Non-Equivalence of Smooth and Nodal Conformal Block Functors in Logarithmic CFT

HAO ZHANG

Abstract

Let $\mathbb{V}$ be an $\mathbb{N}$-graded, C_2-cofinite vertex operator algebra (VOA) admitting a non-lowest generated module in $\mathrm{Mod}(\mathbb{V})$ (e.g., the triplet algebras $\mathcal{W}_p$ for $p \in \mathbb{Z}_{\geqslant 2}$ or the even symplectic fermion VOAs SF_d^+ for $d \in \mathbb{Z}_+$). We prove that, unlike in the rational case, the spaces of conformal blocks associated to certain $\mathbb{V}$-modules do not form a vector bundle on $\overline{\mathcal{M}}_{0,N}$ for $N \geqslant 4$ by showing that their dimensions differ between nodal and smooth curves. Consequently, the sheaf of coinvariants associated to these $\mathbb{V}$-modules on $\overline{\mathcal{M}}_{0,N}$ is not locally free for $N \geqslant 4$. Furthermore, the mode transition algebra $\mathfrak{A}$ introduced in [DGK25a, DGK25b], unlike in the rational case, is not isomorphic to the end $\mathbb{E} = \int_{\mathbb{X} \in \mathrm{Mod}(\mathbb{V})} \mathbb{X} \otimes \mathbb{X}'$ as an object of $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$.

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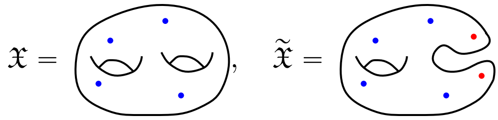
0 Introduction

Let $\mathbb{V}$ be an $\mathbb{N}$-graded, C_2-cofinite vertex operator algebra (VOA). When $\mathbb{V}$ is rational, a remarkable achievement is the factorization property of conformal blocks associated to grading-restricted $\mathbb{V}$-modules [DGT24]. This property establishes isomorphisms relating spaces of conformal blocks of higher genus or fewer marked points (e.g., $\mathfrak{X}$ in (0.1a) or $\mathfrak{Y}$ in (0.1b)) to those of lower genus or more marked points (e.g., $\widetilde{\mathfrak{X}}$ in (0.1a) or $\widetilde{\mathfrak{Y}}$ in (0.1b)):



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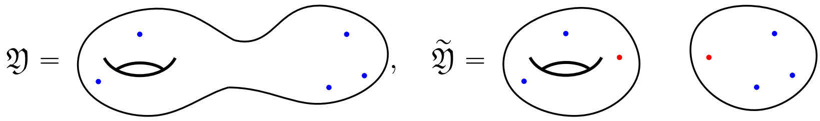

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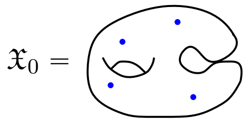
These isomorphisms are typically constructed through conformal blocks for *nodal curves* (e.g., $\mathfrak{X}_0$ in (0.2a) or $\mathfrak{Y}_0$ in (0.2b)) [TUY89, BFM91, Uen97, NT05, Uen08, DGT21, DGT24].



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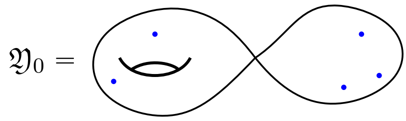

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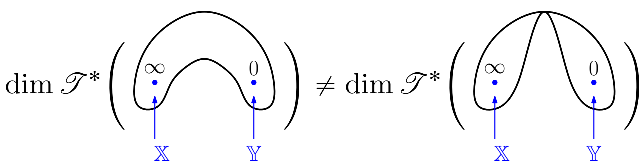
The proof crucially relies on the fact that the dimensions of the spaces of conformal blocks for nodal curves (e.g., $\mathfrak{X}_0$ or $\mathfrak{Y}_0$) and for smooth curves (e.g. $\mathfrak{X}$ or $\mathfrak{Y}$) coincide, a condition that ensures the spaces of conformal blocks form a vector bundle on $\overline{\mathcal{M}}_{g,N}$ (see [DGT24, Sec. 8] for details). In this paper, we show that, once the rationality assumption is removed, this dimension constancy—and hence the vector bundle structure—no longer holds for general $\mathbb{N}$-graded, C_2-cofinite VOAs.

From now on, let $\mathbb{V}$ be an $\mathbb{N}$-graded, C_2-cofinite VOA admitting a module in $\mathrm{Mod}(\mathbb{V})$ that *is not generated by its lowest weight subspace*. (It can be proved that the triplet algebras $\mathcal{W}_p$ for $p \in \mathbb{Z}_{\geqslant 2}$ and the even symplectic fermion VOAs SF_d^+ for $d \in \mathbb{Z}_+$ satisfy this condition; see Cor. 1.13 and 1.14.) Recently, Damiolini-Gibney-Krashen introduced the notion of **strongly unital** property, a sufficient condition that guarantees the vector bundle structure on $\overline{\mathcal{M}}_{g,N}$ [DGK25a, Cor. 5.2.6]. Moreover, it was proved that the triplet algebras $\mathcal{W}_p$ do not satisfy the strongly unital property [DGK25a, Prop. 9.1.4]. This provides evidence, though not directly imply, that the dimensions of spaces of their conformal blocks differ between nodal and smooth curves.

Our main result (Thm. 2.2) states that there exist $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$ such that



(0.3)

where $\mathscr{T}^*(\cdots)$ denotes the space of conformal blocks. Our main idea is a variation of the following argument: the end $\mathbb{E} = \int_{\mathbb{X} \in \mathrm{Mod}(\mathbb{V})} \mathbb{X} \otimes \mathbb{X}'$ and the mode transition algebra $\mathfrak{A}$ [DGK24, DGK25a, DGK25b] are objects in $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$ representing the conformal block functor on the two sides of (0.3). However, $\mathfrak{A}$ is generated by its lowest weight subspace, but $\mathbb{E}$ is not.

There are several further consequences of (0.3), summarized as follows.

- In [DW25, Question 5.6.3], the authors ask about the relationship between the mode transition algebra $\mathfrak{A}$ and the end $\mathbb{E}$. When $\mathbb{V}$ is C_2-cofinite and rational, it is proved in [DGK25b] that $\mathfrak{A} \simeq \mathbb{E}$ as objects in $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$. However, in our case, it follows immediately from (0.3) that

$$\mathfrak{A} \not\simeq \mathbb{E} = \int_{\mathbb{X}\in\mathrm{Mod}(\mathbb{V})} \mathbb{X} \otimes \mathbb{X}' \quad \text{in } \mathrm{Mod}(\mathbb{V}^{\otimes 2}).$$

  See Thm. 2.9 for details.

- In [DGK25b, Question 6.1], the authors ask whether the sheaf of coinvariants forms a vector bundle. In Rem. 2.4, we prove that there exist $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$ such that

  (a) the spaces of conformal blocks associated to $\mathbb{X}, \mathbb{Y}, \mathbb{V}^{\otimes(N-2)}$ do not form a vector bundle on $\overline{\mathcal{M}}_{0,N}$ for $N \geqslant 4$.

  (b) the sheaf of coinvariants associated to $\mathbb{X}, \mathbb{Y}, \mathbb{V}^{\otimes(N-2)}$ is not locally free on $\overline{\mathcal{M}}_{0,N}$ for $N \geqslant 4$.

We remark that it remains possible that (0.3) holds as an equality if the definition of nodal conformal blocks is suitably modified. However, if such a new definition exists, it should not be expected to have a direct connection with the Zhu algebra $A(\mathbb{V})$ [Zhu96], since in the non-semisimple setting $A(\mathbb{V})$ exerts only weak control over modules that are not generated by their lowest weight subspaces.

Acknowledgment

I am grateful to Chiara Damiolini for introducing the connection between nodal conformal blocks and mode transition algebras, to Xu Gao for answering many questions of mode transition algebras, and to Bin Gui for many enlightening conversations and valuable comments on this project.

1 Preliminaries

1.1 Notation

Throughout this paper, we use the following notation.

- $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{Z}_+ = \{1, 2, \cdots\}$, $\mathbb{Z}_{\geqslant 2} = \{2, 3, \cdots\}$.

- Let X be a finite set. Then $\mathrm{Card}(X)$ denotes the cardinate of X.

- Let $\Re(z)$ be the real part of $z \in \mathbb{C}$.

- Let $\mathcal{V}ect$ be the category of finite dimensional vector spaces over $\mathbb{C}$.

