R2

Xuanxi Zhang

1

Recall the indicator function $\mathbf{1}_A(\omega)$ is defined to be $\mathbf{1}$ if $\omega \in A$ and 0 otherwise.

1. For sets $A_1, \ldots, A_n$ with union $A = \cup_{i=1}^n A_i$, show that for all $\omega \in \Omega$,

$$\mathbf{1}_A(\omega) = \sum_{k=1}^n (-1)^{k-1} \sum_{I \subseteq \{1,2,\ldots,n\}:|I|=k} \mathbf{1}_{\cap_{i\in I} A_i(\omega)}.$$

2. Prove the inclusion-exclusion principle: any probability measure P on Ω satisfies

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n (-1)^{k-1} \sum_{I \subseteq \{1,2,\ldots,n\}:|I|=k} \mathbb{P}\left(\bigcap_{i\in I} A_i\right).$$

2 4.74

The number of offspring of an organism is a discrete random variable with mean μ and variance σ^2. Each of its offspring reproduces in the same manner. Find the expected number of offspring in the third generation and its variance.

3

Show that if X_i are independent, identically distributed exponential random variables, $X_i \sim \mathrm{Exp}(\lambda)$, then

$$Y_i = \sum_{i=1}^n X_i \sim \mathrm{Gamma}(n, \lambda).$$

Use the fact that the mgf of $\mathrm{Gamma}(\alpha, \lambda)$ is $M(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$.

4 5.18

Suppose that a company ships packages that are variable in weight, with an average weight of 15 lb and a standard deviation of 10 lb . Assuming that the packages come from a large number of different customers so that it is reasonable to model their weights as independent random variable, find the probability that 100 packages will have a total weight exceeding 1700 lb .

5 5.2

Let X_i be independent random variables with $\mathbb{E}[X_i] = \mu_i. \operatorname{Var}(X_i) = \sigma^2$, and

$$\frac{1}{n}\sum_{i=1}^n \mu_i \longrightarrow \mu.$$

Show that

$$X_n = \frac{1}{n}\sum_{i=1}^n X_i \longrightarrow \mu$$

in probability. (I.e., show that we can still use the Law of Large Numbers when the variables don't all have the same expectation, as long as the mean of the expectations converges.)