Notes on Hodge theory

Hao Zhang

Department of Mathemaical Science, Tsinghua University, China

Contents

Acknowledgement

I would like to thank Professor Bin Gui for his patience. Most parts of this note is taken from Huybrechts' book on complex geometry and Jiaqiang Mei's book on manifold and geometry.

1 Hodge theory for real manifolds

1.1 Hodge $*$-operator

Let (M, g) be a n-dimensional closed oriented Riemann manifold with volume form Ω. Locally, we choose an orthonormal frame $\omega^1, \cdots, \omega^n$ with respect to g for the cotangent bundle and thus we can write $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$. Denote the space of global smooth k-forms as $\Omega^k(M) := \Gamma(M, \wedge^k TM)$.

For $\omega, \eta \in \Omega^k(M)$, we write locally

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \cdots, i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}$$
$$\eta = \sum_{i_1 < \cdots < i_k} \eta_{i_1, \cdots, i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}.$$

Pointwisely, we define

$$\langle \omega, \eta \rangle := \sum_{i_1 < \cdots < i_k} \omega_{i_1, \cdots, i_k} \eta_{i_1, \cdots, i_k}.$$

It is easy to check $\langle \omega, \eta \rangle$ is a globally defined smooth function on M and it is independent of the choice of orthonormal frames. So it is reasonable to define

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle \, \Omega.$$

This gives an inner product on $\Omega^*(M) = \oplus_{k=0}^n \Omega^k(M)$. Note that $(-, -)$ is positive definite because Ω is nowhere vanishing.

Hodge $*$-operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is defined by finding the 'complement' of the k-form in Ω. More precisely, if

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \cdots, i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k},$$

then

$$*\omega = \sum_{i_1<\cdots<i_k} \varepsilon_{i_1,\cdots,i_k}\omega_{i_1,\cdots,i_k}\omega^1 \wedge \cdots \wedge \hat{\omega^{i_1}} \wedge \cdots \wedge \hat{\omega^{i_k}} \wedge \cdots \omega^k,$$

where

$$\varepsilon_{i_1,\cdots,i_k} = (-1)^{i_1+\cdots+i_k+1+\cdots+k}.$$

Proposition 1. *Suppose $\omega, \eta \in \Omega^k(M)$. Then*

1. $*1 = \Omega$,
2. $*\Omega = 1$,
3. $**\omega = (-1)^{k(n-k)}\omega$,
4. $\omega \wedge *\eta = \langle \omega, \eta \rangle \Omega$,
5. $\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle$.

Proof. All is straightforward from definition. We only prove 5 for example:

$$\langle *\omega, *\eta \rangle \Omega = *\omega \wedge ** \eta = (-1)^{k(n-k)} * \omega \wedge \eta = \eta \wedge *\omega = \langle \eta, \omega \rangle \Omega.$$

Then 5 follows from the fact: Ω is nowhere vanishing. □

Recall $d : \Omega^{k-1}(M) \to \Omega^k(M)$. Define $\delta = (-1)^{n(k-1)+1} * d* : \Omega^k(M) \to \Omega^{k-1}(M)$.

Proposition 2. *δ is the adjoint operator of d with respect to $(-,-)$, i.e.,*

$$(d\omega, \eta) = (\omega, \delta\eta),$$

for $\omega \in \Omega^{k-1}(M), \eta \in \Omega^k(M)$.

Proof. By direct computation,

$$\begin{aligned} d(\omega \wedge *\eta) &= d\omega \wedge *\eta + (-1)^{k-1}\omega \wedge d * \eta \\ &= d\omega \wedge *\eta + (-1)^{n(k-1)}\omega \wedge ** d * \eta \\ &= d\omega^*\eta - \omega \wedge *\delta\eta. \end{aligned}$$

By Stoke's theorem,

$$(d\omega, \eta) = \int_M d\omega^*\eta = \int_M \omega \wedge *\delta\eta = (\omega, \delta\eta).$$

□

1.2 Harmonic forms and Hodge decomposition

Definition 1. *$\Delta := d\delta + \delta d : \Omega^k(M) \to \Omega^k(M)$ is called **Hodge-Laplace operator**. If $\omega \in \Omega^k(M)$ satisfies $\Delta\omega = 0$, then ω is called a **harmonic form**.*

Proposition 3. *Hodge-Laplace operator satisfies:*

1. *Δ is self-adjoint, i.e., $(\Delta\omega, \eta) = (\omega, \Delta\eta)$ for all differtial forms.*
2. *Δ is positive, i.e., $(\Delta\omega, \omega) \geq 0$ and the equality holds if and only if $\Delta\omega = 0$.*
3. $*\Delta = \Delta *$.

