Logarithmic conformal field theory: sewing and factorization

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April 2024
Rutgers University
Joint work with Bin Gui
arXiv:2305.10180

Vertex operator algebras

- Vertex operator algebras(VOAs) are mathematically rigorous definition of chiral conformal field theories(CFTs).
- A C_2-cofinite and rational VOA gives a rational conformal field theory.
- A C_2-cofinite VOA gives a logarithmic conformal field theory.
- Mathematically, the lower genus data of chiral CFT can characterize the tensor category $\mathrm{Rep}(\mathbb{V})$.
- All VOAs in this talk are assumed to be C_2-cofinite, positive energy and of CFT type. All modules are assumed to be finitely generated admissible modules. We will NOT emphasize these conditions later.

Tensor category $\mathrm{Rep}(\mathbb{V})$: rational case

- When $\mathbb{V}$ is rational, $\mathrm{Rep}(\mathbb{V})$ is a fusion category with finitely many irreducibles $\mathbb{M}_1, \cdots, \mathbb{M}_n$. The tensor product in $\mathrm{Rep}(\mathbb{V})$, denoted by $\boxtimes$, is defined by

$$\mathbb{U} \boxtimes \mathbb{W} := \bigoplus_{i=1}^{n} \mathcal{I}\binom{\mathbb{M}_i}{\mathbb{U}\mathbb{W}}^* \otimes \mathbb{M}_i,$$

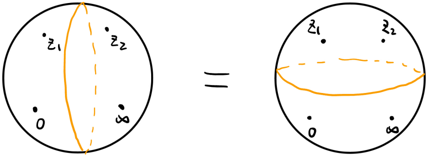
  where $\mathcal{I}\binom{\mathbb{M}_i}{\mathbb{U}\mathbb{W}}$ is the space of intertwining operators of type $\binom{\mathbb{M}_i}{\mathbb{U}\mathbb{W}}$.
- To understand the tensor category structure of $\mathrm{Rep}(\mathbb{V})$, it suffices to understand intertwining operators.

Intertwining operators and genus 0 conformal blocks

- The space of intertwining maps of type $\binom{\mathbb{W}_3}{\mathbb{W}_1\mathbb{W}_2}$ can be understood as the space of conformal blocks $\mathscr{T}^*_{\mathfrak{P}_z}(\mathbb{W}_2 \otimes \mathbb{W}_1 \otimes \mathbb{W}_3')$ associated to a pointed surface with local coordinates $\mathfrak{P}_z = (\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, 1/\zeta)$, which has genus 0.
- If we combine all $\mathfrak{P}_z$ over $z \in \mathbb{C}^\times$ in a proper way, we will get a *family of pointed surfaces* $\mathfrak{P}$ with base manifold $\mathbb{C}^\times$. Then conformal blocks should be 'analytic' on $\mathbb{C}^\times$ in some sense.
- C_2-cofiniteness implies that all spaces of conformal blocks are finite dimensional. $N_{ij}^k = \dim \mathcal{I}\binom{\mathbb{M}_k}{\mathbb{M}_i\mathbb{M}_j}$ is the fusion rule of $\mathrm{Rep}(\mathbb{V})$.

Intertwining operators and genus 0 conformal blocks

- The associativity isomorphism corresponds to sewing and factorization of conformal blocks for the 4-pointed surface $(\mathbb{P}^1; 0, z_1, z_2, \infty)$.



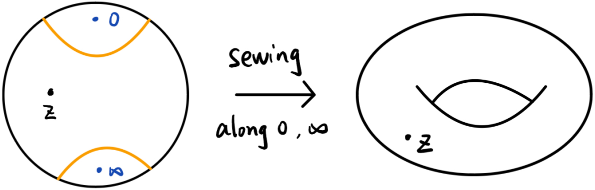
- When $\mathbb{V}$ is rational, intertwining operators $\mathcal{Y}(\cdot, z) : \mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{W}_3\{z\}$ of type $\binom{\mathbb{W}_3}{\mathbb{W}_1 \mathbb{W}_2}$ involve no $\log z$, but the powers of z are not necessarily integers. The same holds for conformal blocks.

