

# Notes on Hodge theory

Hao Zhang

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## 1 Hodge theory for real manifolds

### 1.1 Hodge \*-operator

Let  $(M, g)$  be a  $n$ -dimensional closed oriented Riemann manifold with volume form  $\Omega$ . Locally, we choose an orthonormal frame  $\omega^1, \dots, \omega^n$  with respect to  $g$  for the cotangent bundle and thus we can write  $\Omega = \omega^1 \wedge \dots \wedge \omega^n$ . Denote the space of global smooth  $k$ -forms as  $\Omega^k(M) := \Gamma(M, \wedge^k T^*M)$ .

For  $\omega, \eta \in \Omega^k(M)$ , we write locally

$$\begin{aligned}\omega &= \sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k} \omega^{i_1} \wedge \dots \wedge \omega^{i_k} \\ \eta &= \sum_{i_1 < \dots < i_k} \eta_{i_1, \dots, i_k} \omega^{i_1} \wedge \dots \wedge \omega^{i_k}.\end{aligned}$$

Pointwisely, we define

$$\langle \omega, \eta \rangle := \sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k} \eta_{i_1, \dots, i_k}.$$

It is easy to check  $\langle \omega, \eta \rangle$  is a globally defined smooth function on  $M$  and it is independent of the choice of orthonormal frames. So it is reasonable to define

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle \Omega.$$

This gives an inner product on  $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$ . Note that  $(-, -)$  is positive definite because  $\Omega$  is nowhere vanishing.

Hodge \*-operator  $*$ :  $\Omega^k(M) \rightarrow \Omega^{n-k}(M)$  is defined by finding the 'complement' of the  $k$ -form in  $\Omega$ . More precisely, if

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k} \omega^{i_1} \wedge \dots \wedge \omega^{i_k},$$

then

$$*\omega = \sum_{i_1 < \dots < i_k} \varepsilon_{i_1, \dots, i_k} \omega_{i_1, \dots, i_k} \omega^1 \wedge \dots \wedge \omega^{\hat{i}_1} \wedge \dots \wedge \omega^{\hat{i}_k} \wedge \dots \wedge \omega^n,$$

where

$$\varepsilon_{i_1, \dots, i_k} = (-1)^{i_1 + \dots + i_k + 1 + \dots + n}.$$

**Proposition 1.** Suppose  $\omega, \eta \in \Omega^k(M)$ . Then

1.  $*1 = \Omega$ ,
2.  $*\Omega = 1$ ,
3.  $**\omega = (-1)^{k(n-k)}\omega$ ,
4.  $\omega \wedge *\eta = \langle \omega, \eta \rangle \Omega$ ,
5.  $\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle$ .

*Proof.* All is straightforward from definition. We only prove 5 for example:

$$\langle *\omega, *\eta \rangle \Omega = *\omega \wedge **\eta = (-1)^{k(n-k)} *\omega \wedge \eta = \eta \wedge *\omega = \langle \eta, \omega \rangle \Omega.$$

Then 5 follows from the fact:  $\Omega$  is nowhere vanishing. □

Recall  $d : \Omega^{k-1}(M) \rightarrow \Omega^k(M)$ . Define  $\delta = (-1)^{n(k-1)+1} * d * : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ .

**Proposition 2.**  $\delta$  is the adjoint operator of  $d$  with respect to  $(-, -)$ , i.e.,

$$(d\omega, \eta) = (\omega, \delta\eta),$$

for  $\omega \in \Omega^{k-1}(M), \eta \in \Omega^k(M)$ .

*Proof.* By direct computation,

$$\begin{aligned} d(\omega \wedge *\eta) &= d\omega \wedge *\eta + (-1)^{k-1} \omega \wedge d*\eta \\ &= d\omega \wedge *\eta + (-1)^{n(k-1)} \omega \wedge **d*\eta \\ &= d\omega^* \eta - \omega \wedge *\delta\eta. \end{aligned}$$

By Stoke's theorem,

$$(d\omega, \eta) = \int_M d\omega^* \eta = \int_M \omega \wedge *\delta\eta = (\omega, \delta\eta).$$

□

## 1.2 Harmonic forms and Hodge decomposition

**Definition 1.**  $\Delta := d\delta + \delta d : \Omega^k(M) \rightarrow \Omega^k(M)$  is called **Hodge-Laplace operator**. If  $\omega \in \Omega^k(M)$  satisfies  $\Delta\omega = 0$ , then  $\omega$  is called a **harmonic form**.

**Proposition 3.** Hodge-Laplace operator satisfies:

1.  $\Delta$  is self-adjoint, i.e.,  $(\Delta\omega, \eta) = (\omega, \Delta\eta)$  for all differential forms.
2.  $\Delta$  is positive, i.e.,  $(\Delta\omega, \omega) \geq 0$  and the equality holds if and only if  $\Delta\omega = 0$ .
3.  $*\Delta = \Delta*$ .

