Worksheet 4

1 Norms Equivalency

Two norms in a finite-dimensional linear space X (e.g.: $\mathbb{R}^n$), $\|\cdot\|_a$ and $\|\cdot\|_b$ are called equivalent if there is a constant c such that for all x in X,

$$\|\boldsymbol{x}\|_a \leq c\|\boldsymbol{x}\|_b, \qquad \|\boldsymbol{x}\|_b \leq c\|\boldsymbol{x}\|_a. \tag{1}$$

1.1

Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, and we know that an algorithm produces a sequence of vectors $\{\boldsymbol{x}_n\}_{n\geq 1}$, $\|\boldsymbol{x}_n\|_a \to 0$ as $n \to \infty$. What could we conclude about $\|\boldsymbol{x}_n\|_b$'s behavior for $n \to \infty$?

1.2

Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, $\|\cdot\|_b$ and $\|\cdot\|_c$ are equivalent. Are $\|\cdot\|_a$ and $\|\cdot\|_c$ equivalent?

1.3

We first show that the vector norms on $\mathbb{R}^n$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, are equivalent. To do this prove the inequality:

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \sqrt{n}\|\boldsymbol{x}\|_\infty.$$

1.4

The induced matrix norm on $\mathbb{R}^{n\times n}$: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent as well. Prove the inequality

$$\|A\|_\infty \leq \sqrt{n}\|A\|_2,$$
$$\|A\|_2 \leq \sqrt{n}\|A\|_\infty.$$

1.5

(optional) Prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on $\mathbb{R}^n$.

2 matrix 2-Norms

We are going to show that, if U is a unitary matrix ($U^TU = I$), then

$$\|AU\|_2 = \|A\|_2, \quad \|UA\|_2 = \|A\|_2$$

2.1

Suppose U is a unitary matrix, show that $\|Ux\|_2 = \|x\|_2$ for any vector x.

2.2

recall that the definition of matrix 2-norm is $\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$, show that $\|UA\|_2 = \|A\|_2$ and $\|AU\|_2 = \|A\|_2$ using the definition and above result.

2.3 connection with SVD

For any matrix $A \in \mathbb{R}^{m\times n}$, we can express it by Singular Value Decomposition (SVD)

$$A = U\Sigma V^T$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and $\Sigma \in \mathbb{R}^{m\times n}$ is a diagonal matrix with non-negative entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ on the diagonal. Here $r = \text{rank}(A)$. Based the uni-Invariance of the frobenius norm and 2-norm, we have,

$$\|A\|_2 = \|\Sigma\|_2$$

Show that $\|\Sigma\|_2 = \sigma_1$.

2.4

show that $\|A^{-1}\| = 1/\sigma_r$

3 An ill-conditioned matrix

3.1 review of condition number

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$$

$$A = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$$

This matrix is real and symmetric, so it can be diagonalized as

$$A = Q\Lambda Q^T$$

Solve the eigenvalues of A and using the eigenvalues to compute the condition number of A.

3.2

Now we considet solve the linear system $Ax = b$ with $b = [1, 1]^T$ and $Ax = \tilde{b}$ with $\tilde{b} = [1.1, 0.9]^T$. You can solve the lienar system by LU decomposition or directly compute the inverse of A.

3.3

Compute the relative error of the solution x and input b. What is the theoretical upper bound of ratio of them? Compare the actual error with the theoretical bound.

3.4

Try $b = [1, -1]^T$ and $\tilde{b} = [1.1, -0.9]^T$. Also compute the ratio of errors.

4 Conditional number for the Hilbert matrix

The Hilbert matrix $H \in \mathbb{R}^{n \times n}$ is a matrix with entries

$$h_{ij} = \frac{1}{i + j - 1}.$$

4.1

Using MATLAB, compute the 2-norm-based condition numbers for $n = 3, 5, 10, 20, 25$. You can use MATLAB's built-in function *cond()* to compute the condition number of a matrix.

4.2

Let's consider a relative right hand side perturbation $\delta\boldsymbol{b}$ of a linear system with $\|\delta\boldsymbol{b}\|_2/\|\boldsymbol{b}\|_2 \approx 10^{-15}$. Write down the corresponding bounds $\|\delta\boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2$ from the theory we discussed in class.

4.3

Now, let's compute the actual error. Use the right-hand side vector with entries $b_i = \sum_{j=1}^{n}(j/(i+j-1))$ chosen such that the solution vector has entries $x_i = i$. Now, Compute the numerical solutions[1] $\boldsymbol{x}$, then re-compute $\boldsymbol{b} = H\boldsymbol{x}$ and compare the relative right-hand side error and the relative error in the solutions. How much are these better than the estimates you got from the condition number?

[1]Note that all these computations contain tiny errors due to the final precision of computer computations.