# Session 11

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## 1 Deriving a new quadrature rule

Given $f : [0,1] \to \mathbb{R}$, you want to derive a new quadrature rule that does uses not only function values, but also gradient values:

$$\int_0^1 f(x)\,dx \approx \alpha_0 f(0) + \alpha_1 f'(0) + \alpha_2 f(1). \tag{1}$$

### 1.1

First, find polynomials $J_0, J_1, J_2 \in \mathcal{P}_2$, with the following properties:

$$\begin{aligned} J_0(0) &= 1, \quad J_0'(0) = 0, \quad J_0(1) = 0 \\ J_1(0) &= 0, \quad J_1'(0) = 1, \quad J_1(1) = 0 \\ J_2(0) &= 0, \quad J_2'(0) = 0, \quad J_2(1) = 1. \end{aligned}$$

(Hint: For each $J_i$, make an ansatz for a quadratic polynomial using the monomial basis.) Given $f$, you can now define a polynomial approximation $p \in \mathcal{P}_2$ via

$$p(x) = f(0)J_0(x) + f'(0)J_1(x) + f(1)J_2(x). \tag{2}$$

The polynomial $p$ is an approximation to $f$ in the sense that $p(0) = f(0)$, $p'(0) = f'(0)$ and $p(1) = f(1)$.

### 1.2

Use the polynomial $p$ derived above and the same method used to derive the Newton-Cotess quadrature rules, to find the coefficients $\alpha_0$, $\alpha_1$ and $\alpha_2$ in equation **??**.

### 1.3

Use your new quadrature rule to approximate $\int_0^1 \exp(2x)\sin^2(x)\,dx$, and also compare with Simpson's rule. The exact value of this integral is $1.2668\ldots$. Note $e^2\sin^2(1) \approx 5.232$, $e^1 \sin^2(\frac{1}{2}) \approx 0.6248$

## 2 Trapezoidal rule for smooth periodic functions

We investigate how the (composite) trapezoidal rule performs for smooth, periodic functions. Consider integrating the smooth, periodic function $f(x) = e^{\sin x}$ over a single period. The exact value of the integral is

$$I(f) = \int_0^{2\pi} e^{\sin x}\,dx = 7.95492652101284527\ldots.$$

### 2.1

Write down the composite trapezoidal rule $T_N(f)$ on equispaced nodes $0 = x_0 \le \cdots \le x_N = 2\pi$ for estimating the value of this integral.

### 2.2

Simplify your expression for $T_N(f)$ using the periodicity of $f$. Show that $T_N(f)$ is equivalent to both a left-endpoint Riemann sum and a right-endpoint Riemann sum approximation to $I(f)$.

## 3 Convergence order of quadrature

### 3.1

We would like to integrate a function on $[0,1]$ using the composite trapezoid rule with sub-interval size $h$. We name the result $T_h$. How does the error ($e_h = |T_h - I|$) scale with $h$?

### 3.2

let $f(x) = e^{\sin(x)+\pi\cos(x)} + e^x$ Compute $T_h(f)$ for various progressively larger $N$. Plot the quadrature errors against $h$ on a log-log plot.

### 3.3

In real application, we do not know the true value of the integral. To verify the order of convergence of our code, we can calculate this quantity:

$$\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}.$$

How does this quantity scale with $h$?

### 3.4

show this by code using the same $f$.

# Solutions

## 1.1

The polynomials $J_0, J_1, J_2 \in \mathcal{P}_2$ with the given properties can be found as follows:

$$\begin{aligned} J_0(x) &= 1 - x^2 \\ J_1(x) &= x - x^2 \\ J_2(x) &= x^2. \end{aligned}$$

2/3, 1/6, 1/3
1.744, 1.2885333333