Proof. To show Δ is self-adjoint, it suffices to assume ω and η are both k-forms. Then

$$\begin{aligned} (\Delta\omega, \eta) &= (d\delta\omega, \eta) + (\delta d\omega, \eta) \\ &= (\delta\omega, \delta\eta) + (d\omega, d\eta) \\ &= (\omega, \Delta\eta). \end{aligned}$$

Note that the above identity gives

$$(\Delta\omega, \omega) = (\delta\omega, \delta\omega) + (d\omega, d\omega) \geq 0,$$

and $(\Delta\omega, \omega) = 0$ if and only if $\delta\omega = 0$ and $d\omega = 0$, if and only if $\Delta\omega = 0$.

To show Δ commutes with $*$, assume ω is a k-form, then

$$*\delta\omega = (-1)^{n(k-1)+1} * *d * \omega = (-1)^k d * \omega.$$

Similarly, $\delta * \omega = (-1)^{k+1} * d\omega$. So

$$*d\delta\omega = (-1)^k \delta * \delta\omega = \delta d * \omega.$$

Similarly, $*\delta d = d\delta *$. Thus,

$$*\Delta = *d\delta + *\delta d = \delta d * + d\delta * = \Delta *$$

$\square$

As we can see in the proof, ω is a harmonic form if and only if $d\omega = 0$ and $\delta\omega = 0$.

Denote $\mathcal{H}^k(M)$ as the vector space of harmonic k-forms on M. We will see $\mathcal{H}^k(M)$ is actually isomorphic to the de Rham cohomology of M.

Proposition 4. *Suppose $\omega \in \mathcal{H}^k(M)$.*

1. *ω has minimal norm in the de Rham cohomology class $[\omega]$. More precisely, for any $(k-1)$-form η, $(\omega + d\eta, \omega + d\eta) \geq (\omega, \omega)$, and the equality holds if and only if $d\eta = 0$.*

2. *$*\omega \in \mathcal{H}^{n-k}(M)$.*

Proof. Suppose η is a $(k-1)$-form. Then

$$\begin{aligned}(\omega + d\eta, \omega + d\eta) &= (\omega, \omega) + 2(\omega, d\eta) + (d\eta, d\eta) \\ &= (\omega, \omega) + 2(\delta\omega, \eta) + (d\eta, d\eta) \\ &= (\omega, \omega) + (d\eta, d\eta) \geq (\omega, \omega).\end{aligned}$$

The equality holds if and only if $d\eta = 0$. $*\omega \in \mathcal{H}^{n-k}(M)$ follows immediately from $\Delta * = *\Delta$. $\square$

Theorem 1 (Poincaré duality)**.** *Hodge $*$-operator gives an isomorphism $\mathcal{H}^k(M) \cong \mathcal{H}^{n-k}(M)$.*

Proof. $* : \mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M)$ is an isomorphism because $** = (-1)^{k(n-k)}$. $\square$

To relate $\mathcal{H}^k(M)$ with the usual de Rham cohomology, we introduce our main result in the real case.

Theorem 2 (Hodge decomposition)**.** *There exists an isomorphism of vector spaces:*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M).$$

More precisely, for any k-form ω, there exists a unique decomposition

$$\omega = \omega_h + d\sigma + \delta\tau,$$

where $\omega_h \in \mathcal{H}^k(M), \sigma \in \Omega^{k-1}(M), \tau \in \Omega^{k+1}(M)$. When ω is closed, the decomposition reduces to

$$\omega = \omega_h + d\sigma.$$

Theorem 3 (Poincaré duality for de Rham cohomology)**.** *We have an isomorphism $\mathcal{H}^k(M) \cong H^k_{dR}(M)$, which gives Poincaré duality for de Rham cohomology*

$$H^k_{dR}(M) \cong H^{n-k}_{dR}(M).$$

Proof. Define a linear map $\iota : \mathcal{H}^k(M) \to H^k_{dR}(M)$ by $\omega \mapsto [\omega]$. ι is injective by Proposition 4. ι is surjective by Hodge decomposition. So ι is an isomorphism. $\square$