Genus 1 conformal blocks from intertwining operators

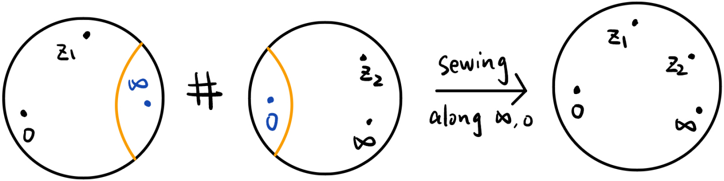
To study rigidity and modularity of $\mathrm{Rep}(\mathbb{V})$, we need to introduce modular invariance of genus 1 conformal blocks. There are (at least) two ways to understand genus 1 conformal blocks.

(1) *Taking q-traces of intertwining operators.* If we choose an intertwining operator $\mathcal{Y}$ of type $\binom{\mathbb{U}}{\mathbb{W}\mathbb{U}}$, we get a formal series $S_{\mathcal{Y}}(w,\tau)=\mathrm{Tr}_{\mathbb{U}}\mathcal{Y}(\mathcal{U}(q_z)w,q_z)q_\tau^{L(0)-\frac{c}{24}}$,where $q_\tau=e^{2\pi i\tau}$. If we prove its convergence in some sense, it may be regarded as a 'conformal blocks' associated to the 1-pointed surface $\mathfrak{T}_\tau=(\mathbb{T}_\tau;z),$. Geometrically, we obtain genus 1 conformal blocks by self sewing genus 0 conformal blocks along $(0,\mathbb{U},\zeta),(\infty,\mathbb{U}',1/\zeta)$.

Sewing along a single pair of points



Self sewing along a single pair of points



Disjoint sewing along a single pair of points

Genus 1 conformal blocks

(2) *Understand conformal blocks of all genera in a uniform way.* For example, for the 1-pointed surface $\mathfrak{T}_\tau = (\mathbb{T}_\tau; z)$, we can define the space of conformal blocks $\mathscr{T}^*_{\mathfrak{T}_\tau}(\mathbb{W})$, consisting of linear maps $\mathbb{W} \to \mathbb{C}$ which vanish on certain action defined by $\mathbb{V}$ and $\mathfrak{T}_\tau$.

(3) If we combine all these $\mathfrak{T}_\tau$ over $\tau \in \mathbb{H}$, we get a family of compact Riemann surfaces $\mathfrak{T}$ with base manifold $\mathbb{H}$. We can also define conformal blocks on $\mathfrak{T}$ and they must be 'analytic' on $\mathbb{H}$ in some sense.

Genus 1 sewing and factorization theorem

Theorem (Sewing and factorization, Huang '03)

Suppose that $\mathbb{V}$ is rational.

- ***Sewing.*** *Taking q-traces of a genus 0 conformal block (intertwining operator) defined in (1) is absolute convergent and gives a genus 1 conformal block on $\mathfrak{X}$ in the sense of (3).*
- ***Factorization.*** *Every conformal block on $\mathfrak{X}$ in the sense of (3) can be written as a linear combination of sewing intertwining operators of type $\binom{\mathbb{M}_i}{\mathbb{W}\mathbb{M}_i}$ for $i = 1, \cdots, n$.*

When $\mathbb{W} = \mathbb{V}$, the above theorem was first given by Zhu in 1996.

Modular invariance of genus 1 conformal blocks

Corollary (Huang '03)

Assume that $\mathbb{V}$ *is rational. For each* $w \in \mathbb{W}$,
$\mathcal{F}_w := \mathrm{span}\{S_{\mathcal{Y}}(w) : \mathcal{Y} \in \mathcal{I}\binom{\mathbb{M}_i}{\mathbb{W}\mathbb{M}_i}, i = 1, \cdots, n\}$ *is* $\mathrm{SL}_2(\mathbb{Z})$*-invariant.*
In particular, $\tau \mapsto -1/\tau$ *gives* S*-matrix.*

Proof.