*Proof.* To show  $\Delta$  is self-adjoint, it suffices to assume  $\omega$  and  $\eta$  are both  $k$ -forms. Then

$$\begin{aligned} (\Delta\omega, \eta) &= (d\delta\omega, \eta) + (\delta d\omega, \eta) \\ &= (\delta\omega, \delta\eta) + (d\omega, d\eta) \\ &= (\omega, \Delta\eta). \end{aligned}$$

Note that the above identity gives

$$(\Delta\omega, \omega) = (\delta\omega, \delta\omega) + (d\omega, d\omega) \geq 0,$$

and  $(\Delta\omega, \omega) = 0$  if and only if  $\delta\omega = 0$  and  $d\omega = 0$ , if and only if  $\Delta\omega = 0$ .

To show  $\Delta$  commutes with  $*$ , assume  $\omega$  is a  $k$ -form, then

$$*\delta\omega = (-1)^{n(k-1)+1} * d * \omega = (-1)^k d * \omega.$$

Similarly,  $\delta * \omega = (-1)^{k+1} * d\omega$ . So

$$*d\delta\omega = (-1)^k \delta * \delta\omega = \delta d * \omega.$$

Similarly,  $*\delta d = d\delta*$ . Thus,

$$*\Delta = *d\delta + *\delta d = \delta d * + d\delta * = \Delta*$$

□

As we can see in the proof,  $\omega$  is a harmonic form if and only if  $d\omega = 0$  and  $\delta\omega = 0$ .

Denote  $\mathcal{H}^k(M)$  as the vector space of harmonic  $k$ -forms on  $M$ . We will see  $\mathcal{H}^k(M)$  is actually isomorphic to the de Rham cohomology of  $M$ .

**Proposition 4.** *Suppose  $\omega \in \mathcal{H}^k(M)$ .*

1.  $\omega$  has minimal norm in the de Rham cohomology class  $[\omega]$ . More precisely, for any  $(k-1)$ -form  $\eta$ ,  $(\omega + d\eta, \omega + d\eta) \geq (\omega, \omega)$ , and the equality holds if and only if  $d\eta = 0$ .
2.  $*\omega \in \mathcal{H}^{n-k}(M)$ .

*Proof.* Suppose  $\eta$  is a  $(k-1)$ -form. Then

$$\begin{aligned} (\omega + d\eta, \omega + d\eta) &= (\omega, \omega) + 2(\omega, d\eta) + (d\eta, d\eta) \\ &= (\omega, \omega) + 2(\delta\omega, \eta) + (d\eta, d\eta) \\ &= (\omega, \omega) + (d\eta, d\eta) \geq (\omega, \omega). \end{aligned}$$

The equality holds if and only if  $d\eta = 0$ .  $*\omega \in \mathcal{H}^{n-k}(M)$  follows immediately from  $\Delta* = *\Delta$ .  $\square$

**Theorem 1** (Poincaré duality). *Hodge  $*$ -operator gives an isomorphism  $\mathcal{H}^k(M) \cong \mathcal{H}^{n-k}(M)$ .*

*Proof.*  $*$  :  $\mathcal{H}^k(M) \rightarrow \mathcal{H}^{n-k}(M)$  is an isomorphism because  $** = (-1)^{k(n-k)}$ .  $\square$

To relate  $\mathcal{H}^k(M)$  with the usual de Rham cohomology, we introduce our main result in the real case.

**Theorem 2** (Hodge decomposition). *There exists an isomorphism of vector spaces:*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M).$$

More precisely, for any  $k$ -form  $\omega$ , there exists a unique decomposition

$$\omega = \omega_h + d\sigma + \delta\tau,$$

where  $\omega_h \in \mathcal{H}^k(M)$ ,  $\sigma \in \Omega^{k-1}(M)$ ,  $\tau \in \Omega^{k+1}(M)$ . When  $\omega$  is closed, the decomposition reduces to

$$\omega = \omega_h + d\sigma.$$

**Theorem 3** (Poincaré duality for de Rham cohomology). *We have an isomorphism  $\mathcal{H}^k(M) \cong H_{dR}^k(M)$ , which gives Poincaré duality for de Rham cohomology*

$$H_{dR}^k(M) \cong H_{dR}^{n-k}(M).$$

*Proof.* Define a linear map  $\iota : \mathcal{H}^k(M) \rightarrow H_{dR}^k(M)$  by  $\omega \mapsto [\omega]$ .  $\iota$  is injective by Proposition 4.  $\iota$  is surjective by Hodge decomposition. So  $\iota$  is an isomorphism.  $\square$

