Detecting Correlation Efficiently in Very Supercritical Stochastic Block Models: Breaking the Otter's Threshold Barrier

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Abstract. Consider a pair of sparse correlated stochastic block models $\mathcal{S}(n, \frac{\lambda}{n}, \epsilon; s)$ subsampled from a common parent stochastic block model with two symmetric communities, average degree $\lambda = O(1)$, divergence parameter $\epsilon \in (0,1)$ and subsampling probability s. For all $\epsilon \in (0,1)$, we construct a statistic based on the combination of two low-degree polynomials and show that there exists a sufficiently small constant $\delta = \delta(\epsilon) > 0$ and a sufficiently large constant $\Delta = \Delta(\epsilon, \delta)$ such that when $\lambda > \Delta$ and $s > \sqrt{\alpha} - \delta$ where $\alpha \approx 0.338$ is Otter's constant, this statistic can distinguish this model and a pair of independent stochastic block models $\mathcal{S}(n, \frac{\lambda s}{n}, \epsilon)$ with probability $1 - o(1)$. We also provide an efficient algorithm that approximates this statistic in polynomial time. Our result is the first detection or matching type algorithm that breaks the Otter's threshold in sparse correlated random graphs. The crux of our statistic's construction lies in a carefully curated family of multigraphs called *decorated trees*, which enables effective aggregation of the community signal and graph correlation from the counts of the same decorated tree while suppressing the undesirable correlations among counts of different decorated trees. We believe such construction may be of independent interest.

1 Introduction Graph matching (also known as network alignment) seeks a correspondence between the vertices of two graphs that maximizes the total number of common edges. When the two graphs are exactly isomorphic to each other, this problem reduces to the classical graph isomorphism problem, for which the best known algorithm runs in quasi-polynomial time [3]. In general, graph matching is an instance of the *quadratic assignment problem* [7], which was known to be NP-hard to solve or even to approximate [31].

Motivated by applications in various applied areas including computational biology [50], social networking [41, 42], and natural language processing [23], a recent line of work is devoted to the study of statistical theory and efficient algorithms for graph matching under statistical models, by assuming the two graphs are randomly generated with correlated edges under a hidden vertex correspondence. From a theoretical perspective, perhaps the most widely studied setting is the correlated Erdős-Rényi graph model [45], in which two correlated Erdős-Rényi graphs are observed, with edges correlated via a latent vertex bijection π_*. Two important and entangling issues regarding this model, i.e., the information-theoretic threshold and the computational phase transition, have been extensively studied recently, and we refer the readers to [12, Section 1.1] for an overview. In particular, on the side of computation, substantial progresses on algorithms were achieved, which may be roughly summarized as follows: in the sparse regime, efficient matching algorithms were designed when the correlation exceeds the square root of Otter's constant (the Otter's constant is approximately 0.338) [18, 19, 34, 33]; in the dense regime, efficient matching algorithms were proposed when the correlation exceeds an arbitrarily small constant [14, 13]. The separation between the sparse and dense regimes above, roughly speaking, depends on whether the average degree grows polynomially or sub-polynomially. In addition, another important direction is to establish lower bounds for computation complexity: while it is challenging to prove hardness for a typical instance of the problem even under the assumption of P $\neq$ NP, in a recent work [12] evidence based on the analysis of low-degree polynomials indicates that the aforementioned algorithms may essentially capture the correct computational thresholds.

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Since our current understanding for the information-computation transition of the correlated Erdős-Rényi model is more or less satisfactory *assuming belief in the low-degree hardness conjecture*, an important future direction is to understand the matching problem in other important correlated random graph models. This research direction is motivated by an important question that whether *additional structural information* in pairs of graphs will enhance algorithmic performance and, consequently, affect the computation threshold. We believe this question holds substantial importance from both applied and theoretical perspectives. On the one hand, realistic networks are usually captured better by more structured graph models, such as the random geometric graph model [51, 22], the random growing graph model [47] and the stochastic block model. On the other hand, the study of algorithmic threshold on structural models enables us to better characterize the *algorithmic universality class* of the correlated Erdős-Rényi model–that is, to identify which models exhibit similar algorithmic behaviors to the correlated Erdős-Rényi model.

In this paper, we consider a pair of correlated sparse stochastic block models (SBMs) that are subsampled from a common parent stochastic block model with two symmetric communities, average degree $\lambda = O(1)$, divergence parameter ϵ and subsampling probability s. Introduced in [24], the SBM provides a framework for understanding when community signal can be extracted from network data, as it exhibits sharp informational and computational phase transitions for various inference tasks (we refer the readers to [1] and [40, Section 1] for an extensive overview for progress on this model). Specifically, denoting by U_n the set of unordered pairs (i,j) where $1 \leq i < j \leq n$, we can define this model as follows.

Definition 1.1 (Stochastic block model). *Given an integer $n \geq 1$ and two parameters $\lambda > 0, \epsilon \in (0,1)$, we define a random graph G on $[n] = \{1, \ldots, n\}$ as follows. First, we select a labeling $\sigma_* \in \{-1,+1\}^n$ uniformly at random. For each distinct pair $(i,j) \in \mathrm{U}_n$, we independently add an edge (i,j) with probability $\frac{(1+\epsilon)\lambda}{n}$ if $\sigma_*(i) = \sigma_*(j)$, and with probability $\frac{(1-\epsilon)\lambda}{n}$ if $\sigma_*(i) \neq \sigma_*(j)$. We denote $\mathcal{S}(n, \frac{\lambda}{n}, \epsilon)$ to be the law of G.*

Definition 1.2 (Correlated stochastic block model). *Given an integer $n \geq 1$ and three parameters $\lambda > 0, \epsilon, s \in (0,1)$, let $J_{i,j}$ and $K_{i,j}$ be independent Bernoulli variables with parameter s for $(i,j) \in \mathrm{U}_n$. In addition, let π_* be an independent uniform permutation on $[n] = \{1, \ldots, n\}$. Then, we define a triple of correlated random graphs (G, A, B) such that G is sampled from a stochastic block model $\mathcal{S}(n, \frac{\lambda}{n}, \epsilon)$, and conditioned on G (note that we identify a graph with its adjacency matrix)*

$$A_{i,j} = G_{i,j} J_{i,j} \,, \quad B_{i,j} = G_{\pi_*^{-1}(i), \pi_*^{-1}(j)} K_{i,j} \,.$$

We denote the joint law of $(\pi_*, \sigma_*, G, A, B)$ by $\mathbb{P}_{*,n} := \mathbb{P}_{*,n,\lambda,\epsilon,s}$, and the marginal law of (A,B) by $\mathbb{P}_n := \mathbb{P}_{n,\lambda,\epsilon,s}$.

First studied in [30], this model of correlated SBMs is a natural generalization of correlated Erdős-Rényi random graphs, where in particular each marginal Erdős-Rényi graph is now replaced by a stochastic block model $\mathcal{S}(n, \frac{\lambda s}{n}, \epsilon)$. This model was then studied in [43, 46, 20, 54, 8, 48, 9] with an emphasis on the interplay between community structure and graph matching. Given two graphs (A,B) on the vertex set $[n]$, our goal is to study the following hypothesis testing problem: determine whether (A,B) is sampled from $\mathbb{P}_n$ or $\mathbb{Q}_n$, where $\mathbb{Q}_n$ is the distribution of two independent stochastic block models $\mathcal{S}(n, \frac{\lambda s}{n}, \epsilon)$. In particular, in [9] (where the authors considered symmetric SBMs with k communities for $k \geq 2$), the authors considered the problem of testing $\mathbb{P}_n$ against $\widetilde{\mathbb{Q}}_n$, where $\widetilde{\mathbb{Q}}_n$ is the law of two independent Erdős-Rényi graphs $\mathcal{G}(n, \frac{\lambda s}{n})$. They showed that in the subcritical regime $\epsilon^2 \lambda s < 1$ where community recovery in a single graph is (information-theoretically when $k=2$) impossible [37], there is evidence suggesting that no efficient algorithm can distinguish $\mathbb{P}_n$ and $\widetilde{\mathbb{Q}}_n$ provided $s < \sqrt{\alpha}$, where α is Otter's threshold. In addition, [37, Theorem 2] implies that when $k=2$ (as we focus on in this paper) $\widetilde{\mathbb{Q}}_n$ and $\mathbb{Q}_n$ are mutually contiguous in the subcritical regime (although their result focuses on a single graph, extending it to two *independent* graphs is straightforward). Altogether, there are evidences suggesting that no efficient algorithm can distinguish any two measures from $\{\mathbb{P}_n, \mathbb{Q}_n, \widetilde{\mathbb{Q}}_n\}$ as long as $s < \sqrt{\alpha}$. This aligns with the lower bound established in [12] for the problem of distinguishing a pair of correlated Erdős-Rényi graphs from a pair of independent Erdős-Rényi graphs. Thus, we see that in the subcritical regime, this model belongs to the same algorithmic universality class with the correlated Erdős-Rényi model. On the contrary, in this paper, we focus on the supercritical regime $\epsilon^2 \lambda s > 1$ where efficient community detection and (weak) recovery in each individual graph are possible [39, 35]. Our main result stated below suggests that the testing problem is strictly easier than that in the subcritical regime in the algorithmic sense, at least when the "community signal strength" $\epsilon^2 \lambda s$ is sufficiently large.

THEOREM 1.3. *Suppose $\epsilon \in (0,1)$. There exist three positive constants $\delta = \delta(\epsilon), \Delta = \Delta(\epsilon, \delta)$ and $C = C(\epsilon, \delta, \Delta)$ such that when $\lambda > \Delta$ and $s > \sqrt{\alpha} - \delta$ where $\alpha \approx 0.338$ is Otter's threshold, there exists a statistic $f = f(A, B)$ that can distinguish $\mathbb{P}_n$ and $\mathbb{Q}_n$ with probability tending to 1 as $n \to \infty$. In addition, this statistic f is a polynomial of degree $\omega(\log n)$ that can be approximately computed in time $O(n^C)$, where the approximation can also distinguish $\mathbb{P}_n$ and $\mathbb{Q}_n$ with probability tending to 1.*

Remark 1.4. Our result suggests that at least when the signal for community structure of a single graph is sufficiently strong, since weak community recovery in a single graph is possible on a large fraction of vertices, the recovered community structure can be leveraged to enhance the task of testing network correlation. We would like to emphasize that this is by all means a highly nontrivial task, and we refer to Sections 1.1 and 2.4 for elaborative discussions on the challenges. In addition, while we believe that our result and many of our arguments extend to smaller λ above the Kesten-Stigum threshold $\epsilon^2 \lambda s > 1$ and to the more general setting of correlated stochastic block models with $k > 2$ communities, certain parts of our proof crucially rely on the assumptions of a two-community model and sufficiently large community signal. We believe that establishing similar results in the general case would require substantial additional efforts and thus we leave this goal for future works.

Remark 1.5. In [8], authors seem to tend to feel that in the logarithmic degree regime the correlated SBMs belong to the same algorithmic universality class of sparse correlated Erdős-Rényi model, and thus all inference tasks are computationally impossible when $s < \sqrt{\alpha}$. However, our result suggests a different possibility by demonstrating a new phase transition phenomenon in this model in the constant degree regime.

The main idea behind our algorithm is to construct a polynomial that is a *combination* of two low-degree polynomials, one is from counting self-avoiding/non-backtracking paths (which was widely used in efficient community recovery [38, 39, 27]) and the other is from counting trees (which was used in [34, 33] for correlation detection/graph matching). Inspired by the sum-of-squares hierarchy, the low-degree polynomial method offers a promising framework for establishing computational lower bounds in high-dimensional inference problems. Roughly speaking, to study the computational efficiency of the hypothesis testing problem between two probability measures $\mathbb{P}$ and $\mathbb{Q}$, the idea is to find a low-degree polynomial f (usually with degree $O(\log n)$ where n is the size of data) that separates $\mathbb{P}$ and $\mathbb{Q}$ in a certain sense. The key quantity is the low-degree advantage $\mathsf{Adv}(f) := \frac{\mathbb{E}_{\mathbb{P}}[f]}{\sqrt{\mathbb{E}_{\mathbb{Q}}[f^2]}}$. The low-degree polynomial method suggests that, if the low-degree advantage of f is uniformly bounded for all f with degree $O(\log n)$, then no polynomial-time algorithm can strongly distinguish $\mathbb{P}$ and $\mathbb{Q}$. This approach is appealing partly because it has yielded tight hardness results for a wide range of problems. Prominent examples include detection and recovery problems such as planted clique, planted dense subgraph, community detection, sparse-PCA and tensor-PCA (see [27, 26, 25, 29, 49, 11, 5, 32, 15, 4, 36, 11, 28]), optimization problems such as maximal independent sets in sparse random graphs [17, 52], and constraint satisfaction problems such as random k-SAT [6].

Another appealing feature of low-degree polynomial method is its ability to provide valuable guidance for designing efficient algorithms. Specifically, if there exists a polynomial $\deg(f) = O(\log n)$ such that $\mathsf{Adv}(f) = \omega(1)$, then it strongly suggests the existence of an efficient algorithm that can distinguish $\mathbb{P}$ and $\mathbb{Q}$ using f. Since this approach has proven successful in both community recovery [27] and graph matching [34, 33], it is tempting to find a low-degree polynomial with a large low-degree advantage. However, our approach is more subtle as the polynomial we construct is a *combination* of two low-degree polynomials: one for community recovery and the other for correlation detection. As a result, unlike in the standard low-degree framework, the polynomial we construct has a degree of $\omega(\log n)$ but can still be computed efficiently.

1.1 Key challenges and algorithmic innovations The major algorithmic innovation of this work is a new construction based on *subgraph counts*. To this end, we first briefly explain how the threshold $\sqrt{\alpha}$ emerges in the detection problem in correlated Erdős-Rényi models. Suppose that marginally both A and B are Erdős-Rényi graphs $\mathcal{G}(n, \frac{\lambda s}{n})$ and their edge correlation is given by $\mathrm{Cov}(A_{\pi_*(i), \pi_*(j)}, B_{i,j}) = [1 + o(1)]s$. A natural attempt is to count the (centered) graphs $\mathbf{H}$ in both A and B for each unlabeled graph $\mathbf{H}$, i.e., we consider the statistics

$$g_{\mathbf{H}} = \Big(\sum_{S \cong \mathbf{H}} \prod_{(i,j) \in E(S)} (A_{i,j} - \tfrac{\lambda s}{n}) \Big) \Big(\sum_{K \cong \mathbf{H}} \prod_{(i,j) \in E(K)} (B_{i,j} - \tfrac{\lambda s}{n}) \Big) ,$$

where $S \cong \mathbf{H}$ denotes that we will sum over all subgraphs of the complete graph $\mathcal{K}_n$ that are isomorphic to $\mathbf{H}$, and $E(S)$ is the edge set of S. A standard calculation yields that (under the law of correlated Erdős-Rényi graphs) the expectation of $g_{\mathbf{H}}$ with suitable standardization is given by $s^{|E(\mathbf{H})|}$. In addition, the signal contained in each $g_{\mathbf{H}}$ is approximately non-repeating, which implies that $g_{\mathbf{H}}$'s are a family of "almost orthogonal" statistics. Thus, if we take our statistic to be a suitable linear combination of $\{g_{\mathbf{H}} : \mathbf{H} \in \mathcal{H}\}$ where $\mathcal{H}$ is a family of unlabeled graphs, the total signal-to-noise ratio is given by

$$\sum_{\mathbf{H} \in \mathcal{H}} s^{2|E(\mathbf{H})|} .$$

The key idea is that since marginally A and B are sparse graphs, we expect a "proper" choice of $\mathcal{H}$ should guarantee that all $\mathbf{H} \in \mathcal{H}$ has low edge density, and thus it can be shown that under this restriction, the cardinality of $|\mathcal{H}|$ should be approximately the same as the cardinality of unlabeled trees, which grows at rate α^{-1} [44]. This informally suggests that $s = \sqrt{\alpha}$ is the "correct" threshold of the detection problem in correlated Erdős-Rényi graphs.

We now focus on our case where marginally A and B are supercritical SBMs such that weak community recovery in A and B is possible. The initial motivation of our result is the following (albeit simple) observation: assuming that all the community labelings in A and B (we denote them as σ_A and σ_B) are known to us, we can count a larger class $\mathcal{H}'$ which consists of all unlabeled trees *decorated with* $\{-1, +1\}$. This is because when counting subgraphs S, K in A, B respectively we can regard the community label σ_A, σ_B as a "decoration" of S, K, and the information from the community labels guarantees that if S and K are isomorphic graphs but have non-isomorphic label decorations, the signal contained in counting S and K are nearly orthogonal. Thus, a naive attempt is to first run the weak community recovery algorithm in both A and B which produces an estimator $(\widehat{\sigma}_A, \widehat{\sigma}_B)$ of (σ_A, σ_B), and then counting $\mathcal{H}'$ in A and B simply by viewing $(\widehat{\sigma}_A, \widehat{\sigma}_B)$ as its "decorations". However, there are several obstacles that impede the rigorous analysis of this naive approach. Firstly, the subgraph counting procedure on $\mathcal{H}'$ is highly fragile and thus we cannot simply ignore those vertices i such that $\widehat{\sigma}_A(i) \neq \sigma_A(i)$ or $\widehat{\sigma}_B(i) \neq \sigma_B(i)$ (which constitutes a small but positive fraction of all vertices). What is more important, the community recovery step will have a strong influence on later subgraph counts, which means that $(\widehat{\sigma}_A, \widehat{\sigma}_B)$ is strongly correlated with $g_{\mathbf{H}}(A, B)$. In fact, it seems that conditioning on $(\widehat{\sigma}_A, \widehat{\sigma}_B)$ will force enumerations of certain types of non-isomorphic graphs to have correlations tending to 1 with each other.

To address these issues, instead of running these two procedures separately, we find a proper way that perform these two procedures *simultaneously*. The key conceptual innovation in our work is to construct a family of subgraphs that *combine* the subgraph counts relevant to both tasks. To be more precise, our approach involves counting a carefully chosen family of unlabeled multigraphs, which (informally speaking) is formed by attaching non-backtracking paths to an unlabeled tree; the flexibility to choose the place of attachment enrich the enumeration of such multigraphs and helps us to gain an exponential factor compared to the number of all unlabeled trees. Furthermore, we can approximate the count of such multigraphs with sufficient accuracy in polynomial time by leveraging the method of color coding [2, 27]. The major difficulty lies in analyzing the behavior of our statistic. While centering the adjacency matrices and counting signed subgraphs are helpful, we still have excessive correlations among different subgraph counts and these correlations are hard to control. To resolve this challenge, another innovation of this work is to count a special family $\mathcal{H}$ of all such multigraphs, which we call *decorated trees*; see Definition 2.3 for the formal definition. As discussed in Section 2.4, this choice plays a crucial role in curbing the undesired correlation between different subgraph counts. Moreover, despite the fact that we pose several structural requirements on such decorated trees, by choosing the parameters appropriately, we can nevertheless ensure that the number of such decorated trees still grows exponentially faster than that of unlabeled trees.

Finally, we point out that our work is significantly more complicated than the detection and matching algorithms for correlated Erdős-Rényi graphs [34, 33] and their extension to correlated SBMs [8] (which are based on counting a specific family of unlabeled trees) at a technical level. Indeed, a key simplification in [34, 33] comes from the fact that the edges within each graph are (marginally) independent, which allows for straightforward cancellations in many parts of their analysis. In contrast, our setting involves edges that are correlated through latent community labels. Thus, we need to carefully analyze the correlation from the latent matching and the latent community labels simultaneously. The authors of [8] made initial progress on this challenge; however, their analysis focus on the logarithmic degree regime, where the strong community signal enables a de-correlation technique on tree counts. In our constant-degree regime, the community signal is inherently weaker, necessitating a far more delicate approach. In addition, as we hope to break the tree-counting threshold, our key innovation is to construct a family of unlabeled graphs that grow faster than unlabeled trees and allow us to leverage the community structure more effectively. Our choice of decorated trees is substantially more intricate than their choice of trees, and our decorated trees have size $\omega(\log n)$, which introduces a significantly larger number of potential overlapping structures between different decorated trees. Handling these overlapping structures poses a major technical challenge, as it requires highly delicate treatment to control their enumeration and correlation. We provide a brief overview of how we control such overlapping structures in Section 2.4.

1.2 Discussions and perspectives Our work reiterates a number of future research directions as we discuss below.

Extension to general block models. Although we believe that our results and many of our arguments can be extended to the more general setting of correlated stochastic block models with $k > 2$ communities, certain parts of our proof rely crucially on the assumption of a two-community model. We expect that establishing analogous results in the general case would require substantial additional effort and thus we leave this direction for future work.

The partial recovery problem. A natural next step is to achieve partial recovery below the Otter threshold by combining our approach with the methods of [33] (which solves the partial recover problem in correlated Erdős-Rényi graphs by counting a specific family of unlabeled trees called chandeliers). A promising strategy would be to adapt these methods by counting decorated chandeliers in the correlated SBM setting. However, additional technical challenges arise in the analysis of this approach due to the more complicated overlapping structures, so we leave this goal for future work.

Settling the sharp computational threshold. We conjecture that, at least for the correlated SBMs with two symmetric communities $\mathcal{S}(n, \frac{\lambda}{n}; \epsilon, s)$, the exact computational threshold depends only on κ, where $\kappa = \kappa(\lambda, \epsilon, s)$ is the optimal fraction of vertices whose community labels can be recovered in a single SBM $\mathcal{S}(n, \frac{\lambda s}{n}; \epsilon)$.[1] Proving this conjecture seems to require ideas beyond the scope of our current work, so we also leave it for future work.

1.3 Notation In this subsection, we record a list of notation that we shall use throughout the paper. Let $\mathrm{Ber}(p)$ be the Bernoulli distribution with parameter p, and let $\mathrm{Binom}(n, p)$ be the binomial distribution with n trials and success probability p. Recall that $\mathbb{P} = \mathbb{P}_n, \mathbb{Q} = \mathbb{Q}_n$ are two probability measures on a pair of random graphs on $[n] = \{1, \ldots, n\}$. Denote $\mathfrak{S}_n$ the set of permutations in $[n]$ and $\mu = \mu_n$ the uniform distribution on $\mathfrak{S}_n$. In addition, denote by $\nu = \nu_n$ the uniform distribution on $\{-1, +1\}^n$. We will use the following notation conventions for graphs.

- *Labeled graphs.* Denote by $\mathcal{K}_n$ the complete graph with vertex set $[n]$ and edge set U_n. For any graph H, let $V(H)$ denote the vertex set of H and let $E(H)$ denote the edge set of H. We say H is a subgraph of G, denoted by $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We say $\varphi : V(H) \to V(S)$ is an injection, if for all $(i, j) \in E(H)$ we have $(\varphi(i), \varphi(j)) \in E(S)$. For $H, S \subset \mathcal{K}_n$, denote by $H \cap S$ the graph with vertex set given by $V(H) \cap V(S)$ and edge set given by $E(H) \cap E(S)$, and denote by $S \cup H$ the graph with vertex set given by $V(H) \cup V(S)$ and edge set given by $E(H) \cup E(S)$. For any graph H, denote the excess of H by $\tau(H) = |E(H)| - |V(H)|$. Given $u \in V(H)$, define $\mathsf{Nei}_H(u)$ to be the set of neighbors of u in H. For two vertices $u, v \in V(H)$, we define $\mathsf{Dist}_H(u, v)$ to be their graph distance. Denote the diameter of a connected graph H by $\mathsf{Diam}(H) = \max_{u,v \in V(H)} \mathsf{Dist}_H(u, v)$.
- *Graph isomorphisms and unlabeled graphs.* Two graphs H and H' are isomorphic, denoted by $H \cong H'$, if there exists a bijection $\pi : V(H) \to V(H')$ such that $(\pi(u), \pi(v)) \in E(H')$ if and only if $(u, v) \in E(H)$. Denote by $\mathcal{H}$ the isomorphism class of graphs; it is customary to refer to these isomorphic classes as unlabeled graphs. Let $\mathrm{Aut}(H)$ be the number of automorphisms of H (graph isomorphisms to itself). For any graph H, define $\mathsf{Fix}(H) = \{u \in V(H) : \varphi(u) = u, \forall \varphi \in \mathrm{Aut}(H)\}$.
- *Induced subgraphs.* For a graph $H = (V, E)$ and a subset $A \subset V$, define $H_A = (A, E_A)$ to be the induced subgraph of H in A, where $E_A = \{(u, v) \in E : u, v \in A\}$. Also, define $H_{\setminus A} = (V, E_{\setminus A})$ to be the subgraph of H obtained by deleting all edges with both endpoints in A. Note that $E_A \cup E_{\setminus A} = E$.
- *Isolated vertices.* For $u \in V(H)$, we say u is an isolated vertex of H if there is no edge in $E(H)$ incident to u. Denote $\mathcal{I}(H)$ as the set of isolated vertices of H.

