Robust Random Graph Matching in Dense Graphs via Vector Approximate Message Passing

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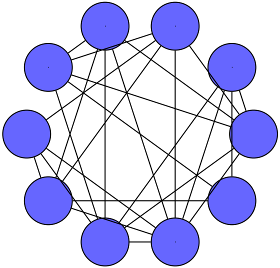
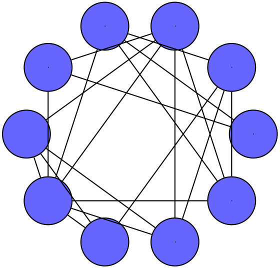
Graph matching (network alignment)





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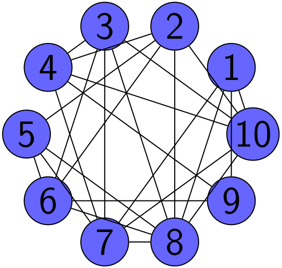

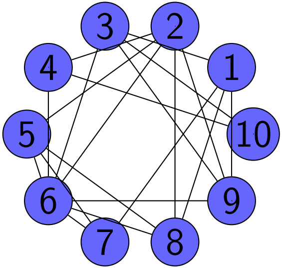

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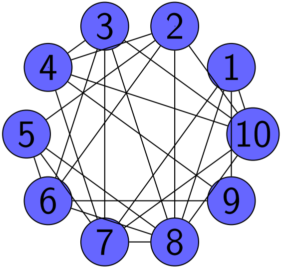

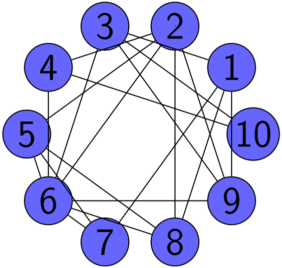

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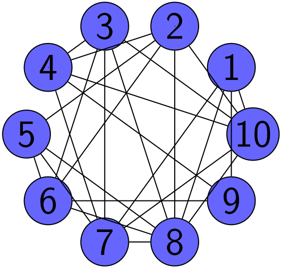

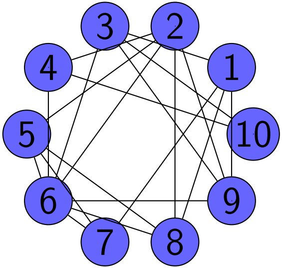

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- NP-hard to solve/approximate in worst case.

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    - Noisy case ($\rho < 1$): little is known for efficient algorithms until recently.

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  - [Ding-L.'22+,23+] polynomial-time iterative algorithm for exact matching when $q \geq n^{-1+\delta}$ and correlation $\rho = \Omega(1)$.
- Evidence in [Ding-Du-L.'23+] suggests that the state-of-the-art algorithms have nearly reached the limit of efficient algorithms.

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- **Motivation from application**: somewhat more "practical" graph matching algorithm for real networks?
- **Motivation from theory**: can we find efficient graph matching algorithms for semi-random models?

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- Observation: the revised matrices $(A', B') = (A + E, B + F)$, where E, F supported on an unknown $\epsilon n * \epsilon n$ principle minor of (A, B).

Our result: a robust Gaussian matching algorithm

ρ: edge correlation; ϵn: size of corruption; π_*: hidden matching.

Theorem (L.'25)

Exact recovery is achieved efficiently by an approximate message passing algorithm w.h.p. if

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- Extends to the case of correlated Erdős-Rényi models when the edge-density q is a constant.

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Usually in the form of the following iteration:

$$\underset{\substack{\uparrow \\ \text{estimator for the hidden signal}}}{f^{(t+1)}} = \underset{\substack{\uparrow \\ \text{entrywise transform by a suitable denoiser}}}{\varphi} \circ \left(\tfrac{1}{\sqrt{n}} A f^{(t)}\right)$$

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- Our strategy: iteratively construct "signatures" using AMP iteration

$$f^{(t+1)} = \varphi \circ \left(\tfrac{1}{\sqrt{n}} A' f^{(t)}\right), \quad f^{(t)} = \left(f_1^{(t)}, \ldots, f_n^{(t)}\right)^\top,$$
$$g^{(t+1)} = \varphi \circ \left(\tfrac{1}{\sqrt{n}} B' g^{(t)}\right), \quad g^{(t)} = \left(g_1^{(t)}, \ldots, g_n^{(t)}\right)^\top.$$

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- Hope: if we choose a suitable denoiser function φ, then at a large time t^* we will have

$$\Pi_* = \arg\max_{\Pi \in \mathfrak{S}_n} \langle f^{(t^*)}, \Pi g^{(t^*)} \rangle,$$

then we can find Π_* by solving a linear assignment problem.

