

Session 11

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1 Deriving a new quadrature rule

Given $f : [0, 1] \rightarrow \mathbb{R}$, you want to derive a new quadrature rule that does uses not only function values, but also gradient values:

$$\int_0^1 f(x) dx \approx \alpha_0 f(0) + \alpha_1 f'(0) + \alpha_2 f(1). \quad (1)$$

1.1

First, find polynomials $J_0, J_1, J_2 \in \mathcal{P}_2$, with the following properties:

$$\begin{aligned} J_0(0) &= 1, & J'_0(0) &= 0, & J_0(1) &= 0 \\ J_1(0) &= 0, & J'_1(0) &= 1, & J_1(1) &= 0 \\ J_2(0) &= 0, & J'_2(0) &= 0, & J_2(1) &= 1. \end{aligned}$$

(Hint: For each J_i , make an ansatz for a quadratic polynomial using the monomial basis.) Given f , you can now define a polynomial approximation $p \in \mathcal{P}_2$ via

$$p(x) = f(0)J_0(x) + f'(0)J_1(x) + f(1)J_2(x). \quad (2)$$

The polynomial p is an approximation to f in the sense that $p(0) = f(0)$, $p'(0) = f'(0)$ and $p(1) = f(1)$.

1.2

Use the polynomial p derived above and the same method used to derive the Newton-Cotes quadrature rules, to find the coefficients α_0 , α_1 and α_2 in equation 1.

1.3

Use your new quadrature rule to approximate $\int_0^1 \exp(2x) \sin^2(x) dx$, and also compare with Simpson's rule. The exact value of this integral is 1.2668.... Note $e^2 \sin^2(1) \approx 5.232$, $e^1 \sin^2(\frac{1}{2}) \approx 0.6248$

2 Trapezoidal rule for smooth periodic functions

We investigate how the (composite) trapezoidal rule performs for smooth, periodic functions. Consider integrating the smooth, periodic function $f(x) = e^{\sin x}$ over a single period. The exact value of the integral is

$$I(f) = \int_0^{2\pi} e^{\sin x} dx = 7.95492652101284527 \dots$$

2.1

Write down the composite trapezoidal rule $T_N(f)$ on equispaced nodes $0 = x_0 \leq \dots \leq x_N = 2\pi$ for estimating the value of this integral.

2.2

Simplify your expression for $T_N(f)$ using the periodicity of f . Show that $T_N(f)$ is equivalent to both a left-endpoint Riemann sum and a right-endpoint Riemann sum approximation to $I(f)$.

3 Convergence order of quadrature

3.1

We would like to integrate a function on $[0, 1]$ using the composite trapezoid rule with sub-interval size h . We name the result T_h . How does the error ($e_h = |T_h - I|$) scale with h ?

3.2

let $f(x) = e^{\sin(x) + \pi \cos(x)} + e^x$ Compute $T_h(f)$ for various progressively larger N . Plot the quadrature errors against h on a log-log plot.

3.3

In real application, we do not know the true value of the integral. To verify the order of convergence of our code, we can calculate this quantity:

$$\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}.$$

How does this quantity scale with h ?

3.4

show this by code using the same f .

Solutions

1.1

The polynomials $J_0, J_1, J_2 \in \mathcal{P}_2$ with the given properties can be found as follows:

$$J_0(x) = 1 - x^2$$

$$J_1(x) = x - x^2$$

$$J_2(x) = x^2.$$

2/3, 1/6, 1/3
1.744, 1.2885333333