2 Hodge theory for complex manifolds

2.1 Dolbeault cohomology

Let X be a n-dimensional complex manifold. Denote Ω_X^p as the sheaf of holomorphic p-forms on X and $\mathcal{A}^k_{X,\mathbb{C}}$ as the sheaf of complex k-forms on X. Recall the decomposition of sheaves

$$\mathcal{A}^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_X,$$

and the differential

$$\begin{aligned} \partial &: \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q}_X \\ \bar{\partial} &: \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X, \end{aligned}$$

where $\mathcal{A}^{p,q}_X$ is the sheaf of forms of type (p,q) on X. Recall these sheaves $\mathcal{A}^{p,q}$ are acyclic, i.e., have trivial higher cohomology from partition of unity. Then Dolbeault cohomology with respect to differential forms is defined by

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,\bullet}(X), \bar{\partial}) = \frac{\operatorname{Ker}(\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\operatorname{Im}(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))}.$$

In fact, the Dolbeault cohomology is isomorphic to sheaf cohomology of Ω^p_X, i.e.,

$$H^{p,q}(X) \cong H^q(X, \Omega^p_X).$$

To see this, it suffices to see the acyclic resolution

$$0 \to \Omega^p_X \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \to \cdots$$

from $\bar{\partial}$-Poincaré lemma.

Let E be a complex vector bundle over X and $\mathcal{A}^{p,q}(E)$ denote the sheaf defined by

$$U \mapsto \mathcal{A}^{p,q}(U, E) := \Gamma(U, \wedge^{p,q} X \otimes E),$$

where the tensor product is taken over $\mathcal{O}_X$. Locally, a section α of $\mathcal{A}^{p,q}(E)$ can be written as $\alpha = \sigma\alpha_i \otimes s_i$ with α and s_i local sections of $\mathcal{A}^{p,q}_X$ and E respectively.

Lemma 1. *Suppose E is a holomorphic vector bundle. There exists a natural $\mathbb{C}$-linear operator $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ with $\bar{\partial}_E^2 = 0$ and which satisfies the Leibniz rule*

$$\bar{\partial}_E(f \cdot \alpha) = \bar{\partial}(f) \wedge \alpha + f\bar{\partial}_E(\alpha).$$

Proof. Choose a local trivialization $s = (s_1, \cdots, s_r)$ of E and write $\alpha \in \mathcal{A}^{p,q}(E)$ locally as $\alpha = \sum \alpha_i \otimes s_i$, where $\alpha \in \mathcal{A}^{p,q}_X$. Define

$$\bar{\partial}_E \alpha := \sum \bar{\partial}(\alpha_i) \otimes s_i.$$

Suppose we choose another holomorphic trivialization $s' = (s'_1, \cdots, s'_r)$ and obtain an operator $\bar{\partial}'_E$. Let $s_i = \sum_j \psi_{ij} s'_j$, where ψ_{ij} is the holomorphic transition function. Then

$$\begin{aligned} \bar{\partial}'_E \alpha &= \bar{\partial}'_E \big(\sum \alpha_i \otimes \sum_j \psi_{ij} s'_j \big) \\ &= \sum_{i,j} \bar{\partial}(\alpha_i \psi_{ij}) \otimes s'_j \\ &= \sum_{i,j} \bar{\partial}(\alpha_i) \psi_{ij} \otimes s'_j = \bar{\partial}_E(\alpha). \end{aligned}$$

So $\bar{\partial}_E = \bar{\partial}'_E$ is independent of the choice of local trivialization. Therefore, $\bar{\partial}_E^2 = 0$ since $\bar{\partial}^2 = 0$. From Leibniz rule of $\bar{\partial}$,

$$\begin{aligned} \bar{\partial}_E(f \cdot \alpha) &= \bar{\partial}_E(\sum f\alpha_i \otimes s_i) \\ &= \sum \bar{\partial}(f\alpha_i) \otimes s_i \\ &= \sum \big(\bar{\partial}(f) \wedge \alpha_i + f\bar{\partial}(\alpha_i) \big) \otimes s_i \\ &= \bar{\partial}(f) \wedge \alpha + f\bar{\partial}_E(\alpha). \end{aligned}$$