## 2 Hodge theory for complex manifolds

### 2.1 Dolbeault cohomology

Let  $X$  be a  $n$ -dimensional complex manifold. Denote  $\Omega_X^p$  as the sheaf of holomorphic  $p$ -forms on  $X$  and  $\mathcal{A}_{X,\mathbb{C}}^k$  as the sheaf of complex  $k$ -forms on  $X$ . Recall the decomposition of sheaves

$$\mathcal{A}_{X,\mathbb{C}}^k = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q},$$

and the differential

$$\begin{aligned} \partial : \mathcal{A}_X^{p,q} &\rightarrow \mathcal{A}_X^{p+1,q} \\ \bar{\partial} : \mathcal{A}_X^{p,q} &\rightarrow \mathcal{A}_X^{p,q+1}, \end{aligned}$$

where  $\mathcal{A}_X^{p,q}$  is the sheaf of forms of type  $(p, q)$  on  $X$ . Recall these sheaves  $\mathcal{A}^{p,q}$  are acyclic, i.e., have trivial higher cohomology from partition of unity. Then Dolbeault cohomology with respect to differential forms is defined by

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,\bullet}(X), \bar{\partial}) = \frac{\text{Ker}(\bar{\partial} : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q+1}(X))}{\text{Im}(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \rightarrow \mathcal{A}^{p,q}(X))}.$$

In fact, the Dolbeault cohomology is isomorphic to sheaf cohomology of  $\Omega_X^p$ , i.e.,

$$H^{p,q}(X) \cong H^q(X, \Omega_X^p).$$

To see this, it suffices to see the acyclic resolution

$$0 \rightarrow \Omega_X^p \rightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \rightarrow \dots$$

from  $\bar{\partial}$ -Poincaré lemma.

Let  $E$  be a complex vector bundle over  $X$  and  $\mathcal{A}^{p,q}(E)$  denote the sheaf defined by

$$U \mapsto \mathcal{A}^{p,q}(U, E) := \Gamma(U, \wedge^{p,q} X \otimes E),$$

where the tensor product is taken over  $\mathcal{O}_X$ . Locally, a section  $\alpha$  of  $\mathcal{A}^{p,q}(E)$  can be written as  $\alpha = \sigma \alpha_i \otimes s_i$  with  $\alpha$  and  $s_i$  local sections of  $\mathcal{A}_X^{p,q}$  and  $E$  respectively.

**Lemma 1.** *Suppose  $E$  is a holomorphic vector bundle. There exists a natural  $\mathbb{C}$ -linear operator  $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$  with  $\bar{\partial}_E^2 = 0$  and which satisfies the Leibniz rule*

$$\bar{\partial}_E(f \cdot \alpha) = \bar{\partial}(f) \wedge \alpha + f \bar{\partial}_E(\alpha).$$

*Proof.* Choose a local trivialization  $s = (s_1, \dots, s_r)$  of  $E$  and write  $\alpha \in \mathcal{A}^{p,q}(E)$  locally as  $\alpha = \sum \alpha_i \otimes s_i$ , where  $\alpha_i \in \mathcal{A}_X^{p,q}$ . Define

$$\bar{\partial}_E \alpha := \sum \bar{\partial}(\alpha_i) \otimes s_i.$$

Suppose we choose another holomorphic trivialization  $s' = (s'_1, \dots, s'_r)$  and obtain an operator  $\bar{\partial}'_E$ . Let  $s_i = \sum_j \psi_{ij} s'_j$ , where  $\psi_{ij}$  is the holomorphic transition function. Then

$$\begin{aligned} \bar{\partial}'_E \alpha &= \bar{\partial}'_E \left( \sum \alpha_i \otimes \sum_j \psi_{ij} s'_j \right) \\ &= \sum_{i,j} \bar{\partial}(\alpha_i \psi_{ij}) \otimes s'_j \\ &= \sum_{i,j} \bar{\partial}(\alpha_i) \psi_{ij} \otimes s'_j = \bar{\partial}_E(\alpha). \end{aligned}$$

So  $\bar{\partial}_E = \bar{\partial}'_E$  is independent of the choice of local trivialization. Therefore,  $\bar{\partial}_E^2 = 0$  since  $\bar{\partial}^2 = 0$ . From Leibniz rule of  $\bar{\partial}$ ,

$$\begin{aligned} \bar{\partial}_E(f \cdot \alpha) &= \bar{\partial}_E \left( \sum f \alpha_i \otimes s_i \right) \\ &= \sum \bar{\partial}(f \alpha_i) \otimes s_i \\ &= \sum (\bar{\partial}(f) \wedge \alpha_i + f \bar{\partial}(\alpha_i)) \otimes s_i \\ &= \bar{\partial}(f) \wedge \alpha + f \bar{\partial}_E(\alpha). \end{aligned}$$