[1] We note that the precise value of κ was determined only for sufficiently large λ [38].

- *Paths, self-avoiding paths and non-backtracking paths.* We say a subgraph $H \subset \mathcal{K}_n$ is a path with endpoints u, v (possibly with $u = v$), if there exist $w_1, \ldots, w_m \in [n] \neq u, v$ such that $V(H) = \{u, v, w_1, \ldots, w_m\}$ and $E(H) = \{(u, w_1), (w_1, w_2) \ldots, (w_m, v)\}$ (we allow the occurrence of multiple vertices or edges). We say H is a self-avoiding path if $w_0, w_1, \ldots, w_m, w_{m+1}$ are distinct (where we denote $w_0 = u$ and $w_{m+1} = v$), and we say H is a non-backtracking path if $w_{i+1} \neq w_{i-1}$ for $1 \leq i \leq m$. Denote $\mathrm{EndP}(P)$ as the set of endpoints of a path P.

- *Cycles and independent cycles.* We say a subgraph H is an m-cycle if $V(H) = \{v_1, \ldots, v_m\}$ and $E(H) = \{(v_1, v_2), \ldots, (v_{m-1}, v_m), (v_m, v_1)\}$. For a subgraph $K \subset H$, we say K is an independent m-cycle of H, if K is an m-cycle and no edge in $E(H) \setminus E(K)$ is incident to $V(K)$. Denote $\mathcal{C}_m(H)$ as the set of independent m-cycles of H. For $H \subset S$, we define $\mathfrak{C}_m(S, H)$ to be the set of independent m-cycles in S whose vertex set is disjoint from $V(H)$.

- *Leaves.* A vertex $u \in V(H)$ is called a leaf of H, if the degree of u in H is 1; denote $\mathcal{L}(H)$ as the set of leaves of H.

- *Trees and rooted trees.* We say a graph $T = (V(T), E(T))$ is a tree, if T is connected and has no cycles. We say a pair $(T, \mathfrak{R}(T))$ is a rooted tree with root $\mathfrak{R}(T)$, if T is a tree and $\mathfrak{R}(T) \in V(T)$. For a rooted tree T and $u \in V(T)$, we define $\mathsf{Dep}_T(u) = \mathsf{Dist}_T(\mathfrak{R}(T), u)$ to be the depth of u in T. For $u, v \in V(T)$, denote by $u \to v$ (or equivalently, $v \leftarrow u$) if v is the child of u, and denote by $u \hookrightarrow v$ (or equivalently, $v \hookleftarrow u$) if v is a descendant of u.

- *Descendant tree.* For a rooted tree $(T, \mathfrak{R}(T))$ and $u \in V(T)$, denote $\mathsf{Des}_T(u)$ the descendant tree of u in T, that is, the rooted tree with root u and with vertices given by the descendants of u in the rooted tree T. We call $\mathsf{Des}_T(u)$ the descendant tree of u in T. For rooted trees T, T' we write $T \hookrightarrow T'$ (or equivalently, $T' \hookleftarrow T$) if T' is a descendant tree of T.

- *Arm-path.* For a rooted tree $(T, \mathfrak{R}(T))$, we say a self-avoiding path $P = (u_1, \ldots, u_m)$ is an arm-path of T starting from u_1, if $P = \mathsf{Des}_T(u_1)$. Given a rooted unlabeled tree $\mathbf{T}$ and $\mathbf{u} \in V(\mathbf{T})$, denote by $\mathbf{T} \oplus \mathcal{L}_{\mathbf{u}}^{(x)}$ the graph obtained by sticking an arm-path with length x starting from $\mathbf{u}$.

- *Multigraphs.* We say S is a multigraph, if $S = \big(V(S), E(S), \{E(S)_{i,j} : (i,j) \in E(S)\}\big)$ where $E(S)_{i,j} \in \mathbb{N}$ denotes the multiplicity of the edge (i,j). For a multigraph S, denote $\widetilde{S} \subset \mathcal{K}_n$ with $E(\widetilde{S}) = E(S)$ and $V(\widetilde{S}) = V(S)$ to be the simple graph (ignore the multiplicity) corresponding to S. For two multigraphs H, S, we say $H \subset S$ if $V(H) \subset V(S)$ and $E(H)_{i,j} \leq E(S)_{i,j}$ for all $i, j \in V(H)$. Also denote $S \cup H$ to be the multigraph with
$$
\begin{aligned}
V(S \cup H) = V(S) \cup V(H)\,,\ E(S \cup H) = E(S) \cup E(H) \\
\text{and } E(S \cup H)_{i,j} = E(S)_{i,j} + E(H)_{i,j}\,.
\end{aligned}
$$
In addition, for a multigraph S, denote $\mathcal{L}(S)$ to be the set of leaves that have degree 1 in S (counting multiplicity).

For two real numbers a and b, we let $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. For two sets A and B, we define $A \sqcup B$ to be the disjoint union of A and B (so the notation $\sqcup$ only applies when A, B are disjoint). We use standard asymptotic notations: for two sequences a_n and b_n of positive numbers, we write $a_n = O(b_n)$, if $a_n < Cb_n$ for an absolute constant C and for all n (similarly we use the notation O_h if the constant C is not absolute but depends only on h); we write $a_n = \Omega(b_n)$, if $b_n = O(a_n)$; we write $a_n = \Theta(b_n)$, if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$; we write $a_n = o(b_n)$ or $b_n = \omega(a_n)$, if $a_n / b_n \to 0$ as $n \to \infty$. In addition, we write $a_n \circeq b_n$ if $a_n = [1 + o(1)] b_n$. For a set A, we will use both $\#\mathsf{A}$ and $|\mathsf{A}|$ to denote its cardinality. We denote $\mathsf{id} \in \mathfrak{S}_n$ as the identity map.

1.4 Organization of this paper The rest of this paper is organized as follows: in Section 2 we describe the construction of decorated trees and the statistic $f(A, B)$, and state our main result (as presented in Theorem 2.12). In Section 3 we bound the first moment of the statistic, and in Section 4 we control the variance of the statistic. In Section 5 we describe an efficient algorithm to approximate the statistic in polynomial running time. Several auxiliary results are moved to the appendix to ensure a smooth flow of presentation.

2 Main results and discussions

2.1 On the assumptions of parameters Before presenting the statistic $f(A,B)$, we first show how to choose δ, Δ in Theorem 1.3 and make several assumptions on various auxiliary parameters that will be used throughout the paper. To this end, we first need the following results on several enumeration problems regarding unlabeled trees which were established in [44].

LEMMA 2.1. *Denote $\mathcal{V}_N$ as the set of unlabeled trees on N vertices and denote $\mathcal{R}_N$ the set of unlabeled rooted trees on N vertices. Then there exists a deterministic constant $1 > C_0 > 0.01$ and a deterministic constant $1 > C_1 \geq 0.81$ such that as $N \to \infty$*

$$|\mathcal{V}_N| \doteq C_0 \alpha^{-N}/N^{2.5}\,, \quad |\mathcal{R}_N| \doteq C_0 C_1 \alpha^{-N}/N^{1.5}\,. \tag{2.1}$$

where (again and throughout the paper) $\alpha \approx 0.338$ is the Otter's constant.

Provided with Lemma 2.1, we can choose a sufficiently large constant $M \geq 100$ such that the following holds: for all $k \geq M$

$$\alpha^{-k}/k^{2.5} \geq |\mathcal{V}_k| \geq \alpha^{-k}/100k^{2.5} \text{ and } \alpha^{-k}/k^{1.5} \geq |\mathcal{R}_k| \geq \alpha^{-k}/200k^{1.5}\,. \tag{2.2}$$

In addition, we choose a sufficiently small constant $\iota = \iota(M, \epsilon)$ such that

$$\log\log\log\log(\iota^{-1}) > \max\left\{M, \tfrac{1}{\epsilon}, \tfrac{1}{\log(\epsilon^{-1})}\right\}. \tag{2.3}$$

Also we choose a sufficiently small $\delta = \delta(M, \epsilon, \iota)$ and a sufficiently large $\Delta = \Delta(M, \epsilon, \iota, \delta)$ such that

$$\alpha^{-1}(\sqrt{\alpha} - \delta)^2 e^{\iota} > 1 \text{ and } \Delta > 100\epsilon^{-4}\iota^{-4}\,. \tag{2.4}$$

We now assume that throughout the rest part of the paper

$$\lambda > \Delta \text{ and } \sqrt{\alpha} - \delta < s\,. \tag{2.5}$$

Since $\epsilon^2 \lambda s > 100$ according to (2.5) and (2.4), we can also choose $\ell = \ell_n \leq \log n$ and choose $\aleph = \aleph_n$ such that $\iota\aleph, \frac{\aleph-1}{2} \in \mathbb{N}$ and

$$(\epsilon^2 \lambda s)^{\ell} > n^4 \text{ and } \omega(1) = \aleph_n = o(\log\log\log n)\,. \tag{2.6}$$

2.2 Construction of decorated trees We first introduce the notion of decorated trees and then discuss the specific selection of trees and their associated vertex pairings. We refer readers to Section 2.4 for underlying intuitions behind these constructions.

DEFINITION 2.2. *Given a tree T with $\aleph$ vertices, if $\xi = \{(u_1, v_1), \ldots, (u_{\iota\aleph}, v_{\iota\aleph})\}$ such that $u_1, \ldots, u_{\iota\aleph}$, $v_1, \ldots, v_{\iota\aleph}$ are distinct vertices in $V(T)$, we say ξ is a pairing of T and we define $\mathsf{Vert}(\xi) = \{u_1, \ldots, u_{\iota\aleph}, v_1, \ldots, v_{\iota\aleph}\}$.*

We can now introduce decorated trees.

DEFINITION 2.3. *For $\mathcal{T}_\aleph \subset \mathcal{V}_\aleph$, define $\mathcal{S}(\mathcal{T}_\aleph) = \{\mathcal{S}(\mathbf{T}) : \mathbf{T} \in \mathcal{T}_\aleph\}$ where*

$$\mathcal{S}(\mathbf{T}) = \left\{\mathtt{W}_1, \ldots, \mathtt{W}_\mathtt{M}\right\} \text{ with } \mathtt{M} = \exp\left(\iota(\log\log(\iota^{-1}))^4 \aleph\right) \text{ and } \mathtt{W}_i\text{'s are pairings of } \mathbf{T}\,.$$

We define the family of decorated trees *associated with $\mathcal{T}_\aleph$ and $\mathcal{S}(\mathcal{T}_\aleph)$, denoted as $\mathcal{H} = \mathcal{H}(\aleph, \iota, M; \mathcal{T}_\aleph, \mathcal{S}(\mathcal{T}_\aleph))$, to be the collection of all $\mathbf{H} = \big(\mathsf{T}(\mathbf{H}), \mathsf{P}(\mathbf{H})\big)$ satisfying the following conditions:*

(1) $\mathsf{T}(\mathbf{H})$ *is an unlabeled $\aleph$-tree in $\mathcal{T}_\aleph$;*

(2) $\mathsf{P}(\mathbf{H}) = \{(u_1, v_1), \ldots, (u_{\iota\aleph}, v_{\iota\aleph})\} \in \mathcal{S}(\mathsf{T}(\mathbf{H}))$.

In addition, we choose two random subsets $\mathcal{J}_A, \mathcal{J}_B \subset [n]$ with $|\mathcal{J}_A| = |\mathcal{J}_B| = \iota^2 n$. We say a multigraph $S \Vdash_A \mathbf{H}$, if (counting multiplicity of edges) S can be decomposed into a tree $\mathsf{T}(S)$ and $\iota\aleph$ self-avoiding paths $\mathsf{L}_1(S), \ldots, \mathsf{L}_{\iota\aleph}(S)$ each with length ℓ such that the following conditions hold (see Figure 2.1 for an illustration):

(1) $V(\mathsf{T}(S)) \subset [n] \setminus \mathcal{J}_A$*;*

(2) $\mathrm{EndP}(\mathsf{L}_i(S)) = \{u_i, v_i\}$ *where* $u_i, v_i \in V(\mathsf{T}(S))$ *and the neighbors of* u_i, v_i *in* $\mathsf{L}_i(S)$ *lie in* $\mathcal{J}_A$*;*

(3) Denoting $\mathsf{P}(S) = \{(u_1, v_1), \dots, (u_{\iota\aleph}, v_{\iota\aleph})\}$*, there exists a bijection* $\varphi : V(\mathsf{T}(S)) \to V(\mathsf{T}(\mathbf{H}))$ *such that* φ *maps* $\mathsf{T}(S)$ *to* $\mathsf{T}(\mathbf{H})$ *and maps* $\mathsf{P}(S)$ *to* $\mathsf{P}(\mathbf{H})$*.*

We define $S \Vdash_B \mathbf{H}$ *with respect to* $\mathcal{J}_B$ *in a similar manner.*

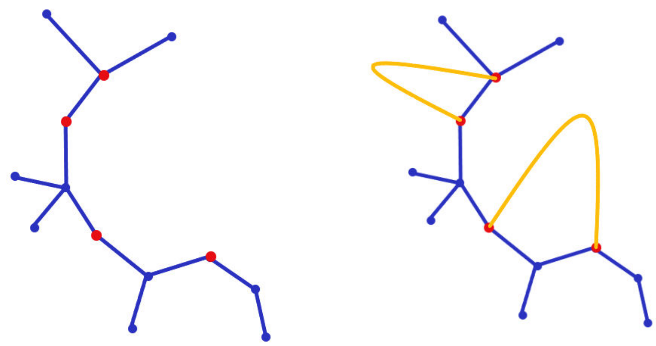


Figure 2.1: An example of a decorated tree. Left: example of an element $\mathbf{H} \in \mathcal{H}$, where the blue part is the tree $\mathsf{T}(\mathbf{H})$ and the red vertices constitute the pairing $\mathsf{P}(\mathbf{H})$; right: example of a multigraph $S \Vdash_A \mathbf{H}$ where the yellow parts are the self-avoiding paths attached to $\mathsf{T}(S)$.

Given $S_1 \Vdash_A \mathbf{H}, S_2 \Vdash_B \mathbf{H}$ where $\mathbf{H}$ is a decorated tree, we may write $\phi_{S_1,S_2}(A,B) = \phi_{S_1}(A)\phi_{S_2}(B)$, where for $S \Vdash \mathbf{H}$

$$\phi_S(X) = \prod_{(i,j)\in E(\mathsf{T}(S))} \frac{X_{i,j} - \frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1 - \frac{\lambda s}{n})}} \cdot \prod_{k=1}^{\iota\aleph} \prod_{(i,j)\in E(\mathsf{L}_k(S))} \frac{X_{i,j} - \frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1 - \frac{\lambda s}{n})}} \,. \tag{2.7}$$

We now define our statistic as (recall the definition of ℓ in (2.6))

$$f = f(A,B) := \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{\aleph-1}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{\aleph+\ell\iota\aleph}} \sum_{\substack{S_1 \Vdash_A \mathbf{H} \\ S_2 \Vdash_B \mathbf{H}}} \phi_{S_1,S_2}(A,B) \,. \tag{2.8}$$

2.3 Choices of trees and pairings Although it might seem natural to choose $\mathcal{T}_\aleph = \mathcal{V}_\aleph$ and $\mathcal{S}(\mathbf{T})$ as the set of all pairings of $\mathbf{T}$, this approach poses several technical challenges when analyzing $f(A,B)$. To address these difficulties, we restrict our choices of $\mathcal{T}_\aleph$ and $\mathcal{S}(\mathbf{T})$ to unlabeled trees and pairings with some desired properties. Before precisely describing our choices of trees and pairings, we introduce some additional notations. Recall that we use $\mathcal{R}_\aleph$ to denote the set of all unlabeled rooted trees with $\aleph$ vertices.

Definition 2.4. *For any rooted trees* $\mathbf{T}, \mathbf{T}'$*, we denote* $\mathbf{T} \sim \mathbf{T}'$ *if there exist*

$$\begin{aligned} &k \le \log^{-1}(\iota^{-1}) \cdot |V(\mathbf{T})|, k' \le \log^{-1}(\iota^{-1}) \cdot |V(\mathbf{T}')| \,, \\ &\mathbf{u}_1, \dots, \mathbf{u}_k \in V(\mathbf{T}) \setminus \mathcal{L}(\mathbf{T}), \mathbf{u}'_1, \dots, \mathbf{u}'_{k'} \in V(\mathbf{T}') \setminus \mathcal{L}(\mathbf{T}') \,, \\ &0 \le x_1, \dots, x_k, x'_1, \dots, x'_{k'} \le \log^2(\iota^{-1}) \end{aligned}$$

such that (recall the definition of arm-path in Section 1.3)

$$\mathbf{T} \oplus \mathcal{L}^{(x_1)}_{\mathbf{u}_1} \oplus \dots \oplus \mathcal{L}^{(x_k)}_{\mathbf{u}_k} \cong \mathbf{T}' \oplus \mathcal{L}^{(x'_1)}_{\mathbf{u}'_1} \oplus \dots \oplus \mathcal{L}^{(x'_{k'})}_{\mathbf{u}'_{k'}} \,.$$

Definition 2.5. *Given a rooted tree* $\mathbf{T}$*, define* $\mathbf{T}_\iota$ *to be the induced subtree of* $\mathbf{T}$ *where the vertex set of* $\mathbf{T}_\iota$ *is defined as*

$$V(\iota) = \left\{ \mathbf{v} \in V(\mathbf{T}) : |V(\mathsf{Des}_{\mathbf{T}}(\mathbf{v}))| \ge \log^2(\iota^{-1}) \right\}.$$

It is straightforward to check that $\mathbf{u} \hookrightarrow \mathbf{v}$ *and* $\mathbf{v} \in V(\iota)$ *imply* $\mathbf{u} \in V(\iota)$*.*

Definition 2.6. *Define* $\widetilde{\mathcal{T}}_\aleph$ *to be the collection of* $\mathbf{T} \in \mathcal{R}_\aleph$ *satisfying the following conditions.*

(1) For every $u \in V(\mathbf{T})$*,* $\mathsf{Deg}_{\mathbf{T}}(u) \le \log^2(\iota^{-1})$*.*

(2) The tree $\mathbf{T}$ *contains no arm-path with length at least* $\log^2(\iota^{-1})$*.*

(3) $|V(\mathbf{T}_\iota)| \ge \frac{\aleph}{\log^4(\iota^{-1})}$*.*

(4) For any descendant trees $\mathbf{T} \hookrightarrow \mathbf{S}, \mathbf{S}'$ *such that* $|V(\mathbf{S})|, |V(\mathbf{S}')| \ge \log^2(\iota^{-1})$*, if* $\mathbf{S} \ne \mathbf{S}'$ *and* $\mathbf{S}, \mathbf{S}'$ *share the same parent in* $\mathbf{T}$*, then* $\mathbf{S} \not\sim \mathbf{S}'$*.*

(5) The root $\mathfrak{R}(\mathbf{T})$ *has exactly two children trees* $\mathbf{T}_1, \mathbf{T}_2$ *with* $V(\mathbf{T}_1) = \lfloor \frac{\aleph-1}{2} \rfloor, V(\mathbf{T}_2) = \lceil \frac{\aleph-1}{2} \rceil$ *respectively such that for all* $\mathbf{T}_2 \hookrightarrow \mathbf{S}$ *we have* $\mathbf{T}_1 \not\sim \mathbf{S}$ *and for all* $\mathbf{T}_1 \hookrightarrow \mathbf{S}'$ *we have* $\mathbf{T}_2 \not\sim \mathbf{S}'$*.*

The following result provides an estimate for the cardinality of $\widetilde{\mathcal{T}}_\aleph$.

Theorem 2.7. *We have*

$$|\widetilde{\mathcal{T}}_\aleph| \ge \alpha^{-\aleph} \cdot \exp\left\{ -10 e^{-\frac{1}{10}\log^2(\iota^{-1})} \aleph \right\}.$$

Thus, denoting by $\mathcal{T}_\aleph$ *the set of* $\mathbf{T} \in \mathcal{V}_\aleph$ *such that there exists a rooted tree* $(\mathbf{T}, \mathbf{u}) \in \widetilde{\mathcal{T}}_\aleph$*, we have*

$$|\mathcal{T}_\aleph| \ge (\alpha + o(1))^{-\aleph} \cdot \exp\left\{ -10 e^{-\frac{1}{10}\log^2(\iota^{-1})} \aleph \right\}. \tag{2.9}$$

The proof of Theorem 2.7 is postponed to Section B.1 of the appendix. Now we discuss the choice of vertex subsets.

Theorem 2.8. *For all* $\mathbf{T} \in \widetilde{\mathcal{T}}_\aleph$*, we can choose* $W_{\mathtt{1}}(\mathbf{T}), \dots, W_{\mathtt{M}}(\mathbf{T}) \subset V(\mathbf{T})$ *to be pairings of* $\mathbf{T}$ *with* $\mathtt{M} = \exp\left(\iota(\log\log(\iota^{-1}))^4 \aleph\right)$ *such that the following conditions hold:*

(1) $\mathsf{Vert}(W_{\mathtt{i}}) \subset V(\mathbf{T}_\iota) \setminus \{\mathfrak{R}(\mathbf{T})\}$ *for all* $\mathtt{1} \le \mathtt{i} \le \mathtt{M}$*.*

(2) For all $\mathtt{1} \le \mathtt{i} \le \mathtt{M}$ *and all descendant trees* $\mathbf{T}' \hookrightarrow \mathbf{T}$ *with* $V(\mathbf{T}') \ge \log^2(\iota^{-1})$*, we have*

$$\left|\mathsf{Vert}(W_{\mathtt{i}}) \cap V(\mathbf{T}')\right| \le \log^{-1}(\iota^{-1}) \cdot |V(\mathbf{T}')|.$$

(3) $|\mathsf{Vert}(W_{\mathtt{i}}) \triangle \mathsf{Vert}(W_{\mathtt{j}})| \ge \frac{\iota\aleph}{2}$ *for all* $\mathtt{i} \ne \mathtt{j}, \mathtt{1} \le \mathtt{i}, \mathtt{j} \le \mathtt{M}$*.*

(4) For all $\mathtt{1} \le \mathtt{i} \le \mathtt{M}$*, we have* $(\log\log(\iota^{-1}))^{10} \le \mathsf{Dist}_{\mathbf{T}}(\mathbf{u}, \mathbf{v}) \le 2(\log\log(\iota^{-1}))^{10}$ *for each* $(\mathbf{u}, \mathbf{v}) \in W_{\mathtt{i}}$ *and* $\mathsf{Dist}_{\mathbf{T}}(\mathbf{u}, \mathbf{v}) \ge 10(\log\log(\iota^{-1}))^{10}$ *for each* $\mathbf{u}, \mathbf{v} \in \mathsf{Vert}(W_{\mathtt{i}}), (\mathbf{u}, \mathbf{v}) \notin W_{\mathtt{i}}$*. Also, for all* $\mathtt{1} \le \mathtt{i} < \mathtt{j} \le \mathtt{M}$ *and* $\mathbf{u} \in \mathsf{Vert}(W_{\mathtt{i}}), \mathbf{v} \in \mathsf{Vert}(W_{\mathtt{j}})$ *such that* $\mathbf{u} \ne \mathbf{v}$*, we have* $\mathsf{Dist}_{\mathbf{T}}(\mathbf{u}, \mathbf{v}) \ge (\log\log(\iota^{-1}))^{10}$*.*

The proof of Theorem 2.8 is postponed to Section B.2 of the appendix. We can now state our choice of trees and vertex sets, as incorporated in the following lemma.

Lemma 2.9. *For each* $\mathbf{T} \in \mathcal{T}_\aleph$*, define* $\mathcal{S}(\mathbf{T}) = \{W_{\mathtt{1}}, \dots, W_{\mathtt{M}}\}$ *where* $W_{\mathtt{1}}, \dots, W_{\mathtt{M}}$ *are given as in Theorem 2.8. Define* $\mathcal{H} = \mathcal{H}(\iota, \aleph, M; \mathcal{T}_\aleph, \mathcal{S}(\mathcal{T}_\aleph))$ *as in Definition 2.3. We then have*

$$|\mathcal{H}| \ge \left(\alpha^{-1} \exp\left(\iota(\log\log(\iota^{-1}))^4\right) \right)^{\aleph}.$$

Proof. The result of Lemma 2.9 follows directly from Theorems 2.7 and 2.8. ∎

We now describe some useful properties for our choice of trees and vertex sets.

