Cohomology theory of complex geometry

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1 Riemann-Roch theorem

1.1 Riemann-Roch theorem for compact Riemann surface

The classical Riemann-Roch theorem is stated by means of divisors. Suppose X is a compact Riemann surface of genus g. D is a divisor of X. Define a sheaf $\mathcal{O}_D$ by setting

$$\mathcal{O}_D(U) = \{f \in \mathcal{M}(U) : \mathrm{ord}_x(f) \geq -D(x), \forall x \in U\}$$

for each open subset U. Note that if $D = 0$, then $\mathcal{O}_0 = \mathcal{O}$ is the structure sheaf.

Theorem 1 (Riemann-Roch theorem)**.** *The cohomology $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite dimensional vector spaces and*

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D.$$

We give a sketch of proof. Suppose P is a point on X. Then there is a short exact sequence of sheaves

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \to \mathbb{C}_P \to 0,$$

where $\mathcal{O}_D \to \mathcal{O}_{D+P}$ is the natural inclusion and $\mathbb{C}_P$ is defined by

$$\mathbb{C}_P(U) = \begin{cases} \mathbb{C}, & P \in U \\ 0, & P \notin U \end{cases}$$

$\beta : \mathcal{O}_{D+P} \to \mathbb{C}_P$ is characterized as follows: Suppose $U \subset X$ is an open set. If $P \notin U$, then β_U is the zero homomorphism. If $P \in U$ and $f \in \mathcal{O}_{D+P}(U)$, then with respect to local coordinate z at P, f can be written as

$$f = \sum_{n=-k-1}^{\infty} c_n z^n,$$

where $k = D(P)$. Set $\beta_U(f) = c_{-k-1} \in \mathbb{C} = \mathbb{C}_P(U)$. The long exact sequence of cohomology group gives

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to \mathbb{C} \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+P}) \to 0.$$

We deduce Riemann-Roch theorem by induction. First Riemann-Roch theorem holds for $D = 0$ because $H^0(X, \mathcal{O}) \cong \mathbb{C}$ and $\dim H^1(X, \mathcal{O}) = g$. Then keeping the notation as above, we set $D' = D + P$. Long exact sequence of cohomology splits into two short exact sequences

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D'}) \to V \to 0,$$
$$0 \to W \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}) \to 0,$$

where $V := \mathrm{Im} H^0(X, \mathcal{O}_{D'}) \to \mathbb{C}$ and $W := \mathbb{C}/V$. So by linear algebra,

$$\begin{aligned}
&\dim H^0(X, \mathcal{O}_{D'}) = \dim H^0(X, \mathcal{O}_D) + \dim V,\\
&\dim H^1(X, \mathcal{O}_D) = \dim W + \dim H^1(X, \mathcal{O}_{D'}),\\
&\dim V + \dim W = 1 = \deg D' - \deg D.
\end{aligned}$$

Then

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) - \deg D = \dim H^0(X, \mathcal{O}_{D'}) - \dim H^1(X, \mathcal{O}_{D'}) - \deg D',$$

which implies: if the Riemann-Roch formula holds for one of D and D', then it must hold for both. For an arbitrary divisor D, we may write

$$D = P_1 + \cdots + P_m - P_{m+1} - \cdots - P_n$$

and Riemann-Roch formula holds by induction.

We give a simple application of Riemann-Roch theorem.

Corollary 1. *Suppose X is a compact Riemann surface of genus g and $a \in X$. Then there exist a non-constant meromorphic function f on X which has a pole of order $\leq g + 1$ at a and is otherwise holomorphic. Moreover, if we view f as a holomorphic function $f : X \to \mathbb{P}^1$, then it is a holomorphic covering map with at most $g + 1$ sheets.*

Proof. Let $D : X \to \mathbb{Z}$ be the divisor with $D(a) = g+1$ and $D(x) = 0$ for $x \neq a$. By Riemann-Roch theorem,

$$\dim H^0(X, \mathcal{O}_D) \geq 1 - g + \deg D = 2.$$

Thus ,there exists a non-constant function $f \in H^0(X, \mathcal{O}_D)$ and this function is what we need. Since the value ∞ has multiplicity $\leq g + 1$, $f : X \to \mathbb{P}^1$ is a covering map with at most $g + 1$ sheets. $\square$

Since 1-sheeted convering map must be a biholomorphism, every Riemann surface of genus 0 is biholomorphic to the Riemann sphere.