- Let ζ be the standard coordinate of $\mathbb{C}$.
- Throughout this paper, we fix an $\mathbb{N}$-graded C_2-cofinite vertex operator algebra (VOA) $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$ with conformal vector $\mathbf{c}$ and vacuum vector $\mathbf{1}$. Each nonzero vector v in $\mathbb{V}(n)$ is **homogeneous** of weight $\mathrm{wt}(v) = n$.
- For each $N \in \mathbb{N}$, let $\mathrm{Mod}(\mathbb{V}^{\otimes N})$ denote the category of grading-restricted generalized $\mathbb{V}^{\otimes N}$-modules. $\mathrm{Mod}(\mathbb{V}^{\otimes N})$ is an abelian category by [Hua09] (see also [MNT10]).
- Since $\mathbb{V}$ is C_2-cofinite, it has only finitely many equivalence classes of irreducible $\mathbb{V}$-modules (see [Hua09, Prop. 4.2]). Denote by Irr a finite set of representatives of these classes.
- Let $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$, and assume that $\mathbb{X}$ is irreducible. Then $[\mathbb{Y} : \mathbb{X}]$ denotes the multiplicity of $\mathbb{X}$ in a composition series of $\mathbb{Y}$.
- If $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ and $v \in \mathbb{V}$, then the i-th vertex operator

$$Y_{\mathbb{W},i}(v,z) = \sum_{n\in\mathbb{Z}} Y_{\mathbb{W},i}(v)_n z^{-n-1}$$

is $Y(\mathbf{1}\otimes\cdots\otimes v\otimes\cdots\otimes\mathbf{1},z)$ with v placed in the i-th component. We abbreviate $Y_{\mathbb{W},i}$ to Y_i when no confusion arises. We also define

$$Y_i'(v,z) = Y_i(\mathcal{U}(\gamma_z)v, z^{-1})$$

where $\mathcal{U}(\gamma_z) = e^{zL(1)}(-z^{-2})^{L(0)}$. Moreover, we expand

$$Y_i'(v,z) = \sum_{n\in\mathbb{Z}} Y_i'(v)_n z^{-n-1}$$

Finally, we set $L_i(n) = Y_i(\mathbf{c})_{n-1}$.
- If $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ and $\lambda_1, \dots, \lambda_N \in \mathbb{C}$, then $\mathbb{W}_{[\lambda_\bullet]}$ is the subspace of all $w \in \mathbb{W}$ such that for all $1 \leqslant i \leqslant N$, w is a generalized eigenvector of $L_i(0)$ with eigenvalue λ_i. Any $w \in \mathbb{W}_{[\lambda_\bullet]}$ is said to be **homogeneous** of weight $\lambda_\bullet$. In this case, we write

$$\begin{aligned}\mathrm{wt}_i(w) &= \lambda_i, \quad 1 \leqslant i \leqslant N,\\ \mathrm{wt}(w) &= \sum_{i=1}^N \mathrm{wt}_i(w) = \sum_{i=1}^N \lambda_i.\end{aligned}$$

The finite dimensional subspace $\mathbb{W}_{[\leqslant\lambda_\bullet]}$ is defined to be the direct sum of all $\mathbb{W}_{[\mu_\bullet]}$ where $\Re(\mu_i) \leqslant \Re(\lambda_i)$ for all $1 \leqslant i \leqslant N$. Then the contragredient $\mathbb{V}^{\otimes N}$-module of $\mathbb{W}$, as a vector space, is

$$\mathbb{W}' = \bigoplus_{\lambda_\bullet\in\mathbb{C}^N} (\mathbb{W}_{[\lambda_\bullet]})^*$$

Then for each $w \in \mathbb{W}, w' \in \mathbb{W}$ we clearly have

$$\langle Y_i(v,z)w, w'\rangle = \langle w, Y_i'(v,z)w'\rangle$$

Finally, the algebraic completion of $\mathbb{W}$ is

$$\overline{\mathbb{W}} = (\mathbb{W}')^* = \prod_{\lambda_\bullet \in \mathbb{C}^N} \mathbb{W}_{[\lambda_\bullet]}$$

- If $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$, we refer to the vertex operators Y_1 and Y_2 as Y_+ and Y_-, respectively, i.e.,

  $$Y_+(v,z) = Y(v \otimes \mathbf{1}, z), \quad Y_-(v,z) = Y(\mathbf{1} \otimes v, z).$$

  for $v \in \mathbb{V}$. We shall also denote $\mathbb{W}$ by $(\mathbb{W}, Y_+, Y_-)$. If $w \in \mathbb{W}$ is homogeneous of weight (λ_+, λ_-), then $\mathrm{wt}_+(w) = \lambda_+, \mathrm{wt}_-(w) = \lambda_-$ and $\mathrm{wt}(w) = \lambda_+ + \lambda_-$.

1.2 Non-lowest generated $\mathbb{V}$-modules in $\mathrm{Mod}(\mathbb{V})$

In this section, we focus on modules in $\mathrm{Mod}(\mathbb{V})$ that are not generated by their lowest weight subspaces.

Definition 1.1. Let $\mathbb{X}$ be a weak $\mathbb{V}$-module. Define

$$\Omega(\mathbb{X}) = \{w \in \mathbb{X} : Y(v)_n w = 0 \text{ for all } v \in \mathbb{V}, n \in \mathbb{Z} \text{ such that } \mathrm{wt}(v) - n - 1 < 0\}$$

We refer to $\Omega(\mathbb{X})$ as the **lowest weight subspace** of $\mathbb{X}$. The module $\mathbb{X}$ is said to be **lowest generated** if it is generated by $\Omega(\mathbb{X})$.

Remark 1.2. Let $\mathbb{X}$ be a weak $\mathbb{V}$-module, and let $\mathcal{A}$ denote the subalgebra of $\mathrm{End}(\mathbb{X})$ generated by all operators $Y(v)_n$ where $v \in \mathbb{V}, n \in \mathbb{Z}$. Let $\mathcal{A}_{\geqslant 0}$ be the unital subalgebra generated by all operators $Y(v)_n$ such that $\mathrm{wt}(v) - n - 1 \geqslant 0$. If $\mathcal{T}$ is a subspace of $\Omega(\mathbb{X})$, then it is easy to see that $\mathcal{A} \cdot \mathcal{T} = \mathcal{A}_{\geqslant 0} \cdot \mathcal{T}$.

Definition 1.3. Let $\mathbb{X} \in \mathrm{Mod}(\mathbb{V})$.

(a) We say that $\mathbb{X}$ is **singly generated** (resp. **singly lowest generated)** if there exists a homogeneous vector $w \in \mathbb{X}$ (resp. $w \in \Omega(\mathbb{X})$) such that $\mathbb{X}$ is generated by w.

(b) Assume that $\mathbb{X}$ is singly generated. The **conformal weight** $\mathrm{wt}(\mathbb{X})$ of $\mathbb{X}$ is defined to be $\mathrm{wt}(x)$ where $x \in \mathbb{X}$ is homogeneous and satisfies $\Re(\mathrm{wt}(y)) \geqslant \Re(\mathrm{wt}(x))$ for all $y \in \mathbb{X}$.

Remark 1.4. Let $\mathbb{X} \in \mathrm{Mod}(\mathbb{V})$.

(a) Assume that $\mathbb{X}$ is singly lowest generated. Let $w \in \Omega(\mathbb{X})$ be any homogeneous vector generating $\mathbb{X}$. Then Rem. 1.2 implies that $\mathrm{wt}(\mathbb{X}) = \mathrm{wt}(w)$.

(b) Assume that $\mathbb{X}$ is singly lowest generated. Let $\mathbb{Y}$ be a nonzero quotient $\mathbb{V}$-module of $\mathbb{X}$. Then $\mathbb{Y}$ is also singly lowest generated, and $\mathrm{wt}(\mathbb{Y}) = \mathrm{wt}(\mathbb{X})$. This is because the homogeneous vector generating $\mathbb{X}$ must be sent by the quotient map to a homogeneous vector generating $\mathbb{Y}$.

Proposition 1.5. *If $\mathbb{X}$ is an irreducible $\mathbb{V}$-module, then it is singly lowest generated with conformal weight $\mathrm{wt}(\mathbb{X}) =: \alpha$. Moreover, $\Omega(\mathbb{X})$ coincides with the subspace $\mathbb{X}_{[\alpha]}$ consisting of eigenvectors of $L(0)$ with eigenvalue α.*

Proof. Since any nonzero homogeneous vector $w \in \Omega(\mathbb{X})$ generates $\mathbb{X}$, it follows from Rem. 1.4-(a) that $\alpha = \mathrm{wt}(w)$. Thus $\Omega(\mathbb{X}) \subset \mathbb{X}_{[\alpha]}$. The reverse inclusion $\Omega(\mathbb{X}) \supset \mathbb{X}_{[\alpha]}$ is obvious. □

Proposition 1.6. *Let $\mathbb{X} \in \mathrm{Mod}(\mathbb{V})$. Then $\Omega(\mathbb{X})$ is finite-dimensional.*

Proof. Let $w \in \Omega(\mathbb{X})$ be an arbitrary homogeneous vector with weight $\mathrm{wt}(w)$. Then w generates a submodule $\mathbb{X}_w \subset \mathbb{X}$ which has a simple quotient $\check{\mathbb{X}}_w$. Since $\mathbb{X}_w$ is singly lowest generated, it follows from Rem. 1.4 that $\mathrm{wt}(\check{\mathbb{X}}_w) = \mathrm{wt}(\mathbb{X}_w) = \mathrm{wt}(w)$. Since $\mathbb{V}$ is C_2-cofinite, there are only finitely many isomorphism classes of irreducible $\mathbb{V}$-modules (cf. [Hua09, Prop. 4.2]). Hence $\Re(\mathrm{wt}(w))$ is bounded above as w ranges over homogeneous vectors in $\Omega(\mathbb{X})$. It follows that $\Omega(\mathbb{X})$ is finite dimensional. □

We need projective covers of irreducible $\mathbb{V}$-modules to study non-lowest generated $\mathbb{V}$-module. The following theorem is due to [Hua09].