By sewing in the previous theorem, for each $\mathcal{Y}$, $S_{\mathcal{Y}}$ is a genus 1 conformal block on $\mathfrak{T}$. It is not hard to check that under $\mathrm{SL}_2(\mathbb{Z})$-transformation, it remains to be a genus 1 conformal block on $\mathfrak{T}$. By factorization in the previous theorem, the theorem is proved. □

Modular invariance of genus 1 conformal blocks

- If we assume $\mathbb{V}$ is not necessarily rational, then the current version of factorization is not true and thus modular invariance given in the above corollary is not true. For $\mathbb{W} = \mathbb{V}$ and $v \in Vbb$, Miyamoto added the mysterious *pseudotraces* in $\mathcal{F}_v$ to make the new $\mathcal{F}_v$ an $\mathrm{SL}_2(\mathbb{Z})$-invariant space.
- In 2023, Huang proved modular invariance involving pseudotraces for nonrational VOAs. The main difficulty in his proof is a modified version of 'factorization' involving pseudotraces. Therefore, genus 1 conformal blocks for C_2-cofinite VOAs are all constructed using intertwining operators and pseudotraces.

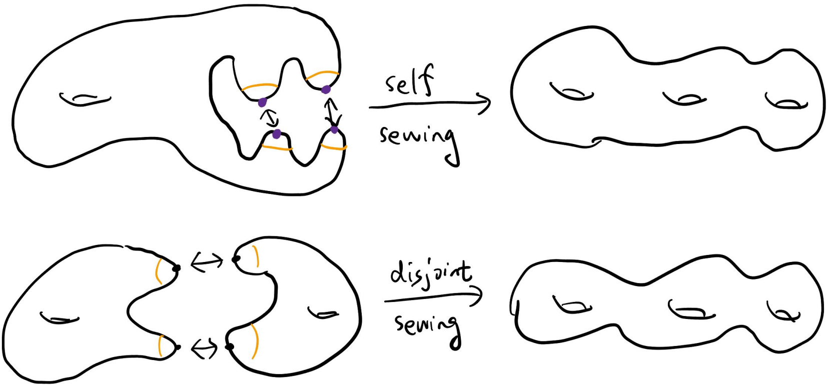
Sewing compact Riemann surfaces, arbitrary genus

- Let $\widetilde{\mathfrak{X}}$ be a (not necessarily connected) $(N+2R)$-pointed compact Riemann surface. Among these points, there are N ordinary points $x_1, \cdots, x_N$ and R pairs of distinguished points $x_1', \cdots, x_R'$ and $x_1'', \cdots, x_R''$. These distinguished points are equipped with local coordinates $\xi_1, \cdots, \xi_R, \varpi_1, \cdots, \varpi_R$.
- Sew $\widetilde{\mathfrak{X}}$ via

$$\xi_i \varpi_i = q_i, \quad i = 1, \cdots, R$$

to get $\mathfrak{X}_{q_\bullet} = \mathfrak{X}_{q_1, \cdots, q_R}$, which is a nodal curve of possibly higher genus.

Sewing compact Riemann surfaces, examples



Sewing compact Riemann surfaces

- Combining all such $\mathfrak{X}_{q_\bullet}$, we get a family of pointed nodal curves $\mathfrak{X}$, whose base manifold is $\mathcal{D}_{r_\bullet\rho_\bullet} = \mathcal{D}_{r_1\rho_1} \times \cdots \times \mathcal{D}_{r_R\rho_R}$ and $q_\bullet$-fiber is exactly $\mathfrak{X}_{q_\bullet}$. If we restrict $\mathfrak{X}$ to $\mathcal{D}^\times_{r_\bullet\rho_\bullet} = \mathcal{D}^\times_{r_1\rho_1} \times \cdots \times \mathcal{D}^\times_{r_R\rho_R}$, then we get a family of pointed compact Riemann surfaces.
- We can define the notion of conformal blocks of $\widetilde{\mathfrak{X}}, \mathfrak{X}$ and $\mathfrak{X}_{q_\bullet}$ in a uniform way.
    - For $\widetilde{\mathfrak{X}}$ (resp. $\mathfrak{X}_{q_\bullet}$), a conformal block is a linear map from a $\mathbb{V}^{\otimes(N+2R)}$-module (resp. $\mathbb{V}^{\otimes N}$-module) to $\mathbb{C}$.
    - For $\mathfrak{X}$, a conformal block is a morphism of $\mathcal{O}_{\mathcal{D}_{r_\bullet\rho_\bullet}}$-modules $\mathbb{W} \otimes \mathcal{O}_{\mathcal{D}_{r_\bullet\rho_\bullet}} \to \mathcal{O}_{\mathcal{D}_{r_\bullet\rho_\bullet}}$, where $\mathbb{W}$ is a $\mathbb{V}^{\otimes N}$-module.

