

# R6

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## 1

Suppose random variable  $X \sim \mathcal{U}[0, \theta]$ , where  $\theta > 0$  is an unknown parameter.  $\{X_1, X_2, \dots, X_n\}$  are i.i.d. samples drawn from the distribution of  $X$ . Consider two estimators of  $\theta$ :

$$\hat{\theta}_1 = 2\bar{X}, \quad \hat{\theta}_2 = \max_i X_i,$$

where  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$  is the sample mean.

1. Find the bias and variance of  $\hat{\theta}_1$  and  $\hat{\theta}_2$ .
2. Find the mean squared error (MSE) of  $\hat{\theta}_1$  and  $\hat{\theta}_2$ .
3. Which estimator would you prefer? Explain your answer.
4. Can we apply the cramer-rao lower bound to this problem? Why or why not?

## 2

4. Suppose that  $X$  is a discrete random variable with

$$\begin{aligned} P(X=0) &= \frac{2}{3}\theta \\ P(X=1) &= \frac{1}{3}\theta \\ P(X=2) &= \frac{2}{3}(1-\theta) \\ P(X=3) &= \frac{1}{3}(1-\theta) \end{aligned}$$

where  $0 \leq \theta \leq 1$  is a parameter. The following 10 independent observations were taken from such a distribution: (3, 0, 2, 1, 3, 2, 1, 0, 2, 1).

1. Find the method of moments estimate of  $\theta$ .
2. Find an approximate standard error for your estimate.
3. What is the maximum likelihood estimate of  $\theta$ ?
4. What is an approximate standard error of the maximum likelihood estimate?
5. If the prior distribution of  $\Theta$  is uniform on  $[0, 1]$ , what is the posterior density? Plot it. What is the mode of the posterior?

## 3 Maximum a Posteriori

$\{X_1, X_2, \dots, X_n\}$  are i.i.d. samples drawn from  $\mathcal{N}(\theta, 1)$ .

1. what is the maximum likelihood estimator (MLE) of  $\theta$ ?
2. If we have a prior distribution  $\theta \sim \mathcal{N}(\theta_0, \sigma^2)$ , what is the maximum a posteriori (MAP) estimator of  $\theta$ ?

## 4 likelihood ratio test

$X = x$  is a single observation with PDF  $f_\theta(x) = \theta x^{\theta-1}$ ,  $0 < x < 1$   
Find the most powerful level-  $\alpha$  (level 5%) test for

$$\begin{aligned} H_0 : \theta &= 3 \\ H_1 : \theta &= 2 \end{aligned}$$