1.2 Hirzebruch-Riemann-Roch theorem

Riemann-Roch formula in previous subsection actually computes the Euler characteristic of line bundle $\mathcal{O}_D$:

$$\chi(X, \mathcal{O}_D) := \dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D).$$

In fact, each line bundle on a compact Riemann surface is isomorphic to some $\mathcal{O}_D$ and so the Euler characteristic of line bundles on a compact Riemann surface is complete. To see this, suppose E is a holomorphic line bundle and choose a global meromorphic section ψ of E which does not vanish identically (must exist). Let D be the divisor of ψ. Then $f \mapsto f\psi$ gives an isomorphism of sheaves $\mathcal{O}_D \simeq E$. The degree of line bundle $E \simeq \mathcal{O}_D$ is defined as $\deg E = \deg D$.

In general, Euler characteristic of vector bundles on a compact complex manifold can be computed by Hirzebruch-Riemann-Roch theorem. Suppose X is a compact complex manifold of dimension n and E is a vector bundle. Define the Euler characteristic

$$\chi(X, E) := \sum_{i=0}^{n} (-1)^i \dim H^i(X, E).$$

Theorem 2 (Hirzebruch-Riemann-Roch)**.** *Euler characteristic is given by*

$$\chi(X, E) = \int_X \mathrm{ch}(E)\mathrm{td}(X),$$

where $\mathrm{ch}(E)$ *is the total Chern character and* $\mathrm{td}(X)$ *is the total Todd class.*

We are not going to discuss this theorem in details.

2 Vanishing of cohomology

2.1 Serre vanishing theorem

In studying cohomology, it is always helpful that certain cohomology vanishes. For simplicity, we will assume our complex manifold is a compact Riemann surface.

Suppose $D = \sum n_i x_i$ is a divisor and $\mathcal{E}$ is a holomorphic vector bundle. Consider the vector bundle $\mathcal{E}(D)$ defined below: for any open set U, $\mathcal{E}(D)(U)$ is the set of all $s \in \mathcal{E}(U - \{x_i\})$ satisfying that for any x_i and any local coordinate η_i near x_i, $\eta_i^{n_i} \cdot s$ has removable singularity at x_i. The sheaf $\mathcal{E}(D)$ is a locally free $\mathcal{O}_X$-module and there is a natural isomorphism of $\mathcal{O}_X$-modules $\mathcal{E}(D) \simeq \mathcal{E} \otimes \mathcal{O}_X(D)$. We will study vanishing theorem of this kind of vector bundles.

In general, the isomorphism classes of line bundles on a complex manifold X form an abelian group $\mathrm{Pic}(X)$ under the tensor product and dual operation. We call $\mathrm{Pic}(X)$ the Picard group of X, which is isomorphic to $H^1(X, \mathcal{O}_X^*)$. One can show $D \mapsto \mathcal{O}_X(D)$ gives a group homomorphism $\mathrm{Div(X)} \to \mathrm{Pic}(X)$, and so $\mathcal{O}_X(D)^* \simeq \mathcal{O}_X(-D), \mathcal{O}_X(D_1 + D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$.

Theorem 3 (Serre vanishing theorem)**.** *Assume X is a connected compact Riemann surface and $\mathcal{E}$ is a vector bundle on X. D is a nonzero effective divisor, i.e., $D \geq 0$ and $D \neq 0$. Then there exists $N \in \mathbb{N}$ such that $H^0(X, \mathcal{E}(-nD)) = 0$ for all $n > N$.*

Proof. Write $D = \sum x_i$. Then for any $x = x_i$, $H^0(X, \mathcal{E}(-nD))$ is a subspace of $H^0(X, \mathcal{E}(-nx))$. So it suffices to prove $H^0(X, \mathcal{E}(-nx)) = 0$ for sufficiently large n.