Theorem 1.7. *For each irreducible $\mathbb{V}$-module $\mathbb{X} \in \mathrm{Mod}(\mathbb{V})$, there exists a* ***projective cover*** *$\varphi_{\mathbb{X}} : P_{\mathbb{X}} \to \mathbb{X}$ (abbreviated as $P_{\mathbb{X}}$ when no confusion arises), i.e.,*

- *(a) $P_{\mathbb{X}}$ is a projective $\mathbb{V}$-module in $\mathrm{Mod}(\mathbb{V})$.*
- *(b) $\varphi_{\mathbb{X}}$ is an epimorphism in $\mathrm{Mod}(\mathbb{V})$.*
- *(c) If $f : \mathbb{M} \to P_{\mathbb{X}}$ is a morphism in $\mathrm{Mod}(\mathbb{V})$ such that the composition*

$$\mathbb{M} \xrightarrow{f} P_{\mathbb{X}} \xrightarrow{\varphi_{\mathbb{X}}} \mathbb{X}$$

  is an epimorphism, then f itself is an epimorphism.

Moreover, $P_{\mathbb{X}}$ is unique up to isomorphisms in $\mathrm{Mod}(\mathbb{V})$.

Remark 1.8. Let $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$, and assume that $\mathbb{X}$ is irreducible. Recall the definition of $[\mathbb{Y} : \mathbb{X}]$ from Sec. 1.1. It is well known from the theory of abelian category that $\dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{X}}, \mathbb{Y}) = [\mathbb{Y} : \mathbb{X}]$ (cf. [EGNO15, (1.7)]).

Proposition 1.9. *Let $\mathbb{X}$ be an irreducible $\mathbb{V}$-module. Then its projective cover $P_{\mathbb{X}}$ is singly generated. Therefore, by Def. 1.3, the conformal weight $\mathrm{wt}(P_{\mathbb{X}})$ is well-defined.*

Proof. Let $w \in P_{\mathbb{X}}$ be a nonzero homogeneous vector such that $\varphi_{\mathbb{X}}(w) \neq 0$. Define $\mathbb{W}$ to be the submodule of $P_{\mathbb{X}}$ generated by w. Then the composition

$$\mathbb{W} \hookrightarrow P_{\mathbb{X}} \twoheadrightarrow \mathbb{X} \tag{1.1}$$

is nonzero. Since $\mathbb{X}$ is irreducible, (1.1) must be surjective. By Thm. 1.7-(c), it follows that the inclusion $\mathbb{W} \hookrightarrow P_{\mathbb{X}}$ is surjective. Hence $P_{\mathbb{X}} = \mathbb{W}$. □

Remark 1.10. Let $\mathbb{X}$ be an irreducible $\mathbb{V}$-module. Then $\mathrm{wt}(\mathbb{X}) \in \mathrm{wt}(P_{\mathbb{X}}) + \mathbb{N}$, and in particular,

$$\Re(\mathrm{wt}(P_{\mathbb{X}})) \leqslant \Re(\mathrm{wt}(\mathbb{X})).$$

Proposition 1.11. *Let $\mathbb{X} \in \mathrm{Mod}(\mathbb{V})$ be an irreducible $\mathbb{V}$-module. The following conditions are equivalent:*

(a) $P_{\mathbb{X}}$ is not lowest generated.

(b) $\Re(\mathrm{wt}(P_{\mathbb{X}})) < \Re(\mathrm{wt}(\mathbb{X}))$.

(c) $P_{\mathbb{X}}$ has a composition factor $\mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$ with $\Re(\mathrm{wt}(\mathbb{Y})) < \Re(\mathrm{wt}(\mathbb{X}))$.

Proof. Assume (b). We show that $P_{\mathbb{X}}$ is not lowest generated. Suppose instead that $P_{\mathbb{X}}$ is lowest generated. By Prop. 1.6, $\Omega(P_{\mathbb{X}})$ is finite dimensional. Let $w_1, \cdots, w_n$ be a homogeneous basis of $\Omega(P_{\mathbb{X}})$, and for each $1 \leqslant i \leqslant n$ let $\mathbb{W}_i$ be the submodule of $P_{\mathbb{X}}$ generated by w_i. Then $\mathbb{W}_i$ is singly lowest generated for each $1 \leqslant i \leqslant n$. Since $P_{\mathbb{X}}$ is lowest generated, there exists an epimorphism

$$\mathbb{W}_1 + \cdots + \mathbb{W}_n \twoheadrightarrow P_{\mathbb{X}}$$

Hence, for some i, the composition $\mathbb{W}_i \hookrightarrow P_{\mathbb{X}} \twoheadrightarrow \mathbb{X}$ is nonzero, and therefore surjective because $\mathbb{X}$ is irreducible. By Thm. 1.7-(c), the inclusion $\mathbb{W}_i \hookrightarrow P_{\mathbb{X}}$ is an epimorphism. It follows from Rem. 1.4-(b) that $P_{\mathbb{X}}$ is singly lowest generated. Noting that $\mathbb{X}$ is a nonzero quotient $\mathbb{V}$-module of $P_{\mathbb{X}}$, it follows from Rem. 1.4-(b) again that $\mathrm{wt}(\mathbb{X}) = \mathrm{wt}(P_{\mathbb{X}})$, contradicting (b). Therefore, (a) holds.

Assume (a). We show that $\Re(\mathrm{wt}(P_{\mathbb{X}})) \neq \Re(\mathrm{wt}(\mathbb{X}))$. Suppose instead that $\Re(\mathrm{wt}(P_{\mathbb{X}})) = \Re(\mathrm{wt}(\mathbb{X}))$. By Rem. 1.10, this forces $\mathrm{wt}(P_{\mathbb{X}}) = \mathrm{wt}(\mathbb{X}) =: \alpha$ and hence $(P_{\mathbb{X}})_{[\alpha]} \subset \Omega(P_{\mathbb{X}})$. Note that $\varphi_{\mathbb{X}}$ restricts to a surjective map

$$\varphi_{\mathbb{X}} : (P_{\mathbb{X}})_{[\alpha]} \twoheadrightarrow \mathbb{X}_{[\alpha]}$$

Since $\mathbb{X}_{[\alpha]}$ is nonzero, there exists $0 \neq w \in (P_{\mathbb{X}})_{[\alpha]}$ with $\varphi_{\mathbb{X}}(w) \neq 0$. Let $\mathbb{W}$ be the submodule of $P_{\mathbb{X}}$ generated by w. Then the composition $\mathbb{W} \hookrightarrow P_{\mathbb{X}} \twoheadrightarrow \mathbb{X}$ is nonzero, hence surjective because $\mathbb{X}$ is irreducible. By Thm. 1.7-(c), the inclusion $\mathbb{W} \hookrightarrow P_{\mathbb{X}}$ is surjective, so $P_{\mathbb{X}}$ is generated by $w \in (P_{\mathbb{X}})_{[\alpha]} \subset \Omega(P_{\mathbb{X}})$. Thus $P_{\mathbb{X}}$ is lowest generated, contradicting (a). Therefore, (b) holds.

In general, for any $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$, the set of weights of $\mathbb{M}$ coincides with the set of weights of its composition factors. Hence

$$\Re(\mathrm{wt}(P_{\mathbb{X}})) = \min_{\mathbb{Y}} \Re(\mathrm{wt}(\mathbb{Y}))$$

where $\mathbb{Y}$ ranges over all composition factors of $P_{\mathbb{X}}$. Therefore, (b) and (c) are equivalent. □

Proposition 1.12. *The following are equivalent:*

(a) There exists $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$ that is not lowest generated.

(b) There exists an irreducible $\mathbb{V}$-module $\mathbb{X} \in \mathrm{Mod}(\mathbb{V})$ such that $P_{\mathbb{X}}$ is not lowest generated.