LEMMA 2.10. *Let $\mathbf{H} \in \mathcal{H}$ and let $S \Vdash_A \mathbf{H}$. We have the following properties:*

(1) For all $\{x_{\mathbf{u}} \geq 0 : \mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))\}$ and all injection $\varphi : \mathsf{T}(\mathbf{H}) \to \mathsf{T}(\mathbf{H}) \oplus (\oplus_{\mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))} \mathcal{L}_{\mathbf{u}}^{(x_{\mathbf{u}})})$, we have φ maps $\mathfrak{R}(\mathsf{T}(\mathbf{H}))$ to $\mathfrak{R}(\mathsf{T}(\mathbf{H}))$.

(2) For all $\{0 \leq x_{\mathbf{u}} \leq \log^2(\iota^{-1}) : \mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))\}$, we have

$$\mathsf{Vert}(\mathsf{P}(\mathbf{H})) \subset V(\mathsf{T}(\mathbf{H})_\iota) \subset \mathsf{Fix}(\mathsf{T}(\mathbf{H}) \oplus (\oplus_{\mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))} \mathcal{L}_{\mathbf{u}}^{(x_{\mathbf{u}})})).$$

(3) Recall in Section 1.3 we denote $\widetilde{S}$ to be the simple graph corresponding to S. We have $\mathcal{L}(\widetilde{S}) = \mathcal{L}(S) \subset \mathcal{L}(\mathsf{T}(S))$.

(4) For all $(\mathbf{u}, \mathbf{v}) \in \mathsf{P}(\mathbf{H})$, we have

$$\#\big\{\mathbf{w} \in V(\mathsf{T}(\mathbf{H})) : \mathsf{Dist}_{\mathsf{T}(\mathbf{H})}(\mathbf{u}, \mathbf{w}) \leq 2(\log\log(\iota^{-1}))^2\big\} \leq \exp\big(4(\log\log(\iota^{-1}))^3\big)\,.$$

(5) For all $(\mathbf{u}, \mathbf{v}) \in \mathsf{P}(\mathbf{H})$, we have

$$\big\{\mathbf{w} \in V(\mathsf{T}(\mathbf{H})) : \mathsf{Dist}_{\mathsf{T}(\mathbf{H})}(\mathbf{u}, \mathbf{w}) \leq 2(\log\log(\iota^{-1}))^2\big\} \cap \mathsf{Vert}(\mathsf{P}(\mathbf{H})) = \{\mathbf{u}, \mathbf{v}\}\,.$$

(6) For all $\{x_{\mathbf{u}} \geq 0 : \mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))\}$, we have

$$\frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}) \oplus (\oplus_{\mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))} \mathcal{L}_{\mathbf{u}}^{(x_{\mathbf{u}})}))}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))} \leq \exp\big(4\iota \log\log(\iota^{-1})\aleph\big)\,.$$

(7) The number of injections from $\mathsf{T}(\mathbf{H})$ to $\mathsf{T}(\mathbf{H}) \oplus (\oplus_{\mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))} \mathcal{L}_{\mathbf{u}}^{(x_{\mathbf{u}})})$ equals

$$\frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}) \oplus (\oplus_{\mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))} \mathcal{L}_{\mathbf{u}}^{(x_{\mathbf{u}})}))}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))}\,.$$

Proof. Denote $\mathbf{T} = \mathsf{T}(\mathbf{H})$ and $\mathbf{T}^{\oplus} = \mathbf{T} \oplus (\oplus_{\mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))} \mathcal{L}_{\mathbf{u}}^{(x_{\mathbf{u}})})$ in the following proof. We first show Property (1). Since $\mathbf{T}$ does not have any arm-path with length at least $\log^2(\iota^{-1})$, we have σ must map $\mathbf{T}$ to $\mathbf{T} \oplus (\oplus_{\mathbf{v} \in \mathsf{P}(\mathbf{H})} \mathcal{L}_{\mathbf{v}}^{(x_{\mathbf{v}} \wedge \log^2(\iota^{-1}))})$. Thus, without loss of generality we may assume that $x_{\mathbf{v}} \leq \log^2(\iota^{-1})$ for all $\mathbf{v}$. Note that in this case if σ maps $\mathfrak{R}(\mathbf{T})$ to another vertex, then either $\mathbf{T}_1$ is mapped into a descendant tree of $\mathbf{T}_2 \oplus (\oplus_{\mathbf{v} \in \mathsf{P}(\mathbf{H}) \cap V(\mathbf{T}_2)} \mathcal{L}_{\mathbf{v}}^{(x_{\mathbf{v}})})$, or $\mathbf{T}_2$ is mapped into a descendant tree of $\mathbf{T}_1 \oplus (\oplus_{\mathbf{v} \in \mathsf{P}(\mathbf{H}) \cap V(\mathbf{T}_1)} \mathcal{L}_{\mathbf{v}}^{(x_{\mathbf{v}})})$. This implies that $\mathbf{T}_1$ has $\sim$ relation with a subtree of $\mathbf{T}_2$ or $\mathbf{T}_2$ has $\sim$ relation with a subtree of $\mathbf{T}_1$, contradicting to Item (5) of Definition 2.6 and thus leading to Property (1).

We now show Property (2). Note that $\mathsf{Vert}(\mathsf{P}(\mathbf{H})) \subset V(\mathbf{T}_\iota)$ holds by Item (1) of Theorem 2.8. It remains to prove $V(\mathbf{T}_\iota) \subset \mathsf{Fix}(\mathbf{T}^{\oplus})$. Suppose that there exists a vertex $v \in V(\mathbf{T}_\iota) \setminus \mathsf{Fix}(\mathbf{T}^{\oplus})$. Then there is an automorphism π of $\mathbf{T}^{\oplus}$ such that $\pi(v) \neq v$. Now denote by $\mathcal{L}_v^*$ the unique self-avoiding path in $\mathbf{T}$ connecting v and $\mathfrak{R}(\mathbf{T})$, and similarly define $\mathcal{L}_{\pi(v)}^*$. Then $\mathcal{L}_v^*, \mathcal{L}_{\pi(v)}^* \subset \mathbf{T}$. By Property (1), $\pi(\mathfrak{R}(\mathbf{T})) = \mathfrak{R}(\mathbf{T})$ and therefore we have $\pi(\mathcal{L}_v^*) = \mathcal{L}_{\pi(v)}^*$ and $\mathcal{L}_v^* \cap \mathcal{L}_{\pi(v)}^* \neq \emptyset$. Denote $v^* = \arg\max_{u \in \mathcal{L}_v^* \cap \mathcal{L}_{\pi(v)}^*} \mathsf{Dep}_{\mathbf{T}^{\oplus}}(u)$. By $\mathsf{Dep}_{\mathbf{T}^{\oplus}}(v^*) = \mathsf{Dep}_{\mathbf{T}^{\oplus}}(\pi(v^*))$ and $\mathsf{Dep}_{\mathbf{T}^{\oplus}}(v) = \mathsf{Dep}_{\mathbf{T}^{\oplus}}(\pi(v))$, we have $\pi(v^*) = v^*$ and therefore $v^* \neq v$. Denote by v^{**} the neighbor of v^* in $\mathcal{L}_v^*$ such that $\mathsf{Dep}_{\mathbf{T}^{\oplus}}(v^{**}) > \mathsf{Dep}_{\mathbf{T}^{\oplus}}(v^*)$. Then $\mathsf{Des}_{\mathbf{T}^{\oplus}}(v^{**})$ and $\mathsf{Des}_{\mathbf{T}^{\oplus}}(\pi(v^{**}))$ are two distinct connected components of $(\mathbf{T}^{\oplus})_{\setminus\{v^*\}} = (\mathbf{T}^{\oplus})_{\setminus\{\pi(v^*)\}}$. Therefore $\pi(\mathsf{Des}_{\mathbf{T}^{\oplus}}(v^{**})) = \mathsf{Des}_{\mathbf{T}^{\oplus}}(\pi(v^{**}))$, and we have $|V(\mathsf{Des}_{\mathbf{T}}(\pi(v^{**})))| = |V(\mathsf{Des}_{\mathbf{T}}(v^{**}))| \geq |V(\mathsf{Des}_{\mathbf{T}}(v))| \geq \log^2(\iota^{-1})$. Since the same parent v^* is shared by $\mathsf{Des}_{\mathbf{T}}(v^{**}) \neq \mathsf{Des}_{\mathbf{T}}(\pi(v^{**}))$, by Item (4) of Definition 2.6 we have $\mathsf{Des}_{\mathbf{T}}(v^{**}) \not\sim \mathsf{Des}_{\mathbf{T}}(\pi(v^{**}))$. Meanwhile, by $\mathsf{Des}_{\mathbf{T}^{\oplus}}(v^{**}) \cong \mathsf{Des}_{\mathbf{T}^{\oplus}}(\pi(v^{**}))$ we have (denote $V_{\mathsf{P}}(v^{**}) = \mathsf{Vert}(\mathsf{P}(\mathbf{H})) \cap V(\mathsf{Des}_{\mathbf{T}}(v^{**}))$ and similarly for $V_{\mathsf{P}}(\pi(v^{**}))$)

$$\mathsf{Des}_{\mathbf{T}}(v^{**}) \oplus (\oplus_{\mathbf{u} \in V_{\mathsf{P}}(v^{**})} \mathcal{L}_{\mathbf{u}}^{(x_{\mathbf{u}})}) \cong \mathsf{Des}_{\mathbf{T}}(\pi(v^{**})) \oplus (\oplus_{\mathbf{v} \in V_{\mathsf{P}}(\pi(v^{**}))} \mathcal{L}_{\mathbf{v}}^{(x_{\mathbf{v}})})\,.$$

Combining this with Item (2) of Theorem 2.8 we have $\mathsf{Des}_{\mathbf{T}}(v^{**}) \sim \mathsf{Des}_{\mathbf{T}}(\pi(v^{**}))$, which is a contradiction. This leads to Property (2).

We now show Property (3). Note that for all $u \in \mathcal{L}(\widetilde{S})$, since each $\mathsf{L}_i(S)$ is self-avoiding, we see that all vertices in $V(\mathsf{L}_i(S)) \setminus \mathrm{EndP}(\mathsf{L}_i(S))$ has degree at least 2 in $\widetilde{S}$ and thus $u \notin V(\mathsf{L}_i(S)) \setminus \mathrm{EndP}(\mathsf{L}_i(S))$. Similarly we have $u \notin V(\mathsf{T}(S)) \setminus \mathcal{L}(\mathsf{T}(S))$. This shows that $\mathcal{L}(\widetilde{S}) \subset \mathcal{L}(\mathsf{T}(S))$. We now show that $\mathcal{L}(\widetilde{S}) = \mathcal{L}(S)$. Clearly we have $\mathcal{L}(S) \subset \mathcal{L}(\widetilde{S})$. In addition, for all $u \in \mathcal{L}(\widetilde{S}) \subset \mathcal{L}(\mathsf{T}(S))$, denote (u, v) to be the edge in $\mathsf{T}(S)$. For all $1 \leq i \leq \iota\aleph$, we must have $u \notin V(\mathsf{L}_i(S)) \setminus \mathrm{EndP}(\mathsf{L}_i(S))$ since otherwise the degree of u in $\widetilde{S}$ is at least 2. This shows that (u, w) has multiplicity 1 in the multigraph S and thus $u \in \mathcal{L}(S)$. This concludes $\mathcal{L}(\widetilde{S}) = \mathcal{L}(S)$.

Also, Property (4) is straightforward due to Item (1) of Definition 2.6. Property (5) can be deduced directly from Item (4) of Theorem 2.8.

As for Property (6), denote $\mathbf{T} = \mathsf{T}(\mathbf{H})$ and $\mathsf{Vert}(\mathsf{P}(\mathbf{H})) = \{\mathbf{w}_1, \ldots, \mathbf{w}_{2\iota\aleph}\}$. If there exists some $x_i > \log^2(\iota^{-1})$, we may assume that $x_i > \log^2(\iota^{-1})$ if and only if $i \leq \Lambda$ for some $\Lambda \geq 1$. By Property (2) of this lemma, we see that for all $\pi \in \mathrm{Aut}(\mathbf{T} \oplus (\oplus_{i \leq 2\iota\aleph} \mathcal{L}_{\mathbf{w}_i}^{(x_i)}))$, π must map $\{\mathbf{w}_1, \ldots, \mathbf{w}_\Lambda\}$ to itself. Thus

$$\mathrm{Aut}(\mathbf{T} \oplus (\oplus_{i \leq 2\iota\aleph} \mathcal{L}_{\mathbf{w}_i}^{(x_i)})) = \mathrm{Aut}(\mathbf{T} \oplus (\oplus_{i > \Lambda} \mathcal{L}_{\mathbf{w}_i}^{(x_i)}))\,.$$

Therefore, we may assume without loss of generality that $x_1, \ldots, x_{2\iota\aleph} \leq \log^2(\iota^{-1})$. Denote $\mathbf{T}(j) = \mathbf{T} \oplus (\oplus_{i<j} \mathcal{L}_{\mathbf{w}_i}^{(x_i)})$. Then we have

$$\frac{\mathrm{Aut}(\mathbf{T} \oplus (\oplus_{i \leq 2\iota\aleph} \mathcal{L}_{\mathbf{w}_i}^{(x_i)}))}{\mathrm{Aut}(\mathbf{T})} = \prod_{j=1}^{2\iota\aleph} \frac{\mathrm{Aut}(\mathbf{T}(j) \oplus \mathcal{L}_{\mathbf{w}_j}^{(x_j)})}{\mathrm{Aut}(\mathbf{T}(j))}\,.$$

By Property (2), we have $\mathbf{w}_j \in \mathsf{Fix}(\mathbf{T}(j) \oplus \mathcal{L}_{\mathbf{w}_j}^{(x_j)})$. Using the fact that $\mathsf{Deg}_{\mathbf{T}}(\mathbf{w}_j) \leq \log^2(\iota^{-1})$, we see that

$$\frac{\mathrm{Aut}(\mathbf{T}(j) \oplus \mathcal{L}_{\mathbf{w}_j}^{(x_j)})}{\mathrm{Aut}(\mathbf{T}(j))} \leq \log^2(\iota^{-1})\,,$$

leading to the desired bound

$$\frac{\mathrm{Aut}(\mathbf{T} \oplus (\oplus_{i \leq 2\iota\aleph} \mathcal{L}_{\mathbf{w}_i}^{(x_i)}))}{\mathrm{Aut}(\mathbf{T})} \leq e^{4\iota \log\log(\iota^{-1})\aleph}\,.$$

As for Property (7), denote $\mathbf{T} = \mathsf{T}(\mathbf{H})$. Note that if there exists $x_{\mathbf{u}} > \log^2(\iota^{-1})$, since $\mathbf{T}$ does not have an arm-path with length at least $\log^2(\iota^{-1})$, then the number of injections from $\mathbf{T}$ to $\mathbf{T} \oplus (\oplus_{\mathbf{v}} \mathcal{L}_{\mathbf{v}}^{(x_{\mathbf{v}})})$ equals the number of injections from $\mathbf{T}$ to $\mathbf{T} \oplus (\oplus_{\mathbf{v} \neq \mathbf{u}} \mathcal{L}_{\mathbf{v}}^{(x_{\mathbf{v}})})$. Thus, (by an inductive application of the preceding observation) without loss of generality we may assume that $x_{\mathbf{u}} \leq \log^2(\iota^{-1})$. Note that using Item (4) in Definition 2.6, the injections from $\mathbf{T}$ to $\mathbf{T} \oplus (\oplus_{\mathbf{u}} \mathcal{L}_{\mathbf{u}})$ must map $\mathbf{T}_\iota$ to itself. Note that we can write

$$\mathbf{T} = \mathbf{T}_\iota \oplus \big(\oplus_{\mathbf{w} \in V(\mathbf{T}_\iota)} \mathbf{T}_{\mathbf{w}} \big)\,,$$

where $\mathbf{T}_{\mathbf{w}} = \mathsf{Des}_{\mathbf{T}}(\mathbf{w})$ are descendant trees rooted at $\mathbf{w}$ with $|V(\mathbf{T}_{\mathbf{w}})| \geq (\log(\iota^{-1}))^2$. Denoting $\mathbf{T}_{\mathbf{w}}^{\oplus} = \mathbf{T}_{\mathbf{w}}$ if $\mathbf{w} \notin \mathsf{Vert}(\mathsf{P}(\mathbf{T}))$ and $\mathbf{T}_{\mathbf{w}}^{\oplus} = \mathbf{T}_{\mathbf{w}} \oplus \mathcal{L}_{\mathbf{w}}^{(x_{\mathbf{w}})}$ if $\mathbf{w} \in \mathsf{Vert}(\mathsf{P}(\mathbf{T}))$, it can be easily checked that the number of injections from $\mathbf{T}_{\mathbf{w}}$ to $\mathbf{T}_{\mathbf{w}}^{\oplus}$ equals $\frac{\mathrm{Aut}(\mathbf{T}_{\mathbf{w}}^{\oplus})}{\mathrm{Aut}(\mathbf{T}_{\mathbf{w}})}$. Thus, the number of injections from $\mathbf{T}$ to $\mathbf{T}^{\oplus}$ equals

$$\prod_{\mathbf{w} \in V(\mathbf{T}_\iota)} \frac{\mathrm{Aut}(\mathbf{T}_{\mathbf{w}}^{\oplus})}{\mathrm{Aut}(\mathbf{T}_{\mathbf{w}})} = \frac{\mathrm{Aut}(\mathbf{T}^{\oplus})}{\mathrm{Aut}(\mathbf{T})}\,,$$

concluding the proof. ∎

Corollary 2.11. *For all $\mathbf{H} \in \mathcal{H}$ and constant $0 \leq \kappa \leq \frac{1}{\iota^2 \epsilon^2 \lambda s}$ (note that $\frac{1}{\iota^2 \epsilon^2 \lambda s} < 1$ from (2.4) and (2.5)) we have*

$$\sum_{x_1, \ldots, x_{2\iota\aleph} \geq 0} \kappa^{x_1 + \ldots + x_{2\iota\aleph}} \cdot \frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}) \oplus (\oplus_{\mathbf{u} \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))} \mathcal{L}_{\mathbf{u}}^{(x_{\mathbf{u}})}))^2}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))^2} \leq \exp\big(10\iota \log\log(\iota^{-1})\aleph\big)\,. \tag{2.10}$$

Proof. Using Property (6) of Lemma 2.10, the left hand side of (2.10) is bounded by

$$e^{8\iota \log\log(\iota^{-1})\aleph} \sum_{x_1,\ldots,x_{\iota\aleph}\geq 0} \kappa^{x_1+\ldots+x_{\iota\aleph}} \overset{(2.3)}{\leq} \exp\left(10\iota\log\log(\iota^{-1})\aleph\right),$$

as desired. ∎

Now we can rigorously state the main result of this paper, which verifies the success of the statistic $f(A,B)$ under the above choices of $\mathcal{T}_\aleph$ and $\mathcal{S}(\mathcal{T}_\aleph)$.

THEOREM 2.12. *Under the assumed conditions* (2.2)–(2.6)*, by choosing*

$$\mathcal{H} = \mathcal{H}(\iota,\aleph,\ell,M;\mathcal{T}_\aleph,\mathcal{S}(\mathcal{T}_\aleph))$$

according to Definition 2.6 and Theorem 2.8, we have

$$\frac{\mathbb{E}_{\mathbb{P}}[f]^2}{\mathbb{E}_{\mathbb{Q}}[f^2]} = \omega(1) \ \textit{and}\ \frac{\mathbb{E}_{\mathbb{P}}[f]^2}{\mathbb{E}_{\mathbb{P}}[f^2]} = 1+o(1)\,.$$

Remark 2.13. It is straightforward by Chebyshev's inequality that Theorem 2.12 implies that the testing error satisfies

$$\mathbb{Q}(f(A,B)\geq\tau) + \mathbb{P}(f(A,B)\leq\tau) = o(1)\,,$$

where the threshold τ is chosen as $\tau = C\mathbb{E}_{\mathbb{P}}[f_{\mathcal{T}}(A,B)]$ for any fixed constant $0<C<1$.

2.4 Discussions Before moving on to the proof of Theorem 2.12, we feel that it is necessary to explain a bit more about the construction of our statistic $f(A,B)$ and the seemingly daunting choices of $\mathcal{T}_\aleph, \mathcal{S}(\mathcal{T}_\aleph)$. Recall that we are in the supercritical region $\epsilon^2\lambda s > 1$ where weak community recovery in A and B is possible. Also, assuming that all the community labelings in A and B (we denote them as σ_A and σ_B) are known to us, it is easy to check that we can achieve detection below Otter's threshold between $\mathbb{P}$ and $\mathbb{Q}$ via the following statistic:

$$g(A,B) = \sum_{\mathbf{H}_c\in\mathcal{H}_c} \frac{s^{\aleph-1}\,\mathrm{Aut}(\mathbf{H}_c)}{n^{\aleph}} \sum_{S_1,S_2\cong\mathbf{H}_c} \phi_{S_1,S_2}(A,B)\,,$$

where $\mathcal{H}_c$ is the set of all unlabeled two-colored trees and for all $S_1,S_2\in\mathcal{K}_n$, and we say $S_1\cong\mathbf{H}_c$ if there exists an isomorphism $\varphi: V(S_1)\to V(\mathbf{H}_c)$ such that $\sigma_A(i) = \sigma_{\mathbf{H}_c}(\varphi(i))$ for all $i\in V(S_1)$. Thus, a naive attempt is to first run the weak community recovery algorithm in both A and B which produces an estimator $(\widehat{\sigma}_A,\widehat{\sigma}_B)$ of (σ_A,σ_B), and then plug this estimator into $g(A,B)$ to obtain a testing variable $\widehat{g}(A,B)$. However, as explained in Section 1.1, it seems of substantial challenge to analyze this naive approach directly.

To overcome this issue, we note that both the task of community recovery (by counting non-backtracking or self-avoiding paths) or correlation detection (by counting trees) can be captured by some low-degree polynomials of (A,B). Thus, instead of running these two algorithms sequentially, we seek for a polynomial which *combines* the low-degree polynomials corresponding to community recovery and correlation detection, and then we try to analyze this polynomial directly. This explains the intuition behind our definition of decorated trees in Definition 2.3, where the self-avoiding paths $\mathsf{L}_i(\mathbf{H})$ record the information from community recovery and the trees $\mathsf{T}(\mathbf{H})$ are used for detecting correlation. Even with this intuition, analyzing the statistic obtained by counting all decorated trees remains challenging. Another key to the success of our detection algorithm is to exploit the correlation of subgraph counts in A and B as much as possible, while suppressing the undesirable correlation between different subgraph counts. We next explain why restricting to the special family $\mathcal{T}_\aleph, \mathcal{S}(\mathcal{T}_\aleph)$ is crucial, and outline some basic guidelines for choosing its parameters. To illustrate, let us consider the second moment under $\mathbb{P}$ (which reduces to calculating $\mathbb{E}_{\mathbb{P}}[\phi_{S_1,S_2}\phi_{K_1,K_2}]$). For simplicity, assume $\pi = \mathsf{id}$. Our baseline analysis focuses on the case in which S_1,S_2,K_1,K_2 are simple (i.e., each edge has multiplicity 1) and $S_1\cap S_2$ (respectively, $K_1\cap K_2$) is a tree containing $\mathsf{T}(S_1)$ (respectively, $\mathsf{T}(K_1)$; see Figure 2.2(a)). In this scenario, $S_1\cap S_2$ must be $\mathsf{T}(S_1)$ with some self-avoiding paths attached. Using Corollary 2.11, we can bound the enumeration of such $(S_1,S_2;K_1,K_2)$ and the expectation in this case can be calculated precisely. Extending from this simple case (referred to as the baseline) to more general cases involves five key observations for bounding the total variance (although the actual proof does not follow this classification exactly):

- If one of the self-avoiding paths (for example, $\mathsf{L}_i(S_1)$) and the trees (for example, $\mathsf{T}(S_1)$) intersect (see Figure 2.2(b)), there will be additional cycles and thus increase the excess of the graph $G_\cup = \widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{K}_1 \cup \widetilde{K}_2$, gaining extra factors of $1/n$ in the second moment bound compared to the contribution of the baseline as (non-rigorously speaking) the enumeration of such graphs depends on the number of vertices of $G_\cup$ and the expectation in each case depends on the number of edges of $G_\cup$.