□

The above lemma gives a complex  $(\mathcal{A}^{p,\bullet}(X, E), \bar{\partial}_E)$ , whose cohomology is called Dolbeault cohomology of the holomorphic vector bundle  $E$ :

$$H^{p,q}(X, E) := H^q(\mathcal{A}^{p,\bullet}(X, E), \bar{\partial}_E) = \frac{\text{Ker}(\bar{\partial}_E : \mathcal{A}^{p,q}(X, E) \rightarrow \mathcal{A}^{p,q+1}(X, E))}{\text{Im}(\bar{\partial}_E : \mathcal{A}^{p,q-1}(X, E) \rightarrow \mathcal{A}^{p,q}(X, E))}.$$

Similarly, Dolbeault cohomology of holomorphic vector bundles is isomorphic to sheaf cohomology:

$$H^{p,q}(X, E) \cong H^q(X, E \otimes \Omega_X^p),$$

which follows from acyclic resolution of  $E \otimes \Omega_X^p$ :

$$0 \rightarrow E \otimes \Omega_X^p \rightarrow \mathcal{A}^{p,0}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,1}(E) \rightarrow \dots$$

To summarize what we obtain:

Dolbeault cohomology of a holomorphic vector bundle  $E$  = sheaf cohomology of  $E \otimes \Omega_X^p$ .

## 2.2 Hermitian and Kähler structure on complex manifolds

We briefly recall some definition in this subsection.

Let  $X$  be a complex manifold with almost complex structure  $I$  and complex dimension  $n$ . A Riemann metric  $g$  on  $X$  is an hermitian structure if for any point  $x \in X$ , the scalar product  $g_x$  is compatible with  $I$ , i.e.,

$$g_x(Iv, Iw) = g_x(v, w).$$

The induced real  $(1, 1)$ -form  $\omega := g(I(), ())$  is called the fundamental form of hermitian manifold  $(X, g)$ . After complexification, the fundamental form  $\omega$  is locally of the form

$$\omega = \frac{i}{2} \sum_{i,j=1}^n h_{ij} dz_i \wedge d\bar{z}_j,$$

where  $(h_{ij}(x))$  is a positive definition matrix for each  $x \in X$ . It is not difficult to see that the hermitian structure is uniquely determined by  $I$  and  $\omega$ . The hermitian structure  $g$  is called a Kähler structure if  $\omega$  is closed. Denote the hermitian extension of  $g$  by  $g_{\mathbb{C}}$ .

The Hodge  $*$ -operator  $*$  :  $\bigwedge_{\mathbb{C}}^k X \rightarrow \bigwedge_{\mathbb{C}}^{2n-k} X$  is similar to the one defined for Riemann manifolds  $(X, g)$  with the natural volume form  $\Omega$ . More precisely,  $*$  is defined by  $\alpha \wedge * \bar{\beta} = g_{\mathbb{C}}(\alpha, \beta) \Omega$ . When restricted to  $\bigwedge_{\mathbb{C}}^k X$ ,  $*$  reduces to the usual Hodge  $*$ -operator for Riemann manifolds.

With Hodge  $*$ -operator, we can define several adjoint operators. Regard  $(X, g)$  as a real Riemann manifold with natural volume form  $\Omega$ . Since  $X$  has even dimension, the adjoint operator  $d^* = \delta$  is exactly  $d^* = - * \circ d \circ *$ . Analogously, one defines  $\partial^*$  and  $\bar{\partial}^*$  as  $\partial^* := - * \circ \partial \circ *$  and  $\bar{\partial}^* := - * \circ \bar{\partial} \circ *$  to make  $d^* = \partial^* + \bar{\partial}^*$  valid.

Therefore, it is natural to define the Laplacian operator associated to  $d, \partial, \bar{\partial}$ :  $\Delta := d^* d + d d^*$ ,  $\Delta_{\partial} = \partial^* \partial + \partial \partial^*$ ,  $\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$ . Note that  $*$  :  $\mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{n-q,n-p}(X)$ ,  $\partial^* : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p-1,q}(X)$ ,  $\bar{\partial}^* : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q-1}(X)$  and  $\Delta_{\partial}, \Delta_{\bar{\partial}}$  perserve bidegrees.

## 2.3 Hodge decomposition for Kähler manifolds

Suppose  $X$  is a complex manifold with an hermitian structure  $g$  and natural fundamental form  $\Omega$ . The hermitian extension of  $g$  is denoted by  $g_{\mathbb{C}}$ . Define an hermitian product on  $\mathcal{A}_{\mathbb{C}}^*(X)$  by

$$(\alpha, \beta) := \int_X g_{\mathbb{C}}(\alpha, \beta) * 1 = \int_X g_{\mathbb{C}}(\alpha, \beta) \Omega.$$

Note that the value of  $g$  on  $\mathcal{A}^*(X)$  is the exactly the inner product defined in the first section.