Proof. It suffices to show that (a) implies (b). Assume (a). Suppose instead that $P_{\mathbb{X}}$ is lowest generated for every irreducible $\mathbb{X} \in \mathrm{Mod}(\mathbb{V})$. Let

$$\mathbb{G} := \oplus_{\mathbb{X} \in \mathrm{Irr}} P_{\mathbb{X}}$$

where Irr is a finite set of representatives of equivalence classes of irreducibles in $\mathrm{Mod}(\mathbb{V})$. Then $\mathbb{G}$ is lowest generated and serves as a projective generator of $\mathrm{Mod}(\mathbb{V})$. Hence for any $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$ there exists $n \in \mathbb{Z}_+$ with an epimorphism $\mathbb{G}^{\oplus n} \twoheadrightarrow \mathbb{M}$, implying that $\mathbb{M}$ is lowest generated. Thus every object in $\mathrm{Mod}(\mathbb{V})$ would be lowest-generated, contradicting (a). □

Corollary 1.13. *Let $p \geqslant 2$ be an integer, and let $\mathcal{W}_p$ denote the triplet W-algebra. Then there exists a module $\mathbb{M} \in \mathrm{Mod}(\mathcal{W}_p)$ that is not lowest generated.*

Proof. Since $\mathcal{W}_p$ is C_2-cofinite [AM08], all of the preceding results for a general C_2-cofinite VOA $\mathbb{V}$ apply. We follow the terminology of [TW13, Sec. 3.1]. Up to isomorphisms, the irreducible $\mathcal{W}_p$-modules are X_s^+ and X_s^- for $1 \leqslant s \leqslant p$. Among them, X_0^- is the unique module with the maximal conformal weight

$$\mathrm{wt}(X_0^-) = \frac{1}{4p}(4p^2 - (p-1)^2).$$

Let P_0^- denote its projective cover. The socle series of P_0^- is given by

$$X_0^- = S_0(P_0^-) \subset S_1(P_0^-) \subset S_2(P_0^-) = P_0^-$$

with $S_1(P_0^-)/S_0(P_0^-) \simeq X_p^+ \oplus X_p^+$ and $S_2(P_0^-)/S_1(P_0^-) \simeq X_0^-$. Since

$$-\frac{1}{4p}(p-1)^2 = \Re(\mathrm{wt}(X_p^+)) < \Re(\mathrm{wt}(X_0^-)) = \frac{1}{4p}(4p^2 - (p-1)^2),$$

condition (c) of Prop. 1.11 holds, and hence P_0^- is not lowest generated. □

Corollary 1.14. *For $d \in \mathbb{Z}_+$, let SF_d^+ denote the even symplectic fermion VOA. Then there exists $\mathbb{M} \in \mathrm{Mod}(SF_d^+)$ that is not lowest generated.*

Proof. Since SF_d^+ is C_2-cofinite [Abe07], all of the preceding results for a general C_2-cofinite VOA $\mathbb{V}$ apply. We follow the terminology of [McR23, Sec. 5]. Noting that $SF_1^+ \simeq \mathcal{W}_2$, it suffices to consider $d \geqslant 2$ in our proof. Let $X_1^\pm, X_2^\pm$ denote the irreducible $\mathcal{W}_2$-modules described in the proof of Cor. 1.13. Recall that SF_d^+ is an extension of $\mathcal{W}_2^{\otimes d}$. As $\mathcal{W}_2^{\otimes d}$-modules, we have the decomposition

$$SF_d^+ = \bigoplus_{\mathrm{Card}(\{1 \leqslant i \leqslant d : \varepsilon_i = -\}) \text{ even}} X_1^{\varepsilon_1} \otimes \cdots \otimes X_1^{\varepsilon_d} \tag{1.2}$$

Up to isomorphisms, the irreducible SF_d^+-modules are $\mathcal{X}_i^\varepsilon$ for $i = 1, 2$ and $\varepsilon = \pm$. By [McR23, Thm. 5.4], the projective cover $\mathcal{P}_1^-$ of $\mathcal{X}_1^-$ has a composition factor isomorphic to the vacuum module $\mathcal{X}_1^+$. Moreover, as $\mathcal{W}_2^{\otimes d}$-modules, we have the equivalence

$$\mathcal{X}_1^- \simeq SF_d^+ \boxtimes (X_1^- \otimes X_1^+ \otimes \cdots \otimes X_1^+) \tag{1.3}$$

Using the decomposition (1.2) and the fusion product $X_1^{\varepsilon_1} \boxtimes X_1^{\varepsilon_2} = X_1^{\varepsilon_1\varepsilon_2}$ in $\mathrm{Mod}(\mathcal{W}_2)$, we deduce that $\mathcal{X}_1^-$ is equivalent to the direct sum of all

$$X_1^+ \otimes X_1^{\varepsilon_2} \otimes \cdots X_1^{\varepsilon_d}, \quad X_1^- \otimes X_1^{\tau_2} \otimes \cdots \otimes X_1^{\tau_d}$$

where $A := \mathrm{Card}(\{2 \leqslant i \leqslant d : \varepsilon_i = -\})$ is odd and $B := \mathrm{Card}(\{2 \leqslant i \leqslant d : \tau_i = -\})$ is even. Since the conformal weights of X_1^+ and X_1^- are 0 and 1, respectively, the conformal weight of $\mathcal{X}_1^-$ is $\min\{A, B+1\}$, which is always $\geqslant 1$. Hence,

$$\mathrm{wt}(\mathcal{X}_1^-) \geqslant 1 > 0 = \mathrm{wt}(\mathcal{X}_1^+)$$

By Prop. 1.11, $\mathcal{P}_1^-$ is not lowest generated. This completes the proof. $\square$

1.3 Smooth conformal block functors

Let $N \in \mathbb{Z}_+$, and let $\mathfrak{X}$ be an N-pointed smooth sphere (with local coordinates), i.e.,

$$\mathfrak{X} = (\mathbb{P}^1 | x_1, \cdots, x_N; \eta_1, \cdots, \eta_N)$$

where $x_1, \cdots, x_N$ are distinct marked points of $\mathbb{P}^1$ and each η_i is a local coordinate at x_i. Let $\mathbb{W}_1, \cdots, \mathbb{W}_N \in \mathrm{Mod}(\mathbb{V})$ and associate $\mathbb{W}_i$ to x_i for each i.

The **space of smooth conformal blocks associated to $\mathfrak{X}$ and** $\mathbb{W}_1, \cdots, \mathbb{W}_N$, denoted

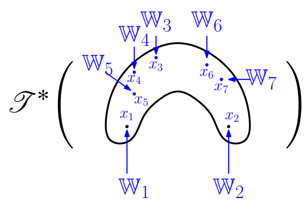
$$\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N)$$

consists of linear functionals

$$\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \to \mathbb{C}$$

that are invariant under the actions of $\mathbb{V}$ and $\mathfrak{X}$ [FBZ04, NT05, DGT21].

Following the graphical calculus of conformal blocks [GZ25b, Ch. 1], the picture representing $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N)$ is

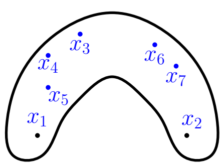


illustrated here for $N = 7$.

Definition 1.15. The contravariant functor

$$\mathscr{T}_{\mathfrak{X}}^* : \mathrm{Mod}(\mathbb{V}) \times \cdots \times \mathrm{Mod}(\mathbb{V}) \to \mathcal{V}ect$$
$$(\mathbb{W}_1, \cdots, \mathbb{W}_N) \mapsto \mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N)$$

is called the **smooth conformal block functor associated to $\mathfrak{X}$**. In the language of graphical calculus, $\mathscr{T}_{\mathfrak{X}}^*$ is also represented as the smooth conformal block functor corresponding to



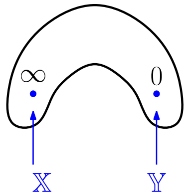
illustrated here for $N = 7$.

Recall that ζ is the standard coordinate of $\mathbb{C}$ (see Sec. 1.1). Let $\mathbb{X},\mathbb{Y}\in\mathrm{Mod}(\mathbb{V})$ and

$$\mathfrak{N}=(\mathbb{P}^1|\infty,0;1/\zeta,\zeta), \tag{1.4}$$

where $\mathbb{P}^1$ is identified with $\mathbb{C}\cup\{\infty\}$ and $1/\zeta,\zeta$ are local coordinates at $\infty,0$. The space of smooth conformal blocks associated to $\mathfrak{N}$ and $\mathbb{X},\mathbb{Y}$

$$\mathscr{T}^*_{\mathfrak{N}}(\mathbb{X}\otimes\mathbb{Y})\equiv\mathscr{T}^*\Bigg(\qquad\Bigg)$$



can be explicitly described by the space of linear functionals $\psi:\mathbb{X}\otimes\mathbb{Y}\to\mathbb{C}$ such that: for each $v\in\mathbb{V},x\in\mathbb{X},y\in\mathbb{Y}$, the relation

$$\psi(Y(v,z)x\otimes y)=\psi(x\otimes Y'(v,z)y)$$

holds in $\mathbb{C}[[z^{\pm1}]]$. It is well known (cf. e.g. [NT05, Prop. 5.9.1] or [GZ25b, Prop. 2.3]) that $\mathscr{T}^*_{\mathfrak{N}}(\mathbb{X}\otimes\mathbb{Y})$ can be identified with $\mathrm{Hom}_{\mathbb{V}}(\mathbb{Y},\mathbb{X}')$ via the isomorphism

$$\mathrm{Hom}_{\mathbb{V}}(\mathbb{Y},\mathbb{X}')\xrightarrow{\simeq}\mathscr{T}^*_{\mathfrak{N}}(\mathbb{X}\otimes\mathbb{Y}),\qquad T\mapsto T^\flat \tag{1.5}$$

where $T^\flat(x\otimes y)=\big\langle x,T(y)\big\rangle$.