□

The above lemma gives a complex $(\mathcal{A}^{p,\bullet}(X,E),\bar\partial_E)$, whose cohomology is called Dolbeault cohomology of the holomorphic vector bundle E:

$$H^{p,q}(X,E) := H^q(\mathcal{A}^{p,\bullet}(X,E),\bar\partial_E) = \frac{\mathrm{Ker}(\bar\partial_E : \mathcal{A}^{p,q}(X,E) \to \mathcal{A}^{p,q+1}(X,E))}{\mathrm{Im}(\bar\partial_E : \mathcal{A}^{p,q-1}(X,E) \to \mathcal{A}^{p,q}(X,E))}.$$

Similarly, Dolbeault cohomology of holomorphic vector bundles is isomorphic to sheaf cohomology:

$$H^{p,q}(X,E) \cong H^q(X, E\otimes \Omega_X^p),$$

which follows from acyclic resolution of $E\otimes\Omega_X^p$:

$$0 \to E\otimes\Omega_X^p \to \mathcal{A}^{p,0}(E) \xrightarrow{\bar\partial_E} \mathcal{A}^{p,1}(E) \to \cdots$$

To summarize what we obtain:

Dolbeault cohomology of a holomorphic vector bundle E = sheaf cohomology of $E\otimes\Omega_X^p$.

2.2 Hermitian and Kähler structure on complex manifolds

We briefly recall some definition in this subsection.

Let X be a complex manifold with almost complex structure I and complex dimension n. A Riemann metric g on X is an hermitian structure if for any point $x\in X$, the scalar product g_x is compatible with I, i.e.,

$$g_x(Iv, Iw) = g_x(v,w).$$

The induced real $(1,1)$-form $\omega := g(I(),())$ is called the fundamental form of hermitian manifold (X,g). After complexification, the fundamental form ω is locally of the form

$$\omega = \frac{i}{2}\sum_{i,j=1}^n h_{ij} dz_i \wedge d\bar z_j,$$

where $(h_{ij}(x))$ is a positive definition matrix for each $x\in X$. It is not difficult to see that the hermitian structure is uniquely determined by I and ω. The hermitian structure g is called a Kähler structure if ω is closed. Denote the hermitian extension of g by $g_{\mathbb{C}}$.

The Hodge $*-$operator $* : \bigwedge_{\mathbb{C}}^k X \to \bigwedge_{\mathbb{C}}^{2n-k} X$ is similar to the one defined for Riemann manifolds (X,g) with the natural volume form Ω. More precisely, $*$ is defined by $\alpha\wedge *\bar\beta = g_{\mathbb{C}}(\alpha,\beta)\Omega$. When restricted to $\bigwedge_{\mathbb{C}}^k X$, $*$ reduces to the usual Hodge $*-$operator for Riemann manifolds.

With Hodge $*-$operator, we can define several adjoint operators. Regard (X,g) as a real Riemann manifold with natural volume form Ω. Since X has even dimension, the adjoint operator $d^* = \delta$ is exactly $d^* = -*\circ d\circ *$. Analogously, one defines ∂^* and $\bar\partial^*$ as $\partial^* := -*\circ\bar\partial\circ *$ and $\bar\partial^* = -*\circ\partial\circ*$ to make $d^* = \partial^* + \bar\partial^*$ valid.

Therefore, it is natural to define the Laplacian operator associated to $d,\partial,\bar\partial$: $\Delta := d^*d + dd^*$, $\Delta_\partial = \partial^*\partial + \partial\partial^*$, $\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$. Note that $* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-q,n-p}(X)$, $\partial^* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p-1,q}(X)$, $\bar\partial^* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q-1}(X)$ and $\Delta_\partial, \Delta_{\bar\partial}$ perserve bidegrees.

2.3 Hodge decomposition for Kähler manifolds

Syppose X is a complex manifold with an hermitian structure g and natural fundamental form Ω. The hermitian extension of g is denoted by $g_{\mathbb{C}}$. Define an hermitian product on $\mathcal{A}_{\mathbb{C}}^*(X)$ by

$$(\alpha,\beta) := \int_X g_{\mathbb{C}}(\alpha,\beta) * 1 = \int_X g_{\mathbb{C}}(\alpha,\beta)\Omega.$$

Note that the value of g on $\mathcal{A}^*(X)$ is the exactly the inner product defined in the first section.

With respect to the hermitian product $(-,-)$. The degree decomposition

$$\mathcal{A}_{\mathbb{C}}^*(X) = \bigoplus_k \mathcal{A}_{\mathbb{C}}^k(X)$$

and the bidegree decomposition

$$\mathcal{A}_{\mathbb{C}}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

are both orthogonal decompositions and each component $\mathcal{A}^{p,q}(X)$ is an infinite dimensional normed vector space with scalar product $(-,-)$ and the induced norm $\|\alpha\|^2 = (\alpha,\alpha)$.