With respect to the hermitian product  $(-, -)$ . The degree decomposition

$$\mathcal{A}_{\mathbb{C}}^*(X) = \bigoplus_k \mathcal{A}_{\mathbb{C}}^k(X)$$

and the bidegree decomposition

$$\mathcal{A}_{\mathbb{C}}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

are both orthogonal decompositions and each component  $\mathcal{A}^{p,q}(X)$  is an infinite dimensional normed vector space with scalar product  $(-, -)$  and the induced norm  $\|\alpha\|^2 = (\alpha, \alpha)$ .

**Proposition 5.** *Suppose  $X$  is a closed hermitian manifold. Then with respect to  $(-, -)$ , the operators  $d^*, \partial^*, \bar{\partial}^*$  are actually adjoint operators of  $d, \partial, \bar{\partial}$ .*

*Proof.* We give a proof for  $\partial$  for example. For  $\alpha \in \mathcal{A}^{p-1,q}(X)$  and  $\beta \in \mathcal{A}^{p,q}(X)$ ,

$$\begin{aligned} (\partial \alpha, \beta) &= \int_X g_{\mathbb{C}}(\partial \alpha, \beta) * 1 = \int_X \partial \alpha \wedge * \bar{\beta} \\ &= \int_X \partial(\alpha \wedge * \bar{\beta}) - (-1)^{p+q-1} \int_X \alpha \wedge \partial(* \bar{\beta}) \end{aligned}$$

Since  $\alpha \wedge * \bar{\beta}$  is of bidegree  $(n-1, n)$ ,  $\partial(\alpha \wedge * \bar{\beta}) = d(\alpha \wedge * \bar{\beta})$ . So

$$\int_X \partial(\alpha \wedge * \bar{\beta}) = 0$$

by Stokes' theorem. Using  $*^2 = (-1)^k$  on  $\mathcal{A}^k(X)$ , we compute

$$\int_X \alpha \wedge \partial(*\bar{\beta}) = (-1)^{2n-(p+q)+1} \int_X g_{\mathbb{C}}(\alpha, -\partial^*\beta) * 1 = (-1)^{2n-(p+q)}(\alpha, \partial^*\beta).$$

So  $(\partial\alpha, \beta) = (\alpha, \partial^*\beta)$ .  $\square$

In the case of real manifolds, we interpret de Rham cohomology by harmonic forms. In the complex case, we will apply similar approach. For the differential  $d$ , the spaces of harmonic  $k$ -forms and  $(p, q)$ -forms (which are defined similarly in the real case) are denoted by  $\mathcal{H}^k(X, g)$  and  $\mathcal{H}^{p,q}(X, g)$ . For  $\partial$  and  $\bar{\partial}$ , we have analogous definition.

**Definition 2.** A  $k$ -form is called  $\bar{\partial}$ -harmonic if  $\Delta_{\bar{\partial}}\alpha = 0$  and define the spaces of  $\bar{\partial}$ -harmonic  $k$ -forms and  $(p, q)$ -forms by  $\mathcal{H}_{\bar{\partial}}^k(X, g)$  and  $\mathcal{H}_{\bar{\partial}}^{p,q}(X, g)$ .  $\partial$ -harmonic forms are analogous.

**Proposition 6.** Suppose  $(X, g)$  is a closed hermitian manifold. A form  $\alpha$  is  $\bar{\partial}$ -harmonic (resp.  $\partial$ -harmonic) if and only if  $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$  (resp.  $\partial\alpha = \partial^*\alpha = 0$ ).

*Proof.* The  $\bar{\partial}$  case follows from the identity

$$\begin{aligned} (\Delta_{\bar{\partial}}\alpha, \alpha) &= (\bar{\partial}^*\bar{\partial}\alpha + \bar{\partial}\bar{\partial}^*\alpha, \alpha) \\ &= \|\bar{\partial}^*(\alpha)\|^2 + \|\bar{\partial}\|^2. \end{aligned}$$

$\square$

**Proposition 7.** Suppose  $(X, g)$  is an hermitian manifold. Then

1.  $\mathcal{H}_{\bar{\partial}}^k(X, g) = \bigoplus_{\bar{\partial}} \mathcal{H}^{p,q}(X, g)$  and  $\mathcal{H}_{\partial}^k(X, g) = \bigoplus_{\partial} \mathcal{H}^{p,q}(X, g)$ .
2. If  $(X, g)$  is Kähler, then both decompositions coincide with  $\mathcal{H}^k(X, g)_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g)$ . In particular,  $\mathcal{H}^k(X, g)_{\mathbb{C}} = \mathcal{H}_{\bar{\partial}}^k(X, g) = \mathcal{H}_{\partial}^k(X, g)$ .

*Proof.* Suppose  $\alpha = \sum \alpha^{p,q}$  is the bidegree decomposition of a  $\bar{\partial}$ -harmonic form  $\alpha$ . Then

$$0 = \sum \Delta_{\bar{\partial}}(\alpha^{p,q})$$

is also a bidegree decomposition, which implies  $\Delta_{\bar{\partial}}(\alpha^{p,q}) = 0$  for all  $p, q$ . The proof of  $\partial$  decomposition is analogous.