1.4 Nodal conformal block functors

Let $N\in\mathbb{Z}_{\geqslant2}$, and let $\mathfrak{Y}$ be an N-pointed nodal sphere (with local coordinates), i.e.,

$$\mathfrak{Y}=(\mathcal{P}\big|x_1,\cdots,x_N;\eta_1,\cdots,\eta_N)$$

where $\mathcal{P}$ is a nodal sphere (i.e., a nodal curve of genus 0) with one node, $x_1,\cdots,x_N$ are marked points of C distinct from the nodes and each η_i is a local coordinate at x_i. Moreover, we assume that the two irreducible components of $\mathcal{P}$ contain x_1 and x_2, respectively. Let $\mathbb{W}\in\mathrm{Mod}(\mathbb{V}^{\otimes2})$ and $\mathbb{W}_3,\cdots,\mathbb{W}_N\in\mathrm{Mod}(\mathbb{V})$. Associate $\mathbb{W}=(\mathbb{W},Y_+,Y_-)$ to x_1,x_2 via the default ordering (cf. [GZ25b])

$$\varepsilon:\{+,-\}\to\{x_1,x_2\},\qquad\varepsilon(+)=x_1,\quad\varepsilon(-)=x_2$$

and $\mathbb{W}_i$ to x_i for each $3\leqslant i\leqslant N$.

The **space of nodal conformal blocks associated to $\mathfrak{Y}$ and $\mathbb{W},\mathbb{W}_3,\cdots,\mathbb{W}_N$ via the default ordering** ε, denoted

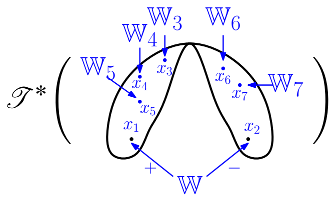
$$\mathscr{T}^*_{\mathfrak{Y}}(\mathbb{W}\otimes\mathbb{W}_3\otimes\cdots\otimes\mathbb{W}_N)$$

consists of linear functionals

$$\mathbb{W}\otimes\mathbb{W}_3\otimes\cdots\otimes\mathbb{W}_N\to\mathbb{C}$$

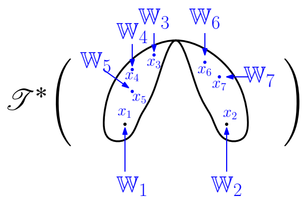
that are invariant under the action of $\mathbb{V}$ and $\mathfrak{Y}$ [DGT21].

Following the graphical calculus of conformal blocks [GZ25b, Ch. 1], the picture representing $\mathscr{T}_{\mathfrak{Y}}^*(\mathbb{W}\otimes\mathbb{W}_3\otimes\cdots\otimes\mathbb{W}_N)$ is



illustrated here for $N=7$.

In the special case where $\mathbb{W}=\mathbb{W}_1\otimes\mathbb{W}_2$ with $\mathbb{W}_1,\mathbb{W}_2\in\mathrm{Mod}(\mathbb{V})$, the picture representing $\mathscr{T}_{\mathfrak{Y}}^*(\mathbb{W}\otimes\mathbb{W}_3\otimes\cdots\otimes\mathbb{W}_N)\equiv\mathscr{T}_{\mathfrak{Y}}^*(\mathbb{W}_1\otimes\cdots\otimes\mathbb{W}_N)$ is given by

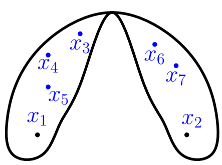


illustrated here for $N=7$.

Definition 1.16. The contravariant functor

$$\begin{aligned}\mathscr{T}_{\mathfrak{Y}}^*:\mathrm{Mod}(\mathbb{V})\times\cdots\times\mathrm{Mod}(\mathbb{V})&\to\mathcal{V}ect\\(\mathbb{W}_1,\cdots,\mathbb{W}_N)&\mapsto\mathscr{T}_{\mathfrak{Y}}^*(\mathbb{W}_1\otimes\cdots\otimes\mathbb{W}_N)\end{aligned}$$

is called the **nodal conformal block functor associated to** $\mathfrak{X}$. In the language of graphical calculus, $\mathscr{T}_{\mathfrak{Y}}^*$ is also represented as the nodal conformal block functor corresponding to



Consider the 2-pointed nodal sphere (with local coordinates)

$$\mathfrak{B}=(\mathcal{B}\big|\infty,0;1/\zeta,\zeta). \tag{1.6}$$

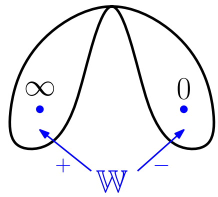
Here $\mathcal{B}$ is the nodal sphere obtained by gluing two copies of

$$\mathfrak{N}=(1.4)=(\mathbb{P}^1|\infty,0;1/\zeta,\zeta)$$

along the point 0 on the first copy and the point ∞ on the second. The nodal sphere $\mathcal{B}$ carries two marked points, ∞ and 0, both distinct from the node: the point ∞ is inherited from the first copy of $\mathfrak{N}$, while 0 is inherited from the second. The local coordinates $1/\zeta$ and ζ are likewise inherited from the two copies, respectively. Thus, each irreducible component of $\mathfrak{B}$ contains exactly one marked point.

Let $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$. The space of nodal conformal blocks associated to $\mathfrak{B}$ and $\mathbb{W}$ via ε

$$\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W}) \equiv \mathscr{T}^*\Bigg(\qquad \Bigg)$$



can be explicitly described by the space of linear functionals $\psi : \mathbb{W} \to \mathbb{C}$ such that: for each $n \in \mathbb{Z}$, $w \in \mathbb{W}$ and homogeneous $v \in \mathbb{V}$, we have

$$\psi(Y_+(v)_{\mathrm{wt}(v)-1}w) = \psi(Y'_-(v)_{\mathrm{wt}(v)-1}w) \tag{1.7a}$$

$$\psi(Y_+(v)_n w) = \psi(Y_-(v)_n w) = 0, \quad \text{if } \mathrm{wt}(v) - n - 1 > 0. \tag{1.7b}$$

Proposition 1.17. *For each $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$, $\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$, a priori a subspace of $\mathbb{W}^*$, is actually inside $\mathbb{W}'$.*

It follows that $\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W}')$ is a subspace of $\mathbb{W}$ for each $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$.

Proof. It follows from [Miy04, Lem. 2.4] that there exists an integer $\nu \in \mathbb{N}$ such that any homogeneous vector $w \in \mathbb{W}$ with $\Re(\mathrm{wt}(w)) > \nu$ can be expressed as a finite sum of vectors of the form $Y_+(u_+)_{-l}w_+$ and $Y_-(u_-)_{-k}w_-$, where $l > 1, k > 1, w_+, w_- \in \mathbb{W}$ and $u_+, u_- \in \mathbb{V}$ homogeneous (see also [GZ23, Lem. 3.24] for a detailed explanation). Let $\psi \in \mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$. By (1.7b), ψ vanishes on all $w \in \mathbb{W}$ with $\Re(\mathrm{wt}(w)) > \nu$. Hence, ψ can be regarded as a linear functional

$$\bigoplus_{\Re(\lambda+\mu) \leqslant \nu} \mathbb{W}_{[\lambda,\mu]} \to \mathbb{C}$$

Thus, $\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$ embeds into $(\oplus_{\Re(\lambda+\mu)\leqslant\nu}\mathbb{W}_{[\lambda,\mu]})^*$, which is finite dimensional. Consequently, $\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$ itself is finite dimensional.

Define the action of $L(0)$ on $\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$ by

$$\big\langle L(0)\psi, w \big\rangle = \big\langle \psi, L_+(0)w \big\rangle = \big\langle \psi, L_-(0)w \big\rangle, \quad \text{for all } \psi \in \mathscr{T}^*_{\mathfrak{B}}(\mathbb{W}), w \in \mathbb{W}$$

where the last equality is due to (1.7a). We claim that $\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$ is invariant under $L(0)$. To see this, for each $\psi \in \mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$, $w \in \mathbb{W}$ and homogeneous $v \in \mathbb{V}$, noting that $L_+(0)$ commutes with $Y_+(v)_{\mathrm{wt}(v)-1}$, we have

$$\begin{aligned}\big\langle L(0)\psi, Y_+(v)_{\mathrm{wt}(v)-1}w \big\rangle &= \big\langle \psi, L_+(0)Y_+(v)_{\mathrm{wt}(v)-1}w \big\rangle = \big\langle \psi, Y_+(v)_{\mathrm{wt}(v)-1}L_+(0)w \big\rangle \\ = \big\langle \psi, Y'_-(v)_{\mathrm{wt}(v)-1}L_+(0)w \big\rangle &= \big\langle \psi, L_+(0)Y'_-(v)_{\mathrm{wt}(v)-1}w \big\rangle = \big\langle L(0)\psi, Y'_-(v)_{\mathrm{wt}(v)-1}w \big\rangle\end{aligned}$$

This proves that $L(0)\psi$ satisfies (1.7a). Clearly, $L(0)\psi$ satisfies (1.7b). Thus $\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$ is $L(0)$-invariant.