Proposition 5. *Suppose X is a closed hermitian manifold. Then with respect to $(-,-)$, the operators $d^*, \partial^*, \bar{\partial}^*$ are actually adjoint operators of $d, \partial, \bar{\partial}$.*

Proof. We give a proof for ∂ for example. For $\alpha \in \mathcal{A}^{p-1,q}(X)$ and $\beta \in \mathcal{A}^{p,q}(X)$,

$$\begin{aligned}(\partial\alpha, \beta) &= \int_X g_{\mathbb{C}}(\partial\alpha, \beta) * 1 = \int_X \partial\alpha \wedge *\bar{\beta} \\ &= \int_X \partial(\alpha \wedge *\bar{\beta}) - (-1)^{p+q-1} \int_X \alpha \wedge \partial(*\bar{\beta})\end{aligned}$$

Since $\alpha \wedge *\bar{\beta}$ is of bidegree $(n-1, n)$, $\partial(\alpha \wedge *\bar{\beta}) = d(\alpha \wedge *\bar{\beta})$. So

$$\int_X \partial(\alpha \wedge *\bar{\beta}) = 0$$

by Stokes' theorem. Using $*^2 = (-1)^k$ on $\mathcal{A}^k(X)$, we compute

$$\int_X \alpha \wedge \partial(*\bar{\beta}) = (-1)^{2n-(p+q)+1} \int_X g_{\mathbb{C}}(\alpha, -\partial^*\beta) * 1 = (-1)^{2n-(p+q)}(\alpha, \partial^*\beta).$$

So $(\partial\alpha, \beta) = (\alpha, \partial^*\beta)$. □

In the case of real manifolds, we interpret de Rham cohomology by harmonic forms. In the complex case, we will apply similar approach. For the differential d, the spaces of harmonic k-forms and (p,q)-forms (which are defined similarly in the real case) are denoted by $\mathcal{H}^k(X,g)$ and $\mathcal{H}^{p,q}(X,g)$. For ∂ and $\bar{\partial}$, we have analogous definition.

Definition 2. *A k-form is called $\bar{\partial}$-harmonic if $\Delta_{\bar{\partial}}\alpha = 0$ and define the spaces of $\bar{\partial}$-harmonic k-forms and (p,q)-forms by $\mathcal{H}^k_{\bar{\partial}}(X,g)$ and $\mathcal{H}^{p,q}_{\bar{\partial}}(X,g)$. ∂-harmonic forms are analogous.*

Proposition 6. *Suppose (X,g) is a closed hermitian manifold. A form α is $\bar{\partial}$-harmonic (resp. ∂-harmonic) if and only if $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$ (resp. $\partial\alpha = \partial^*\alpha = 0$).*

Proof. The $\bar{\partial}$ case follows from the identity

$$\begin{aligned}(\Delta_{\bar{\partial}}\alpha, \alpha) &= (\bar{\partial}^*\bar{\partial}\alpha + \bar{\partial}\bar{\partial}^*\alpha, \alpha) \\ &= \|\bar{\partial}^*(\alpha)\|^2 + \|\bar{\partial}\|^2.\end{aligned}$$

□

Proposition 7. *Suppose (X,g) is an hermitian manifold. Then*

1. $\mathcal{H}^k_{\bar{\partial}}(X,g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X,g)$ *and* $\mathcal{H}^k_{\partial}(X,g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\partial}(X,g)$.
2. *If (X,g) is Kähler, then both decompositions coincide with $\mathcal{H}^k(X,g)_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X,g)$. In particular, $\mathcal{H}^k(X,g)_{\mathbb{C}} = \mathcal{H}^k_{\bar{\partial}}(X,g) = \mathcal{H}^k_{\partial}(X,g)$.*

Proof. Suppose $\alpha = \sum \alpha^{p,q}$ is the bidegree decomposition of a $\bar{\partial}$-harmonic form α. Then

$$0 = \sum \Delta_{\bar{\partial}}(\alpha^{p,q})$$

is also a bidegree decomposition, which implies $\Delta_{\bar{\partial}}(\alpha^{p,q}) = 0$ for all p, q. The proof of ∂ decomposition is analogous.