The second assertion follows from the identity  $\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$  on a Kähler manifold.  $\square$

**Theorem 4** (Serre duality for harmonic forms). Suppose  $(X, g)$  is a compact connected hermitian manifold. Then the pairing

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \times \mathcal{H}_{\bar{\partial}}^{n-p, n-q}(X, g) \rightarrow \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

is non-degenerate. This yields an isomorphism

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \cong \mathcal{H}_{\bar{\partial}}^{n-p, n-q}(X, g)^*.$$

*Proof.* Suppose  $0 \neq \alpha \in \mathcal{H}_{\bar{\partial}}^{p,q}(X, g)$ . Then

$$\int_X \alpha \wedge *\bar{\alpha} = \|\alpha\|^2 > 0$$

implies the pairing is non-degenerate.  $\square$

Our main result is:

**Theorem 5** (Hodge decomposition of harmonic forms). Let  $(X, g)$  be a compact hermitian manifold. Then there exists two natural orthogonal decompositions

$$\begin{aligned} \mathcal{A}^{p,q}(X) &= \partial\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \oplus \partial^*\mathcal{A}^{p+1,q}(X), \\ \mathcal{A}^{p,q}(X) &= \bar{\partial}\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \oplus \bar{\partial}^*\mathcal{A}^{p+1,q}(X). \end{aligned}$$

Moreover,  $\mathcal{H}^{p,q}(X, g)_{\mathbb{C}}$  are all finite dimensional and if  $X$  is a Kähler manifold, then  $\mathcal{H}^{p,q}(X, g)_{\mathbb{C}} = \mathcal{H}_{\bar{\partial}}^{p,q}(X, g) = \mathcal{H}_{\partial}^{p,q}(X, g)$ .

The most nontrivial part is the existence of decomposition. We will not prove it in our note. The significance of Kähler condition is that we can forget all about ' $\partial$  or  $\bar{\partial}$ -harmonic', and replace them by 'harmonic'.

**Corollary 1.** *Suppose  $(X, g)$  is a compact hermitian manifold. Then the canonical map  $\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \rightarrow H^{p,q}(X)$  is an isomorphism.*

*Proof.* The canonical map  $\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \rightarrow H^{p,q}(X)$  is given by  $\alpha \mapsto [\alpha]$ , where  $[\alpha]$  is a cohomology class of  $\alpha$ . This map is injective because any harmonic coboundary  $\bar{\partial}\beta$  must satisfy  $\bar{\partial}^*\bar{\partial}\beta = 0$ . But  $0 = (\bar{\partial}^*\bar{\partial}\beta, \beta) = (\bar{\partial}\beta, \bar{\partial}\beta) = \|\bar{\partial}\beta\|^2 = 0$  implies  $\bar{\partial}\beta = 0$ . To show this map is surjective, it suffices to show any  $\bar{\partial}$ -closed  $(p, q)$ -form  $\beta$  must be cohomological to a  $\bar{\partial}$ -harmonic form. By Hodge decomposition, write  $\beta = \bar{\partial}\beta_1 + \beta_h + \bar{\partial}^*\beta_2$ , where  $\beta_h$  is harmonic. Since  $\bar{\partial}\beta = 0$ , it follows  $\bar{\partial}\bar{\partial}^*\beta_2 = 0$ , which implies  $\bar{\partial}^*\beta_2$  by a similar argument in the injective case.  $\square$

The proof shows the Hodge decomposition for closed forms does not contain the terms  $\partial^*\mathcal{A}^{p+1,q}(X)$  or  $\bar{\partial}^*\mathcal{A}^{p+1,q}(X)$ , just as the case in real manifolds.

**Corollary 2.** *Let  $(X, g)$  be a compact Kähler manifold. Then there exists a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

*which does not depend on the Kähler structure.*

*Proof.* Since  $X$  is Kähler,

$$H^k(X, \mathbb{C}) = \mathcal{H}^k(X, g)_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, g) = \bigoplus_{p+q=k} H^{p,q}(X).$$

To show the decomposition is independent of the Kähler structure, we choose two Kähler metric  $g, g'$  and  $\alpha \in \mathcal{H}^{p,q}(X, g), \alpha' \in \mathcal{H}^{p,q}(X, g')$ , which induce same elements in  $H^{p,q}(X)$ . So  $\alpha$  and  $\alpha'$  differ by some  $\bar{\partial}\gamma$ , i.e.,  $\alpha' = \alpha + \bar{\partial}\gamma$ . Then  $d\bar{\partial}\gamma = 0$ . By Hodge decomposition for  $d$ ,

$$\bar{\partial}\gamma = d\beta + \beta_h.$$

But  $0 = (\gamma, \bar{\partial}^*\beta_h) = (\bar{\partial}\gamma, \beta_h) = (\beta_h, \beta_h)$  implies  $\beta_h = 0$ . So  $\bar{\partial}\gamma \in d(\mathcal{A}_{\mathbb{C}}^{k-1}(X))$  and  $\alpha, \alpha'$  induces the same de Rham cohomology class in  $H^k(X, \mathbb{C})$ .  $\square$