The generalized eigenspace decomposition of $\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})$ with respect to $L(0)$ is

$$\mathscr{T}^*_{\mathfrak{B}}(\mathbb{W}) = \bigoplus_{i=1}^{N} \mathscr{T}^*_{\mathfrak{B}}(\mathbb{W})_{[\lambda_i]}, \quad \text{where } \lambda_i \in \mathbb{C}.$$

For each $1 \leqslant i \leqslant N$, $\mathscr{T}_{\mathfrak{B}}^{*}(\mathbb{W})_{[\lambda_i]}$ is contained in $\mathbb{W}^{*}_{[\lambda_i,\lambda_i]}$. Hence

$$\mathscr{T}_{\mathfrak{B}}^{*}(\mathbb{W}) \subset \bigoplus_{i=1}^{N} \mathbb{W}^{*}_{[\lambda_i,\lambda_i]} \subset \mathbb{W}'.$$

This completes the proof. $\square$

By [DSPS19, Cor. 1.10], every left exact linear functor from a finite $\mathbb{C}$-linear category to $\mathcal{V}ect$ is representable. Therefore, there exists a $\mathbb{W}$-natural linear isomorphism

$$\varphi_{\mathbb{W}} : \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{A}, \mathbb{W}) \xrightarrow{\simeq} \mathscr{T}_{\mathfrak{B}}^{*}(\mathbb{W}') \tag{1.8}$$

for some $\mathbb{A} = (\mathbb{A}, Y_+, Y_-) \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$. It is clear that the $\mathbb{A}$ realizing such an isomorphism as (1.8) are unique up to isomorphisms. We fix such an isomorphism $\varphi_{\mathbb{W}}$. We define

$$\omega = \varphi_{\mathbb{A}}(\mathrm{id}_{\mathbb{A}}) \in \mathscr{T}_{\mathfrak{B}}^{*}(\mathbb{A}') \subset \mathbb{A}$$

where the last inclusion is due to Prop. 1.17. The object $\mathbb{A}$ is called the **(genus 0) nodal fusion product**, and the element ω is called the **canonical conformal block**. The isomorphism $\varphi_{\mathbb{W}} = $ (1.8) is therefore implemented by

$$\begin{aligned} \varphi_{\mathbb{W}} : \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{A}, \mathbb{W}) &\xrightarrow{\simeq} \mathscr{T}_{\mathfrak{B}}^{*}(\mathbb{W}') \\ f &\mapsto f(\omega) \end{aligned} \tag{1.9}$$

where $f(\omega)$, a priori an element of $\mathbb{W}$, actually lies in $\mathscr{T}_{\mathfrak{B}}^{*}(\mathbb{W}')$. This is because $f(\omega)$, viewed as a linear functional $\mathbb{W}' \to \mathbb{C}$, also satisfies a similar property as ψ does in (1.7).

Proposition 1.18. *The canonical conformal block $\omega \in \mathbb{A}$ generates $\mathbb{A}$ as a $\mathbb{V}^{\otimes 2}$-module.*

Proof. Let $\mathbb{U} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$ be the submodule of $\mathbb{A}$ generated by ω. We claim that $\mathbb{U} = \mathbb{A}$. Suppose, for contradiction, that $\mathbb{U} \neq \mathbb{A}$. Take $\mathbb{W} = \mathbb{A}/\mathbb{U}$ in (1.9). Then (1.9) specializes to

$$\begin{aligned} \varphi_{\mathbb{A}/\mathbb{U}} : \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{A}, \mathbb{A}/\mathbb{U}) &\xrightarrow{\simeq} \mathscr{T}_{\mathfrak{B}}^{*}\big((\mathbb{A}/\mathbb{U})'\big) \\ f &\mapsto f(\omega) \end{aligned}$$

Let $\pi : \mathbb{A} \to \mathbb{A}/\mathbb{U}$ denote the canonical projection. Since $\mathbb{A}/\mathbb{U}$ is nonzero, π is a nonzero morphism. However,

$$\varphi_{\mathbb{A}/\mathbb{U}}(\pi) = \pi(\omega) = 0$$

contradicting the injectivity of $\varphi_{\mathbb{A}/\mathbb{U}}$. Therefore, $\mathbb{U} = \mathbb{A}$, and hence ω generates $\mathbb{A}$. $\square$

Corollary 1.19. *$(\mathbb{A}, Y_+)$ is lowest generated as a weak $\mathbb{V}$-module.*

Proof. Let $\Omega(\mathbb{A})$ denote the lowest weight subspace of $(\mathbb{A}, Y_+)$. The submodule $\mathbb{A}_-$ of $(\mathbb{A}, Y_-)$ generated by ω is contained in $\Omega(\mathbb{A})$. By Prop. 1.18, $\mathbb{A}_-$ generates $(\mathbb{A}, Y_+)$. Thus, $\Omega(\mathbb{A})$ also generates $(\mathbb{A}, Y_+)$. $\square$

2 Smooth and nodal conformal blocks

2.1 A dimension criterion

Proposition 2.1. *Let* $\varphi_{\mathbb{X}} : P_{\mathbb{X}} \twoheadrightarrow \mathbb{X}$ *be the projective cover of an irreducible* $\mathbb{V}$*-module* $\mathbb{X}$ *in* $\mathrm{Mod}(\mathbb{V})$*. Let* $\mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$*. If* $\dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{X}}, \mathbb{M}) = \dim \mathrm{Hom}_{\mathbb{V}}(\mathbb{Y}, \mathbb{M})$ *for all* $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$*, then* $P_{\mathbb{X}} \simeq \mathbb{Y}$ *in* $\mathrm{Mod}(\mathbb{V})$*.*

Proof. Since

$$\dim \mathrm{Hom}_{\mathbb{V}}(\mathbb{Y}, \mathbb{X}) = \dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{X}}, \mathbb{X}) = 1$$

there exists a nonzero morphism $\alpha : \mathbb{Y} \to \mathbb{X}$. As $\mathbb{X}$ is irreducible, α must be an epimorphism. By the projectivity of $P_{\mathbb{X}}$, there exists $\beta : P_{\mathbb{X}} \to \mathbb{Y}$ such that $\varphi_{\mathbb{X}} = \alpha \circ \beta$.

We first show that β is surjective. Suppose, to the contrary, that β is not surjective. Then the nonzero quotient $\mathbb{V}$-module $\mathbb{Y}/\beta(P_{\mathbb{X}})$ admits an epimorphism onto some irreducible $\mathbb{V}$-module $\mathbb{U}$.

- If $\mathbb{U} \not\simeq \mathbb{X}$, then $\dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{X}}, \mathbb{U}) = 0$. However, since $\mathbb{U}$ is a quotient of $\mathbb{Y}/\beta(P_{\mathbb{X}})$, we have $\dim \mathrm{Hom}_{\mathbb{V}}(\mathbb{Y}, \mathbb{U}) > 0$, contradicting our assumption.
- If $\mathbb{U} \simeq \mathbb{X}$, then the composition

  $$\mathbb{Y} \twoheadrightarrow \mathbb{Y}/\beta(P_{\mathbb{X}}) \twoheadrightarrow \mathbb{U} \simeq \mathbb{X} \tag{2.1}$$

  yields a nonzero morphism $\gamma : \mathbb{Y} \to \mathbb{X}$. We claim that α, γ are linearly independent in $\mathrm{Hom}_{\mathbb{V}}(\mathbb{Y}, \mathbb{X})$. To see this, suppose instead that they are linearly dependent. Then, up to a nonzero scalar multiplication, $\varphi_{\mathbb{X}} = \alpha \circ \beta$ coincides with

  $$P_{\mathbb{X}} \xrightarrow{\beta} \mathbb{Y} \twoheadrightarrow \mathbb{Y}/\beta(P_{\mathbb{X}}) \twoheadrightarrow \mathbb{U} \simeq \mathbb{X}$$

  which is zero, a contradiction. Hence α, γ are linearly independent, and so

  $$\dim \mathrm{Hom}_{\mathbb{V}}(\mathbb{Y}, \mathbb{X}) \geqslant 2 > 1 = \dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{X}}, \mathbb{X})$$

  which is again a contradiction.

Therefore, β must be surjective.

