Research Statement

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1 Background

My research primarily focuses on the theory of conformal blocks in logarithmic conformal field theory (CFT). By logarithmic CFT, I mean the case where the underlying vertex operator algebra (VOA) is $\mathbb{N}$-graded and C_2-cofinite, but not necessarily rational. The representation categories of such VOAs are finite abelian categories, but not necessarily semisimple. In Segal's framework, conformal block theory plays a central role in the geometric understanding of the representation theory of VOA and CFT. The construction of the tensor category of a VOA relies on the deep understanding of genus 0 and genus 1 conformal blocks.

In Segal's framework, sewing conformal blocks provides an efficient method to construct higher genus conformal blocks from those of lower genus. For instance, both the product/iterate of intertwining operators and their q-traces can be viewed as examples of sewing genus 0 conformal blocks. The proofs of associativity (genus 0) and modular invariance (genus 1) in rational CFT [Hua05a, Hua05b] can be reformulated in the following way:

(1) **Convergence of sewing**. Sewing conformal blocks converges to a higher genus conformal block.

(2) **Factorization**. Sewing conformal blocks gives all higher genus conformal blocks.

(3) **Connections**. The spaces of smooth conformal blocks form a holomorphic vector bundle together with a connection ∇.

(4) **Parallelism**. Sewing conformal blocks is parallel with respect to ∇, up to the projective term depending on the central charge c.

For rational CFT of arbitrary genus, Bin Gui proved the convergence of sewing conformal blocks [Gui24]. Thanks to the significant result [DGT24] in algebraic geometry, factorization property (2) of arbitrary genus is proved. The general theory of (3) and (4) is also discussed in [Gui24].

In logarithmic CFT, the situation becomes more intricate even for torus (genus 1) conformal blocks. The most subtle point lies in the factorization property (2): self-sewing conformal blocks does *not* produce all higher genus conformal blocks. To solve this problem, there are two parallel approaches to recover torus conformal blocks:

- **Pseudo-q-traces (VOA approach)**. Pseudo-q-traces were first introduced by Miyamoto as a tool to establish modular invariance of vertex operators. HIn this framework, pseudo-q-traces replace the usual q-traces, which arise naturally from self-sewing constructions. The general modular invariance theorem in logarithmic CFT was later proved by Huang [Hua24].

- **Ends/coends (TFT approach)**. Ends and coends provide an alternative method to study factorization properties in logarithmic CFT from the perspective of topological field theory. This method was originally introduced by Lyubashenko in [Lyu96] in the context of topological modular functors. (See also [FS17, HR24] for discussions.)

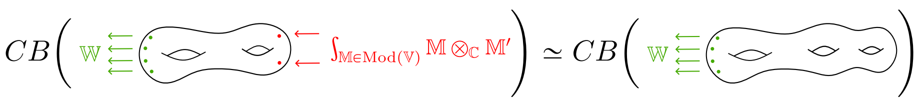
Therefore, it is natural to ask:

Question 1. *How are pseudo-q-traces (VOA approach) related to ends/coends (TFT approach)?*

2 Sewing-Factorization Theorem and (co)ends

In joint work with Bin Gui [GZ23, GZ24, GZ25a], we provide a systematic treatment of (1)-(4) for logarithmic CFT of *arbitrary* genus. The answers to (1) and (2) are encapsulated in the **Sewing–Factorization Theorem**, established in [GZ24, GZ25a]. The general theory addressing (3) and (4) is also developed in [GZ24].

Since self-sewing does not produce all higher genus conformal blocks, we instead use disjoint sewing along several points to formulate Sewing–Factorization Theorem, and then transform self-sewing to disjoint sewing, as illustrated in Figure 1. A special case of Sewing-Factorization Theorem for Figure 1 is



where both sides denote the space of conformal blocks. This point of view is closely related to the construction of nonsemisimple modular functors in topological field theory [Lyu95, Lyu96], especially the horizontal composition of profunctors, as in [HR24]. Therefore, in our formulation, the ends and coends naturally appear as in TFT approaches. This leads to a connection between the space of conformal blocks and topological modular functors [GZ25a].

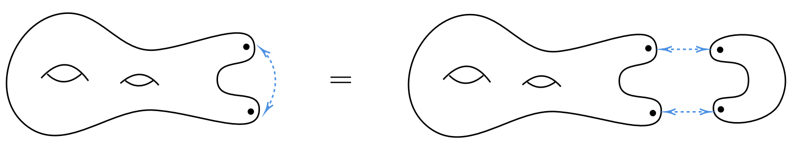


Figure 1 Transforming self-sewing to disjoint sewing

3 Gainutdinov-Runkel conjecture

In joint work with Bin Gui [GZ25c, GZ25b], we explain how this perspective relates to pseudo-q-traces, thereby resolving Question 1. In particular, we prove a conjecture of Gainutdinov-Runkel [GR19], which asserts that the space of vacuum torus conformal blocks is isomorphic to the space of symmetric linear functionals (SLF) on $\mathrm{End}_{\mathbb{V}}(\mathbb{G})$ via the pseudo-q-trace construction, where $\mathbb{G} \in \mathrm{Mod}(\mathbb{V})$ is a projective generator.