- If the two trees $\mathsf{T}(K_1), \mathsf{T}(K_2)$ (or $\mathsf{T}(S_1), \mathsf{T}(S_2)$) do not completely overlap (see Figure 2.2(c)), there will also be additional cycles and thus increase the excess of the graph $G_\cup = S_1 \cup S_2 \cup K_1 \cup K_2$, gaining extra factors of $1/n$ in the second moment bound compared to the contribution of the baseline.

- If (S_1, S_2) and (K_1, K_2) have overlapping vertices (see Figure 2.2(d)), this will decrease the enumeration of (S_1, S_2, K_1, K_2), gaining extra factors of $1/n$ in the second moment bound compared to the contribution of the baseline.

- If $S_1 \cap S_2$ or $K_1 \cap K_2$ contains a whole length-ℓ self-avoiding path (see Figure 2.2(e)), this will decrease $|E(G_\cup)|$ by ℓ and decrease $\tau(G_\cup)$ by 1, gaining extra factors of $n(\epsilon^2\lambda s)^{-\ell} < 1/n$ (thanks to (2.6)) in the second moment bound compared to the contribution of the baseline (note that the contribution from the pair of paths is $O(1)$ when they completely overlap with each other, compared to the baseline case where the contribution is $\frac{(\epsilon^2\lambda s)^\ell}{n^\ell} \cdot n^{\ell-1}$)).

- If $\mathsf{T}(S_1) = \mathsf{T}(K_1)$ and $\mathsf{T}(S_2) = \mathsf{T}(K_2)$ (see Figure 2.2(f)), as we shall see using Item (3) in Theorem 2.8 the expectation is smaller, gaining extra factors $\epsilon^{\iota(\log\log(\iota^{-1}))^2 \aleph}$ in the second moment bound compared to the contribution of the baseline.

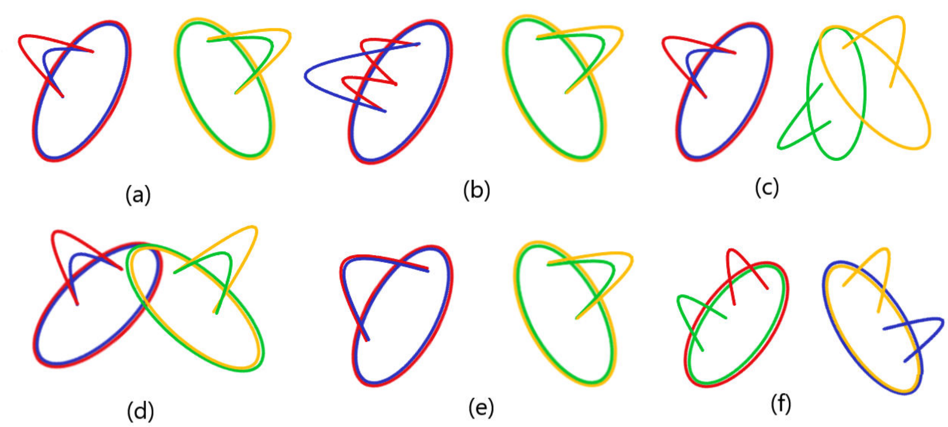


Figure 2.2: Examples of overlapping patterns of S_1, S_2 and K_1, K_2, shown in red/blue and green/orange respectively; the ellipses represent the trees and the curves represent the paths attached to it. (a) $S_1 \cap S_2, T_1 \cap T_2$ are non-intersecting trees containing $\mathsf{T}(S_1), \mathsf{T}(K_1)$, respectively; (b) The self-avoiding paths and the trees intersect, creating cycle(s); (c) The two trees $\mathsf{T}(K_1), \mathsf{T}(K_2)$ (or $\mathsf{T}(S_1), \mathsf{T}(S_2)$) do not completely overlap, creating cycle(s); (d) The four trees $\mathsf{T}(S_1), \mathsf{T}(S_2), \mathsf{T}(K_1), \mathsf{T}(K_2)$ have non-empty common part; (e) Two self-avoiding paths completely overlap; (f) $\mathsf{T}(S_1) = \mathsf{T}(K_1)$ and $\mathsf{T}(S_2) = \mathsf{T}(K_2)$.

3 Estimation of first moment

The main goal of this section is to prove the following proposition.

PROPOSITION 3.1. *We have the following estimation:*

$$s^{2(\aleph-1)}|\mathcal{H}|\cdot\Big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph}e^{-4\iota^2\aleph}\le\mathbb{E}_{\mathbb{P}}[f] \tag{3.1}$$

$$\le s^{2(\aleph-1)}|\mathcal{H}|\cdot\Big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph}e^{5\iota(\log\log(\iota^{-1}))\aleph} \tag{3.2}$$

To prove Proposition 3.1, we first need the following two lemmas:

LEMMA 3.2. *For any $r+t\ge1$, there exist $u_{r,t},v_{r,t}$ such that*

$$\mathbb{E}_{\mathbb{P}_{\sigma,\pi}}\bigg[\bigg(\frac{A_{i,j}-\frac{\lambda s}{n}}{\big(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\big)^{1/2}}\bigg)^r\bigg(\frac{B_{\pi(i),\pi(j)}-\frac{\lambda s}{n}}{\big(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\big)^{1/2}}\bigg)^t\bigg]=u_{r,t}+v_{r,t}\sigma_i\sigma_j\,. \tag{3.3}$$

In addition, we have the estimation

$$0\le u_{r,t},v_{r,t}\le\big(\tfrac{n}{\epsilon^2\lambda s}\big)^{(r+t-2)/2}\,. \tag{3.4}$$

LEMMA 3.3. *For a path P with $V(P)=\{v_0,\dots,v_l\}$ and $\mathrm{EndP}(P)=\{v_0,v_l\}$, we have that for all $a_i,b_i\in\mathbb{R}$*

$$\mathbb{E}_{\sigma\sim\nu}\Big[\prod_{i=1}^{l}\big(a_i+b_i\cdot\sigma_{i-1}\sigma_i\big)\mid\sigma_0,\sigma_l\Big]=\prod_{i=1}^{l}a_i+\sigma_0\sigma_l\cdot\prod_{i=1}^{l}b_i\,. \tag{3.5}$$

The proofs of Lemmas 3.2 and 3.3 are incorporated in Sections C.1 and C.2, respectively. We now state several useful bounds controlling the expectation of ϕ_{S_1,S_2}.

LEMMA 3.4. *We have $\mathbb{E}_{\mathbb{P}_\pi}[\phi_{S_1,S_2}]\ge0$ for all $\pi\in\mathfrak{S}_n$ and thus $\mathbb{E}_{\mathbb{P}_\pi}[f]\ge0$ for all $\pi\in\mathfrak{S}_n$.*

LEMMA 3.5. *For connected multigraphs S_1,S_2, we have*

$$\mathbb{E}_{\mathbb{P}_\pi}[\phi_{S_1,S_2}]=0\ \textit{if}\ \mathcal{L}(\pi(S_1))\cup\mathcal{L}(S_2)\not\subset V(\pi(S_1))\cap V(S_2)\,. \tag{3.6}$$

In addition, for all $(i,j)\in E(S_1)\cup E(S_2)$, denote

$$\chi(i,j)=(\chi_1(i,j),\chi_2(i,j))=(E(S_1)_{i,j},E(S_2)_{i,j})\,. \tag{3.7}$$

Then for all $V\subset V(S_1)\setminus\mathcal{L}(S_1)$ we have that $\mathbb{E}_{\mathbb{P}_\pi}\Big[\phi_{S_1,S_2}\cdot\prod_{i\in V}\sigma_i\Big]$ is bounded by (recall that $\tau(H)=|E(H)|-|V(H)|$ and recall (3.3)*)*

$$2^{5\tau(\pi(S_1)\cup S_2)+5|V|+10\aleph}\cdot\prod_{(i,j)\in E(S_1)\cup E(S_2)}\max\big\{u_{\chi_1(i,j),\chi_2(i,j)},v_{\chi_1(i,j),\chi_2(i,j)}\big\}\,. \tag{3.8}$$

LEMMA 3.6. *Given a tree T and a subset $\mathtt{U}\subset V(T)$ such that $\mathsf{Dist}_T(\mathtt{u},\mathtt{v})\ge\mathsf{d}$ for all $\mathtt{u},\mathtt{v}\in\mathtt{U}$ ($\mathtt{u}\neq\mathtt{v}$), we have*

$$\mathbb{E}\Big[\phi_T(A)^2\cdot\prod_{\mathtt{u}\in\mathtt{U}}\sigma_{\mathtt{u}}\Big]\le2^{6|\mathtt{U}|}\cdot\epsilon^{\mathsf{d}|\mathtt{U}|/2}\,. \tag{3.9}$$

The proofs of Lemmas 3.4, 3.5 and 3.6 are incorporated in Sections C.3, C.4 and C.5 of the appendix, respectively. We now return to the proof of Proposition 3.1. Since f is symmetric under permutations, it suffices to prove the bound for $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[f]$ in place of $\mathbb{E}_{\mathbb{P}}[f]$ in Proposition 3.1. We may also fix $\mathcal{J}_A$ and $\mathcal{J}_B$ and simply write $S\Vdash\mathbf{H}$ instead of $S\Vdash_A\mathbf{H}$ or $S\Vdash_B\mathbf{H}$ to further simplify the notations. Define $\mathfrak{R}^*_{\mathbf{H}}$ to be the subset of $\mathfrak{R}_{\mathbf{H}}:=\{S:S\Vdash\mathbf{H}\}$ such that each vertex in $V(S)$ appears exactly once in

$$V(\mathsf{T}(S)),V(\mathsf{L}_1(S))\setminus\mathrm{EndP}(\mathsf{L}_1(S)),\dots,V(\mathsf{L}_{\iota\aleph}(S))\setminus\mathrm{EndP}(\mathsf{L}_{\iota\aleph}(S))\,.$$

Note that for $S\in\mathfrak{R}^*_{\mathbf{H}}$ all of the edges in $\mathsf{T}(S)$ and $\mathsf{L}_i(S)$ for $i=1,\dots,\iota\aleph$ are not multi-edges, and thus $\widetilde{S}=S$. In addition, define $\mathcal{A}$ to be a map from a pair of subgraphs (S_1,S_2) to subsets of permutations by

$$\mathcal{A}(S_1,S_2)=\begin{cases}\big\{\pi\in\mathfrak{S}_n:\pi(S_1)\cap S_2\text{ is a tree containing }\pi(\mathsf{T}(S_1))\cup\mathsf{T}(S_2)\big\}\,, & S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}\,;\\ \emptyset\,, & \text{otherwise}\,.\end{cases} \tag{3.10}$$

We now introduce the following lemma.

LEMMA 3.7. *If $S, K \in \mathfrak{R}^*_{\mathbf{H}}$ (recall that in this case S, K are simple graphs), $S \cap K = T$ and $S \cup K = T \sqcup (\sqcup_{i=1}^m \mathcal{L}_i)$ where $\{\mathcal{L}_i : 1 \le i \le m\}$ are disjoint self-avoiding paths with $V(\mathcal{L}_i) \cap V(T) = \mathrm{EndP}(\mathcal{L}_i)$. Then we have the following:*

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\phi_S(A)\phi_K(B)\big] \circeq \big(\tfrac{\epsilon^2 \lambda s}{n}\big)^{\frac{1}{2}\sum_{i\le m}|E(\mathcal{L}_i)|}\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\Big[\phi_T(A)\phi_T(B)\cdot \prod_{i\le m}\prod_{u\in \mathrm{EndP}(\mathcal{L}_i)}\sigma_u\Big]\,, \tag{3.11}$$

$$\mathbb{E}_{\mathbb{Q}}\big[\phi_S(A)\phi_K(B)\big] \circeq \big(\tfrac{\epsilon^2 \lambda s}{n}\big)^{\frac{1}{2}\sum_{i\le m}|E(\mathcal{L}_i)|}\mathbb{E}_{\mathbb{Q}}\Big[\phi_T(A)\phi_T(B)\cdot \prod_{i\le m}\prod_{u\in \mathrm{EndP}(\mathcal{L}_i)}\sigma_u\Big]\,. \tag{3.12}$$

The proof of Lemma 3.7 is provided in Section C.6. For any $\{\mathcal{A}(S_1,S_2) \subset \mathfrak{S}_n : S_1, S_2 \in \cup_{\mathbf{H}\in\mathcal{H}}\mathfrak{R}_{\mathbf{H}}\}$ define

$$f_{\mathcal{A}} := \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{\aleph-1}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{\aleph+\ell\iota\aleph}} \sum_{S_1,S_2\in\mathfrak{R}_{\mathbf{H}}} \phi_{S_1,S_2}(A,B)\cdot \mathbf{1}_{\{\pi_*\in\mathcal{A}(S_1,S_2)\}}\,. \tag{3.13}$$

We first deal with $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[f_{\mathcal{A}}]$, as incorporated in the following lemma.

LEMMA 3.8. *For all $S_1, S_2 \in \mathfrak{R}^*_{\mathbf{H}}$, we have (recall (3.10))*

$$\begin{aligned} s^{2(\aleph-1)}|\mathcal{H}|\cdot\Big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph} e^{-4\iota^2\aleph} &\le \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[f_{\mathcal{A}}\big] \\ &\le s^{2(\aleph-1)}|\mathcal{H}|\cdot\Big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph} e^{5\iota(\log\log(\iota^{-1}))\aleph}\,. \end{aligned}$$

Proof. On the one hand, define $\mathcal{A}_\diamond$ to be a map from any pair (S_1,S_2) to subsets of permutations by

$$\mathcal{A}_\diamond(S_1,S_2) = \begin{cases} \big\{\pi\in\mathfrak{S}_n : \pi(S_1)\cap S_2 = \pi(\mathsf{T}(S_1)) = \mathsf{T}(S_2)\big\}\,, & S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}\,;\\ \emptyset\,, & \text{otherwise}\,.\end{cases} \tag{3.14}$$

It is clear that $\mathcal{A}_\diamond(S_1,S_2)\subset\mathcal{A}(S_1,S_2)$ for all (S_1,S_2). Recall (3.13). Using Lemma 3.4, we have

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[f_{\mathcal{A}}\big] \ge \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[f_{\mathcal{A}_\diamond}\big]\,. \tag{3.15}$$

In addition, we have that for $S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}$, $\mathsf{id}\in\mathcal{A}_\diamond(S_1,S_2)$ if and only if $\mathsf{T}(S_1)=\mathsf{T}(S_2)=S_1\cap S_2$. Thus

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[f_{\mathcal{A}_\diamond}\big] = \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{\aleph-1}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{\aleph+\ell\iota\aleph}} \sum_{\substack{S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}\\ S_1\cap S_2=\mathsf{T}(S_1)=\mathsf{T}(S_2)}} \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\phi_{S_1,S_2}(A,B)\big]\,.$$

We now calculate $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}(A,B)]$ for $S_1\cap S_2=\mathsf{T}(S_1)=\mathsf{T}(S_2)$ and $S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}$. Note that from Theorems 2.7 and 2.8, we see that $\mathsf{Vert}(\mathsf{P}(S_1))\subset\mathsf{Fix}(\mathsf{T}(S_1))$, and thus $S_1\cap S_2=\mathsf{T}(S_1)=\mathsf{T}(S_2)$ and $S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}$ imply that $\mathsf{Vert}(\mathsf{P}(S_1))=\mathsf{Vert}(\mathsf{P}(S_2))$, as illustrated in Figure 3.1. Thus, applying Lemma 3.7 we have $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}(A,B)]$ equals

$$\big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph}\cdot\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\Bigg[\prod_{(i,j)\in E(\mathsf{T}(S_1))}\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\prod_{(i,j)\in E(\mathsf{T}(S_2))}\frac{B_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\Bigg] = \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph}s^{\aleph-1}\,. \tag{3.16}$$

Thus, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[f_{\mathcal{A}_\diamond}\big] &= \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{\aleph-1}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{\aleph+\ell\iota\aleph}} \sum_{\substack{S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}\\ S_1\cap S_2=\mathsf{T}(S_1)=\mathsf{T}(S_2)}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph}s^{\aleph-1} \\ &= \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{2(\aleph-1)}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{2\ell\iota\aleph}}{n^{\aleph+2\ell\iota\aleph}}\#\big\{(S_1,S_2) : S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}, S_1\cap S_2=\mathsf{T}(S_1)=\mathsf{T}(S_2)\big\} \\ &\circeq \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{2(\aleph-1)}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{2\ell\iota\aleph}}{n^{\aleph+2\ell\iota\aleph}}\cdot\frac{(1-\iota^2)^{2\aleph}n^{\aleph}}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))}\big(\iota^4 n^{\ell-1}\big)^{2\iota\aleph} \\ &\ge e^{-4\iota^2\aleph}|\mathcal{H}|s^{2(\aleph-1)}\big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\big)^{2\iota\aleph}\,, \end{aligned} \tag{3.17}$$

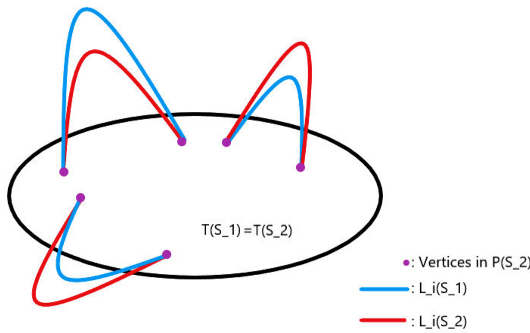


Figure 3.1: Intersection pattern for $\mathsf{id} \in \mathcal{A}_\diamond(S_1, S_2)$

where in the last inequality we used the fact that $1 - \iota^2 \geq e^{-2\iota^2}$. Plugging (3.17) into (3.15) yields the desired lower bound for $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[f_{\mathcal{A}}]$.

For the upper bound, recall that for all $S_1, S_2 \in \mathfrak{R}^*_{\mathbf{H}}$ such that $\mathsf{id} \in \mathcal{A}(S_1, S_2)$, we have $S_1 \cap S_2$ is a tree containing $\mathsf{T}(S_1) \cup \mathsf{T}(S_2)$. Thus, denoting $\mathsf{P}(S_1) = \{(u_1, u_2), \ldots, (u_{2\iota\aleph-1}, u_{2\iota\aleph})\}$, there must exist paths $\mathcal{L}_1, \ldots, \mathcal{L}_{2\iota\aleph}$ with $u_i \in \mathrm{EndP}(\mathcal{L}_i)$ such that

$$S_1 \cap S_2 = \mathsf{T}(S_1) \oplus \Big(\oplus_{u \in \mathsf{Vert}(\mathsf{P}(S_1))} \mathcal{L}_u \Big) ,$$

as illustrated in Figure 3.2. Denote $\mathcal{O}(\{x_u : u \in \mathsf{Vert}(\mathsf{P}(S_1))\})$ as the set of $S_2 \in \mathfrak{R}^*_{\mathbf{H}}$ such that there exist paths

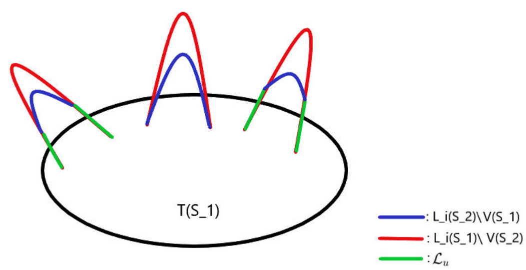


Figure 3.2: Intersection pattern for $\mathsf{id} \in \mathcal{A}(S_1, S_2)$

$\mathcal{L}_u$ with $u \in \mathrm{EndP}(\mathcal{L}_u)$ and $|E(\mathcal{L}_u)| = x_u$ such that $S_1 \cap S_2 = \mathsf{T}(S_1) \oplus (\oplus_{u \in \mathsf{Vert}(\mathsf{P}(S_1))} \mathcal{L}_u)$. We first bound the cardinality of $\mathcal{O}(\{x_u : u \in \mathsf{Vert}(\mathsf{P}(S_1))\})$. First note that we must have $\mathsf{T}(S_2) \subset \mathsf{T}(S_1) \oplus (\oplus_{u \in \mathsf{Vert}(\mathsf{P}(S_1))} \mathcal{L}_u)$, and thus the number of possible choices of $\mathsf{T}(S_2)$ is bounded by

$$\#\big\{T \subset \mathsf{T}(S_1) \oplus (\oplus_{u \in \mathsf{Vert}(\mathsf{P}(S_1))} \mathcal{L}_u) : T \cong \mathsf{T}(\mathbf{H})\big\} \leq \frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}) \oplus (\oplus_{u \in \mathsf{Vert}(\mathsf{P}(\mathbf{H}))} \mathcal{L}_u))}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))} ,$$

where the inequality follows from Item (7) in Lemma 2.10. In addition, since $S_2 \Vdash \mathbf{H}$ and $\mathsf{Vert}(\mathsf{P}(S_2)) \subset \mathsf{Fix}(\mathsf{T}(S_2))$, we have that the relative position of $\mathsf{P}(S_2)$ on the tree $\mathsf{T}(S_2)$ is determined, which implies that the choice of $\mathsf{P}(S_2)$ is fixed once we have chosen $\mathsf{T}(S_2)$. Finally, it is straightforward to check that

$$\begin{aligned} &\#\Big(\cup_{1 \leq i \leq \iota\aleph} V(\mathsf{L}_i(S_2)) \setminus V(S_1) \Big) = (\ell - 1)\iota\aleph - \sum x_u , \\ &\#\Big(\cup_{1 \leq i \leq \iota\aleph} \cup_{u \in \mathrm{EndP}(\mathsf{L}_i(S_2))} \mathsf{Nei}_{\mathsf{L}_i(S_2)}(u) \setminus V(S_1) \Big) = \#\{u : x_u = 0\} . \end{aligned}$$

Thus given $\mathsf{T}(S_2)$, the number of choices for $\{\mathsf{L}_i(S_2) : 1 \leq i \leq \iota\aleph\}$ is bounded by

$$n^{(\ell-1)\iota\aleph - \sum x_u} \iota^{2\#\{u : x_u = 0\}} .$$

As a result, we have

$$\#\mathcal{O}(\{x_u : u \in \mathsf{Vert}(\mathsf{P}(S_1))\}) \leq \frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}) \oplus (\oplus_{u\in\mathsf{Vert}(\mathsf{P}(\mathbf{H}))}\mathcal{L}_u))n^{(\ell-1)\iota\aleph-\sum x_u}\iota^{2\#\{u:x_u=0\}}}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))} .$$

Also, by (3.11) we obtain

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\phi_{S_1,S_2}(A,B)\big] \stackrel{\circ}{=} s^{\aleph-1+\sum_{u\in\mathsf{Vert}(\mathsf{P}(S_2))} x_u}\Big(\tfrac{\epsilon^2\lambda s}{n}\Big)^{\ell\iota\aleph-\sum_{u\in\mathsf{Vert}(\mathsf{P}(S_2))} x_u} . \tag{3.18}$$

Thus (writing $\mathbf{x}\geq 0$ for $x_u \geq 0$ for all $u \in \mathsf{P}(S_2)$),

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[f_{\mathcal{A}}\big] &= \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{\aleph-1}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{\aleph+\ell\iota\aleph}} \sum_{S_1\in\mathfrak{R}^*_{\mathbf{H}}}\sum_{\mathbf{x}\geq 0}\sum_{S_2\in\mathcal{O}(\{x_u\})} \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\phi_{S_1,S_2}(A,B)\big] \\
&\leq \sum_{\mathbf{H}\in\mathcal{H}} \mathrm{Aut}(\mathsf{T}(\mathbf{H}))\frac{s^{2(\aleph-1)}\iota^{4\iota\aleph}(\epsilon^2\lambda s)^{2\ell\iota\aleph}}{n^{\aleph+(\ell+1)\iota\aleph}} \sum_{S_1\in\mathfrak{R}^*_{\mathbf{H}}}\sum_{\mathbf{x}\geq 0} \frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}) \oplus (\oplus_{u\in\mathsf{Vert}(\mathsf{P}(\mathbf{H}))}\mathcal{L}_u))}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))\iota^{2\sum x_u}(\epsilon^2\lambda s)^{\sum x_u}} \\
&\leq \sum_{\mathbf{H}\in\mathcal{H}} \mathrm{Aut}(\mathsf{T}(\mathbf{H}))\frac{s^{2(\aleph-1)}\iota^{4\iota\aleph}(\epsilon^2\lambda s)^{2\ell\iota\aleph}}{n^{\aleph+(\ell+1)\iota\aleph}} e^{4\iota(\log\log\iota^{-1})\aleph}(1-\tfrac{1}{\iota^2\epsilon^2\lambda s})^{-2\iota\aleph}|\mathfrak{R}^*_{\mathbf{H}}| \\
&\leq \sum_{\mathbf{H}\in\mathcal{H}} s^{2(\aleph-1)}e^{4\iota(\log\log\iota^{-1})\aleph}\Big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph}(1-\tfrac{1}{\iota^2\epsilon^2\lambda s})^{-2\iota\aleph} \\
&\leq s^{2(\aleph-1)}e^{5\iota(\log\log\iota^{-1})\aleph}\Big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph}|\mathcal{H}| .
\end{aligned} \tag{3.19}$$

where the first inequality follows from $\#\{u : x_u = 0\} \geq 2\iota\aleph - \sum x_u$, the second inequality follows from Item (6) in Lemma 2.10 and (2.5), the third inequality follows from

$$|\mathfrak{R}^*_{\mathbf{H}}| \leq \frac{n^{\aleph}}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))}\cdot\big(\iota^4 n^{\ell-1}\big)^{\iota\aleph} , \tag{3.20}$$

and the fourth inequality follows from $\lambda > \Delta$ and (2.4). This leads to the desired upper bound of $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[f_{\mathcal{A}}]$. ∎

Based on Lemma 3.8, it remains to deal with the part $\pi_* \notin \mathcal{A}(S_1,S_2)$, as incorporated in the next lemma.