Sewing conformal blocks

- Suppose $\mathbb{W}$ is a $\mathbb{V}^{\otimes N}$-module and $\mathbb{M}$ is a $\mathbb{V}^{\otimes R}$-module. Associate $\mathbb{W}$ to $x_1, \cdots, x_N$, $\mathbb{M}$ to $x'_1, \cdots, x'_R$ and $\mathbb{M}'$ to $x''_1, \cdots, x''_R$.
- It is interesting to remark that: if $\mathbb{V}$ is C_2-cofinite and rational, then $\mathbb{W}$ and $\mathbb{M}$ can both be written as tensor products of $\mathbb{V}$-modules.
- If $\psi : \mathbb{W} \otimes \mathbb{M} \otimes \mathbb{M}' \to \mathbb{C}$ is a conformal blocks on $\widetilde{\mathfrak{X}}$, then we sew ψ to get $\mathcal{S}\psi : \mathbb{W} \to \mathbb{C}\{q_\bullet\}[\log q_\bullet]$ by setting

$$\mathcal{S}\psi(w) = \psi(q_\bullet^{L_\bullet(0)} \blacktriangleright \otimes \blacktriangleleft)$$

  Sometimes, we may use $\mathcal{S}_{\mathbb{M}}\psi$ to emphasize $\mathbb{M}$.
- When $\mathbb{V}$ is rational, $\mathcal{S}\psi$ actually involves no $\log$ terms.

Higher genus sewing and factorization, the rational case

Assume that $\mathbb{V}$ is rational. Suppose $\mathfrak{X}$ is a family of N-pointed nodal curves obtained by (self or disjoint) sewing an $(N+2)$-pointed surface $\widetilde{\mathfrak{X}}$ along x' and x'', i.e., $R = 1$. Since $\mathbb{V}$ is rational, we may write $\mathbb{W} = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$.

Theorem (Gui '20, Gui-Z. to appear, in a more general version)

(1) *$\mathcal{S}\psi(w_1 \otimes \cdots \otimes w_N)$ is absolutely and locally uniformly convergent for each $w_1 \otimes \cdots \otimes w_N \in \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$ and $\mathcal{S}\psi$ is a conformal block on $\mathfrak{X}$ outside the nodes. This property does NOT rely on rationality.*

(2) *Each conformal block on $\mathfrak{X}$ outside the nodes can be written as a linear combination of*
$\{\mathcal{S}_{\mathbb{M}}\psi : \mathbb{M} \in \mathrm{Irr}, \psi$ *is a conformal block on* $\widetilde{\mathfrak{X}}\}$.

Idea of proof for convergence

- Using complex analytic geometry, we can lift the vector field $\mathfrak{y} = q\partial_q$ on the base manifold $\mathcal{D}_{r\rho}$ to the nodal family. The resulting vector field is denoted by $\widetilde{\mathfrak{y}}$. However, there will be possible poles at N punctures.
- We use $\widetilde{\mathfrak{y}}$ to define a projective flat connection $\nabla_{q\partial_q}$ on the sheaf of conformal blocks and this will give a linear differential equations with simple poles for $\mathcal{S}\psi$. The convergence hence follows.
- This method can also be used to prove that: conformal blocks of C_2*-cofinite VOAs* give rise to holomorphic vector bundles over families of compact Riemann surfaces (Gui-Z. to appear).

Higher genus sewing and factorization, the rational case

Corollary

For each $q \in \mathcal{D}_{r\rho}^{\times}$, sewing conformal blocks gives an isomorphism of vector spaces

$$\mathcal{S}_q : \bigoplus_{\mathbb{M} \in \mathrm{Irr}} \mathscr{T}_{\widetilde{\mathfrak{X}}}^{*}(\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \otimes \mathbb{M} \otimes \mathbb{M}') \xrightarrow{\simeq} \mathscr{T}_{\mathfrak{X}_q}^{*}(\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N).$$

Theorem (Nodal factorization, Damiolini-Gibney-Tarasca '19)