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- Problem in iteration: for any $n * K$ matrix $f^{(t)}, g^{(t)}$

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one can check that the covariance between $f_i^{(t)}$ and $g_{\pi_*(i)}^{(t)}$ must decreases in t (i.e., entrywise signal is decreasing).

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  - $\langle f_i^{(t)}, g_j^{(t)} \rangle$ have expectation 0 and variance K_t (i.e., no signal in fake pairs).
  - The signal-to-noise ratio $\frac{(\epsilon_t K_t)^2}{K_t} = \epsilon_t^2 K_t$ grows rapidly in t.

Dealing with adversarial corruption

Input: a spectral cleaning procedure proposed in [Ivkov-Schramm'24].

Theorem (Ivkov-Schramm'24)

Given a matrix $M' = M + E$ with $\|M\|_{\mathrm{op}} = O(\sqrt{n})$ and E supported on an $\epsilon n * \epsilon n$ minor of M, there exists a polynomial-time algorithm that zeros-out $O(\epsilon n)$ rows and columns of M' such that the "cleaned" matrix $\widehat{M}$ satisfies $\|\widehat{M}\|_{\mathrm{op}} = O(\sqrt{n})$.

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- Our method: apply the spectral cleaning procedure to A', B' respectively to obtain $\widehat{A}, \widehat{B}$. Then run the iteration w.r.t. $(\widehat{A}, \widehat{B})$:

$$\widehat{f}^{(0)} = \varphi \circ \left(\widehat{A}_{[n]\times U}\right), \quad \widehat{f}^{(t+1)} = \varphi \circ \left(\tfrac{1}{\sqrt{n}} \widehat{A}\widehat{f}^{(t)} \Xi^{(t)}\right),$$
$$\widehat{g}^{(0)} = \varphi \circ \left(\widehat{B}_{[n]\times V}\right), \quad \widehat{g}^{(t+1)} = \varphi \circ \left(\tfrac{1}{\sqrt{n}} \widehat{B}\widehat{g}^{(t)} \Xi^{(t)}\right),$$

Intuitions behind the spectral cleaning

$\widehat{f}^{(t)}$: iterations w.r.t. $\widehat{A}$; $f^{(t)}$: iterations w.r.t. A;
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$$\widehat{A}\widehat{f}^{(t)}\Xi^{(t)} - Af^{(t)}\Xi^{(t)} = \widehat{A}(\widehat{f}^{(t)} - f^{(t)})\Xi^{(t)} + (\underbrace{\widehat{A} - A}_{\uparrow \atop E_{Q\setminus S\times Q\setminus S} + A_{[n]\times Q\setminus S} + A_{Q\setminus S\times[n]}})f^{(t)}\Xi^{(t)}$$

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$$\widehat{A}\widehat{f}^{(t)}\Xi^{(t)} - Af^{(t)}\Xi^{(t)} = \widehat{A}(\widehat{f}^{(t)} - f^{(t)})\Xi^{(t)} + (\widehat{A} - A)f^{(t)}\Xi^{(t)}$$

$$\uparrow$$

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Induction hypothesis $\Longrightarrow (*) \approx 0$;

Intuitions behind the spectral cleaning

$\widehat{f}^{(t)}$: iterations w.r.t. $\widehat{A}$; $f^{(t)}$: iterations w.r.t. A;

S: index are zeroed-out; E: adversarial corruption; Q: support of E;

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- Then $\widehat{f}^{(t+1)} = \varphi \circ (\frac{1}{\sqrt{n}}\widehat{A\widehat{f}^{(t)}}\Xi^{(t)}) \approx \varphi \circ (\frac{1}{\sqrt{n}}A\widehat{f}^{(t)}\Xi^{(t)}) \approx f^{(t+1)}$.

Conclusions and open problems

- We found a poly-time algorithm that matches two correlated Gaussian matrices with constant correlation even when two $\frac{n}{\mathrm{poly}(\log n)}$ size submatrices are adversarially corrupted.
- Our method: construct "signatures" by iteratively running an vector AMP on two matrices.
- A few open problems:
    - Other ways of corruption (e.g., corruption on arbitrary small edge set).
    - Robust algorithm for sparse graphs (edge density $q = n^{-\alpha+o(1)}$ when $\alpha > 0$)?

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Thank you!