n\|_\infty \to 0$ as $n \to \infty$.

## 3 Lagrange interpolation polynomial example

Let $x_0, \ldots, x_n$ be distinct interpolation nodes, and let

$$p_n(x) = \sum_{k=0}^{n} L_k(x)(x_k)^j,$$

where $j$ is an integer and $n \ge j > 0$. What is the $p_n(x)$ function? What are the values of $p_n(0)$ and $p_n(1)$?

# 4 Hermite interpolation polynomial example

Recall that the Hermite interpolation of a function $f$ at the points $x_0, x_1, x_2$ has the form

$$p(x) = \sum_{j=0}^{2} H_j(x)f(x_j) + \sum_{j=0}^{2} K_j(x)f'(x_j).$$

where

$$H_j(x) = (1 - 2(x - x_j)L'_j(x_j))L_j^2(x), \quad K_j(x) = (x - x_j)L_j^2(x).$$

## 4.1

Show that the polynomial

$$-\frac{1}{\pi}x^2 + x$$

is the Hermite interpolation polynomial of $f(x) := \sin(x)$ based on the nodes $x_0 = 0$, $x_1 = \pi$.

## 4.2

Calculate all 4 Hermite basis for the nodes $x_0 = 0$, $x_1 = 1$. This might be useful for the next section.

# 5 spline interpolation

Regular interpolation methods suffer from several issues:

- They cannot guarantee that the interpolation error will go to zero for all functions as we use more interpolation points. For example, the Runge phenomenon.
- Higher degree polynomials require more computations.

To address these issues, we often use piecewise polynomial interpolation, known as spline interpolation. There are two common types of spline interpolation: Lagrange and Hermite splines. The former is the simplest $C$ interpolant, while the latter is the simplest $C^1$ interpolant.