

Sewing and Factorization of Smooth and Nodal Conformal Blocks in Logarithmic CFT

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Algebraic, Topological and Probabilistic approaches in CFT
Orsay, France
October 2025

The goal of my talk

- This talk is based on the following papers.

GZ1 arXiv:2305.10180

GZ2 arXiv:2411.07707 to appear in CCM

★ GZ3 arXiv:2503.23995

★ Zhang 25 arXiv:2509.07720

- The goal is to introduce **sewing-factorization (SF) theorem** in logarithmic CFT (GZ1-GZ3) and the non-equivalence of smooth and nodal conformal block functors (Zhang 25).
- Throughout my talk, I will fix a C_2 -cofinite \mathbb{N} -graded VOA \mathbb{V} , which is not necessarily self dual or rational. The representation category of \mathbb{V} is denoted by $\text{Mod}(\mathbb{V})$.

Smooth conformal block functors

- Let $\mathfrak{X} = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$ be an N -pointed compact Riemann surface with local coordinates. The **smooth conformal block (CB) functor** is the left exact contravariant functor

$$CB(\mathfrak{X}, -) : \text{Mod}(\mathbb{V}^{\otimes N}) \rightarrow \mathcal{V}ect$$
$$\mathbb{W} \mapsto CB(\mathfrak{X}, \mathbb{W}),$$

where $CB(\mathfrak{X}, \mathbb{W})$ is the **space of smooth conformal blocks (CB)** described as follows.

- Associate $\mathbb{W} \in \text{Mod}(\mathbb{V}^{\otimes N})$ to the ordered marked points x_1, \dots, x_N . Then $CB(\mathfrak{X}, \mathbb{W})$ consists of linear functionals $\mathbb{W} \rightarrow \mathbb{C}$ invariant under certain intertwining properties.

Graphical calculus

- The picture for $CB(\mathfrak{X}, \mathbb{W})$ is

$$CB\left(\begin{array}{c} \text{diagram with blue arrows } w \\ \text{diagram with red arrows } w' \end{array}\right) = CB\left(\begin{array}{c} \text{diagram with blue arrows } w' \\ \text{diagram with red arrows } w \end{array}\right)$$

- The marked points is typically partitioned into several subsets.

$$CB\left(\begin{array}{c} \text{diagram with blue arrows } \mathfrak{X} \\ \text{diagram with red arrows } \mathbb{Y} \end{array}\right) = CB\left(\begin{array}{c} \text{diagram with blue arrows } \mathfrak{X} \\ \text{diagram with red arrows } \mathbb{Y}' \end{array}\right)$$

Any CB $\phi : \mathbb{X} \otimes \mathbb{Y}' \rightarrow \mathbb{C}$ in the above space can also be viewed as a linear map $\phi^\sharp : \mathbb{X} \rightarrow \overline{\mathbb{Y}} = (\mathbb{Y}')^*$ satisfying certain intertwining properties.

Towards higher genus: sewing/composing CB

Let $\mathbb{X} \in \text{Mod}(\mathbb{V}^{\otimes N})$, $\mathbb{Y} \in \text{Mod}(\mathbb{V}^{\otimes K})$, $\mathbb{M} \in \text{Mod}(\mathbb{V}^{\otimes L})$ and

$$\phi \in CB(\text{diagram with } \mathbb{X} \text{ and } \mathbb{Y}), \quad \psi \in CB(\text{diagram with } \mathbb{Y} \text{ and } \mathbb{M})$$

The **sewing/composition** of ϕ and ψ is defined as

$$(\psi \circ \phi)^\#(w) := \sum_{\lambda_\bullet \in \mathbb{C}^K} \psi^\#(P_{\lambda_\bullet}(\phi^\#(w)))$$

Theorem (GZ2, to appear in CCM)

$\psi \circ \phi$ converges to a CB in $CB(\text{diagram with } \mathbb{X} \text{ and } \mathbb{M})$.

SF theorem A: horizontal composition

Fix $\mathbb{X} \in \text{Mod}(\mathbb{V}^{\otimes N})$, $\mathbb{M} \in \text{Mod}(\mathbb{V}^{\otimes L})$. For each $\mathbb{Y} \in \text{Mod}(\mathbb{V}^{\otimes K})$, sewing CB gives a linear map

$$\mathcal{S}_{\mathbb{Y}} : CB(\text{diagram 1}) \otimes CB(\text{diagram 2}) \rightarrow CB(\text{diagram 3})$$

The diagrams represent cobordisms with inputs and outputs labeled by \mathbb{X} , \mathbb{Y} , and \mathbb{M} . Diagram 1 is a genus-0 surface with two pairs of blue input/output ports labeled \mathbb{X} and two pairs of red input/output ports labeled \mathbb{Y} . Diagram 2 is a genus-0 surface with two pairs of red input/output ports labeled \mathbb{Y} and two pairs of purple input/output ports labeled \mathbb{M} . Diagram 3 is a genus-0 surface with two pairs of blue input/output ports labeled \mathbb{X} and two pairs of purple input/output ports labeled \mathbb{M} , where the \mathbb{Y} labels have been sewn together.