LEMMA 3.9. *Recall* (3.13) *and let* $\mathcal{A}^c$ *be a map such that* $\mathcal{A}^c(S_1,S_2) = \mathfrak{S}_n \setminus \mathcal{A}(S_1,S_2)$ *for* $S_1,S_2 \in \cup_{\mathbf{H}\in\mathcal{H}}\mathfrak{R}_{\mathbf{H}}$. *We have*

$$\mathbb{E}_{\mathbb{P}}\big[f_{\mathcal{A}^c}\big] = o(1)\cdot s^{2(\aleph-1)}e^{-4\iota^2\aleph}\Big(\tfrac{\iota^2(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph}|\mathcal{H}| . \tag{3.21}$$

Proof of Proposition 3.1. It suffices to note that Proposition 3.1 follows directly from combining Lemmas 3.8 and 3.9. ∎

The rest of this section is devoted to the proof of Lemma 3.9. To this end, note that by symmetry we have $\mathbb{E}[f_{\mathcal{A}^c}] = \mathbb{E}_{\mathbb{P}_\pi}[f_{\mathcal{A}^c}]$ for any $\pi \in \mathfrak{S}_n$. Thus, it suffices to show that

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[f_{\mathcal{A}^c}] = o(1)\cdot s^{2(\aleph-1)}e^{-4\iota^2\aleph}\Big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph}|\mathcal{H}| . \tag{3.22}$$

For $S_1,S_2 \in \mathfrak{R}_{\mathbf{H}}$, define $G_\cup$ to be the (simple) graph obtained by

$$G_\cup = \widetilde{S}_1 \cup \widetilde{S}_2 .$$

From Property (2) in Lemma 2.10, we know that there must exist a leaf in $\widetilde{S}_1 \triangle \widetilde{S}_2$ if $\mathcal{L}(\widetilde{S}_1) \neq \mathcal{L}(\widetilde{S}_2)$, which implies that $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}(A,B)] = 0$ from Lemma 3.5. In addition, we have $\mathcal{L}(G_\cup) \subset \mathcal{L}(\widetilde{S}_1)\cap\mathcal{L}(\widetilde{S}_2)$ and thus $|\mathcal{L}(G_\cup)| \leq \aleph$.

Also we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\Bigg[\prod_{(i,j)\in E(\widetilde{S}_1)}\Big(\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S_1)_{i,j}}\prod_{(i,j)\in E(\widetilde{S}_2)}\Big(\frac{B_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S_2)_{i,j}}\Bigg]\\
&=\mathbb{E}_{\sigma\sim\nu}\Bigg\{\prod_{(i,j)\in E(G_\cup\setminus\widetilde{S}_2)}\mathbb{E}_{\mathbb{P}_{\mathsf{id},\sigma}}\Bigg[\Big(\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S_1)_{i,j}}\Bigg]\prod_{(i,j)\in E(G_\cup\setminus\widetilde{S}_1)}\mathbb{E}_{\mathbb{P}_{\mathsf{id},\sigma}}\Bigg[\Big(\frac{B_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S_2)_{i,j}}\Bigg]\\
&\qquad\prod_{(i,j)\in E(\widetilde{S}_1\cap\widetilde{S}_2)}\mathbb{E}_{\mathbb{P}_{\mathsf{id},\sigma}}\Bigg[\Big(\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S_1)_{i,j}}\Big(\frac{B_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S_2)_{i,j}}\Bigg]\Bigg\}.
\end{aligned}
\tag{3.23}
$$

Using Lemmas 3.2 and 3.5 (with $V=\emptyset$), we see that (3.23) is bounded by (below we denote $E(S)_{i,j}=0$ if $(i,j)\notin E(S)$)

$$
2^{5\tau(G_\cup)+10\aleph}\prod_{(i,j)\in E(G_\cup)}\Big(\frac{\sqrt{n}}{\sqrt{\epsilon^2\lambda s}}\Big)^{E(S_1)_{i,j}+E(S_2)_{i,j}-2}.
$$

Using the fact that

$$
\sum_{(i,j)\in E(G_\cup)}(E(S_1)_{i,j}+E(S_2)_{i,j}-2)=2(\aleph-1+\iota\ell\aleph)-2|E(G_\cup)|\,,
$$

we obtain that (3.23) is further bounded by

$$
2^{5\tau(G_\cup)+10\aleph}n^{\aleph-1+\iota\ell\aleph-|E(G_\cup)|}/(\epsilon^2\lambda s)^{\aleph-1+\iota\ell\aleph-|E(G_\cup)|}\,. \tag{3.24}
$$

Combined with (2.8), it yields that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}[f_{\mathcal{A}^c}]&\le\sum_{\mathbf{H}\in\mathcal{H}}\sum_{\substack{S_1,S_2\Vdash\mathbf{H}\\ \mathsf{id}\notin\mathcal{A}(S_1,S_2)}}\frac{2^{5\tau(G_\cup)+10\aleph}s^{\aleph-1}(\epsilon^2\lambda s)^{\ell\iota\aleph}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))}{n^{\aleph+\ell\iota\aleph}}\cdot\big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(G_\cup)|-\ell\iota\aleph-\aleph+1}\\
&=n^{-1+o(1)}\sum_{\substack{|\mathcal{L}(G_\cup)|\le\aleph\\ |E(G_\cup)|\le2\ell\iota\aleph}}2^{5\tau(G_\cup)}\big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(G_\cup)|}\,\mathrm{ENUM}(G_\cup)\,,
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
\mathrm{ENUM}(G_\cup)=\#\Big(\cup_{\mathbf{H}\in\mathcal{H}}\big\{&(S_1,S_2)\in\mathfrak{R}_{\mathbf{H}}\times\mathfrak{R}_{\mathbf{H}}:\widetilde{S}_1\cup\widetilde{S}_2=G_\cup,\\
&\mathcal{L}(S_1)\cup\mathcal{L}(S_2)\subset V(S_2)\cap V(S_1),\mathsf{id}\notin\mathcal{A}(S_1,S_2)\big\}\Big)\,.
\end{aligned}
\tag{3.26}
$$

Now it suffices to control (3.25), for which the following lemma will be useful.

LEMMA 3.10. *For all $S_1,S_2\in\mathfrak{R}_{\mathbf{H}}$ such that $\mathsf{id}\notin\mathcal{A}(S_1,S_2)$ and $\mathcal{L}(S_1)\cup\mathcal{L}(S_2)\subset V(S_1)\cap V(S_2)$, we have*

$$
2\iota\aleph-\tau(G_\cup)\le\tfrac{2\iota\ell\aleph+2\aleph-|E(G_\cup)|}{\ell/2}\,. \tag{3.27}
$$

Proof. We first deal with the case that $S_1,S_2\in\mathfrak{R}^*_{\mathbf{H}}$ and $\mathsf{id}\notin\mathcal{A}(S_1,S_2)$. Recall in this case S_1,S_2 are just simple graphs. Denote $H=S_1\cap S_2$. Note that $\tau(S_1)=\tau(S_2)=\iota\aleph-1$ and $|E(S_1)|=|E(S_2)|=\aleph-1+\ell\iota\aleph$. Then we have

$$
\tau(G_\cup)=2\iota\aleph-2-\tau(H)\text{ and }|E(G_\cup)|=2(\aleph-1)+2\ell\iota\aleph-|E(H)|\,,
$$

and thus it suffices to show that in this case we have

$$
\tau(H)\le\tfrac{|E(H)|}{\ell/2}-2<\tfrac{|E(H)|+2}{\ell/2}-2\,. \tag{3.28}
$$

To this end, denote $H_0 = H \cap \mathsf{T}(S_1)$ and $H_i = H_{i-1} \cup (H \cap \mathsf{L}_i(S_1))$ for $i = 1, \dots, \iota\aleph$ (thus $H_{\iota\aleph} = H$). Since $S_1 \in \mathfrak{R}^*_{\mathbf{H}}$ implies that $V(\mathsf{L}_{i+1}(S_1)) \cap V(H_i) = \mathrm{EndP}(\mathsf{L}_{i+1}(S_1))$, we have $\tau(H_{i+1}) \le \tau(H_i) + 1$, with equality holding if and only if $\mathsf{L}_{i+1}(S_1) \subset H$. In addition, note that

$$\mathsf{L}_{i+1}(S_1) \subset H \Longrightarrow |E(H_{i+1})| = |E(H_i)| + \ell\,.$$

Thus, we have

$$\tau(H_{i+1}) - \tau(H_i) \le \tfrac{|E(H_{i+1})| - |E(H_i)|}{\ell/2} - \mathbf{1}_{\{\mathsf{L}_{i+1}(S_1) \subset H\}}\,. \tag{3.29}$$

Rearranging the above inequality and summing over i yields

$$\begin{aligned}\tau(H) - \tfrac{|E(H)|}{\ell/2} = \tau(H_{\iota\aleph}) - \tfrac{|E(H_{\iota\aleph})|}{\ell/2} &\le \tau(H_0) - \tfrac{|E(H_0)|}{\ell/2} - \#\{i : \mathsf{L}_i(S_1) \subset H\} \\ &\le \tau(H_0) - \#\{i : \mathsf{L}_i(S_1) \subset H\}\,. \end{aligned}\tag{3.30}$$

By Item (3) of Lemma 2.10 and the assumption that $\mathcal{L}(S_1) \cup \mathcal{L}(S_2) \subset V(S_1) \cap V(S_2)$, we have that $\mathcal{L}(\mathsf{T}(S_1)) = \mathcal{L}(S_1) \subset V(H_0)$. Also, $\mathsf{id} \notin \mathcal{A}(S_1, S_2)$ implies that

$$\exists j \in \{1,2\},\ \text{either } \mathsf{T}(S_j) \neq H_0 \text{ or } \{i : \mathsf{L}_i(S_j) \subset H\} \neq \emptyset\,; \tag{3.31}$$

this is because otherwise $H = H_{\iota\aleph}$ will be a tree containing $\mathsf{T}(S_1)$ and $\mathsf{T}(S_2)$, which contradicts to $\mathsf{id} \notin \mathcal{A}(S_1, S_2)$. Without loss of generality, assume $j = 1$ in (3.31). When $\mathsf{T}(S_1) \neq H_0$, by $H_0 \subset \mathsf{T}(S_1)$ and $\mathcal{L}(T(S_1)) \subset V(H_0)$, all connected components of H_0 must be trees and H_0 cannot be connected, which yields $\tau(H_0) \le -2$. When $\mathsf{T}(S_1) = H_0$, by (3.31), we have that $\tau(H_0) = -1$ and $\{i : \mathsf{L}_i(S_1) \subset H\} \neq \emptyset$. Therefore, when $\mathsf{id} \notin \mathcal{A}(S_1, S_2)$ either $\tau(H_0) \le -2$ or $\tau(H_0) = -1, \{i : \mathsf{L}_i(S_1) \subset H\} \neq \emptyset$ holds, leading to

$$\tau(H_0) - \#\{i : \mathsf{L}_i(S_1) \subset H\} \le -2\,. \tag{3.32}$$

Then (3.28) follows by plugging the above estimate into (3.30). Now we deal with the case that $S_1 \notin \mathfrak{R}^*_{\mathbf{H}}$ (the case that $S_2 \notin \mathfrak{R}^*_{\mathbf{H}}$ can be treated similarly). In this case we have $\tau(\widetilde{S}_1) \ge \iota\aleph$. Denoting $H = \widetilde{S}_1 \cap \widetilde{S}_2$, we claim the following

$$\tau(H) - \tfrac{|E(H)| + \aleph}{\ell} \le \min\Big\{ \tau(\widetilde{S}_1) - \tfrac{|E(\widetilde{S}_1)|}{\ell}, \tau(\widetilde{S}_2) - \tfrac{|E(\widetilde{S}_2)|}{\ell} \Big\}\,. \tag{3.33}$$

To prove (3.33), denote $H_{\mathsf{T}} = H \cup \mathsf{T}(S_1)$. Since $\mathcal{L}(\widetilde{S}_1) \subset V(H)$, we then have

$$\tau(H_{\mathsf{T}}) \ge \tau(H) \text{ and } |E(H_{\mathsf{T}})| \le |E(H)| + \aleph\,. \tag{3.34}$$

In addition, since $\mathcal{L}(\widetilde{S}_1) \subset V(H) \subset V(H_{\mathsf{T}})$, by using Lemma A.4, we see that $\widetilde{S}_1 \setminus H_{\mathsf{T}}$ can be decomposed into $\mathtt{t} = \tau(\widetilde{S}_1) - \tau(H_{\mathsf{T}})$ self-avoiding paths $P_1, \dots, P_{\mathtt{t}}$ satisfying Items (1)–(4) of Lemma A.4. Clearly we have $|E(P_{\mathtt{i}})| \le \ell$, and thus

$$|E(\widetilde{S}_1)| - |E(H_{\mathsf{T}})| \le \ell \mathtt{t} \le \ell(\tau(\widetilde{S}_1) - \tau(H_{\mathsf{T}}))\,. \tag{3.35}$$

In conclusion, we have shown that

$$\tau(H) - \tfrac{|E(H)| + \aleph}{\ell} \overset{(3.34)}{\le} \tau(H_{\mathsf{T}}) - \tfrac{|E(H_{\mathsf{T}})|}{\ell} \overset{(3.35)}{\le} \tau(\widetilde{S}_1) - \tfrac{|E(\widetilde{S}_1)|}{\ell}\,.$$

Similar results also hold for $\widetilde{S}_2$, leading to (3.33). Thus,

$$\begin{aligned}\tau(G_\cup) - \tfrac{|E(G_\cup)| - \aleph}{\ell} &= \big(\tau(\widetilde{S}_1) - \tfrac{|E(\widetilde{S}_1)|}{\ell}\big) + \big(\tau(\widetilde{S}_2) - \tfrac{|E(\widetilde{S}_2)|}{\ell}\big) - \big(\tau(H) - \tfrac{|E(H)| + \aleph}{\ell}\big) \\ &\overset{(3.33)}{\ge} \max\Big\{\tau(\widetilde{S}_1) - \tfrac{|E(\widetilde{S}_1)|}{\ell}, \tau(\widetilde{S}_2) - \tfrac{|E(\widetilde{S}_2)|}{\ell}\Big\} \ge \tau(\widetilde{S}_1) - \tfrac{|E(\widetilde{S}_1)|}{\ell} \ge -\tfrac{\aleph}{\ell}\,,\end{aligned}$$

where the second inequality follows from $\tau(\widetilde{S}_1) \geq \iota\aleph$ and $|E(\widetilde{S}_1)| \leq \ell\iota\aleph + \aleph$. Using the fact that $|E(G_\cup)| \leq |E(\widetilde{S}_1)| + |E(\widetilde{S}_2)| \leq 2\ell\iota\aleph + 2\aleph$, we obtain

$$\begin{aligned}\tau(G_\cup) - \tfrac{|E(G_\cup)|}{\ell/2} &= \Big(\tau(G_\cup) - \tfrac{|E(G_\cup)|-\aleph}{\ell}\Big) - \tfrac{|E(G_\cup)|+\aleph}{\ell} \\ &\geq -\tfrac{2\aleph}{\ell} - \tfrac{2\ell\iota\aleph+2\aleph}{\ell} = -2\iota\aleph - \tfrac{4\aleph}{\ell}\,,\end{aligned}$$

as desired. □

We can now complete the proof of Lemma 3.9.

Proof of Lemma 3.9. Based on Lemma 3.10, we have (in what follows we say $\mathbf{G}$ satisfies (3.27) if (3.27) holds after replacing $G_\cup$ by $\mathbf{G}$ and we denote by $\mathrm{ENUM}(G)$ as (3.26) after replacing $G_\cup$ with G))

$$(3.25) = n^{-1+o(1)} \sum_{\substack{|\mathcal{L}(\mathbf{G})|\leq\aleph, |E(\mathbf{G})|\leq 2\ell\iota\aleph+2\aleph \\ \mathbf{G} \text{ satisfying (3.27)}}} \sum_{G\cong\mathbf{G}} 2^{5\tau(G)} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(G)|} \mathrm{ENUM}(G)\,, \tag{3.36}$$

where the summation is taken over all unlabeled graph $\mathbf{G}$ such that $\mathrm{ENUM}(G) > 0$ for $G \cong \mathbf{G}$. In addition, using Lemma A.5, for G satisfying (3.27), we have

$$\mathrm{ENUM}(G) \leq \binom{|E(G)|}{\aleph} \cdot \Big(|V(G)| \cdot \tau(G)!\Big)^{2\iota\aleph} \leq (2\ell\iota\aleph)^{2\aleph} \cdot \Big(\tau(G)!\Big)^{2\iota\aleph} \leq (2\ell\aleph)^{4\aleph} \leq n^{o(1)}\,.$$

Thus, we have

$$\begin{aligned}(3.36) &\leq n^{-1+o(1)} \cdot \sum_{\substack{|\mathcal{L}(\mathbf{G})|\leq\aleph, |E(\mathbf{G})|\leq 2\ell\iota\aleph+2\aleph \\ \mathbf{G} \text{ satisfying (3.27)} \\ \mathrm{ENUM}(\mathbf{G})>0}} 2^{5\tau(\mathbf{G})} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(\mathbf{G})|} \cdot \#\big\{G \subset \mathcal{K}_n : G \cong \mathbf{G}\big\} \\ &\leq n^{-1+o(1)} \cdot \sum_{\substack{|\mathcal{L}(\mathbf{G})|\leq\aleph, |E(\mathbf{G})|\leq 2\ell\iota\aleph+2\aleph \\ \mathbf{G} \text{ satisfying (3.27)} \\ \mathrm{ENUM}(\mathbf{G})>0}} 2^{5\tau(\mathbf{G})} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(\mathbf{G})|} n^{|V(\mathbf{G})|} \\ &\leq n^{-1+o(1)} (\epsilon^2\lambda s)^{2\ell\iota\aleph} \sum_{\substack{|\mathcal{L}(\mathbf{G})|\leq\aleph, |E(\mathbf{G})|\leq 2\ell\iota\aleph+2\aleph \\ \mathbf{G} \text{ satisfying (3.27)} \\ \mathrm{ENUM}(\mathbf{G})>0}} 2^{5\tau(\mathbf{G})} n^{-\tau(\mathbf{G})} (\epsilon^2\lambda s)^{|E(\mathbf{G})|-2\ell\iota\aleph} \\ &\leq n^{-1+o(1)} (\epsilon^2\lambda s)^{2\ell\iota\aleph} \sum_{\substack{0\leq x\leq 2\ell\iota\aleph+2\aleph \\ 2(2\ell\iota\aleph+2\aleph-x)/\ell\geq 2\iota\aleph-y}} \big(\tfrac{32(\ell\aleph)^3}{n}\big)^{y} (\epsilon^2\lambda s)^{x-2\ell\iota\aleph} \\ &\leq n^{-1+o(1)} (\epsilon^2\lambda s)^{2\ell\iota\aleph} \sum_{0\leq x\leq 2\ell\iota\aleph+2\aleph} \big(\tfrac{32(\ell\aleph)^3}{n}\big)^{2\iota\aleph} \big(\tfrac{(\epsilon^2\lambda s)^\ell}{n^2}\big)^{(x-2\ell\iota\aleph)/\ell} \\ &\leq n^{-1-2\iota\aleph+o(1)} (\epsilon^2\lambda s)^{2\ell\iota\aleph}\,, \end{aligned} \tag{3.37}$$

where the second inequality follows from $\#\{G \subset \mathcal{K}_n : G \cong \mathbf{G}\} = \frac{n^{|V(\mathbf{G})|}}{\mathrm{Aut}(\mathbf{G})}$, the fourth inequality holds by Lemma A.6 and the last inequality follows from $(\ell\aleph)^\aleph = n^{o(1)}$. Plugging (3.37) into (3.25), we obtain that

$$(3.25) \leq n^{-1+o(1)} \cdot \Big(\tfrac{(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph} \overset{(2.6)}{=} o(1) \cdot s^{2(\aleph-1)} |\mathcal{H}| \cdot \Big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{2\iota\aleph}\,,$$

as desired. □

4 Estimation of second moment

The main goal of this section is to prove the following proposition.

Proposition 4.1. *We have the following estimates:*

(1) $\mathbb{E}_{\mathbb{Q}}[f^2] = o(1) \cdot \mathbb{E}_{\mathbb{P}}[f]^2$;

(2) $\mathrm{Var}_{\mathbb{P}}[f] = o(1) \cdot \mathbb{E}_{\mathbb{P}}[f]^2$.

In particular, combining Proposition 4.1 with Proposition 3.1 (which implies that $\mathbb{E}_{\mathbb{P}}[f] \to \infty$) yields Theorem 2.12 (note that Item (1) also implies that $\mathrm{Var}_{\mathbb{Q}}[f], \mathbb{E}_{\mathbb{Q}}[f]^2 \le \mathbb{E}_{\mathbb{Q}}[f^2] = o(1) \cdot \mathbb{E}_{\mathbb{P}}[f]^2$). The remainder of this section is devoted to proving Proposition 4.1.

4.1 Proof of Item (1) Recall (2.8). By the independence of A and B under $\mathbb{Q}$, we have

$$\mathbb{E}_{\mathbb{Q}}[f^2] = \sum_{\mathbf{H},\mathbf{I} \in \mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2 \lambda s)^{2\ell\iota\aleph} \operatorname{Aut}(\mathsf{T}(\mathbf{H})) \operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \sum_{\substack{S_1, S_2 \Vdash \mathbf{H} \\ K_1, K_2 \Vdash \mathbf{I}}} \mathbb{E}_{\mathbb{Q}}[\phi_{S_1,S_2} \phi_{K_1,K_2}]$$

$$= \sum_{\mathbf{H},\mathbf{I} \in \mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2 \lambda s)^{2\ell\iota\aleph} \operatorname{Aut}(\mathsf{T}(\mathbf{H})) \operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \Big(\sum_{S \Vdash \mathbf{H}, K \Vdash \mathbf{I}} \mathbb{E}_{\mathbb{Q}}[\phi_S(A) \phi_K(A)] \Big)^2 . \tag{4.1}$$

Denote $\mathfrak{P}_{\mathbf{H},\mathbf{I}} = \big\{ (S,K) : S \in \mathfrak{R}_{\mathbf{H}}, K \in \mathfrak{R}_{\mathbf{I}} \big\}$. In addition, denote

$$\mathfrak{P}^*_{\mathbf{H},\mathbf{I}} = \big\{ (S,K) : S \in \mathfrak{R}^*_{\mathbf{H}}, K \in \mathfrak{R}^*_{\mathbf{I}}, S \cap K \text{ is a tree containing } \mathsf{T}(S) \cup \mathsf{T}(K) \big\} . \tag{4.2}$$

Then (4.1) is bounded by 2 times

$$\sum_{\mathbf{H},\mathbf{I} \in \mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2 \lambda s)^{2\ell\iota\aleph} \operatorname{Aut}(\mathsf{T}(\mathbf{H})) \operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \Big(\sum_{(S,K) \in \mathfrak{P}^*_{\mathbf{H},\mathbf{I}}} \mathbb{E}_{\mathbb{Q}}[\phi_S(A) \phi_K(A)] \Big)^2 \tag{4.3}$$

$$+ \sum_{\mathbf{H},\mathbf{I} \in \mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2 \lambda s)^{2\ell\iota\aleph} \operatorname{Aut}(\mathsf{T}(\mathbf{H})) \operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \Big(\sum_{(S,K) \not\in \mathfrak{P}^*_{\mathbf{H},\mathbf{I}}} \mathbb{E}_{\mathbb{Q}}[\phi_S(A) \phi_K(A)] \Big)^2 . \tag{4.4}$$

We first control (4.3) via the following lemma:

Lemma 4.2. *We have*

$$(4.3) \le s^{2(\aleph-1)} \big(1 - \tfrac{1}{\iota^2 \epsilon^2 \lambda s}\big)^{-4\iota\aleph} \Big(\tfrac{\iota^4 (\epsilon^2 \lambda s)^\ell}{n} \Big)^{4\iota\aleph} |\mathcal{H}| (1 + \epsilon^{2\iota(\log\log(\iota^{-1})^2)\aleph}) . \tag{4.5}$$

In particular, combining Lemma 4.2 with Proposition 3.1 and Lemma 2.9, we see that

$$(4.3) = o(1) \cdot \mathbb{E}_{\mathbb{P}}[f]^2 . \tag{4.6}$$

Now we present the proof of Lemma 4.2.