### 3 Serre duality

In this section, we will give a generalized version of Hodge decomposition and Serre duality on a holomorphic vector bundle. Serre duality, together with Riemann-Roch theorem and Kodaira vanishing theorem, is significant in controlling the cohomology of holomorphic vector bundles. Most parts of this section are routine.

#### 3.1 Hermitian structure on vector bundles

Suppose  $M$  is a real manifold and  $E \rightarrow M$  is a complex vector bundle.

**Definition 3.** *An hermitian structure  $h$  on  $E \rightarrow M$  is an hermitian scalar product  $h_x$  on each fiber  $E(x)$  which depends differtably on  $x$ . The pair  $(E, h)$  is called an hermitian vector bundle.*

If  $\psi : E|_U \cong U \times \mathbb{C}^r$  is a trivialization over some open subset  $U$ , then  $h_x$  is given by a positive-definite hermitian matrix  $(h_{ij}(x))$  for each  $x \in U$ . The definition says  $(h_{ij}(x))$  relies differtably on  $x \in U$ .

**Example 1.** *If  $(X, g)$  is an hermitian manifold, then the tangent, cotangent and form bundles have natural hermitian structures. Moreover, if  $(E, h)$  is an hermitian vector bundle over  $(X, g)$ , then  $\bigwedge^{p,q} X \otimes E$  have natural hermitian structures.*

**Proposition 8.** *Every complex vector bundle admits an hermitian structure.*

*Proof.* Choose an open covering  $X = \bigcup U_i$  trivializing  $E$  and glue the constant hermitian structure on the trivial bundles  $U_i \times \mathbb{C}^r$  by means of partition of unity. The resulting product is hermitian because positive linear combination of hermitian product is again hermitian.  $\square$

Now let  $(X, g)$  be an hermitian manifold of complex dimension  $n$  and  $(E, h)$  is an hermitian vector bundle. Denote the induced hermitian structure on  $\bigwedge^{p,q} X \otimes E$  by  $(-, -)$ . We may interpret  $h$  as a  $\mathbb{C}$ -antilinear isomorphism  $h : E \cong E^*$ .

**Definition 4.** Hodge  $*$ -operator  $\bar{*}_E : \bigwedge^{p,q} X \otimes E \rightarrow \bigwedge^{n-p, n-q} X \otimes E^*$  is defined by

$$\bar{*}_E(\varphi \otimes s) = \bar{*}_E(\varphi) \otimes h(s) = \overline{*(\varphi)} \otimes h(s) = *( \bar{\varphi}) \otimes h(s).$$

Hodge  $*$ -operator  $\bar{*}_E$  is a  $\mathbb{C}$ -antilinear isomorphism that depends on  $g$  and  $h$ . Similarly, we check easily

$$(\alpha, \beta) * 1 = \alpha \wedge \bar{*}_E(\beta)$$

where  $\wedge$  means taking usual wedge products in the form part and evaluation map in the bundle part. Moreover,  $\bar{*}_E \circ \bar{*}_E = (-1)^{p+q}$  on  $\bigwedge^{p,q} X \otimes E$ . From now on, denote  $\mathcal{A}^{p,q}(E)$  by sheaf of sections of  $\bigwedge^{p,q} X \otimes E$ . We will not distinguish  $\mathcal{A}^{p,q}(E)$  and  $\bigwedge^{p,q} X \otimes E$  from now on.

Define  $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$  by  $\bar{\partial}_E(\alpha \otimes s) = \bar{\partial}(\alpha) \otimes s$  and its adjoint operator  $\bar{\partial}_E^* : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q-1}(E)$  by  $\bar{\partial}_E^* = -\bar{*}_E \circ \bar{\partial}_E \circ \bar{*}_E$ . The Laplacian operator  $\Delta_E : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q}(E)$  is defined by  $\Delta_E = \bar{\partial}_E^* \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E^*$ .

**Definition 5.** A section  $\alpha \in \mathcal{A}^{p,q}(E)$  is called harmonic if  $\Delta_E(\alpha) = 0$ . The space of harmonic forms is denoted by  $\mathcal{H}^{p,q}(X, E)$ .

Since  $\bar{*}_E$  commutes with  $\Delta_E$ ,  $\bar{*}_E$  restricts to a  $\mathbb{C}$ -antilinear isomorphism  $\bar{*}_E : \mathcal{H}^{p,q}(X, E) \rightarrow \mathcal{H}^{p,q}(X, E^*)$ .