We now show that β is injective. Suppose otherwise. Then the transpose of

$$P_{\mathbb{X}} \xrightarrow{\beta} \mathbb{Y} \xrightarrow{\alpha} \mathbb{X}$$

is the sequence

$$\mathbb{X}' \xrightarrow{\alpha^t} \mathbb{Y}' \xrightarrow{\beta^t} P'_{\mathbb{X}} \tag{2.2}$$

where α^t is injective and β^t is injective but not surjective. Thus we may regard (2.2) as

$$\mathbb{X}' \subset \mathbb{Y}' \subsetneq P'_{\mathbb{X}}$$

It follows that the composition series of $P'_{\mathbb{X}}$ is strictly longer than that of $\mathbb{Y}'$. Hence there exists an irreducible $\mathbb{V}$-module $\mathbb{W}$ such that $[\mathbb{Y}' : \mathbb{W}] < [P'_{\mathbb{X}} : \mathbb{W}]$, equivalently,

$$\dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{W}}, \mathbb{Y}') < \dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{W}}, P'_{\mathbb{X}})$$

(See Rem. 1.8). This implies

$$\dim \mathrm{Hom}_{\mathbb{V}}(\mathbb{Y}, P'_{\mathbb{W}}) \neq \dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{X}}, P'_{\mathbb{W}})$$

contradicting our assumption. Therefore, β must be injective, and we conclude that β is an isomorphism, i.e., $P_{\mathbb{X}} \simeq \mathbb{Y}$. □

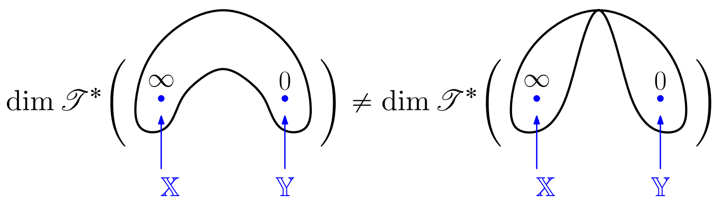
2.2 Non-equivalence of nodal and smooth conformal block functors

Recall the 2-pointed smooth sphere $\mathfrak{N} = (1.4) = (\mathbb{P}^1|\infty, 0; 1/\zeta, \zeta)$ and the 2-pointed nodal sphere $\mathfrak{B} = (1.6) = (\mathcal{B}|\infty, 0; 1/\zeta, \zeta)$, together with their conformal block functor $\mathscr{T}^*_{\mathfrak{N}}, \mathscr{T}^*_{\mathfrak{B}}$ described in Sec. 1.3 and 1.4.

Theorem 2.2. *Assume that there exists a module in* $\mathrm{Mod}(\mathbb{V})$ *that is not lowest generated. Then there exist* $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$ *such that*

$$\dim \mathscr{T}^*_{\mathfrak{N}}(\mathbb{X} \otimes \mathbb{Y}) \neq \dim \mathscr{T}^*_{\mathfrak{B}}(\mathbb{X} \otimes \mathbb{Y}). \tag{2.3}$$

The picture for (2.3) *is*



Proof. Suppose, to the contrary, that we have

$$\dim \mathscr{T}^*_{\mathfrak{N}}(\mathbb{X} \otimes \mathbb{Y}) = \dim \mathscr{T}^*_{\mathfrak{B}}(\mathbb{X} \otimes \mathbb{Y}), \text{ for all } \mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V}) \tag{2.4}$$

By Prop. 1.12, there exists an irreducible $\mathbb{W} \in \mathrm{Mod}(\mathbb{V})$ such that $P_{\mathbb{W}}$ is not lowest generated. By (1.8), we have

$$\dim \mathscr{T}^*_{\mathfrak{B}}(\mathbb{U}' \otimes P_{\mathbb{W}}) = \dim \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{A}, \mathbb{U} \otimes P'_{\mathbb{W}}), \text{ for all } \mathbb{U} \in \mathrm{Mod}(\mathbb{V}). \tag{2.5}$$

By [DSPS19, Cor. 1.10], every left exact linear functor from a finite $\mathbb{C}$-linear category to $\mathcal{V}ect$ is representable. Thus, there exists $\mathbb{D} \in \mathrm{Mod}(\mathbb{V})$ such that we have a $\mathbb{U}$-natural linear isomorphism

$$\psi_{\mathbb{U}} : \mathrm{Hom}_{\mathbb{V}}(\mathbb{D}, \mathbb{U}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{A}, \mathbb{U} \otimes P'_{\mathbb{W}}). \tag{2.6}$$

We fix such an isomorphism $\psi_{\mathbb{U}}$. It follows by (2.5) and (2.6) that

$$\dim \mathscr{T}^*_{\mathfrak{B}}(\mathbb{U}' \otimes P_{\mathbb{W}}) = \dim \mathrm{Hom}_{\mathbb{V}}(\mathbb{D}, \mathbb{U}), \text{ for all } \mathbb{U} \in \mathrm{Mod}(\mathbb{V}). \tag{2.7}$$

On the other hand, by (1.5), $\mathscr{T}^*_{\mathfrak{N}}(\mathbb{U}' \otimes P_{\mathbb{W}})$ can be identified with $\mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{W}}, \mathbb{U})$, so

$$\dim \mathscr{T}^*_{\mathfrak{N}}(\mathbb{U}' \otimes P_{\mathbb{W}}) = \dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{W}}, \mathbb{U}), \text{ for all } \mathbb{U} \in \mathrm{Mod}(\mathbb{V}). \tag{2.8}$$

By (2.4), (2.7) and (2.8), we have $\dim \mathrm{Hom}_{\mathbb{V}}(\mathbb{D}, \mathbb{U}) = \dim \mathrm{Hom}_{\mathbb{V}}(P_{\mathbb{W}}, \mathbb{U})$ for all $\mathbb{U} \in \mathrm{Mod}(\mathbb{V})$. This together with Prop. 2.1 implies that $\mathbb{D} \simeq P_{\mathbb{W}}$.

We claim that $\mathbb{D}$ is lowest generated. If this claim is true, then it will contradict our assumption that $P_{\mathbb{W}}$ is not lowest generated. This completes our proof. To see the claim, let

$$\widetilde{\alpha} \in \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{A}, \mathbb{D} \otimes P'_{\mathbb{W}})$$

be the element such that $\widetilde{\alpha} = \psi_{\mathbb{D}}(\mathrm{id}_{\mathbb{D}})$. The isomorphism $\psi_{\mathbb{U}} = $ (2.6) is therefore implemented by

$$\begin{aligned} \psi_{\mathbb{U}} : \mathrm{Hom}_{\mathbb{V}}(\mathbb{D}, \mathbb{U}) &\xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{A}, \mathbb{U} \otimes P'_{\mathbb{W}}) \\ f &\mapsto (f \otimes \mathrm{id}_{P'_{\mathbb{W}}}) \circ \widetilde{\alpha} \end{aligned} \tag{2.9}$$

We view $\widetilde{\alpha}$ as a linear map

$$\alpha : \mathbb{A} \otimes P_{\mathbb{W}} \to \mathbb{D}, \qquad a \otimes x \mapsto \langle \widetilde{\alpha}(a), x \rangle.$$

Since $\widetilde{\alpha}$ intertwines the actions of $\mathbb{V}^{\otimes 2}$, α satisfies the following property: for each $v \in \mathbb{V}$, $a \in \mathbb{A}$ and $x \in P_{\mathbb{W}}$, the relations

$$\alpha(Y_+(v,z)a \otimes x) = Y_{\mathbb{D}}(v,z)\alpha(a \otimes x) \tag{2.10a}$$
$$\alpha(Y_-(v,z)a \otimes x) = \alpha(a \otimes Y'_{P_{\mathbb{W}}}(v,z)x) \tag{2.10b}$$

hold in $\mathbb{C}[[z^{\pm 1}]]$. Suppose that α is not surjective. By (2.10a), the image $\alpha(\mathbb{A} \otimes P_{\mathbb{W}})$ is a proper left $\mathbb{V}$-submodule of $\mathbb{D}$. Set

$$\mathbb{U} = \mathbb{D}/\alpha(\mathbb{A} \otimes P_{\mathbb{W}}) \tag{2.11}$$

in (2.9). Then $\mathbb{U}$ is nonzero and (2.9) takes the form

$$\begin{aligned} \mathrm{Hom}_{\mathbb{V}}(\mathbb{D}, \mathbb{D}/\alpha(\mathbb{A} \otimes P_{\mathbb{W}})) &\xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}\big(\mathbb{A}, (\mathbb{D}/\alpha(\mathbb{A} \otimes P_{\mathbb{W}})) \otimes P'_{\mathbb{W}}\big) \\ f &\mapsto (f \otimes \mathrm{id}_{P'_{\mathbb{W}}}) \circ \widetilde{\alpha} \end{aligned} \tag{2.12}$$

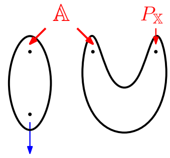
The map $(f \otimes \mathrm{id}_{P'_{\mathbb{W}}}) \circ \widetilde{\alpha}$, viewed as a linear map $\mathbb{A} \otimes P_{\mathbb{W}} \to \mathbb{D}/\alpha(\mathbb{A} \otimes P_{\mathbb{W}})$, is equal to $f \circ \alpha$. Therefore, the image of the nonzero canonical projection $\pi : \mathbb{D} \to \mathbb{D}/\alpha(\mathbb{A} \otimes P_{\mathbb{W}})$ under (2.12) is zero, contradicting the injectivity of (2.12). Therefore, α must be surjective.