Proof of Lemma 4.2. Recall (4.2). Define

$$\mathfrak{P}_{\mathbf{H}} = \big\{ (S,K) \in \mathfrak{P}^*_{\mathbf{H},\mathbf{H}} : \mathsf{T}(S) = \mathsf{T}(K), \mathsf{P}(S) = \mathsf{P}(K) \big\} ; \tag{4.7}$$

$$\mathfrak{Q}_{\mathbf{H},\mathbf{I}} = \big\{ (S,K) \in \mathfrak{P}^*_{\mathbf{H},\mathbf{I}} : \mathsf{T}(S) = \mathsf{T}(K), \mathsf{P}(S) \neq \mathsf{P}(K) \big\} . \tag{4.8}$$

We first show that

$$\mathfrak{P}^*_{\mathbf{H},\mathbf{I}} \subset \mathfrak{P}_{\mathbf{H}} \cup \mathfrak{Q}_{\mathbf{H},\mathbf{I}} . \tag{4.9}$$

In fact, for all $(S,K) \in \mathfrak{P}^*_{\mathbf{H},\mathbf{I}}$, since $S \cap K$ is a tree containing $\mathsf{T}(S)$, there must exist self-avoiding paths $\{\mathcal{L}_u : u \in \mathsf{Vert}(\mathsf{P}(S))\}$ such that (for each $u \in \mathsf{Vert}(\mathsf{P}(S))$, denote $i(u) \in \{1, \ldots, \iota\aleph\}$ to be the index such that $u \in \mathrm{EndP}(\mathsf{L}_{i(u)}(S))$)

$$\mathcal{L}_u \subset \mathsf{L}_{i(u)}(S), \mathrm{EndP}(\mathcal{L}_u) \cap \mathrm{EndP}(\mathsf{L}_{i(u)}(S)) = \{u\} \text{ and } S \cap K = \mathsf{T}(S) \cup \big(\cup_{u \in \mathsf{Vert}(\mathsf{P}(S))} \mathcal{L}_u \big) .$$

However, from Definition 2.3 we see that the neighbor of u in $\mathcal{L}_u$ belongs to $\mathcal{J}_A$, and thus the neighbor of u in $\mathcal{L}_u$ does not belong to $\mathsf{T}(K)$. This shows that $E(\mathcal{L}_u) \cap E(\mathsf{T}(K)) = \emptyset$ for each $u \in \mathsf{Vert}(\mathsf{P}(S))$ and thus

$E(\mathsf{T}(K)) \subset E(\mathsf{T}(S))$. Similarly we have $E(\mathsf{T}(S)) \subset E(\mathsf{T}(K))$, leading to $\mathsf{T}(S) = \mathsf{T}(K)$ and concluding (4.9). Based on (4.9), we have

$$(4.3) \leq \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))^2}{n^{2(\aleph+\ell\iota\aleph)}}\Big(\sum_{(S,K)\in\mathfrak{P}_{\mathbf{H}}} \mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)]\Big)^2 \tag{4.10}$$

$$+ \sum_{\mathbf{H},\mathbf{I}\in\mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))\,\mathrm{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}}\Big(\sum_{(S,K)\in\mathfrak{Q}_{\mathbf{H},\mathbf{I}}} \mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)]\Big)^2 . \tag{4.11}$$

We first control (4.10). Note that for $(S,K)\in\mathfrak{P}_{\mathbf{H}}$, there exist self-avoiding paths $\{\mathcal{L}_u : u \in \mathsf{Vert}(\mathsf{P}(S))\}$ satisfying $u \in \mathrm{EndP}(\mathcal{L}_u)$ for all $u\in\mathsf{Vert}(\mathsf{P}(S))$ such that

$$S\cap K = \mathsf{T}(S) \oplus (\oplus_{u\in\mathsf{Vert}(\mathsf{P}(S))}\mathcal{L}_u)$$

(see Figure 4.1 for an illustration). For each $S\in\mathfrak{R}^*_{\mathbf{H}}$ and each non-negative sequence $\{x_u\}$, define $\mathfrak{P}_{\mathbf{H}}(S,\{x_u\})$

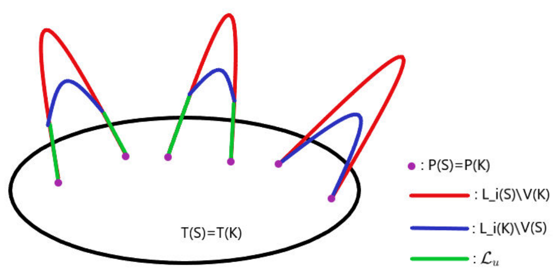


Figure 4.1: Intersection pattern for $(S,K)\in\mathfrak{P}_{\mathbf{H}}$

to be the set of all $K\subset\mathcal{K}_n$ such that

$$(S,K)\in\mathfrak{P}_{\mathbf{H}}, K\cap S = \mathsf{T}(S)\oplus(\oplus_{u\in\mathsf{Vert}(\mathsf{P}(S))}\mathcal{L}_u) \text{ with } |E(\mathcal{L}_u)| = x_u .$$

We have (recall Definition 2.3 and recall that the neighbor of u in $\mathsf{L}_{i(u)}(K)$ must belong to $\mathcal{J}_A$)

$$\#\mathfrak{P}_{\mathbf{H}}(S,\{x_u\}) \doteq n^{(\ell-1)\iota\aleph-\sum x_u}\iota^{2\#\{u:x_u=0\}} .$$

In addition, by (3.12), for $K\in\mathfrak{P}_{\mathbf{H}}(S,\{x_u\})$ we have

$$\mathbb{E}_{\mathbb{Q}}\big[\phi_S(A)\phi_K(A)\big] \doteq \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph-\sum x_u} .$$

Thus, we have (writing $\mathbf{x}\geq 0$ for $x_u\geq 0$ for all $u\in\mathsf{P}(S)$)

$$\begin{aligned}
&\sum_{S,K\in\mathfrak{P}_{\mathbf{H}}} \mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)] = \sum_{S\in\mathfrak{R}^*_{\mathbf{H}}}\sum_{\mathbf{x}\geq 0}\sum_{K\in\mathfrak{P}_{\mathbf{H}}(S,\{x_u\})} \mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)] \\
&\doteq \sum_{S\in\mathfrak{R}^*_{\mathbf{H}}}\sum_{\mathbf{x}\geq 0} n^{(\ell-1)\iota\aleph-\sum x_u}\cdot\big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph-\sum x_u}\iota^{2\#\{u:x_u=0\}} = |\mathfrak{R}^*_{\mathbf{H}}|\cdot n^{-\iota\aleph}\sum_{\mathbf{x}\geq 0}\big(\epsilon^2\lambda s\big)^{\ell\iota\aleph-\sum x_u}\cdot\iota^{2\#\{u:x_u=0\}} \\
&\leq \big|\mathfrak{R}^*_{\mathbf{H}}\big|\cdot n^{-\iota\aleph}\sum_{\mathbf{x}\geq 0}\big(\epsilon^2\lambda s\big)^{\ell\iota\aleph-\sum x_u}\cdot\iota^{4\iota\aleph-2\sum x_u} \\
&\leq \frac{n^{\aleph}}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))}\cdot\big(\iota^4 n^{\ell-1}\big)^{\iota\aleph}\cdot n^{-\iota\aleph}(\epsilon^2\lambda s)^{\ell\iota\aleph}\Big(\iota^{4\iota\aleph}\sum_{\mathbf{x}\geq 0}(\iota^2\epsilon^2\lambda s)^{-x_u}\Big) \\
&\leq \frac{n^{\aleph+(\ell-1)\iota\aleph}\iota^{8\iota\aleph}}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))}n^{-\iota\aleph}\big(1-\tfrac{1}{\iota^2\epsilon^2\lambda s}\big)^{-2\iota\aleph}(\epsilon^2\lambda s)^{\ell\iota\aleph} ,
\end{aligned}$$

where the second inequality follows from (3.20) and the third inequality follows from $\lambda > \Delta$ and (2.4). Plugging this bound into (4.10), we get that

$$(4.10) \le \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))^2}{n^{2(\aleph+\ell\iota\aleph)}}\Big(\frac{n^{-\iota\aleph}\iota^{8\iota\aleph}\big(1-\frac{1}{\iota^2\epsilon^2\lambda s}\big)^{-2\iota\aleph}(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{-\aleph-(\ell-1)\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))}\Big)^2$$

$$= s^{2(\aleph-1)}\big(1-\tfrac{1}{\iota^2\epsilon^2\lambda s}\big)^{-4\iota\aleph}\Big(\frac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{4\iota\aleph}|\mathcal{H}|\,. \tag{4.12}$$

We then deal with (4.11). For $(S,K)\in\mathfrak{Q}_{\mathbf{H},\mathbf{I}}$, let

$$\mathtt{U} = \mathsf{Vert}(\mathsf{P}(S))\cup\mathsf{Vert}(\mathsf{P}(K)) \text{ and } \mathtt{V} = \mathsf{Vert}(\mathsf{P}(S))\cap\mathsf{Vert}(\mathsf{P}(K))\,.$$

Then there exist $\{x_v\ge 0 : v\in\mathtt{V}\}$ and self-avoiding paths $\mathcal{L}_v$ satisfying $v\in\mathrm{EndP}(\mathcal{L}_v)$ for all v such that $|E(\mathcal{L}_v)| = x_v$ and

$$S\cap K = \mathsf{T}(S)\oplus\big(\oplus_{v\in\mathtt{V}}\mathcal{L}_v\big)$$

(see Figure 4.2 for an illustration). Define $\mathfrak{Q}_{\mathbf{H},\mathbf{I}}(\{x_v\})$ to be the set of $K\in\mathfrak{R}^*_{\mathbf{I}}$ such that

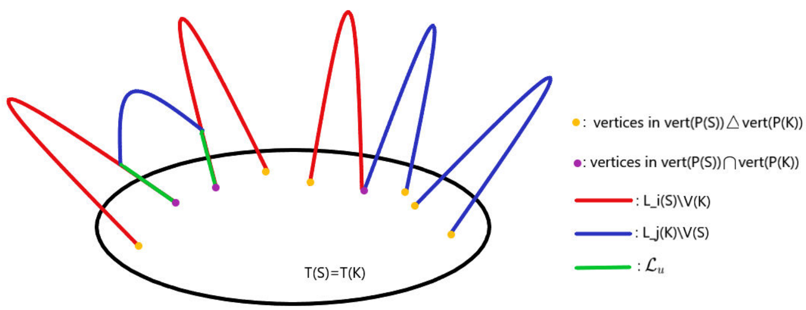


Figure 4.2: Intersection pattern for $(S,K)\in\mathfrak{Q}_{\mathbf{H},\mathbf{I}}(\{x_v\})$

$$S\cap K = \mathsf{T}(S)\oplus\big(\oplus_{v\in\mathtt{V}}\mathcal{L}_v\big) \text{ with } |E(\mathcal{L}_v)| = x_v\,.$$

We then have (below the factor of $2^{2\iota N}$ counts the enumeration for $\mathtt{V}$)

$$\#\mathfrak{Q}_{\mathbf{H},\mathbf{I}}(\{x_v\}) \le \mathbf{1}_{\{\mathsf{T}(\mathbf{H})\cong\mathsf{T}(\mathbf{I})\}}\cdot 2^{2\iota\aleph}n^{(\ell-1)\iota\aleph-\sum x_v}\,.$$

In addition, conditioning on $\sigma_{\mathtt{U}} = \{\sigma_u : u\in\mathtt{U}\}$, we have that

$$\Bigg\{\prod_{(i,j)\in E(\mathsf{T}(S))}\Big(\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\Big)^2, \prod_{m\le\iota\aleph}\prod_{(i,j)\in E(\mathsf{L}_m(S))}\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\prod_{(i,j)\in E(\mathsf{L}_m(K))}\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\Bigg\}$$

is a collection of conditionally independent variables. Also, similar as in (3.16), we have

$$\begin{aligned}&\mathbb{E}\Bigg\{\prod_{m\le\iota\aleph}\prod_{(i,j)\in E(\mathsf{L}_m(S))}\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\prod_{(i,j)\in E(\mathsf{L}_m(K))}\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\mid\sigma_{\mathtt{U}}\Bigg\}\\&=\big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph-\sum x_v}\prod_{u\in\mathsf{Vert}(\mathsf{P}(S))}\sigma_u\prod_{u\in\mathsf{Vert}(\mathsf{P}(K))}\sigma_u = \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph-\sum x_v}\prod_{u\in\mathtt{U}\setminus\mathtt{V}}\sigma_u\,.\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)] &= \mathbb{E}_{\sigma_{\mathtt{U}}\sim\nu_{\mathtt{U}}}\Big\{\mathbb{E}_{\mathbb{Q}}\big[\phi_S(A)\phi_K(A)\mid\sigma_{\mathtt{U}}\big]\Big\} \\
&= \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph-\sum x_v}\mathbb{E}_{\sigma_{\mathtt{U}}\sim\nu_{\mathtt{U}}}\Bigg\{\mathbb{E}\Big[\prod_{(i,j)\in E(\mathsf{T}(S))}\Big(\tfrac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\Big)^2\mid\sigma_{\mathtt{U}}\Big]\prod_{u\in\mathtt{U}\setminus\mathtt{V}}\sigma_u\Bigg\} \\
&= \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph-\sum x_v}\mathbb{E}_{\sigma_{\mathtt{U}}\sim\nu_{\mathtt{U}}}\Bigg\{\prod_{(i,j)\in E(\mathsf{T}(S))}\Big(\tfrac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\Big)^2\prod_{u\in\mathtt{U}\setminus\mathtt{V}}\sigma_u\Bigg\}. \qquad (4.13)
\end{aligned}$$

From Items (3), (4) and (5) in Theorem 2.8, we see that $|\mathtt{U}\setminus\mathtt{V}|\geq\iota\aleph/2$ and $\mathsf{Dist}_{\mathsf{T}(S)}(u,v)\geq(\log\log(\iota^{-1}))^{10}$ for all $u,v\in\mathtt{U}\setminus\mathtt{V}$ ($u\neq v$). Using Lemma 3.6, we obtain

$$(4.13)\leq\big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph-\sum x_v}\epsilon^{\iota(\log\log\iota^{-1})^{10}\aleph}.$$

Thus, we have that (4.11) is bounded by (writing $\mathbf{x}\geq 0$ for $x_v\geq 0$ for all $v\in\mathtt{V}$)

$$\begin{aligned}
&\sum_{\substack{\mathsf{T}(\mathbf{H})\cong\mathsf{T}(\mathbf{I})\\ \mathbf{H},\mathbf{I}\in\mathcal{H}}}\frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))^2}{n^{2(\aleph+\ell\iota\aleph)}}\Big(\sum_{S\in\mathfrak{R}^*_{\mathbf{H}}}\sum_{\mathbf{x}\geq 0}\sum_{K\in\mathfrak{Q}_{\mathbf{H},\mathbf{I}}(\{x_v\})}\mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)]\Big)^2 \\
\leq&\sum_{\substack{\mathsf{T}(\mathbf{H})\cong\mathsf{T}(\mathbf{I})\\ \mathbf{H},\mathbf{I}\in\mathcal{H}}}\frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\,\mathrm{Aut}(\mathsf{T}(\mathbf{H}))^2}{n^{2(\aleph+\ell\iota\aleph)}}\Big(\frac{\iota^{4\iota\aleph}n^{(\ell-1)\iota\aleph+\aleph}}{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))} * \\
&\sum_{\mathbf{x}\geq 0}2^{\iota\aleph}\iota^{2\#\{u:x_u=0\}}n^{(\ell-1)\iota\aleph-\sum x_v}\cdot\big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\ell\iota\aleph-\sum x_v}\epsilon^{\iota(\log\log(\iota^{-1})^{10})\aleph}\Big)^2 \\
\leq&\ s^{2(\aleph-1)}|\mathcal{H}|e^{\iota(\log\log(\iota^{-1}))^4\aleph}\big(\tfrac{\iota^2(\epsilon^2\lambda s)^\ell}{n}\big)^{4\iota\aleph}\big(\iota^{4\iota\aleph}\sum_{\mathbf{x}\geq 0}(\iota^2\epsilon^2\lambda s)^{-\sum x_v}\big)^2\cdot\epsilon^{2\iota(\log\log(\iota^{-1})^{10})\aleph} \\
\overset{(2.3)}{\leq}&\ s^{2(\aleph-1)}|\mathcal{H}|\big(\tfrac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\big)^{4\iota\aleph}\big(1-\tfrac{1}{\iota^2\epsilon^2\lambda s}\big)^{-4\iota\aleph}\cdot\epsilon^{5\iota(\log\log(\iota^{-1})^{10})\aleph}, \qquad (4.14)
\end{aligned}$$

where the second inequality holds because $\#\{u:x_u\geq 0\}\geq 2\iota\aleph-\sum x_u$. Combining (4.12) and (4.14) yields the desired result. ◻

Now it remains to deal with (4.4). Denote $G_\cup=\widetilde{S}\cup\widetilde{K}$. We then have $\mathcal{L}(G_\cup)\subset\mathcal{L}(\widetilde{S})\cup\mathcal{L}(\widetilde{K})$ and thus $|\mathcal{L}(G_\cup)|\leq 2\aleph$. In addition, if $\mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)]\neq 0$ we must have $\mathcal{L}(S)\cup\mathcal{L}(K)\subset V(S\cap K)$. Applying Lemmas 3.2 and 3.3, we have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)] &= \mathbb{E}_{\mathbb{Q}}\Bigg[\prod_{(i,j)\in E(\widetilde{S})}\Big(\tfrac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S)_{i,j}}\prod_{(i,j)\in E(\widetilde{K})}\Big(\tfrac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(K)_{i,j}}\Bigg] \\
&= \mathbb{E}_{\mathbb{Q}}\Bigg[\prod_{(i,j)\in E(\widetilde{S}\cap\widetilde{T})}\Big(\tfrac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S)_{i,j}+E(K)_{i,j}}\prod_{(i,j)\in E(\widetilde{S}\triangle\widetilde{T})}\Big(\tfrac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Big)^{E(S)_{i,j}+E(K)_{i,j}}\Bigg] \\
&\leq \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{-(\aleph-1+\ell\iota\aleph)+|E(G_\cup)|}. \qquad (4.15)
\end{aligned}$$

LEMMA 4.3. *For all $(S,K)\notin\mathfrak{P}^*_{\mathbf{H},\mathbf{I}}$ such that $\widetilde{S}\cup\widetilde{K}=G_\cup$ and $\mathcal{L}(S)\cup\mathcal{L}(K)\subset V(S\cap K)$, we have*

$$2\iota\aleph-\tau(G_\cup)\leq\tfrac{2\ell\iota\aleph+2\aleph-|E(G_\cup)|}{\ell/2}. \qquad (4.16)$$

Proof. The proof is highly similar to the proof of Lemma 3.10, and we omit further details here for simplicity. ◻

Using Lemma 4.3, we have

$$\sum_{S,K\not\in\mathfrak{P}^*_{\mathbf{H},\mathbf{I}}}\mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)]\le\sum_{\substack{|E(G_\cup)|\le 2\ell\iota\aleph+2\aleph\\ G_\cup\text{ satisfies (4.16)}}}\sum_{S\cup K=G_\cup}\mathbb{E}_{\mathbb{Q}}[\phi_S(A)\phi_K(A)]$$

$$\le\sum_{\substack{|E(G_\cup)|\le 2\ell\iota\aleph+2\aleph\\ G_\cup\text{ satisfies (4.16)}}}\left(\tfrac{\epsilon^2\lambda s}{n}\right)^{-(\aleph-1+\ell\iota\aleph)+|E(G_\cup)|}\cdot\mathrm{ENUM}'(G_\cup)\,, \tag{4.17}$$

where for a subgraph G (e.g., $G=G_\cup$)

$$\mathrm{ENUM}'(G)=\#\Big(\cup_{\mathbf{H}\in\mathcal{H}}\left\{(S,K)\in\mathfrak{P}_{\mathbf{H},\mathbf{I}}\setminus\mathfrak{P}^*_{\mathbf{H},\mathbf{I}}:\widetilde{S}\cup\widetilde{K}=G,\mathcal{L}(S)\cup\mathcal{L}(K)\subset V(S\cap K)\right\}\Big)\,. \tag{4.18}$$

Using Lemma A.5, we have

$$\begin{aligned}\mathrm{ENUM}'(G_\cup)&\le\binom{|E(G_\cup)|}{\aleph}\cdot\Big(|V(G_\cup)|\cdot\tau(G_\cup)!\Big)^{2\iota\aleph}\\&\le(2\ell\iota\aleph)^{2\aleph}\cdot\Big(\tau(G_\cup)!\Big)^{2\iota\aleph}\le(2\ell\aleph)^{4\aleph}\le n^{o(1)}\,.\end{aligned}$$

Plugging this estimation into (4.17), then we have (below we write G satisfies (4.16) if (4.16) holds with $G_\cup$ replaced by G)

$$\begin{aligned}(4.17)&\le n^{o(1)}\cdot\sum_{\substack{|E(G)|\le 2\ell\iota\aleph+2\aleph,|\mathcal{L}(G)|\le 2\aleph\\ G\text{ satisfying (4.16)}\\ \mathrm{ENUM}'(G)>0}}\left(\tfrac{\epsilon^2\lambda s}{n}\right)^{-(\aleph-1+\ell\iota\aleph)+|E(G)|}\\&=n^{o(1)}\cdot\sum_{\substack{|E(\mathbf{G})|\le 2\ell\iota\aleph+2\aleph,|\mathcal{L}(\mathbf{G})|\le 2\aleph\\ \mathbf{G}\text{ satisfying (4.16)}\\ \mathrm{ENUM}'(\mathbf{G})>0}}\left(\tfrac{\epsilon^2\lambda s}{n}\right)^{-(\aleph-1+\ell\iota\aleph)+|E(\mathbf{G})|}\cdot\#\{G\subset\mathcal{K}_n:G\cong\mathbf{G}\}\\&\le n^{\aleph-1+\ell\iota\aleph+o(1)}\cdot\sum_{\substack{0\le x\le 2\aleph+2\ell\iota\aleph\\ \ell(2\iota\aleph-y)\le 2(2\ell\iota\aleph+2\aleph-x)}}(\epsilon^2\lambda s)^{-(\aleph-1+\ell\iota\aleph)+x}n^{-y}(\ell\aleph)^{2y}\,,\end{aligned} \tag{4.19}$$

where G is summed over $G\cong\mathbf{G}$, $\mathbf{G}$ is summed over $|E(\mathbf{G})|=x,\tau(\mathbf{G})=y$ and then over x,y, and the last inequality holds by Lemma A.6. Thus, we have

$$\begin{aligned}(4.17)&\le n^{\aleph-1+\ell\iota\aleph+o(1)}\sum_{0\le x\le 2\aleph+2\ell\iota\aleph}(\epsilon^2\lambda s)^{-(\aleph-1+\ell\iota\aleph)+x}\left(\tfrac{(\ell\aleph)^3}{n}\right)^{-2\iota\aleph+(2(x-2\aleph)/l)}\\&\doteq n^{\aleph-1+\ell\iota\aleph+o(1)}\sum_{0\le x\le 2\aleph+2\ell\iota\aleph}\left(\tfrac{(\epsilon^2\lambda s)^\ell}{n^2}\right)^{-\iota\aleph+x/\ell}\\&\doteq n^{\aleph-1+\ell\iota\aleph-2\iota\aleph+o(1)}(\epsilon^2\lambda s)^{\ell\iota\aleph}\,.\end{aligned} \tag{4.20}$$

Plugging this bound into (4.4) yields

$$\begin{aligned}(4.4)&\le\sum_{\mathbf{H},\mathbf{I}\in\mathcal{H}}\frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2+4\iota\aleph+o(1)}}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\\&\le o(1)\cdot s^{2(\aleph-1)}\left(\tfrac{(\epsilon^2\lambda s)^\ell}{n}\right)^{4\iota\aleph}\,,\end{aligned} \tag{4.21}$$

giving the desired bound on (4.4). In particular, by combining (4.21) with Proposition 3.1 and Lemma 2.9, we obtain

$$(4.4)=o(1)\cdot\mathbb{E}_{\mathbb{P}}[f]^2\,. \tag{4.22}$$

Finally, combining (4.6) and (4.22) completes the proof of Item (1) of Proposition 4.1.