The second assertion follows from the identity $\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$ on a Kähler manifold. □

Theorem 4 (Serre duality for harmonic forms)**.** *Suppose (X,g) is a compact connected hermitian manifold. Then the pairing*

$$\mathcal{H}^{p,q}_{\bar{\partial}}(X,g) \times \mathcal{H}^{n-p,n-q}_{\bar{\partial}}(X,g) \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

is non-degenerate. This yields an isomorphism

$$\mathcal{H}^{p,q}_{\bar{\partial}}(X,g) \cong \mathcal{H}^{n-p,n-q}_{\bar{\partial}}(X,g)^*.$$

Proof. Suppose $0 \neq \alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X,g)$. Then

$$\int_X \alpha \wedge *\bar\alpha = \|\alpha\|^2 > 0$$

implies the pairing is non-degenerate. □

Our main result is:

Theorem 5 (Hodge decomposition of harmonic forms)**.** *Let (X,g) be a compact hermitian manifold. Then there exists two natural orthogonal decompositions*

$$\mathcal{A}^{p,q}(X) = \partial\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\partial}(X,g) \oplus \partial^*\mathcal{A}^{p+1,q}(X),$$
$$\mathcal{A}^{p,q}(X) = \bar\partial\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\bar\partial}(X,g) \oplus \bar\partial^*\mathcal{A}^{p+1,q}(X).$$

Moreover, $\mathcal{H}^{p,q}(X,g)_{\mathbb{C}}$ are all finite dimensional and if X is a Kähler manifold, then $\mathcal{H}^{p,q}(X,g)_{\mathbb{C}} = \mathcal{H}^{p,q}_{\partial}(X,g) = \mathcal{H}^{p,q}_{\bar\partial}(X,g)$.

The most nontrivial part is the existence of decomposition. We will not prove it in our note. The significance of Kähler condition is that we can forget all about '$\bar\partial$ or $\bar\partial$-harmonic', and replace them by 'harmonic'.

Corollary 1. *Suppose (X,g) is a compact hermitian manifold. Then the canonical map $\mathcal{H}^{p,q}_{\bar\partial}(X,g) \to H^{p,q}(X)$ is an isomorphism.*

Proof. The canonical map $\mathcal{H}^{p,q}_{\bar\partial}(X,g) \to H^{p,q}(X)$ is given by $\alpha \mapsto [\alpha]$, where $[\alpha]$ is a cohomology class of α. This map is injective because any harmonic coboundary $\bar\partial\beta$ must satisfy $\bar\partial^*\bar\partial\beta = 0$. But $0 = (\bar\partial^*\bar\partial\beta, \beta) = (\bar\partial\beta, \bar\partial\beta) = \|\bar\partial\beta\|^2 = 0$ implies $\bar\partial\beta = 0$. To show this map is surjective, it suffices to show any $\bar\partial-$closed $(p,q)-$form β must be cohomological to a $\bar\partial-$harmonic form. By Hodge decomposition, write $\beta = \bar\partial\beta_1 + \beta_h + \bar\partial^*\beta_2$, where β_h is harmonic. Since $\bar\partial\beta = 0$, it follows $\bar\partial\bar\partial^*\beta_2 = 0$, which implies $\bar\partial^*\beta_2$ by a similar argument in the injective case. □

The proof shows the Hodge decomposition for closed forms does not contain the terms $\partial^*\mathcal{A}^{p+1,q}(X)$ or $\bar\partial^*\mathcal{A}^{p+1,q}(X)$, just as the case in real manifolds.

Corollary 2. *Let (X,g) be a compact Kähler manifold. Then there exists a decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

which does not depend on the Kähler structure.

Proof. Since X is Kähler,

$$H^k(X,\mathbb{C}) = \mathcal{H}^k(X,g)_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X,g) = \bigoplus_{p+q=k} H^{p,q}(X).$$

To show the decomposition is independent of the Kähler structure, we choose two Kähler metric g, g' and $\alpha \in \mathcal{H}^{p,q}(X,g), \alpha' \in \mathcal{H}^{p,q}(X,g')$, which induce same elements in $H^{p,q}(X)$. So α and α' differ by some $\bar\partial\gamma$, i.e., $\alpha' = \alpha + \bar\partial\gamma$. Then $d\bar\partial\gamma = 0$. By Hodge decomposition for d,

$$\bar\partial\gamma = d\beta + \beta_h.$$

But $0 = (\gamma, \bar\partial^*\beta_h) = (\bar\partial\gamma, \beta_h) = (\beta_h, \beta_h)$ implies $\beta_h = 0$. So $\bar\partial\gamma \in d(\mathcal{A}^{k-1}_{\mathbb{C}}(X))$ and α, α' induces the same de Rham cohomology class in $H^k(X,\mathbb{C})$. □

