Session 8

Xuanxi Zhang

1 Gershgorin disks and the power method

Consider the matrix

$$A = \begin{bmatrix} -6 & 2 & 0.3 & 0 & -0.7 \\ 2 & -4 & 0.1 & 0.05 & 0 \\ 0.3 & 0.1 & 2 & 0.1 & 0.1 \\ 0 & 0.05 & 0.1 & 4 & 0 \\ -0.7 & 0 & 0.1 & 0 & 6 \end{bmatrix}$$

and recall the definition of the Gershgorin disks:

$$D_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}.$$

Review of Complex Numbers: $z \in \mathbb{C}$ can be represented as $z = x + iy$, where $x, y \in \mathbb{R}$ is called real part and image part, i is the imaginary unit, $i * i = -1$. The modulus of z is defined as $|z| = \sqrt{x^2 + y^2}$.

1.1

Argue that all eigenvalues of A are real.

1.2

What are the Gershgorin disks for A? Give a set, $D \subset \mathbb{R}$, that contains all eigenvalues of A.

1.3

Can you conclude that the eigenvalue with the largest absolute value is simple?

1.4

Argue that A is invertible. Conclude that all diagonally dominant matrix is invertible.

1.5

True or False? Let $A \in \mathbb{R}^{n \times n}$ and D_i, $i = 1, 2, \ldots, n$, be the Gerschgorin disks of A. If $0 \in \bigcup_{i=1}^{n} D_i$ then A is singular.

1.6

Write down the first iteration of the power method starting from $\boldsymbol{x}_0 = (0, 0, 0, 0, 1)^T$. You don't need to normalize. Explain why $\boldsymbol{x}_0 = \mathbf{0}$ is not a suitable starting point.

1.7

The eigenvalues of A, after rounding, are $\{-7, -3, 2, 4, 6\}$. Which eigenvalue direction will the sequence of the previous question converge to?

2 Computing eigenvalues via the Power Iteration

Given the following matrix:

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix},$$

It has eigenvalues and eigenvectors:

$$\lambda_1 = 0,\ \boldsymbol{v}_1 = \begin{bmatrix} 0.41 \\ -0.82 \\ 0.41 \end{bmatrix}, \quad \lambda_2 = -6,\ \boldsymbol{v}_2 = \begin{bmatrix} 0.71 \\ 0.0 \\ -0.71 \end{bmatrix}, \quad \lambda_3 = 3,\ \boldsymbol{v}_3 = \begin{bmatrix} -0.58 \\ -0.58 \\ -0.58 \end{bmatrix}.$$

2.1

Calculate the first iterate of the power method when $\boldsymbol{x}_0 = (0, 1, 1)^T$.

2.2

Which eigenvalue direction will the sequence start from above x_0 converge to?

2.3

Give an initialization vector such that the power method does *not* converge to the direction of the largest (in absolute value) eigenvalue.

2.4

Write a simple program implementing the power method for the matrix A.

3 The Inverse Iteration

Take A to be the matrix above, and let $\theta \in \mathbb{R}$ and let $\boldsymbol{x}_0 \in \mathbb{R}^3$.

3.1

If $\theta = 2$, where will the sequence defined in (i) converge to and why?

3.2

If $\theta = -2$, where will the sequence defined in (i) converge to and why?

3.3

Write a simple program implementing the inverse power method.

4

What is the flops for Computing

- $(I - 2vv^T)x$
- $x - (2vv^T)x$
- $x - 2 * v(v^T x)$