4.2 Proof of Item (2) This subsection is devoted to the proof of Item (2) of Proposition 4.1. We define $\mathfrak{R}_{\mathbf{H},\mathbf{I}} = \mathfrak{R}_{\mathbf{H}} \times \mathfrak{R}_{\mathbf{I}}$, and introduce

$$\mathfrak{R}^*_{\mathbf{H},\mathbf{I}} = \left\{ (S_1, S_2; K_1, K_2) \in \mathfrak{R}^*_{\mathbf{H}} \times \mathfrak{R}^*_{\mathbf{I}} : V(S_1) \cap V(K_1) = V(S_2) \cap V(K_2) = \emptyset \right\}. \tag{4.23}$$

$$\mathfrak{R}^{**}_{\mathbf{H},\mathbf{I}} = \left\{ (S_1, S_2; K_1, K_2) \in \mathfrak{R}^*_{\mathbf{H}} \times \mathfrak{R}^*_{\mathbf{I}} : \mathsf{T}(S_1) = \mathsf{T}(K_1), \mathsf{T}(S_2) = \mathsf{T}(K_2) \right\}. \tag{4.24}$$

Denote

$$G_\cup = \pi(\widetilde{S}_1) \cup \pi(\widetilde{K}_1) \cup \widetilde{S}_2 \cup \widetilde{K}_2 \text{ and } \mathsf{T}_\cup = \pi(\mathsf{T}(S_1)) \cup \pi(\mathsf{T}(K_1)) \cup \mathsf{T}(S_2) \cup \mathsf{T}(K_2).$$

In addition, let $G_{\geq 2} \subset G_\cup$ be the (simple) subgraph whose vertex set $V(G_{\geq 2})$ consists of vertices that appear in at least two of the sets $\{V(S_1), V(S_2), V(K_1), V(K_2)\}$, and whose edge set $E(G_{\geq 2})$ is defined analogously with respect to the edges. We define $\mathcal{A}_\star = \mathcal{A}_\star(S_1, S_2; K_1, K_2)$ as follows:

- If $(S_1, S_2; K_1, K_2) \in \mathfrak{R}^*_{\mathbf{H},\mathbf{I}}$, define $\mathcal{A}_\star$ to be the subset of $\mathfrak{S}_n$ such that $\mathcal{L}(G_\cup) \subset V(G_{\geq 2})$ and $G_{\geq 2}$ is a union of two disjoint trees and contains $\mathsf{T}_\cup$.
- If $(S_1, S_2; K_1, K_2) \in \mathfrak{R}^{**}_{\mathbf{H},\mathbf{I}}$, define $\mathcal{A}_\star$ to be the subset of $\mathfrak{S}_n$ such that $\mathcal{L}(G_\cup) \subset V(G_{\geq 2})$ and $\pi(V(S_1 \cup K_1)) \cap V(S_2 \cup K_2) = \emptyset$.
- Otherwise define $\mathcal{A}_\star = \emptyset$.

Recalling (2.8) and noting that symmetry ensures $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[f^2] = \mathbb{E}_{\mathbb{P}}[f^2], \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[f] = \mathbb{E}_{\mathbb{P}}[f]$, we may write the variance as

$$\begin{aligned} \mathrm{Var}_{\mathbb{P}}[f] = \sum_{\mathbf{H},\mathbf{I} \in \mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2 \lambda s)^{2\ell\iota\aleph} \operatorname{Aut}(\mathsf{T}(\mathbf{H})) \operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \cdot \\ \sum_{(S_1,S_2;K_1,K_2) \in \mathfrak{R}_{\mathbf{H},\mathbf{I}}} \Big(\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}] - \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}]\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,K_2}] \Big). \end{aligned} \tag{4.25}$$

Thus, (recalling (3.10)) we may further decompose (4.25) into three parts, where the first part equals

$$\begin{aligned} \sum_{\mathbf{H},\mathbf{I} \in \mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2 \lambda s)^{2\ell\iota\aleph} \operatorname{Aut}(\mathsf{T}(\mathbf{H})) \operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \cdot \sum_{(S_1,S_2;K_1,K_2) \in \mathfrak{R}^*_{\mathbf{H},\mathbf{I}}} \\ \Big(\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}_\star(S_1,S_2;K_1,K_2)}] - \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}(S_1,S_2)}]\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,K_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}(K_1,K_2)}] \Big), \end{aligned} \tag{4.26}$$

the second part equals

$$\begin{aligned} \sum_{\mathbf{H},\mathbf{I} \in \mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2 \lambda s)^{2\ell\iota\aleph} \operatorname{Aut}(\mathsf{T}(\mathbf{H})) \operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \cdot \sum_{(S_1,S_2;K_1,K_2) \in \mathfrak{R}^{**}_{\mathbf{H},\mathbf{I}}} \\ \Big(\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}_\star(S_1,S_2;K_1,K_2)}] - \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}(S_1,S_2)}]\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,K_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}(K_1,K_2)}] \Big), \end{aligned} \tag{4.27}$$

and the third part equals

$$\begin{aligned} \sum_{\mathbf{H},\mathbf{I} \in \mathcal{H}} \frac{s^{2(\aleph-1)}(\epsilon^2 \lambda s)^{2\ell\iota\aleph} \operatorname{Aut}(\mathsf{T}(\mathbf{H})) \operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \cdot \\ \sum_{(S_1,S_2;K_1,K_2) \in \mathfrak{R}_{\mathbf{H},\mathbf{I}}} \Big(\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}\mathbf{1}_{\mathsf{id}\notin\mathcal{A}_\star(S_1,S_2;K_1,K_2)}] - h(S_1,S_2;K_1,K_2) \Big), \end{aligned} \tag{4.28}$$

where

$$h(S_1,S_2;K_1,K_2) = \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}]\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,K_2}] - \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}(S_1,S_2)}]\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,K_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}(K_1,K_2)}]. \tag{4.29}$$

It suffices to show that all terms in (4.26), (4.27) and (4.28) are upper-bounded by $o(1) \cdot \mathbb{E}_{\mathbb{P}}[f]^2$. We first consider (4.26). Note that when $(S_1, S_2, K_1, K_2) \in \mathfrak{R}^*_{\mathbf{H},\mathbf{I}}$ and $\mathsf{id} \in \mathcal{A}_\star(S_1, S_2; K_1, K_2)$, we have $G_{\geq 2} \subset \widetilde{S}_1 \cup \widetilde{S}_2$ (since $V(K_1) \cap V(K_2) = \emptyset$) is the union of two disjoint trees. Thus, from $\mathsf{T}(S_1), \mathsf{T}(S_2) \subset \mathsf{T}_\cup \subset G_{\geq 2}$ and $V(\mathsf{T}(S_1)) \cap V(\mathsf{T}(S_2)) = \emptyset$ we have that one of the trees in $G_{\geq 2}$ contains $\mathsf{T}(S_1)$ and the other contains $\mathsf{T}(S_2)$. Similarly, we can show that one of the trees in $G_{\geq 2}$ contains $\mathsf{T}(K_1)$ and the other contains $\mathsf{T}(K_2)$. In conclusion, when $(S_1, S_2, K_1, K_2) \in \mathfrak{R}^*_{\mathbf{H},\mathbf{I}}$ and $\mathsf{id} \in \mathcal{A}_\star(S_1, S_2; K_1, K_2)$ one of the two following conditions must hold: (i) $S_1 \cap S_2, K_1 \cap K_2$ are two disjoint trees containing $\mathsf{T}(S_1) \cup \mathsf{T}(S_2), \mathsf{T}(K_1) \cup \mathsf{T}(K_2)$ respectively; (ii) $S_1 \cap K_2, K_1 \cap S_2$ are two disjoint trees containing $\mathsf{T}(S_1) \cup \mathsf{T}(K_2), \mathsf{T}(K_1) \cup \mathsf{T}(S_2)$ respectively. Thus, if we denote

$$S \Bumpeq K \text{ if and only if } S \cap K \text{ is a tree containing } \mathsf{T}(S) \cup \mathsf{T}(K)\,, \tag{4.30}$$

we have

$$\mathbf{1}_{\mathsf{id} \in \mathcal{A}_\star} \leq \mathbf{1}_{S_1 \Bumpeq S_2} \mathbf{1}_{K_1 \Bumpeq K_2} + \mathbf{1}_{S_1 \Bumpeq K_2} \mathbf{1}_{S_2 \Bumpeq K_1}\,.$$

Thus, we have (recalling Lemma 3.4)

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}_\star}] \leq \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}(\mathbf{1}_{S_1 \Bumpeq S_2}\mathbf{1}_{K_1 \Bumpeq K_2} + \mathbf{1}_{S_1 \Bumpeq K_2}\mathbf{1}_{K_1 \Bumpeq S_2})] \\
&= \mathbf{1}_{S_1 \Bumpeq S_2}\mathbf{1}_{K_1 \Bumpeq K_2}\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}]\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,K_2}] + \mathbf{1}_{S_1 \Bumpeq K_2}\mathbf{1}_{K_1 \Bumpeq S_2}\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,K_2}]\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,S_2}]\,.
\end{aligned}$$

Combined with the fact that

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}(S_1,S_2)}] = \mathbf{1}_{S_1 \Bumpeq S_2}\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}] \text{ and } \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,K_2}\mathbf{1}_{\mathsf{id}\in\mathcal{A}(K_1,K_2)}] = \mathbf{1}_{K_1 \Bumpeq K_2}\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,K_2}]\,,$$

it yields that (note that $S_1 \Bumpeq K_2, S_2 \Bumpeq K_1$ implies $\mathsf{T}(\mathbf{H}) \sim \mathsf{T}(\mathbf{I})$)

$$(4.26) \leq \sum_{\substack{\mathbf{H},\mathbf{I}\in\mathcal{H} \\ \mathsf{T}(\mathbf{H})\sim\mathsf{T}(\mathbf{I})}} \frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \sum_{\substack{(S_1,S_2;K_1,K_2)\in\mathfrak{R}^*_{\mathbf{H},\mathbf{I}} \\ S_1 \Bumpeq K_2, S_2 \Bumpeq K_1}} \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,K_2}]\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{K_1,S_2}]\,. \tag{4.31}$$

Similarly as in (3.18) and (3.19), we get that (4.26) is bounded by

$$\begin{aligned}
&\sum_{\substack{\mathbf{H},\mathbf{I}\in\mathcal{H} \\ \mathsf{T}(\mathbf{H})\sim\mathsf{T}(\mathbf{I})}} \frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^{2(\aleph+\ell\iota\aleph)}} \cdot |\mathfrak{R}^*_{\mathbf{H}}||\mathfrak{R}^*_{\mathbf{I}}|\iota^{8\iota\aleph} \cdot \\
&\Big(\sum_{\mathbf{x}\geq 0} \frac{\operatorname{Aut}(\mathsf{T}(\mathbf{H}) \oplus (\oplus_{u\in\mathsf{Vert}(\mathsf{P}(\mathbf{H}))}\mathcal{L}_u))}{\operatorname{Aut}(\mathsf{T}(\mathbf{I}))\iota^{2\sum x_u}(\epsilon^2\lambda s)^{\sum x_u}}\Big)\Big(\sum_{\mathbf{y}\geq 0} \frac{\operatorname{Aut}(\mathsf{T}(\mathbf{I}) \oplus (\oplus_{v\in\mathsf{Vert}(\mathsf{P}(\mathbf{I}))}\mathcal{L}_v))}{\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\iota^{2\sum y_v}(\epsilon^2\lambda s)^{\sum y_v}}\Big) \\
&\leq s^{4(\aleph-1)}e^{4\iota(\log\log\iota^{-1})\aleph}(1-\tfrac{1}{\iota^2\epsilon^2\lambda s})^{-4\iota\aleph}\Big(\frac{\iota^4(\epsilon^2\lambda s)^\ell}{n}\Big)^{4\iota\aleph} \cdot \#\{\mathbf{H},\mathbf{I}\in\mathcal{H} : \mathbf{H}\sim\mathbf{I}\} \\
&= o(1)\cdot\mathbb{E}_{\mathbb{P}}[f]^2\,,
\end{aligned} \tag{4.32}$$

where in the inequality we used Item (7) in Lemma 2.10 and in the equality we used Lemma A.1 and Proposition 3.1. Combining (4.31) and (4.32), we have

$$(4.26) = o(1)\cdot\mathbb{E}_{\mathbb{P}}[f]^2\,. \tag{4.33}$$

Now we consider (4.27). It is clear that for $(S_1, S_2; K_1, K_2) \in \mathfrak{R}^{**}_{\mathbf{H},\mathbf{I}}$ and $\pi \in \mathcal{A}_\star$, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_\pi}\big[\phi_{S_1,S_2}\phi_{K_1,K_2}\big] &= \mathbb{E}_{\mathbb{P}_\pi}\big[\phi_{S_1}(A)^2\phi_{S_2}(B)^2\big] \\
&= \mathbb{E}_{\mathbb{P}_\pi}\big[\phi_{S_1}(A)^2\big]\mathbb{E}_{\mathbb{P}_\pi}\big[\phi_{S_2}(B)^2\big] = \mathbb{E}_{\mathbb{Q}}\big[\phi_{S_1}(A)^2\big]\mathbb{E}_{\mathbb{Q}}\big[\phi_{S_2}(B)^2\big]\,.
\end{aligned}$$

Thus, we obtain the following bound

$$\begin{aligned}
(4.27) &\leq \sum_{(S_1,S_2;K_1,K_2)\in\mathfrak{R}^{**}_{\mathbf{H},\mathbf{I}}} \frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\operatorname{Aut}(\mathsf{T}(\mathbf{I}))\mathbb{E}_{\mathbb{Q}}[\phi^2_{S_1}]\mathbb{E}_{\mathbb{Q}}[\phi^2_{S_2}]}{n^{2(\aleph+\ell\iota\aleph)}} \\
&\leq \mathbb{E}_{\mathbb{Q}}[f^2] = o(1)\cdot\mathbb{E}_{\mathbb{P}}[f]^2\,,
\end{aligned} \tag{4.34}$$

where the equality follows from Item (1) of Proposition 4.1. Now we turn to (4.28), which can be written as

$$\sum_{(S_1,S_2;K_1,K_2)\in\mathfrak{R}_{\mathbf{H},\mathbf{I}}} \frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\operatorname{Aut}(\mathsf{T}(\mathbf{I}))\mathbb{E}_{\mathbb{P}}[\phi_{S_1,S_2}\phi_{K_1,K_2}\mathbf{1}_{\pi\notin\mathcal{A}_\star}]}{n^{2(\aleph+\ell\iota\aleph)}} \tag{4.35}$$

$$-\sum_{(S_1,S_2;K_1,K_2)\in\mathfrak{R}_{\mathbf{H},\mathbf{I}}} \frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\operatorname{Aut}(\mathsf{T}(\mathbf{I}))h(S_1,S_2;K_1,K_2)}{n^{2(\aleph+\ell\iota\aleph)}}\,. \tag{4.36}$$

We first deal with (4.36). Recalling (4.29) and applying Lemma 3.4, we can bound $h(S_1,S_2;K_1,K_2)$ by

$$h(S_1,S_2;K_1,K_2)\le\mathbb{E}_{\mathbb{P}}[\phi_{S_1,S_2}\mathbf{1}_{\pi\notin\mathcal{A}(S_1,S_2)}]\mathbb{E}_{\mathbb{P}}[\phi_{K_1,K_2}]+\mathbb{E}_{\mathbb{P}}[\phi_{S_1,S_2}]\mathbb{E}_{\mathbb{P}}[\phi_{K_1,K_2}\mathbf{1}_{\pi\notin\mathcal{A}(K_1,K_2)}]\,.$$

Thus, by Lemma 3.9, we see that

$$\begin{aligned}(4.36)&=o(1)\cdot\sum_{\substack{\mathbf{H},\mathbf{I}\in\mathcal{H}\\(S_1,S_2;K_1,K_2)\in\mathfrak{R}_{\mathbf{H},\mathbf{I}}}}\frac{s^{2(\aleph-1)}(\epsilon^2\lambda s)^{2\ell\iota\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\operatorname{Aut}(\mathsf{T}(\mathbf{I}))\mathbb{E}_{\mathbb{P}}[\phi_{S_1,S_2}]\mathbb{E}_{\mathbb{P}}[\phi_{K_1,K_2}]}{n^{2(\aleph+\ell\iota\aleph)}}\\ &=o(1)\cdot\mathbb{E}_{\mathbb{P}}[f]^2\,.\end{aligned} \tag{4.37}$$

Finally, we consider (4.35). By symmetry, we can write (for each $\pi\in\mathfrak{S}_n$)

$$\sum_{(S_1,S_2;K_1,K_2)\in\mathfrak{R}_{\mathbf{H},\mathbf{I}}}\mathbb{E}_{\mathbb{P}}[\phi_{S_1,S_2}\phi_{K_1,K_2}\mathbf{1}_{\pi\notin\mathcal{A}_\star}]=\sum_{(S_1,S_2;K_1,K_2)\in\mathfrak{R}_{\mathbf{H},\mathbf{I}}}\mathbb{E}_{\mathbb{P}_\pi}[\phi_{S_1,S_2}\phi_{K_1,K_2}\mathbf{1}_{\pi\notin\mathcal{A}_\star}]\,.$$

We may assume $\pi=\mathsf{id}$ without loss of generality and continue to denote $G_\cup=\widetilde{S}_1\cup\widetilde{S}_2\cup\widetilde{K}_1\cup\widetilde{K}_2$. Then we have $|\mathcal{L}(G_\cup)|\le 4\aleph$. We establish a lemma which serves as a general estimate first:

Lemma 4.4. *Suppose S_1,S_2,K_1,K_2 are multigraphs and $G_\cup=\widetilde{S}_1\cup\widetilde{S}_2\cup\widetilde{K}_1\cup\widetilde{K}_2$. Then we have*

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}]\le\big(\tfrac{n}{\epsilon^2\lambda s}\big)^{\frac{\mathtt{m}}{2}-|E(G_\cup)|}\,, \tag{4.38}$$

where $\mathtt{m}=\sum_{(i,j)\in G_\cup}(E(S_1)_{i,j}+E(S_2)_{i,j}+E(K_1)_{i,j}+E(K_2)_{i,j})$.

Proof. Recall the definition of χ in (3.7). By Lemmas 3.2 and 3.3 we have (for simplicity of notations we assume $E(S_1)_{i,j}=0$ for $(i,j)\notin E(S_1)$ and similarly for S_2,K_1,K_2)

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}]&=\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\Bigg\{\prod_{(i,j)\in E(G_\cup)}\Bigg(\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Bigg)^{\chi_{S_1\cup K_1}(i,j)}\Bigg(\frac{B_{i,j}-\frac{\lambda s}{n}}{\sqrt{\lambda s/n}}\Bigg)^{\chi_{S_2\cup K_2}(i,j)}\Bigg\}\\ &\le\big(\tfrac{n}{\epsilon^2\lambda s}\big)^{\frac12(\sum_{(i,j)\in E(G_\cup)}(E(S_1)_{i,j}+E(S_2)_{i,j}+E(K_1)_{i,j}+E(K_2)_{i,j}-2)}=\big(\tfrac{n}{\epsilon^2\lambda s}\big)^{\frac{\mathtt{m}}{2}-|E(G_\cup)|}\,,\end{aligned}$$

which yields our desired result. ∎

By Lemma 4.4, we have

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}]\le\Big(\frac{n}{\epsilon^2\lambda s}\Big)^{2(\aleph-1+\ell\iota\aleph)-|E(G_\cup)|}\,, \tag{4.39}$$

where we use the fact that

$$\sum_{(i,j)\in E(G_\cup)}(E(S_1)_{i,j}+E(S_2)_{i,j}+E(K_1)_{i,j}+E(K_2)_{i,j})=4(\aleph-1+\ell\iota\aleph)\,.$$

Thus, we have

$$\begin{aligned}(4.35)&=\sum_{\mathbf{H},\mathbf{I}\in\mathcal{H}}\sum_{\substack{(S_1,S_2;K_1,K_2)\\ \mathsf{id}\notin\mathcal{A}_\star(S_1,S_2;K_1,K_2)}}\frac{s^{2\aleph}(\epsilon^2\lambda s)^{2\aleph}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))\operatorname{Aut}(\mathsf{T}(\mathbf{I}))}{n^2}\cdot\Big(\frac{\epsilon^2\lambda s}{n}\Big)^{|E(G_\cup)|}\\ &=\sum_{\mathbf{H},\mathbf{I}\in\mathcal{H}}\sum_{\substack{(S_1,S_2;K_1,K_2)\\ \mathsf{id}\notin\mathcal{A}_\star(S_1,S_2;K_1,K_2)}}\frac{1}{n^{2-o(1)}}\cdot\Big(\frac{\epsilon^2\lambda s}{n}\Big)^{|E(G_\cup)|}\,.\end{aligned} \tag{4.40}$$

Recall the definition of $G_{\ge2}$ as in the paragraph below (4.24). The following claim is crucial for our proof.

Lemma 4.5. *We have*

$$4\iota\aleph - 1 - \tau(G_\cup) \le \tfrac{4\ell\iota\aleph + 8\aleph - |E(G_\cup)|}{\ell/2} \tag{4.41}$$

for $\mathsf{id} \notin \mathcal{A}_\star(S_1, S_2; K_1, K_2)$ *and* $\mathcal{L}(G_\cup) \subset V(G_{\ge 2})$.

Remark 4.6. Lemma 4.5 remains unchanged if the self-avoiding paths $\{\mathsf{L}_j(S_i), \mathsf{L}_j(K_i) : 1 \le j \le \iota\aleph, i = 1, 2\}$ are replaced by non-backtracking paths. The proof is similar and we omit further details.

Proof of Lemma 4.5. Define

$$G_{\mathsf{T}} = G_{\ge 2} \cup \mathsf{T}(S_1) \cup \mathsf{T}(S_2) \cup \mathsf{T}(K_1) \cup \mathsf{T}(K_2) \,.$$

Note that we have assumed $\mathcal{L}(G_\cup) \subset V(G_{\ge 2})$. Define Γ to be the number of elements in

$$\{\mathsf{L}_i(S_1), \mathsf{L}_i(S_2), \mathsf{L}_i(K_1), \mathsf{L}_i(K_2) : 1 \le i \le \iota\aleph\}$$

that are included in G_{T}. It is straightforward to check that $\tau(G_\cup) - \tau(G_{\mathsf{T}}) \ge 4\iota\aleph - \Gamma$ and $|E(G_{\mathsf{T}})| \ge \ell\Gamma$. Thus,

$$\tau(G_\cup) - \tau(G_{\mathsf{T}}) + \tfrac{|E(G_{\mathsf{T}})|}{\ell} \ge 4\iota\aleph \,.$$

In addition, we have

$$|E(G_\cup)| \le 4\ell\iota\aleph + 4\aleph - |E(G_{\ge 2})| \le 4\ell\iota\aleph + 4\aleph - (|E(G_{\mathsf{T}})| - 4\aleph) = 4\ell\iota\aleph + 8\aleph - |E(G_{\mathsf{T}})| \,.$$

Thus, if $\tau(G_{\mathsf{T}}) \ge -1$ we immediately get that (4.41) holds.