By Hodge theory, $H^0(X, \mathcal{E})$ is finite dimensional, so we choose a basis $s_1, s_2, \cdots, s_M$. With respect to local chart (U, z) of x, $\mathcal{E}|_U \simeq E \otimes \mathcal{O}_U$ where $E = \mathcal{E}_x$ and all s_k has expansion

$$s_k(z) = \sum_{j=0}^{\infty} v_{k,j} z^j$$

where $v_{k,j} \in E$. For any $n \in \mathbb{N}$, let

$$\boldsymbol{v}_k^n = (v_{k,0}, v_{k,1}, \cdots, v_{k,n}) \in E \otimes \mathbb{C}^{n+1}$$

and

$$F_n := \mathrm{span}\{\boldsymbol{v}_k^n : k = 1, 2, \cdots, M\} \subset E \otimes \mathbb{C}^{n+1}.$$

$\boldsymbol{v}_k^n$ is understood as the first n coeffients of s_k. Then $\{F_n : n \in \mathbb{N}\}$ is a sequence of finite dimensional vector spaces with increasing and bounded dimension (note that the dimension must be bounded by M). Thus, $\dim F_n$ must be constant when n is sufficiently large. Choose $N \in \mathbb{N}$ such that $\dim F_n = K$ for $n \geq N$. Without loss of generality, assume $\boldsymbol{v}_1^N, \cdots, \boldsymbol{v}_K^N$ are linearly independent. Then by linear algebra, $\boldsymbol{v}_1^n, \cdots, \boldsymbol{v}_K^n$ are also linearly independent, and so form a basis of F_n for each $n \geq N$. So for any k and $n \geq N$, there exist unique $c_{1,n}, \cdots, c_{K,n} \in \mathbb{C}$, such that

$$\boldsymbol{v}_k^n = c_{1,n} \boldsymbol{v}_1^n + \cdots + c_{K,n} \boldsymbol{v}_K^n.$$

By uniqueness, $c_{1,n} = c_{1,N}, \cdots, c_{K,n} = c_{K,N}$ for all $n \geq N$. This implies $s_k = c_{1,N} s_1 + \cdots + c_{K,N} s_K$ locally for any fixed $k = 1, \cdots, M$. By connectedness, this equation holds globally. But $s_1, \cdots, s_M$ is a basis, so $K = M = \dim H^0(X, \mathcal{E})$.

Choose any $n \geq N$ and $\sigma \in H^0(X, \mathcal{E}(-nx)) \subset H^0(X, \mathcal{E})$. Then there exist $c_1, \cdots, c_K \in \mathbb{C}$ such that $\sigma = c_1 s_1 + \cdots + c_K s_K$. Expand σ locally near x that

$$\sigma(z) = \sum_{j=0}^{\infty} \nu_j z^j.$$

Then by our pervious discussion, $(\nu_1, \cdots, \nu_N) = c_1 \boldsymbol{v}_1^N + \cdots + c_K \boldsymbol{v}_K^N$. Since $z^{-n}\sigma(z)$ has removable singularity near x for any $n \geq N$, $\nu_1 = \cdots = \nu_N = 0$. Therefore $c_1 = \cdots = c_K = 0$ and $\sigma = 0$. $\square$

This theorem says that nonzero global (holomorphic) sections cannot have zeros of arbitrary orders.

Corollary 2 (Serre vanishing theorem)**.** *Assume X is a connected compact Riemann surface. $\mathcal{E}$ is a vector bundle and D is a nonzero effective divisor. Then there exists $N \in BN$ such that $H^1(X, \mathcal{E}(nD)) = 0$ for any $n > N$.*

Proof. Denote the tangent sheaf as Θ_X and the cotangent sheaf as ω_X. It is well known these two sheaves are dual to each other. Then

$$\mathcal{E}(nD) \simeq \mathcal{E} \otimes \Theta_X \otimes \mathcal{O}_X(nD) \otimes \omega_X \simeq (\mathcal{E}^* \otimes \omega_X(-nD))^* \otimes \omega_X.$$

By Serre duality,

$$H^1(X, \mathcal{E}(nD)) \cong H^1(X, (\mathcal{E}^* \otimes \omega_X(-nD))^* \otimes \omega_X) \cong H^0(X, \mathcal{E}^* \otimes \omega_X(-nD)).$$

By our previous Serre vanishing theorem, $H^1(X, \mathcal{E}(nD)) = 0$ for sufficiently large n. □

2.2 Kodaira vanishing theorem

Kodaira vanishing theorem helps to approximate the value of N in Serre vanishing theorem when $\mathcal{E}$ is a line bundle.