Let $\Omega(\mathbb{A})$ denote the lowest weight subspace of $(\mathbb{A}, Y_+)$, and let $\Omega(\mathbb{D})$ denote the lowest weight subspace of $\mathbb{D}$. By (2.10a), we have

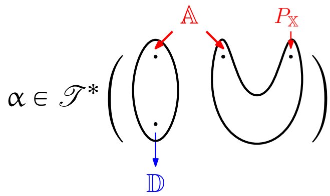
$$\alpha(\Omega(\mathbb{A}) \otimes P_{\mathbb{W}}) \subset \Omega(\mathbb{D})$$

Moreover, by Cor. 1.19, $\Omega(\mathbb{A})$ generates $\mathbb{A}$ as a left $\mathbb{V}$-module. Since α is surjective and satisfies (2.10a), it follows that $\mathbb{D}$ is generated by $\alpha(\Omega(\mathbb{A}) \otimes P_{\mathbb{W}})$. Consequently, $\mathbb{D}$ is generated by $\Omega(\mathbb{D})$, and hence is lowest generated. This completes the proof. □

Remark 2.3. In the proof of Thm. 2.2, the module $\mathbb{D}$ is in fact the fusion product of

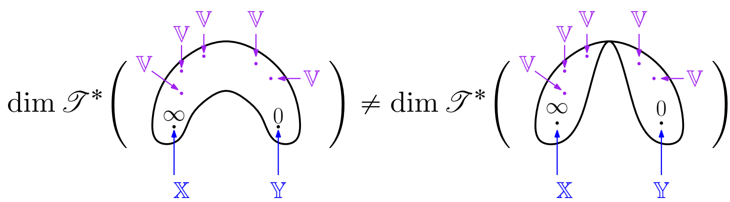


and α is the canonical conformal block of $\mathbb{D}$ [GZ23]:



The surjectivity of α is precisely the partial injectivity property of canonical conformal blocks (cf. [GZ23, Ch. 3] or [GZ24, Rem. 3.17]). Nevertheless, since we do not assume that the reader is familiar with the notion of fusion products introduced in [GZ23], we provide here a self-contained proof.

Remark 2.4. Assume that there exists a module in $\mathrm{Mod}(\mathbb{V})$ that is not lowest generated. Let $N \geqslant 2$. By Thm. 2.2 and propagation of conformal blocks (cf. Rem. 2.6), there exist $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$ such that



(2.13)

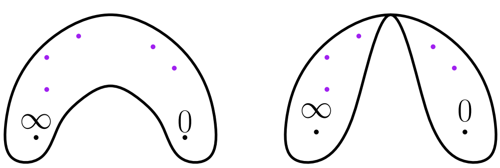
In (2.13), both spheres carry N marked points, partitioned into two groups. The first group consists of two blue marked points, ∞ and 0, inherited from $\mathfrak{N}, \mathfrak{B}$ respectively. The second group consists of $N-2$ purple marked points, all distinct from the nodes, each associated with a copy of $\mathbb{V}$.

We conclude that:

(a) The spaces of conformal blocks associated to $\mathbb{X}, \mathbb{Y}, \mathbb{V}, \cdots, \mathbb{V}$ do not form a vector bundle on $\overline{\mathcal{M}}_{0,N}$ for $N \geqslant 4$.

(b) The sheaf of coinvariants associated to $\mathbb{X}, \mathbb{Y}, \mathbb{V}, \cdots, \mathbb{V}$ on $\overline{\mathcal{M}}_{0,N}$ is not locally free for $N \geqslant 4$.

(c) The two conformal block functors (cf. Def. 1.15 and 1.16) corresponding to



are not equivalent.

2.3 The end is not isomorphic to the mode transition algebra

The mode transition algebra $\mathfrak{A}$ was first introduced in [DGK25b]. $\mathfrak{A}$ is a quotient of $\mathbb{X} \otimes \mathbb{Y}$ by a $\mathbb{V} \times \mathbb{V}$-invariant subspace, where $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$. Thus, $\mathfrak{A}$ is an object in $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$. Moreover, $\mathfrak{A}$ contains a distinguished element, denoted by 1.

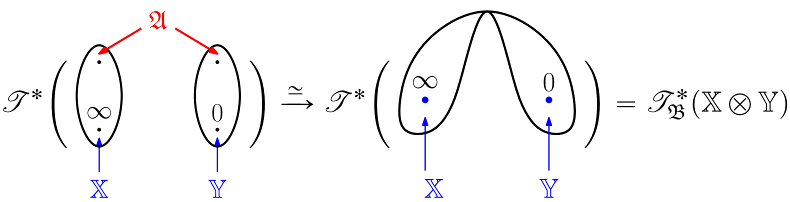
Recall the nodal conformal block functor $\mathscr{T}^*_{\mathfrak{B}}$ and the smooth conformal block functor $\mathscr{T}^*_{\mathfrak{N}}$ described in Sec. 1.3 and 1.4.

Theorem 2.5 ([DGK25b, Prop. 3.3])**.** *Let* $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$. *The linear map*

$$\begin{gathered}\mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathfrak{A}, \mathbb{X}' \otimes \mathbb{Y}') \simeq \mathscr{T}^*_{\mathfrak{B}}(\mathbb{X} \otimes \mathbb{Y}) \\ T \mapsto T(1)\end{gathered}$$

is an isomorphism.

Remark 2.6. By [DGK25b, Prop. 3.3] and propagation of conformal blocks [Zhu94, Cod19, DGT21, GZ23], there are natural equivalences



(2.14)

Though the curves are not stable, they are affine when the marked points are removed. Therefore, the proof of propagation in [DGT21, Thm. 6.2] (using Riemann-Roch theorem) still applies to the present situation.

The space of smooth conformal blocks on the left hand side of (2.14) can be identified with $\mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathfrak{A}, \mathbb{X}' \otimes \mathbb{Y}')$ (see [GZ25b, Prop. 2.3] for details). Therefore, Thm. 2.5 follows.

The following remark indicates the relation between the mode transition algebra $\mathfrak{A}$ and the nodal fusion product $\mathbb{A}$. It will not be used in this paper.

Remark 2.7. By Thm. 2.5, the mode transition algebra $\mathfrak{A}$ represents the nodal conformal block functor. Recall from Sec. 1.4 that the nodal fusion product $\mathbb{A}$ represents the nodal conformal block functor. Thus, $\mathfrak{A} \simeq \mathbb{A}$ as objects of $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$.

On the other hand, we consider the end

$$\mathbb{E} = \int_{\mathbb{M} \in \mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}' \quad \in \mathrm{Mod}(\mathbb{V}^{\otimes 2}).$$

For each $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$, the dinatural transformation of $\mathbb{E}$ gives a morphism $\varphi_{\mathbb{M}} : \mathbb{E} \to \mathbb{M} \otimes \mathbb{M}'$ in $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$.

Proposition 2.8. *Let* $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$. *We have an isomorphism*

$$\mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{E}, \mathbb{X}' \otimes \mathbb{Y}') \simeq \mathscr{T}^*_{\mathfrak{N}}(\mathbb{X} \otimes \mathbb{Y}) \tag{2.15}$$

Proof. By [FSS20, Cor. 2.9], the linear map

$$\begin{aligned} \mathrm{Hom}_{\mathbb{V}}(\mathbb{Y}, \mathbb{X}') &\to \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathbb{E}, \mathbb{X}' \otimes \mathbb{Y}') \\ T &\mapsto (\mathrm{id}_{\mathbb{X}'} \otimes T^t) \circ \varphi_{\mathbb{X}'} \end{aligned} \tag{2.16}$$

is an isomorphism. See [GZ25a, GZ25b] for details. By (1.5) and (2.16), we have the isomorphism (2.15). □

Theorem 2.9. *Assume that there exists a module in* $\mathrm{Mod}(\mathbb{V})$ *that is not lowest generated. Then* $\mathbb{E} \not\simeq \mathfrak{A}$ *in* $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$.

Proof. By Thm. 2.2, there exist $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$ such that

$$\dim \mathscr{T}^*_{\mathfrak{N}}(\mathbb{X} \otimes \mathbb{Y}) \neq \dim \mathscr{T}^*_{\mathfrak{B}}(\mathbb{X} \otimes \mathbb{Y}). \tag{2.17}$$

By Thm. 2.5, we have an isomorphism

$$\mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\mathfrak{A}, \mathbb{X}' \otimes \mathbb{Y}') \simeq \mathscr{T}^*_{\mathfrak{B}}(\mathbb{X} \otimes \mathbb{Y}). \tag{2.18}$$

Suppose, to the contrary, that $\mathbb{E} \simeq \mathfrak{A}$ in $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$. By Prop. 2.8 and (2.18), there exists an isomorphism

$$\mathscr{T}^*_{\mathfrak{N}}(\mathbb{X} \otimes \mathbb{Y}) \simeq \mathscr{T}^*_{\mathfrak{B}}(\mathbb{X} \otimes \mathbb{Y}).$$

contradicting (2.17). Therefore, $\mathbb{E} \not\simeq \mathfrak{A}$ in $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$. □

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YAU MATHEMATICAL SCIENCES CENTER AND DEPARTMENT OF MATHEMATICS, TSINGHUA UNIVERSITY, BEIJING, CHINA.
E-mail: zhanghao1999math@gmail.com   h-zhang21@mails.tsinghua.edu.cn