3 Serre duality

In this section, we will give a generalized version of Hodge decomposition and Serre duality on a holomorphic vector bundle. Serre duality, together with Riemann-Roch theorem and Kodaira vanishing theorem, is significant in controlling the cohomology of holomorphic vector bundles. Most parts of this section are routine.

3.1 Hermitian structure on vector bundles

Suppose M is a real manifold and $E \to M$ is a complex vector bundle.

Definition 3. *An hermitian structure h on $E \to M$ is an hermitian scalar product h_x on each fiber $E(x)$ which depends differtiably on x, The pair (E,h) is called an hermitian vector bundle.*

If $\psi : E|_U \cong U \times \mathbb{C}^r$ is a trivialization over some open subset U, then h_x is given by a positive-definite hermitian matrix $(h_{ij}(x))$ for each $x \in U$. The definition says $(h_{ij}(x))$ relies differtiably on $x \in U$.

Example 1. *If (X,g) is an hermitian manifold, then the tangent, cotangent and form bundles have natural hermitian structures. Moreover, if (E,h) is an hermitian vector bundle over (X,g), then $\bigwedge^{p,q} X \otimes E$ have natural hermitian structures.*

Proposition 8. *Every complex vector bundle admits an hermitian structure.*

Proof. Choose an open covering $X = \bigcup U_i$ trivializing E and glue the constant hermitian structure on the trivial bundles $U_i \times \mathbb{C}^r$ by means of partition of unity. The resulting product is hermitian because positive linear combination of hermitian product is again hermitian. □

Now let (X,g) be an hermitian manifold of complex dimension n and (E,h) is an hermitian vector bundle. Denote the induced hermitian structure on $\bigwedge^{p,q} X \otimes E$ by $(-,-)$. We may interpret h as a $\mathbb{C}-$antilinear isomorphism $h : E \cong E^*$.

Definition 4. *Hodge $*$-operator $\bar{*}_E : \bigwedge^{p,q} X \otimes E \to \bigwedge^{n-p,n-q} X \otimes E^*$ is defined by*

$$\bar{*}_E(\varphi \otimes s) = \bar{*}_E(\varphi) \otimes h(s) = \overline{*(\varphi)} \otimes h(s) = *(\bar{\varphi}) \otimes h(s).$$

Hodge $*$-operator $\bar{*}_E$ is a $\mathbb{C}$-antilinear isomorphism that depends on g and h. Similarly, we check easily

$$(\alpha, \beta) * 1 = \alpha \wedge \bar{*}_E(\beta)$$

where $\wedge$ means taking usual wedge products in the form part and evaluation map in the bundle part. Moreover, $\bar{*}_E \circ \bar{*}_E = (-1)^{p+q}$ on $\bigwedge^{p,q} X \otimes E$. From now on, denote $\mathcal{A}^{p,q}(E)$ by sheaf of sections of $\bigwedge^{p,q} X \otimes E$. We will not distinguish $\mathcal{A}^{p,q}(E)$ and $\bigwedge^{p,q} X \otimes E$ from now on.

Define $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ by $\bar{\partial}_E(\alpha \otimes s) = \bar{\partial}(\alpha) \otimes s$ and its adjoint operator $\bar{\partial}_E^* : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q-1}(E)$ by $\bar{\partial}_E^* = -\bar{*}_{E^*} \circ \bar{\partial}_{E^*} \circ \bar{*}_E$. The Laplacian operator $\Delta_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q}(E)$ is defined by $\Delta_E = \bar{\partial}_E^* \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E^*$.

Definition 5. *A section $\alpha \in \mathcal{A}^{p,q}(E)$ is called harmonic if $\Delta_E(\alpha) = 0$. The space of harmonic forms is denoted by $\mathcal{H}^{p,q}(X,E)$.*

Since $\bar{*}_E$ commutes with Δ_E, $\bar{*}_E$ restricts to a $\mathbb{C}-$antilinear isomorphism $\bar{*}_E : \mathcal{H}^{p,q}(X,E) \to \mathcal{H}^{p,q}(X,E^*)$.