Theorem (GZ3, SF theorem A)

As $\mathbb{Y} \in \text{Mod}(\mathbb{V}^{\otimes K})$ varies, the dinatural transform $\mathcal{S}_{\mathbb{Y}}$ is a coend, i.e.,

$$\int^{\mathbb{Y} \in \text{Mod}(\mathbb{V}^{\otimes K})} CB(\text{diagram 1}) \otimes CB(\text{diagram 2}) \simeq CB(\text{diagram 3})$$

The diagrams are the same as in the previous equation block, representing the coend formula for the dinatural transform $\mathcal{S}_{\mathbb{Y}}$.

Genus 0: Huang-Lepowsky-Zhang, Moriwaki.

Fusion products and canonical CB

Fix $\mathbb{X} \in \text{Mod}(\mathbb{V}^{\otimes N})$ and $\mathfrak{X} = \text{[torus with 4 blue points]}$. Associate \mathbb{X} to the blue points of \mathfrak{X} .

- Since every left exact linear functor from a finite \mathbb{C} -linear category to $\mathcal{V}ect$ is representable (Douglas-SchommerPries-Snyder 19), there exists a \mathbb{Y} -natural isomorphism

$$\text{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}(\mathbb{X}), \mathbb{Y}) \simeq CB(\text{[torus with 4 blue points]} \boxtimes \text{[torus with 4 red points]})$$

for some unique $\boxtimes_{\mathfrak{X}}(\mathbb{X}) \in \text{Mod}(\mathbb{V}^{\otimes K})$ (called **fusion product**).

- The CB $\mathbb{J}_{\mathfrak{X}} \in CB(\text{[torus with 4 blue points]} \boxtimes \boxtimes_{\mathfrak{X}}(\mathbb{X}))$ corresponding to $\text{id} \in \text{End}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}(\mathbb{X}))$ is called the **canonical CB**.

To summarize: fusion products represent smooth CB functors.

SF theorem B: fusion products

Recall the canonical CB $\mathbb{I}_X \in CB(\text{diagram}, \mathbb{I}_X(X))$.

Theorem (GZ3, SF theorem B)

The linear map $\psi \mapsto \psi \circ \mathbb{I}_X$ gives an isomorphism

$$CB(\text{diagram}_1, \mathbb{I}_X(X)) \xrightarrow{\cong} CB(\text{diagram}_2, \mathbb{I}_X(X))$$

This isomorphism is called the **SF isomorphism**.

In short: replace the red part with the fusion product.

SF theorem A implies B


We have

$$\begin{aligned}
 & CB(\text{diagram with 4 handles, blue input, red dashed line, purple output}) \\
 & \simeq \int^{\mathbb{Y} \in \text{Mod}(\mathbb{V}^{\otimes K})} CB(\text{diagram with 2 handles, blue input, red dashed line, red output}) \otimes CB(\text{diagram with 2 handles, red input, purple output}) \\
 & \simeq \int^{\mathbb{Y} \in \text{Mod}(\mathbb{V}^{\otimes K})} \text{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathbb{X}}(\mathbb{X}), \mathbb{Y}) \otimes CB(\text{diagram with 2 handles, red input, purple output}) \\
 & \simeq CB(\text{diagram with 2 handles, red input, purple output})
 \end{aligned}$$

The last isomorphism is due to Lyubashenko 96, Fuchs-Schweigert 17.

Application: self-sewing via the end \mathbb{E}

The end $\mathbb{E} := \int_{\mathbb{X} \in \text{Mod}(\mathbb{V})} \mathbb{X} \otimes \mathbb{X}' \in \text{Mod}(\mathbb{V}^{\otimes 2})$ is a fusion product of \mathbb{C} :

 $\boxtimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{E}$. Let $\omega \in CB(\text{C-cup with two red dots} \boxtimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{E})$ be the canonical CB.

Corollary (GZ3)

The linear map $\psi \mapsto \omega \circ \psi$ gives an SF isomorphism

$$CB(\text{Diagram 1}) \xrightarrow{\cong} CB(\text{Diagram 2})$$

Diagram 1: A genus-2 surface with a blue 'X' on the left, blue arrows on the first handle, and a red 'E' on the right boundary.

Diagram 2: A genus-2 surface with a blue 'X' on the left, blue arrows on the first handle, and a red shaded region on the right boundary.

To summarize: factorization of smooth CB is given by the end \mathbb{E} .

Remark: When \mathbb{V} is \mathbb{N} -graded, C_2 -cofinite and rational, factorization of smooth CB is given by Damiolini-Gibney-Tarasca.

Factorization of CB in rational CFT

We briefly recall how algebraic geometers obtain factorization of smooth CB via nodal CB in rational CFT. We assume that \mathbb{V} is \mathbb{N} -graded, C_2 -cofinite and *rational* in the following two pages.