Theorem 4 (Kodaira vanishing theorem). *Assume X is a connected compact Riemann surface. D is a divisor with $\deg D > 0$. Then*

$$H^0(X, \mathcal{O}_X(-D)) = 0, \quad H^1(X, \omega_X(D)) = 0.$$

Proof. Suppose on the contrary $H^0(X, \mathcal{O}_X(-D))$ is nonzero. Choose a nonzero global section $f \in H^0(X, \mathcal{O}_X(-D))$. By residue theorem, $\deg(f) = 0$, where (f) is the divisor of f. But $f \in H^0(X, \mathcal{O}_X(-D))$ implies $(f) - D \geq 0$. So $\deg(f) \geq \deg D > 0$ is a contradiction. $H^1(X, \omega_X(D)) = 0$ follows from Serre duality. □

From now on, we consider line bundles.

Proposition 1. *Suppose X is a connected compact Riemann surface of genus g. Then*

$$\deg \omega_X = 2g - 2, \quad \deg \Theta_X = 2 - 2g.$$

Proof. By Riemann-Roch theorem and Serre duality,

$$\begin{aligned} 1 - g + \deg \omega_X = \chi(X, \omega_X) &= \dim H^0(X, \omega_X) - \dim H^1(X, \omega_X) \\ &= \dim H^1(X, \mathcal{O}_X) - \dim H^0(X, \mathcal{O}_X) = g - 1. \end{aligned}$$

So $\deg \omega_X = 2g - 2$ and $\deg \Theta_X = 2 - 2g$. □

The following theorem gives an explicit description of vanishing condition of line bundles.

Theorem 5. *Assume X is a connected compact Riemann surface of genus g. D is a divisor and $\mathcal{L}$ is a line bundle. Then $H^1(X, \mathcal{L}(D)) = 0$ when $\deg D > 2g - 2 - \deg \mathcal{L}$.*

Proof. Choose divisors T, L such that $\Theta_X \simeq \mathcal{O}_X(T)$ and $\mathcal{L} \simeq \mathcal{O}(L)$. Then

$$\mathcal{L}(D) \simeq \omega_X \otimes \Theta_X \otimes \mathcal{L}(D) \simeq \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_X(D) \otimes \mathcal{O}_X(T) \simeq \omega_X(T + L + D).$$

When $\deg D > 2g - 2 - \deg ML$,

$$\deg(T + L + D) > 2g - 2 - \deg L + \deg L + \deg T = 0.$$

By Kodaira vanishing theorem, $H^1(X, \mathcal{L}(D)) = 0$. □

Corollary 3. *Assume X is a connected compact Riemann surface of genus g. D is a divisor and $n \in \mathbb{Z}$. Then $H^1(X, \Theta_X^{\otimes n}(D)) = 0$ when $\deg D > (n+1)(2g-2)$.*

Proof. It is because $\deg \Theta_X^{\otimes n} = n(2 - 2g)$. □

Note that $(n+1)(2g-2)$ relies only on the topology of Riemann surface because g does. To see this, use Hodge theory and Serre duality

$$H^1(X, \mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X) \cong H^0(X, \omega_X) \oplus H^1(X, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X) \oplus H^1(X, \mathcal{O}_X).$$

We see $g = \dim H^1(X, \mathcal{O}_X) = \dim H^1(X, \mathbb{C})/2$. The de Rham cohomology is isomorphic to the singular cohomology, which relies only on the topology of Riemann surface.

2.3 General case

In fact, Serre and Kodaira vanishing theorem is still valid for a genral compact Kähler manifold. We only some results in this subsection. The main idea is to give an alternative definition of positivity without metioning divisors.

We assume X is a n-dimensional compact Kähler manifold.

Definition 1. *A line bundle $\mathcal{L}$ is called positive if its first Chern class $c_1(\mathcal{L}) \in H^2(X, \mathbb{R})$ can be represented by a closed positive real (1,1)-form.*

Theorem 6 (Kodaira vanishing theorem)**.** *Let $\mathcal{L}$ be a positive line bundle. Then*

$$H^q(X, \omega_X^p \otimes \mathcal{L}) = 0, \quad for p + q > n.$$

Note that Theorem 4 is the special case $q = 1, p = 1, n = 1$.

Theorem 7 (Serre vanishing theorem)**.** *Let $\mathcal{L}$ be a positive line bundle on X and $\mathcal{E}$ is a holomorphic vector bundle. Then there exists a constant m_0 such that $H^q(X, \mathcal{E} \otimes \mathcal{L}^m) = 0$ for $m \geq m_0$ and $q > 0$.*

Note that Theorem 2 is the special case $q = n = 1$.

Kodaira and Serre vanishing theorem, together with Hirzebruch-Riemann-Roch theorem, are important tools in studying cohomology of vector bundles.