From now on, we suppose (X,g) is a compact hermitian manifold. Define a hermitian scalar product on $\mathcal{A}^{p,q}(X,E)$ by

$$(\alpha, \beta) = \int_X (\alpha, \beta) * 1,$$

where $(-,-)$ inside the integral is the pointwise hermitian inner product on $\mathcal{A}^{p,q}(X,E)$.

Proposition 9. *1. $\bar{\partial}_E^*$ is actually the adjoint operator of $\bar{\partial}_E$ and Δ_E is self-adjoint with respect to $(-,-)$.*

2. $\alpha \in \mathcal{A}^{p,q}(X,E)$ is harmonic if and only if $\bar{\partial}_E(\alpha) = \bar{\partial}_E^*(\alpha) = 0$.

Proof. Analogous to Proposition 2 and 3. □

3.2 Serre duality on vector bundles

To give Serre duality on vector bundles, we have to give a generalized version of Hodge decomposition for vector bundles.

Theorem 6 (Hodge decomposition for vector bundles)**.** *Suppose (X,g) is a compact hermitian manifold and (E,h) is an hermitian vector bundle. We have Hodge decomposition*

$$\mathcal{A}^{p,q}(X,E) = \bar{\partial}_E \mathcal{A}^{p,q-1}(X,E) \oplus \mathcal{H}^{p,q}(X,E) \oplus \bar{\partial}_E^* \mathcal{A}^{p,q+1}(X,E)$$

and $\mathcal{H}^{p,q}(X,E)$ is finite dimensional for each p and q.

If we choose $E = \mathcal{O}_X$, the above theorem reduces to the usual Hodge decomposition.

Corollary 3. *The natural map $\mathcal{H}^{p,q}(X,E) \to H^{p,q}(X,E)$ is an isomorphism. In particular, Dolbeault cohomology $H^{p,q}(X,E) \cong H^q(X, E \otimes \Omega_X^p)$ is finite dimensional.*

Proof. To show the map is injective, choose a harmonic coboundary $\bar{\partial}_E \beta$. So

$$0 = (\bar{\partial}_E^* \bar{\partial}_E \beta, \beta) = (\bar{\partial}_E \beta, \bar{\partial}_E \beta) = \|\bar{\partial}_E \beta\|^2$$

implies $\bar{\partial}_E \beta = 0$.

To show the map is surjective, choose a closed element $\alpha \in \mathcal{A}^{p,q}(X,E)$ and by Hodge decomposition,

$$\alpha = \bar{\partial}_E \alpha_1 + \alpha_h + \bar{\partial}_E^* \alpha_2.$$

Since $\bar{\partial}_E \alpha = 0$, $\bar{\partial}_E \bar{\partial}_E^* \alpha_2 = 0$ and then $\bar{\partial}_E^* \alpha_2 = 0$. So $\alpha = \bar{\partial}_E \alpha_1 + \alpha_h$, i.e., the image of $\alpha_h \in \mathcal{H}^{p,q}(X,E)$ under the natural map is the cohomology class $[\alpha]$. □

By this corollary, we see any (nonzero) Dolbeault cohomology class can be represented by a (nonzero) harmonic element.

Theorem 7 (Serre duality for vector bundles)**.** *Suppose X is a compact complex manifold. For any holomorphic vector bundle E on X, define a pairing*

$$H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \to \mathbb{C}$$

by

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

Then the pairing is non-degenerate.

Proof. This pairing is well-defined by Stokes' theorem. Choose hermitian structures h and g on E and X. For any nonzero cohomology class in $H^{p,q}(X,E)$, we can find a nonzero harmonic element $\alpha \in \mathcal{H}^{p,q}(X,E)$ representing the given class. Define $\beta = \bar{*}_E \alpha \in \mathcal{H}^{n-p,n-q}(X,E^*)$. We check

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge \bar{*}_E \alpha = \int_X (\alpha, \alpha) * 1 = \|\alpha\|^2 \neq 0.$$

So this pairing is non-degenerate. □

Corollary 4. *For any holomorphic vector bundle E over a compact complex manifold X, there exist $\mathbb{C}-$linear isomorphisms:*

$$H^q(X, E \otimes \Omega^p) \cong H^{p,q}(X,E) \cong H^{n-p,n-q}(X,E^*)^* \cong H^{n-q}(X, E^* \otimes \Omega^{n-p})^*$$