From now on, we suppose  $(X, g)$  is a compact hermitian manifold. Define a hermitian scalar product on  $\mathcal{A}^{p,q}(X, E)$  by

$$(\alpha, \beta) = \int_X (\alpha, \beta) * 1,$$

where  $(-, -)$  inside the integral is the pointwise hermitian inner product on  $\mathcal{A}^{p,q}(X, E)$ .

**Proposition 9.** 1.  $\bar{\partial}_E^*$  is actually the adjoint operator of  $\bar{\partial}_E$  and  $\Delta_E$  is self-adjoint with respect to  $(-, -)$ .

2.  $\alpha \in \mathcal{A}^{p,q}(X, E)$  is harmonic if and only if  $\bar{\partial}_E(\alpha) = \bar{\partial}_E^*(\alpha) = 0$ .

*Proof.* Analogous to Proposition 2 and 3. □

### 3.2 Serre duality on vector bundles

To give Serre duality on vector bundles, we have to give a generalized version of Hodge decomposition for vector bundles.

**Theorem 6** (Hodge decomposition for vector bundles). Suppose  $(X, g)$  is a compact hermitian manifold and  $(E, h)$  is an hermitian vector bundle. We have Hodge decomposition

$$\mathcal{A}^{p,q}(X, E) = \bar{\partial}_E \mathcal{A}^{p,q-1}(X, E) \oplus \mathcal{H}^{p,q}(X, E) \oplus \bar{\partial}_E^* \mathcal{A}^{p,q+1}(X, E)$$

and  $\mathcal{H}^{p,q}(X, E)$  is finite dimensional for each  $p$  and  $q$ .

If we choose  $E = \mathcal{O}_X$ , the above theorem reduces to the usual Hodge decomposition.

**Corollary 3.** The natural map  $\mathcal{H}^{p,q}(X, E) \rightarrow H^{p,q}(X, E)$  is an isomorphism. In particular, Dolbeault cohomology  $H^{p,q}(X, E) \cong H^q(X, E \otimes \Omega_X^p)$  is finite dimensional.

*Proof.* To show the map is injective, choose a harmonic coboundary  $\bar{\partial}_E \beta$ . So

$$0 = (\bar{\partial}_E^* \bar{\partial}_E \beta, \beta) = (\bar{\partial}_E \beta, \bar{\partial}_E \beta) = \|\bar{\partial}_E \beta\|^2$$

implies  $\bar{\partial}_E \beta = 0$ .

To show the map is surjective, choose a closed element  $\alpha \in \mathcal{A}^{p,q}(X, E)$  and by Hodge decomposition,

$$\alpha = \bar{\partial}_E \alpha_1 + \alpha_h + \bar{\partial}_E^* \alpha_2.$$

Since  $\bar{\partial}_E \alpha = 0$ ,  $\bar{\partial}_E \bar{\partial}_E^* \alpha_2 = 0$  and then  $\bar{\partial}_E^* \alpha_2 = 0$ . So  $\alpha = \bar{\partial}_E \alpha_1 + \alpha_h$ , i.e., the image of  $\alpha_h \in \mathcal{H}^{p,q}(X, E)$  under the natural map is the cohomology class  $[\alpha]$ . □

By this corollary, we see any (nonzero) Dolbeault cohomology class can be represented by a (nonzero) harmonic element.



**Theorem 7** (Serre duality for vector bundles). *Suppose  $X$  is a compact complex manifold. For any holomorphic vector bundle  $E$  on  $X$ , define a pairing*

$$H^{p,q}(X, E) \times H^{n-p, n-q}(X, E^*) \rightarrow \mathbb{C}$$

by

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

*Then the pairing is non-degenerate.*

*Proof.* This pairing is well-defined by Stokes' theorem. Choose hermitian structures  $h$  and  $g$  on  $E$  and  $X$ . For any nonzero cohomology class in  $H^{p,q}(X, E)$ , we can find a nonzero harmonic element  $\alpha \in \mathcal{H}^{p,q}(X, E)$  representing the given class. Define  $\beta = \bar{*}_E \alpha \in \mathcal{H}^{n-p, n-q}(X, E^*)$ . We check

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge \bar{*}_E \alpha = \int_X (\alpha, \alpha) * 1 = \|\alpha\|^2 \neq 0.$$

So this pairing is non-degenerate. □

**Corollary 4.** *For any holomorphic vector bundle  $E$  over a compact complex manifold  $X$ , there exist  $\mathbb{C}$ -linear isomorphisms:*

$$H^q(X, E \otimes \Omega^p) \cong H^{p,q}(X, E) \cong H^{n-p, n-q}(X, E^*)^* \cong H^{n-q}(X, E^* \otimes \Omega^{n-p})^*$$