The strategy is as follows:

- By applying the Sewing–Factorization Theorem, we show that the end

$$\mathbb{E} = \int_{\mathbb{X} \in \mathrm{Mod}(\mathbb{V})} \mathbb{X} \otimes \mathbb{X}' \in \mathrm{Mod}(\mathbb{V} \otimes \mathbb{V})$$

  carries the structure of an associative algebra, and that the category of suitable $\mathbb{E}$-modules is linearly isomorphic to $\mathrm{Mod}(\mathbb{V})$.

- Again using the Sewing–Factorization Theorem, we establish that the space of vacuum torus conformal blocks is isomorphic to $\mathrm{SLF}(\mathbb{E})$.

- The pseudo-trace construction with respect to the $\mathbb{E} - \mathrm{End}_{\mathbb{V}}(\mathbb{G})^{\mathrm{op}}$ bimodule $\mathbb{G}$ [GZ25c] yields an isomorphism

$$\mathrm{SLF}(\mathrm{End}_{\mathbb{V}}(\mathbb{G})) \xrightarrow{\simeq} \mathrm{SLF}(\mathbb{E}). \tag{1}$$

  Composing the Sewing–Factorization isomorphism with (1) gives the desired isomorphism.

4 Non vector-bundle structure for nodal conformal blocks in logarithmic CFT

In algebraic geometry, conformal blocks for nodal curves play a crucial role in the study of vector bundles over the moduli space [TUY89, NT05, DGT24]. The most recent advance, due to [DGT24], establishes that the spaces of conformal blocks form a vector bundle over $\overline{\mathcal{M}}_{g,N}$. Their proof relies on a factorization result for smooth conformal blocks obtained via infinitesimal smoothing of nodal curves.

In my recent work [Zha25], I showed that the above result on vector bundles does not hold in the irrational/non-semisimple case. More precisely, if $\mathbb{V}$ is an $\mathbb{N}$-graded, C_2-cofinite VOA admitting a module that is not generated by its lowest weight subspace, then the spaces of conformal blocks associated to certain $\mathbb{V}$-modules do *not* form a vector bundle over $\overline{\mathcal{M}}_{0,N}$. In particular, the end $\mathbb{E}$ is not isomorphic to the mode transition algebra defined by [DGK25a, DGK25b] as an object of $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$.

Many C_2-cofinite VOAs admit modules that are not generated by their lowest weight subspaces—for instance, the triplet algebras $\mathcal{W}_p$ for $p \in \mathbb{Z}_{\geqslant 2}$ and the even symplectic fermion VOAs SF_d^+ for $d \in \mathbb{Z}_+$. Consequently, my result applies to these examples.

5 Proposed research

It is well-known that: the proof of Verlinde conjecture and rigidity in rational CFT relies heavily on the fact that the modular S-transform coincides with the categorical S-transform [Hua08a, Hua08b]. In fact, this is the most difficult part in Huang's proof. Therefore, it is natural to consider the following open problem in logarithmic CFT:

Question 2. *Prove that at least when $\mathbb{V}$ is strongly finite and rigid, the categorical S-transform coincides with the modular S-transform.*

To answer Question 2, we need to solve the following problems.

- **Describe the space of torus conformal blocks via (co)ends.** This is done by Sewing-Factorization Theorem: the space of torus conformal blocks is isomorphic to $\mathrm{Hom}_{\mathbb{V}}(\mathbb{L}, \mathbb{M})$, where

$$\mathbb{L} = \boxtimes_{\mathrm{HLZ}} \Big(\int_{\mathbb{X} \in \mathrm{Mod}(\mathbb{V})} \mathbb{X} \otimes \mathbb{X}' \Big)$$

  is called the **Lyubashenko's construction**, which is isomorphic to Lyubashenko's coend when $\mathbb{V}$ is strongly finite and rigid.

- **Define the modular and categorical S-transform.** This is done in the rigid setting. The modular S-transform is defined by $\tau \mapsto -1/\tau$. The categorical S-transform is defined by the Hopf pairing of $\mathbb{L}$.

- **Check that the modular and categorical S-transforms are equivariant with respect to Sewing-Factorization isomorphism.** This is not done yet, even when $\mathbb{V}$ is strongly finite and rigid.

Bin Gui and I are now working on Question 2.

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