- Virasoro algebras, higher genus: Beilinson-Feigin-Mazur 91.
- Affine Lie algebras, higher genus: Tsuchiya-Ueno-Yamada 89, Bakalov-Kirillov 01. Looijenga 13.
- General VOA, genus 0: Nagatomo-Tsuchiya 05.
- General VOA, higher genus: Damiolini-Gibney-Tarasca 19.

I will use the setting of Damiolini-Gibney-Tarasca 19 to give an introduction.

Factorization of CB in rational CFT

- The definition of CB can be generalized to nodal curves.
- Factorization of nodal CB is given by irreducible \mathbb{V} -modules.

$$\bigoplus_{\mathbb{M} \in \text{Irr}} CB(\text{Diagram 1}) \simeq CB(\text{Diagram 2})$$



- By infinitesimal smoothing of the above isomorphism, the spaces of conformal blocks form a vector bundle over $\overline{\mathcal{M}}_{g,N}$.
- In particular, we have factorization of smooth CB given by irreducible \mathbb{V} -modules.

$$\bigoplus_{\mathbb{M} \in \text{Irr}} CB(\text{Diagram 1}) \simeq CB(\text{Diagram 2})$$



Factorization of nodal CB in logarithmic CFT

We return to the assumption that \mathbb{V} is \mathbb{N} -graded and C_2 -cofinite. The mode transition algebra (MTA) \mathfrak{A} was introduced by Damiolini-Gibney-Krashen in 2022. As an object in $\text{Mod}(\mathbb{V}^{\otimes 2})$, \mathfrak{A} is defined by the two-sided induction of Zhu algebra.

Theorem (Damiolini-Gibney-Krashen 22)

We have the factorization of nodal CB:



$$CB(\text{diagram 1}) \simeq CB(\text{diagram 2})$$

The diagram on the left shows a genus-2 surface (a torus with a handle) with blue arrows indicating a flow from left to right. A red dashed line labeled \mathfrak{A} is shown on the right side, representing a nodal cut. The diagram on the right shows the same genus-2 surface, but the red dashed line is now a solid red line, indicating a different configuration of the nodal cut.


To summarize: factorization of nodal CB is given by the MTA \mathfrak{A} .

Genus 0 CB via the end and the MTA

For each $\mathbb{X}, \mathbb{Y} \in \text{Mod}(\mathbb{V})$, the theorem of Damilolini-Gibney-Krashen implies the factorization of genus 0 nodal CB:

$$CB(\text{nodal}) \simeq CB(\text{two loops}) \simeq \text{Hom}_{\mathbb{V}^{\otimes 2}}(\mathfrak{A}, \mathbb{X}' \otimes \mathbb{Y}')$$



On the other hand, by Fuchs-Schaumann-Schweigert 16, we have the factorization of genus 0 smooth CB:

$$CB(\text{smooth}) \simeq \text{Hom}_{\mathbb{V}}(\mathbb{Y}, \mathbb{X}') \simeq \text{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{E}, \mathbb{X}' \otimes \mathbb{Y}').$$


Non-equivalence of genus 0 smooth and nodal CB

In the rest of this talk, let \mathbb{V} be a C_2 -cofinite \mathbb{N} -graded VOA admitting a module that is not generated by its lowest weight subspace (e.g., the triplet algebra \mathcal{W}_p and the even symplectic fermion VOA).

Theorem (Zhang 25)

There exist $\mathbb{X}, \mathbb{Y} \in \text{Mod}(\mathbb{V})$ such that

$$\dim CB(\text{nodal}) \neq \dim CB(\text{smooth})$$

By propagation of CB, the spaces of CB associated to $\mathbb{X}, \mathbb{Y}, \mathbb{V}, \dots, \mathbb{V}$ do not form a vector bundle on $\overline{\mathcal{M}}_{0,N}$ for $N \geq 4$.

The choice of \mathbb{X} and \mathbb{Y}

- In the proof of the above theorem, we choose \mathbb{X} to be an indecomposable projective \mathbb{V} -module that is not generated by its lowest weight subspace, and \mathbb{Y} to be an indecomposable projective module or irreducible module.
- If \mathbb{V} is the triplet algebra \mathcal{W}_p , then \mathbb{X} can be chosen to be the projective cover of X_1^- , where X_1^- is the unique irreducible module with maximal conformal weight.

The end \mathbb{E} is not isomorphic to the MTA \mathfrak{A}

Recall that

$$CB(\text{cup}) \simeq \text{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{E}, \mathbb{X}' \otimes \mathbb{Y}')$$

$$CB(\text{cap}) \simeq \text{Hom}_{\mathbb{V}^{\otimes 2}}(\mathfrak{A}, \mathbb{X}' \otimes \mathbb{Y}')$$

for each $\mathbb{X}, \mathbb{Y} \in \text{Mod}(\mathbb{V})$. Therefore,

Corollary (Zhang 25)

The end $\mathbb{E} = \int_{\mathbb{M} \in \text{Mod}(\mathbb{V})} \mathbb{M} \otimes \mathbb{M}'$ is not isomorphic to the MTA \mathfrak{A} as an object in $\text{Mod}(\mathbb{V}^{\otimes 2})$.