'Formal sewing' gives an isomorphism of vector spaces

$$\mathcal{S}_0 : \bigoplus_{\mathbb{M} \in \mathrm{Irr}} \mathscr{T}_{\widetilde{\mathfrak{X}}}^{*}(\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \otimes \mathbb{M} \otimes \mathbb{M}') \xrightarrow{\simeq} \mathscr{T}_{\mathfrak{X}_0}^{*}(\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N).$$

Logarithmic story

Now, our VOA $\mathbb{V}$ is not assumed to be rational. Our goal is to give an analog of sewing and factorization theorem in the nonrational case. The main differences between logarithmic and rational stories are the following.

- Intertwining operators and conformal blocks will all involve $\log$ terms. This is mainly because $L(0)$ is no longer semisimple on $\mathbb{V}$-modules.
- Instead of associating $\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$ to market points $x_1, \cdots, x_N$ separatedly, we associate a $\mathbb{V}^{\otimes N}$-module $\mathbb{W}$ to $x_1, \cdots, x_N$.
- We need to consider disjoint sewing along several pairs of points.

Sewing conformal blocks, nonrational case

Recall that $\mathfrak{X}$ is obtained by sewing a (not necessarily connected) $(N+2R)$-pointed compact Riemann surface $\widetilde{\mathfrak{X}}$ along R pairs of punctures and local coordinates. In this case, R is not necessarily 1. The sewing theorem is not hard to generalize:

Theorem (Gui-Z. to appear)

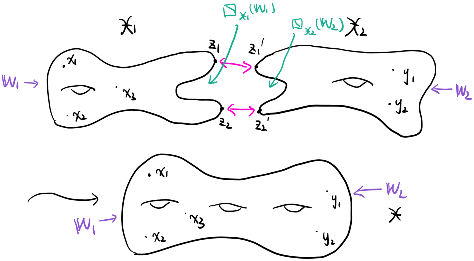
For each $w \in \mathbb{W}$, $\mathcal{S}\psi(w)$ is absolutely and locally uniformly convergent and $\mathcal{S}\psi$ is a conformal block on $\mathfrak{X}$ outside the nodes.

Proof.

For $1 \leqslant k \leqslant R$, use complex analytic geometry to construct the connection $\nabla_{q_k \partial_{q_k}}$. This will give linear differential equations with simple poles for $\mathcal{S}\psi$. □

Disjoint sewing along several pairs of points

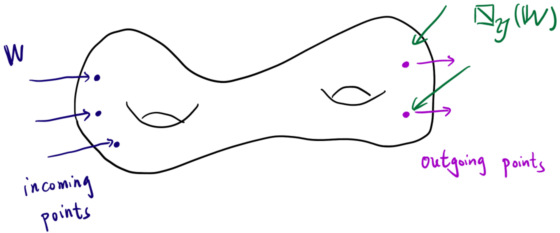
We only need to consider the disjoint sewing of the following type:



To construct factorization, it suffices to figure out what modules can be associated to points to be sewn. We will use these modules to replace irreducible modules in the sewing-factorization theorem.

Dual fusion products

Suppose $\mathfrak{Y} = (y_1, \cdots, y_M; \theta_1, \cdots, \theta_M | C | x_1, \cdots, x_N)$ is a pointed surface. $x_1, \cdots, x_N$ are incoming points, $y_1, \cdots, y_M$ are outgoing points and $\theta_1, \cdots, \theta_M$ are local coordinates at $y_1, \cdots, y_M$. We call $\mathfrak{Y}$ an (M, N)-**pointed surface** for simplicity.



Dual fusion products

Definition

Associate a $\mathbb{V}^{\otimes N}$-module $\mathbb{W}$ to $x_1, \cdots, x_N$. A **dual fusion product associated to** $\mathbb{W}$ **and** $\mathfrak{Y}$ is a $\mathbb{V}^{\otimes M}$-module $\boxbslash_{\mathfrak{Y}}(\mathbb{W})$ contained in $\mathbb{W}^*$ such that: if we associate $\boxbslash_{\mathfrak{Y}}(\mathbb{W})$ to $y_1, \cdots, y_M$, then

(1) the natural pairing $\omega : \mathbb{W} \otimes \boxbslash_{\mathfrak{Y}}(\mathbb{W}) \to \mathbb{C}$ is a conformal block in $\mathscr{T}^*_{\mathfrak{Y}}\big(\mathbb{W} \otimes \boxbslash_{\mathfrak{Y}}(\mathbb{W})\big)$. It is called a **canonical conformal block**.

