Session 9

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1 SVD

We know that every symmetric matrix A can be diagonalized by orthogonal matrix Q:

$$A = Q\Lambda Q^T$$

We will use this property to show that every matrix $A \in \mathbb{R}^{m\times n}$ can be decomposed into $U\Sigma V^T$ where U, V are orthogonal and Σ is diagonal.

1.1

Show that for any matrix $A \in \mathbb{R}^{m\times n}$, $A^T A$ is real symmetric. So we have $A^T A = VDV^T$, where D is diagonal and V is orthogonal.

1.2

Let $B = AV$. What can you infer about the columns of B?

1.3

Finally conclusion that $A = U\Sigma V^T$ where U, V are orthogonal and Σ is diagonal. **Note:** Based on above analyzing, If we want to obtain the SVD of A, we can first compute the eigenvalue decomposition of $A^T A$ to obtain V then compute U from AV.

2 Low-rank approximation

Low-rank approximation is valuable for three main reasons. First, data compression: it reduces the data size by capturing only essential information, making storage and processing more efficient. Second, noise reduction: it smooths out random variations, leaving the core structure intact. Third, feature extraction: it highlights the most significant patterns in the data, which is particularly useful in machine learning for reducing the number of features. Together, these benefits make low-rank approximation a powerful tool in simplifying and analyzing complex datasets.

We want to solve the problem

$$\begin{aligned} \min_B \quad & \|A - B\|_2^2 \\ s.t. \quad & rank(B) \le k \end{aligned}$$

suppose $A \in \mathbb{R}^{n\times n}$, A has SVD $A = U\Sigma V^T$, $|\sigma_1| > |\sigma_2| > \cdots > |\sigma_n|$ and corresponding singular vectors $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ which are all normalized.

2.1

Show that in the space spanned by $v_1, \cdots, v_{k+1}$, there exsits a vector $w = \alpha v_1 + \cdots + \alpha_{k+1} v_{k+1}$ such that $Bw = 0$

2.2

What is the result of Av_i? what is the result of Aw?

2.3

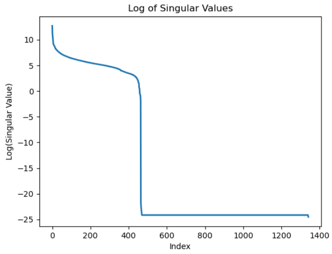
Suppose w is normalized, $\sum \alpha_i^2 = 1$. Then conduct that for any matrix B with rank less than k, $\|A - B\|_2^2 \ge \sigma_{k+1}$. Hint: consider the definition of 2-norm.

2.4

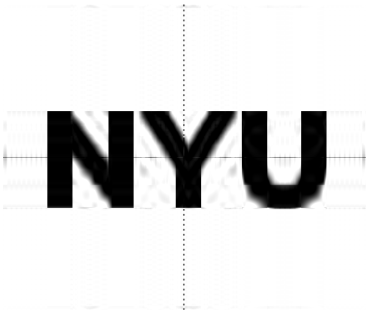
Show that if we take $B = \sum_{i=1}^k \sigma_i u_i v_i^T$, then the minimum is achieved.

3 Image compression

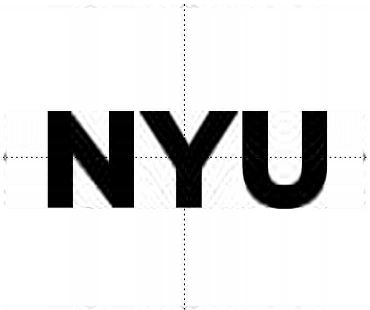
(a) original picture



(b) log plot of singular values



(a) rank 10



(b) rank 20

(c) rank 30

4 QR

We simplify the demonstration by assuming A is full rank and has eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$.

4.1

For any matrix $A \in \mathbb{R}^{n\times n}$, with $A = QR$. $a_1, a_2, \cdots, a_n$ are columns of A and $q_1, q_2, \cdots, q_n$ are columns of Q. Show that for any k, space spann by $a_1, \cdots, a_k$ is the same as $q_1, \cdots, q_k$.

4.2

try to undersand the following iteration, Argure that Q_k converges to Q

$$
\begin{aligned}
&\hat{Q}_0 = I, \\
&B_1 = A\hat{Q}_0, \\
&\hat{Q}_1\hat{R}_1 = B_1, \\
&B_2 = A\hat{Q}_1, \\
&\hat{Q}_2\hat{R}_2 = B_2, \\
&B_2 = A\hat{Q}_2, \\
&\cdots
\end{aligned}
$$

4.3

compare above iteration with QR iteration, Argue that $\hat{Q}_k = Q_k Q_{k-1} \cdots Q_1$

$$
\begin{aligned}
&A_1 = A, \\
&A_1 = Q_1 R_1 \\
&A_2 = R_1 Q_1 \\
&A_2 = Q_2 R_2 \\
&A_3 = R_2 Q_2 \\
&\cdots
\end{aligned}
$$