(2) for any $\mathbb{V}^{\otimes M}$ module $\mathbb{M}$ and any conformal block $\Psi : \mathbb{W} \otimes \mathbb{M} \to \mathbb{C}$ associated to $\mathfrak{Y}$, there exists a unique homomorphism $\Phi : \mathbb{M} \to \boxbslash_{\mathfrak{Y}}(\mathbb{W})$ such that $\Psi = \omega \circ (\mathbf{1} \otimes \Phi)$.

The contragredient module $\boxtimes_{\mathfrak{Y}}(\mathbb{W})$ of $\boxbslash_{\mathfrak{Y}}(\mathbb{W})$ is called a **fusion product associated to** $\mathfrak{Y}$ **and** $\mathbb{W}$.

Dual fusion products, universal property

Theorem (Gui-Z. '23)

Assume $\mathbb{V}$ is C_2-cofinite. Associate a $\mathbb{V}^{\otimes N}$-module $\mathbb{W}$ to $x_1, \cdots, x_N$. There exists a unique dual fusion product $\boxslash_{\mathfrak{Y}}(\mathbb{W})$ associated to $\mathbb{W}$ and $\mathfrak{Y}$.

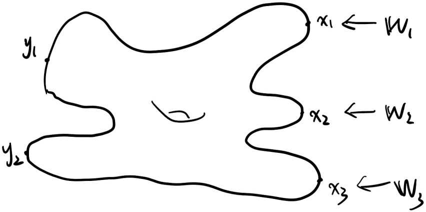
The main difficulty in this theorem is the module structure of $\boxslash_{\mathfrak{Y}}(\mathbb{W})$. In the rational case, Zhu, Damiolini-Gibney-Tarasca defined the module structure using level 0 Zhu's algebras. This method does not work in the nonrational case. We introduce **propagation of dual fusion products** to define the module structure.

Examples of (dual) fusion products

- If we choose $\mathfrak{P} = (0, \infty; \zeta, 1/\zeta \big| \mathbb{P}^1 \big| z)$ and associate $\mathbb{V}$ to z, then $\boxbslash_{\mathfrak{P}}(\mathbb{V})$ covers (higher level) Zhu's algebras, due to Zhu ('96) and Dong-Li-Mason ('98). Caution: this does NOT mean higher level Zhu's algebras are the same as $\boxbslash_{\mathfrak{P}}(\mathbb{V})$.
- If we choose $\mathfrak{Y} = (\infty; 1/\zeta \big| \mathbb{P}^1 \big| 0, 1)$ and associate $\mathbb{W}_1, \mathbb{W}_2$ to $0, 1$, then $\boxtimes_{\mathfrak{Y}}(\mathbb{W}_1 \otimes \mathbb{W}_2)$ is the usual fusion product in $\mathrm{Rep}(\mathbb{V})$.

Examples of (dual) fusion products

- Assume $\mathbb{V}$ is rational. If

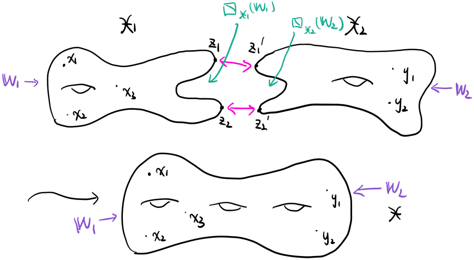


then

$$\boxslash_{\mathfrak{Y}}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3)$$
$$\simeq \bigoplus_{\mathbb{M}_1,\mathbb{M}_2 \in \mathcal{E}} \mathbb{M}_1 \otimes \mathbb{M}_2 \otimes \mathscr{T}^*_{\mathfrak{Y}}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3 \otimes \mathbb{M}_1 \otimes \mathbb{M}_2)$$

Sewing and factorization, nonrational case

We sew $\mathfrak{X}_1$ and $\mathfrak{X}_2$ along R pairs of points $z_\bullet, z'_\bullet$.



The canonical conformal block on $\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$) is denoted by ω_1 (resp. ω_2).

Factorization of conformal blocks, nonrational case

Theorem (Gui-Z. to appear)

Any conformal block on $\mathfrak{X}$ outside the nodes can be written as (a linear combination of) sewing conformal blocks on $\mathfrak{X}_1$ and $\mathfrak{X}_2$ along $(z_\bullet, \boxbslash_{\mathfrak{X}_1}(\mathbb{W}_1)), (z'_\bullet, \boxtimes_{\mathfrak{X}_1}(\mathbb{W}_1))$.

This can be viewed as the surjective part of sewing and factorization theorem in the nonrational case. It is the most difficult part in our proof.

Factorization of conformal blocks, nonrational case

Corollary (Gui-Z. to appear)

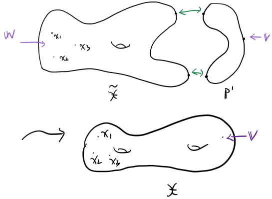
For each $q_\bullet \in \mathcal{D}^\times_{r_\bullet \rho_\bullet}$, the linear map

$$\mathrm{Hom}_{\mathbb{V}\otimes R}\Big(\boxtimes_{\mathfrak{X}_2}(\mathbb{W}_2), \boxbackslash_{\mathfrak{X}_1}(\mathbb{W}_1)\Big) \rightarrow \mathscr{T}^*_{\mathfrak{X}_{q_\bullet}}(\mathbb{W}_1 \otimes \mathbb{W}_2)$$
$$T \mapsto \sum_i \omega_1(w_1 \otimes T e_i^\vee)\omega_2(w_2 \otimes q_\bullet^{L_\bullet(0)} e_i)$$

is an isomorphism. Here $\{e_i\}$ is a basis of $\boxbackslash_{\mathfrak{X}_2}(\mathbb{W}_2)$ and $\{e_i^\vee\}$ is a dual basis of $\boxtimes_{\mathfrak{X}_2}(\mathbb{W}_2)$.

Factorization of conformal blocks, nonrational case

- Suppose $\mathbb{V}$ is C_2-cofinite. Consider



Then

$$\mathscr{T}^*_{\mathfrak{X}_{q_\bullet}}(\mathbb{W}) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}\Big(\boxtimes_{\mathfrak{P}}(\mathbb{V}), \boxbslash_{\widetilde{\mathfrak{X}}}(\mathbb{W})\Big)$$

$\boxtimes_{\mathfrak{P}}(\mathbb{V})$ is crucial and is closely related to coends and Longo-Rehren subfactors.

Recover the rational case from the nonrational case

- If $\mathbb{V}$ is rational, then

$$\boxtimes_{\mathfrak{P}}(\mathbb{V}) \simeq \bigoplus_{\mathbb{M}\in \mathrm{Irr}} \mathbb{M}\otimes\mathbb{M}'.$$

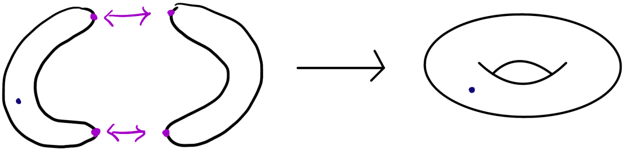
So

$$\mathscr{T}^*_{\mathfrak{X}_{q_\bullet}}(\mathbb{W}) \simeq \bigoplus_{\mathbb{M}\in \mathrm{Irr}} \mathscr{T}^*_{\widetilde{\mathfrak{X}}}(\mathbb{W}\otimes\mathbb{M}\otimes\mathbb{M}')$$

This covers sewing and factorization theorem in the rational case.

Genus 1 factorization, nonrational case

Consider the following disjoint sewing along two pairs of points.



Then we have an isomorphism of vector spaces

$$\mathscr{T}^*_{\mathfrak{T}_{q_1,q_2}}(\mathbb{W}) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}\big(\boxtimes_{\mathfrak{P}}(\mathbb{V}), \boxbslash_{\mathfrak{P}}(\mathbb{W})\big).$$

It is natural to expect: Huang's result on modular invariance of pseudotraces can be related dual fusion product and the above isomorphism.

Thank you for listening to my talk!