Now we assume that $\tau(G_{\mathsf{T}}) \le -2$. Note that in this case G_{T} has at least two connected components. If one of the connected components of G_{T} contains only vertices in S_1 (or $S_2/K_1/K_2$), we see that $S_1 \in \mathfrak{R}_{\mathbf{H}} \setminus \mathfrak{R}^*_{\mathbf{H}}$ (or similarly for $S_2/K_1/K_2$). Denote $1 \le \kappa \le 4$ the number of such components, and in this case it is straightforward to verify that

$$\tau(G_\cup) - \tau(G_{\mathsf{T}}) + \tfrac{|E(G_{\mathsf{T}})|}{\ell} \ge 4\iota\aleph + 2\kappa \,,$$

which combined with the fact that $\tau(G_{\mathsf{T}}) \ge -1 - \kappa$ yields (4.41).

The only remaining case is that $\tau(G_{\mathsf{T}}) = -2$ and G_{T} has exactly two connected tree components and each intersects with at least two of S_1, S_2, K_1, K_2. Suppose that one of the components intersects with S_1, S_2 and the other component intersects with K_1, K_2. Denote $G_S = \widetilde{S}_1 \cup \widetilde{S}_2$. If $(S_1, S_2) \in \mathfrak{R}^*_{\mathbf{H}}$ and $S_1 \cap S_2$ is a tree containing $\mathsf{T}(S_1) \cup \mathsf{T}(S_2)$, it is straightforward to verify that

$$2\iota\aleph - 1 - \tau(G_S) \le \tfrac{2\ell\iota\aleph + 2\aleph - |E(G_S)|}{\ell} \,;$$

if the aforementioned condition does not hold, then recall (3.10) we have $\mathsf{id} \notin \mathcal{A}(S_1, S_2)$. In addition, from $\mathcal{L}(G_\cup) \subset V(G_{\ge 2})$ we have $\mathcal{L}(S_1), \mathcal{L}(S_2) \subset V(S_1) \cap V(S_2)$. Thus, using Lemma 3.10 we see that

$$2\iota\aleph - \tau(G_S) \le \tfrac{2\ell\iota\aleph + 2\aleph - |E(G_S)|}{\ell} \,.$$

The version replacing S by K above holds by similarity. Thus, since $\mathsf{id} \notin \mathcal{A}_\star$ we see that

$$\text{either} \begin{cases} 2\iota\aleph - 1 - \tau(G_S) \le \frac{2\ell\iota\aleph + 2\aleph - |E(G_S)|}{\ell} \\ 2\iota\aleph - \tau(G_K) \le \frac{2\ell\iota\aleph + 2\aleph - |E(G_K)|}{\ell} \end{cases} \text{or} \begin{cases} 2\iota\aleph - \tau(G_S) \le \frac{2\ell\iota\aleph + 2\aleph - |E(G_S)|}{\ell} \\ 2\iota\aleph - 1 - \tau(G_K) \le \frac{2\ell\iota\aleph + 2\aleph - |E(G_K)|}{\ell} \end{cases} .$$

This leads to (4.41), since $\tau(G_\cup) = \tau(G_S) + \tau(G_K)$ and $|E(G_\cup)| = |E(G_S)| + |E(G_K)|$. ∎

Based on Lemma 4.5, we now claim our bound on (4.40). Note that we can write (4.40) as

$$n^{-2+o(1)} \sum_{\substack{|E(G)| \le 4\ell\iota\aleph, |\mathcal{L}(G)| \le 4\aleph \\ G \text{ satisfying (4.41)} \\ \mathrm{ENUM}''(G) > 0}} \big(\tfrac{\epsilon^2 \lambda s}{n}\big)^{|E(G)|} \cdot \mathrm{ENUM}''(G) \,, \tag{4.42}$$

where G is summed over $G \subset \mathcal{K}_n$ with $|E(G)| \le 4\ell\iota\aleph, |\mathcal{L}(G)| \le 4\aleph$ and the graphs satisfying inequality (4.41) with $G_\cup$ replaced by G, and

$$\mathrm{ENUM}''(G) = \#\big\{(S_1, S_2; T_1, T_2) : S_1 \cup S_2 \cup T_1 \cup T_2 = G, \mathcal{L}(G) \subset G_{\ge 2}(S_1, S_2, T_1, T_2)\big\}\,.$$

By Lemma A.5, we have

$$\mathrm{ENUM}''(G) \le \binom{4\ell\aleph + 4\aleph}{\aleph}^4 \Big((\ell\aleph)^2 2^{5\tau(G)+5}\Big)^{4\iota\aleph}\,.$$

Therefore, we have (below $\mathbf{G}$ is summed over $|E(\mathbf{G})| = x, \tau(\mathbf{G}) = y$ and then also over x, y)

$$\begin{aligned}
(4.42) \le{}& n^{-2+o(1)} \sum_{\substack{|E(G)|\le 4\ell\iota\aleph\\ G \text{ satisfying (4.41)}\\ \mathrm{ENUM}''(G)>0}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(G)|} \binom{4\ell\aleph + 4\aleph}{\aleph}^4 \Big((\ell\aleph)^2 2^{5\tau(G)+5}\Big)^{4\iota\aleph}\\
\le{}& n^{-2+o(1)} \sum_{\substack{|E(\mathbf{G})|\le 4\ell\iota\aleph\\ \mathbf{G} \text{ satisfying (4.41)}\\ \mathrm{ENUM}''(G)>0}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(\mathbf{G})|} (\ell\aleph)^{\tau(\mathbf{G})} \#\{G \subset \mathcal{K}_n : G \cong \mathbf{G}\}\\
\le{}& n^{-2+o(1)} \sum_{\substack{0\le x\le 4\ell\iota\aleph\\ \ell(4\iota\aleph-1-y)\le 2(4\ell\iota\aleph+8\aleph-x)}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{x} (\ell\aleph)^y n^{x-y} (\ell\aleph)^{2y}\\
\le{}& n^{-2+o(1)} \sum_{\substack{0\le x\le 4\ell\iota\aleph\\ 4\iota\aleph-1-y\le 2(4\ell\iota\aleph+8\aleph-x)/\ell}} (\epsilon^2\lambda s)^x \big(\tfrac{(\ell\aleph)^3}{n}\big)^y\\
={}& n^{-2+o(1)} (\epsilon^2\lambda s)^{4\ell\iota\aleph} \sum_{0\le x\le 4\ell\iota\aleph} \big(\tfrac{(\ell\aleph)^3}{n}\big)^{4\iota\aleph-1} \big(\tfrac{(\epsilon^2\lambda s)^\ell}{n^2}\big)^{(x-4\ell\iota\aleph)/\ell}
\end{aligned}$$

$$= \frac{(\epsilon^2\lambda s)^{4\ell\iota\aleph}}{n^{2-o(1)}} \cdot n^{-(4\iota\aleph-1)+o(1)} \overset{\text{Proposition 3.1}}{=} o(1) \cdot \mathbb{E}_{\mathbb{P}}[f]^2\,, \tag{4.43}$$

where the second inequality holds by $\binom{4\ell\aleph+4\aleph}{\aleph}^4 (\ell\aleph)^{8\iota\aleph} 2^{20\iota\aleph} = n^{o(1)}$ and $2^{20\iota\aleph} \le \ell\aleph$, and the third inequality holds by Lemmas A.6 and 4.5. Combining (4.40) and (4.43), we see that

$$(4.35) = o(1) \cdot \mathbb{E}_{\mathbb{P}}[f]^2\,. \tag{4.44}$$

Combined with (4.37), this completes the proof of Item (2) in Proposition 4.1.

5 Approximating the statistic In this section, we put together an algorithm to approximately compute the statistic $f = f(A, B)$ defined in (2.8). The key starting point is to replace self-avoiding paths with non-backtracking paths, simplifying the computation while retaining essential structural properties.

Definition 5.1. *For each $\mathbf{H} \in \mathcal{H}$, we say a multigraph $S \vdash_A \mathbf{H}$, if (counting edge multiplicities) S can be decomposed into a tree $\mathsf{T}(S)$ and $\iota\aleph$ non-backtracking paths $\mathsf{M}_1(S), \dots, \mathsf{M}_{\iota\aleph}(S)$ such that the following conditions hold:*

(1) $V(\mathsf{T}(S)) \subset [n] \setminus \mathcal{J}_A$;

(2) $\mathrm{EndP}(\mathsf{M}_i(S)) = \{u_i, v_i\}$ *where* $u_i, v_i \in V(\mathsf{T}(S))$ *and the neighbors of* u_i, v_i *in* $\mathsf{M}_i(S)$ *are in* $\mathcal{J}_A$;

(3) Denoting $\mathsf{P}(S) = \{(u_1, v_1), \dots, (u_{\iota\aleph}, v_{\iota\aleph})\}$, there exists a graph isomorphism $\varphi : S \to \mathbf{H}$ such that φ maps $\mathsf{T}(S)$ to $\mathsf{T}(\mathbf{H})$, maps $\mathsf{P}(S)$ to $\mathsf{P}(\mathbf{H})$ and maps $\mathsf{M}_k(S)$ to $\mathsf{M}_k(\mathbf{H})$ for $1 \le k \le \iota\aleph$.

Similarly, We define $S \vdash_B \mathbf{H}$ with respect to $\mathcal{J}_B$. When the context is clear, we simply write $S_1 \vdash \mathbf{H}$ instead of $S_1 \vdash_A \mathbf{H}$, and similarly for $S_2 \vdash \mathbf{H}$. Given $S_1, S_2 \vdash \mathbf{H} \in \mathcal{H}$, we similarly write $\phi_{S_1,S_2}(A, B) = \phi_{S_1}(A)\phi_{S_2}(B)$, where for $S \vdash \mathbf{H}$

$$\phi_S(X) = \prod_{(i,j)\in E(\mathsf{T}(S))} \frac{X_{i,j} - \frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1 - \frac{\lambda s}{n})}} \cdot \prod_{k=1}^{\iota\aleph} \prod_{(i,j)\in E(\mathsf{M}_k(S))} \frac{X_{i,j} - \frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1 - \frac{\lambda s}{n})}}\,.$$

Finally, we define

$$\widetilde{f} = \widetilde{f}(A,B) = \sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{\aleph-1}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{\aleph+\ell\iota\aleph}} \sum_{S_1,S_2\vdash\mathbf{H}} \phi_{S_1,S_2}(A,B)\,. \tag{5.1}$$

Lemma 5.2. *We have that*

$$\frac{\widetilde{f}(A,B)-f(A,B)}{\mathbb{E}_{\mathbb{P}}[f(A,B)]} \longrightarrow 0 \tag{5.2}$$

in probability under both $\mathbb{P}$ and $\mathbb{Q}$.

The proof of Lemma 5.2 is provided in Section C.7. We now show that $\widetilde{f}(A,B)$ can be approximated efficiently. To this end, we proceed as follows: Given a standardized adjacency matrix M of a graph on $[n]$, we generate a random coloring $\mu:[n]\to[\aleph]$ that assigns a color to each vertex of M from the color set $[\aleph]$ independently and uniformly at random. Given any $V\subset[n]$, let $\chi_\mu(V)$ be the indicator for the event that $\mu(V)$ is colorful, i.e., $\mu(x)\neq\mu(y)$ for any distinct $x,y\in V$. In particular, if $|V|=\aleph$, then $\chi_\mu(V)=1$ with probability

$$r = \frac{\aleph!}{\aleph^\aleph}\,. \tag{5.3}$$

For any graph $\mathbf{H}$ with $\aleph$ vertices, we define

$$X_{\mathbf{H}}(M,\mu) := \sum_{S\vdash\mathbf{H}} \chi_\mu(V(\mathsf{T}(S))) \prod_{(i,j)\in E(S)} M_{i,j}\,. \tag{5.4}$$

This construction ensures that $\frac{1}{r}X_{\mathbf{H}}(M,\mu)$ is an unbiased estimator of

$$\sum_{S\vdash\mathbf{H}} \prod_{(i,j)\in E(S)} M_{i,j}\,.$$

For $t\geq 1$, we generate $2t$ random colorings $\{\mu_i : 1\leq i\leq t\}$ and $\{\nu_j : 1\leq j\leq t\}$ that are independent copies of μ. Then, we define (denote $\overline{A}_{i,j}=A_{i,j}-\frac{\lambda s}{n}$ and $\overline{B}_{i,j}=B_{i,j}-\frac{\lambda s}{n}$ respectively)

$$\overline{f}(A,B) = \frac{1}{r^2}\sum_{\mathbf{H}\in\mathcal{H}} \frac{s^{\aleph-1}\operatorname{Aut}(\mathsf{T}(\mathbf{H}))(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{\aleph+\ell\iota\aleph}} \Big(\frac{1}{t}\sum_{i=1}^{t} X_{\mathbf{H}}(\overline{A},\mu_i)\Big)\Big(\frac{1}{t}\sum_{j=1}^{t} X_{\mathbf{H}}(\overline{B},\nu_i)\Big)\,. \tag{5.5}$$

Similarly as [34, Proposition 3], we can show that when $t=\frac{1}{r}$

$$\frac{\overline{f}(A,B)-\widetilde{f}(A,B)}{\mathbb{E}_{\mathbb{P}}[f(A,B)]} \xrightarrow{L^2} 0 \tag{5.6}$$

under both $\mathbb{P}$ and $\mathbb{Q}$. Combining (5.2) and (5.6), we obtain

$$\mathbb{Q}(\overline{f}(A,B)\geq\tau) + \mathbb{P}(\overline{f}(A,B)\leq\tau) = o(1)\,, \tag{5.7}$$

where the threshold τ is chosen as $\tau = C\mathbb{E}_{\mathbb{P}}[f_{\mathcal{T}}(A,B)]$ for any fixed constant $0<C<1$. It remains to show that $\overline{f}(A,B)$ can be computed efficiently, as in the next lemma.

Lemma 5.3. *There exists an algorithm with running time $O(n^C)$ (see Algorithm 5.2) to compute $X_{\mathbf{H}}(M,\mu)$ given any $\mathbf{H}\in\mathcal{H}$, a weighted graph M on $[n]$, and a coloring $\mu:[n]\to[\aleph]$.*

Provided with Lemma 5.3, we can calculate $\overline{f}_{A,B}$ using Algorithm 5.1 below.

Algorithm 5.1 Computation of $\overline{f}_{A,B}$

1: **Input:** Adjacency matrices A and B, correlation parameter s, divergence parameter ϵ, edge-density parameter λ.

Choose $\aleph, \ell$ and ι according to (2.6) and (2.3). Let $t = \frac{\aleph^{\aleph}}{\aleph!}$.
Apply the constant-time free tree generation algorithm in [16, 53] to list all non-isomorphic, unrooted unlabeled trees with $\aleph$ edges and return $\mathcal{T}'$ as the resulting set.
For each $\mathbf{T} \in \mathcal{T}'$, check if there exists $\mathbf{o} \in V(\mathbf{T})$ such that $(\mathbf{T}, \mathbf{o})$ satisfies Definition 2.6. Collect these satisfying trees in $\mathcal{T}$.
For each $\mathbf{T} \in \mathcal{T}$, compute $\mathrm{Aut}(\mathbf{H})$ using the algorithm in [10].
For each $\mathbf{T} \in \mathcal{T}$, enumerate all $V \subset V(\mathbf{T})$ and select $\mathcal{S}(\mathbf{T})$ according to Theorem 2.8.
Construct $\mathcal{H} = \mathcal{H}(\mathcal{T}, \mathcal{S}(\mathcal{T}))$ following Definition 2.3.
Generate i.i.d. random colorings $\{\mu_i : 1 \leq i \leq t\}$ and $\{\nu_j : 1 \leq j \leq t\}$ that map $[n]$ to $[\aleph]$ uniformly at random.
for each $1 \leq i \leq t$ **do**
    For each $\mathbf{H} \in \widetilde{\mathcal{H}}$, compute $X_{\mathbf{H}}(\overline{A}, \mu_i)$ and $X_{\mathbf{H}}(\overline{B}, \nu_i)$ via Algorithm 5.2.
end for
Compute $\overline{f}(A, B)$ according to (5.5).
Output: $\overline{f}(A, B)$.

Now we show that Algorithm 5.1 runs in polynomial time.

LEMMA 5.4. *The running time of Algorithm 5.1 is $O(n^{C+o(1)})$.*

Proof. The constant-time free tree generation algorithm provided in [16, 53] returns a list $\mathcal{T}'$ of all non-isomorphic unrooted trees in time linear in the total number of trees. [10] provided an algorithm to compute the automorphism group of a given tree in time linear in the size of the tree. For all $\mathbf{T} \in \mathcal{T}'$, determining whether $\mathbf{T} \in \mathcal{T}$ takes time $O(\aleph!) = n^{o(1)}$ and selecting $\mathcal{S}(\mathbf{T})$ takes time $O(\aleph^{\aleph}) = n^{o(1)}$. For each $1 \leq i \leq t$, it takes time $O(n^{C+o(1)})$ to calculate $X_{\mathbf{H}}(\overline{A}, \mu_i)$ and $X_{\mathbf{H}}(\overline{B}, \nu_i)$. Thus, the total running time of Algorithm 5.1 is bounded by

$$n^{o(1)} + n^{o(1)} \cdot n^{o(1)} \cdot t \cdot O(n^{C+o(1)}) = n^{C+o(1)},$$

as desired. ∎

We point out that the main result Theorem 1.3 immediately follows from Theorem 2.12, Equation (5.7) and Lemma 5.4. It remains to prove Lemma 5.3. We now describe how to calculate $X_{\mathbf{H}}(M, \mu)$ via dynamical programming. For $l \geq 1$, denote by $\mathrm{NB}_l(x, y)$ the collection of non-backtracking paths from x to y with l edges. Denote $\mathcal{H}'$ to be the set of all $\mathbf{H}' = (\mathbf{T}', \mathsf{P}'(\mathbf{T}'))$ such that there exists $(\mathbf{T}, \mathsf{P}(\mathbf{T})) \in \mathcal{H}$ with $\mathbf{T}' \subset \mathbf{T}$ and $\mathsf{P}'(\mathbf{T}')$ is the restriction of $\mathsf{P}(\mathbf{T})$ in $\mathbf{T}'$.

Algorithm 5.2 Computation of $X_{\mathbf{H}}(M, \mu)$

Input: A weighted host graph M on $[n]$ with its vertices colored by μ, an element $\mathbf{H} \in \mathcal{H}'$, and a given subset $\mathcal{J}_M$ of $[n]$.
For each $x, y \in [n] \setminus \mathcal{J}_M$ use [39, Section 3.2] to calculate

$$\mathcal{L}(x, y) = \sum_{\substack{\gamma \in \mathrm{NB}_\ell(x,y) \\ (N_\gamma(x) \cup N_\gamma(y)) \cap ([n] \setminus \mathcal{J}_M) = \emptyset}} \Big(\prod_{e \in \gamma} M_e \Big).$$

Choose an arbitrary root $\mathbf{o} \in V(\mathsf{T}(\mathbf{H}))$. If $\mathbf{H}$ is a single point we simply let

$$Y(x, \mathbf{H}, \{c\}, \mu) = \mathbf{1}_{\{\mu_x = c\}} \text{ for each } x \in [n].$$

if $\mathbf{o} \notin \mathsf{Vert}(\mathsf{P}(\mathbf{H}))$ **then**
    Using Item (1) of Definition 2.6, by removing all the edges adjacent to $\mathbf{o}$ in $\mathbf{H}$, $\mathbf{H}$ is partitioned into $K \leq \log^2(\iota^{-1})$ rooted subgraphs $\{\mathbf{H}_{a_1}, \mathbf{H}_{a_2}, \ldots, \mathbf{H}_{a_K}\}$ with $\mathfrak{R}(\mathbf{H}_{a_i}) = \mathbf{o}_i$ adjacent to $\mathbf{o}$ in $\mathbf{H}$.
    For every $x \in [n] \setminus \mathcal{J}_M$ and every subset $C \subset [\aleph]$ of colors with $|C| = |V(\mathsf{T}(\mathbf{H_o}))|$ (note that $\mathbf{H_o} = \mathbf{H}$), compute recursively for each $x \in [n]$

$$Y(x, \mathbf{H_o}, C, \mu) = \sum_{\substack{y_1, \ldots, y_K \in [n] \setminus \mathcal{J}_M \\ y_i \neq y_j, y_i \neq x \text{ for all } i \neq j \in [K]}} \sum_{C = C_0 \sqcup \ldots \sqcup C_K} \prod_{i=1}^{K} \Big(Y(y_i, \mathbf{H}_{a_i}, C_i, \mu) M_{x, y_i} \Big). \tag{5.8}$$

7: **else**

8: There exists $\mathbf{u}$ such that $(\mathbf{o}, \mathbf{u}) \in \mathsf{P}(\mathbf{H})$. Denote $V_{\mathbf{o}} = \{\mathbf{v} \in V(\mathbf{H}) : \mathsf{Dist}_{\mathbf{H}}(\mathbf{o}, \mathbf{v}) \leq 2(\log\log(\iota^{-1}))^2\}$. From Property (2) of Lemma 2.10 we see that $|V_{\mathbf{o}}| \leq e^{4(\log\log(\iota^{-1}))^3}$. In addition, from Property (3) of Lemma 2.10, we see that $V_{\mathbf{o}} \cap \mathsf{Vert}(\mathsf{P}(\mathbf{H})) = \{\mathbf{o}, \mathbf{u}\}$. Denote the subgraph of $\mathbf{H}_{\mathbf{o}}$ induced by $V_{\mathbf{o}}$ as $\mathbf{H}_{a_0}$. By removing $\mathbf{H}_{a_0}$ from $\mathbf{H}$, $\mathbf{H}$ is partitioned into at most $K \leq \exp(10(\log\log(\iota^{-1}))^3)$ rooted subgraphs $\{\mathbf{H}_{a_1}, \mathbf{H}_{a_2}, \ldots, \mathbf{H}_{a_K}\}$ with $\mathfrak{R}(\mathbf{H}_{a_i}) = \mathbf{o}_i$ connected to $\mathbf{H}_{a_0}$. The unique vertex in $\mathbf{H}_{a_0}$ connected to $\mathbf{o}_i$ in $\mathbf{H}$ is denoted by $\mathbf{w}_i$.

9: For every $x \in [n] \setminus \mathcal{J}_M$ and every subset $C \subset [\aleph]$ of colors with $|C| = |V(\mathsf{T}(\mathbf{H}_{\mathbf{o}}))|$, compute recursively

$$
\begin{aligned}
Y(x, \mathbf{H}_{\mathbf{o}}, C, \mu) = & \sum_{\substack{B_x \subset [n]\setminus \mathcal{J}_M : x \in B_x \\ |B_x| = |V_{\mathbf{o}}|}} \sum_{C = C_0 \sqcup \ldots \sqcup C_K} \sum_{\substack{((w_i), x, z, O) : O \cong \mathbf{H}_{a_0}, V(O) = B_x \\ ((w_i), x, z, O) \hookrightarrow ((\mathbf{w}_i), \mathbf{o}, \mathbf{u}, \mathsf{T}(\mathbf{H}))}} \mathcal{L}(x, z) \prod_{(i,j) \in O} M_{i,j} \\
& \times \sum_{\substack{y_1, \ldots, y_K \in [n]\setminus \mathcal{J}_M \\ y_i \neq y_j, y_i \notin B_x}} \prod_{i=0}^{K} \Big(M_{w_i, y_i} Y(y_i, \mathbf{H}_{a_i}, C_i, \mu) \Big) ,
\end{aligned} \tag{5.9}
$$

where

$$
Y(x, \mathbf{H}_{a_0}, C_0, \mu) = \sum_{\substack{(x, z, O) : O \cong \mathbf{H}_{a_0}, V(O) = B_x \\ (x, z, O) \hookrightarrow (\mathbf{o}, \mathbf{u}, \mathsf{T}(\mathbf{H}))}} \mathcal{L}(x, z) \prod_{(i,j) \in O} M_{i,j}
$$

(above $((w_i), x, z, O) \hookrightarrow ((\mathbf{w}_i), \mathbf{o}, \mathbf{u}, \mathsf{T}(\mathbf{H}))$ means that there exists a bijection which is an isomorphism from O to $\mathsf{T}(\mathbf{H})$ and maps $(w_i), x, z \in V(O)$ to $(\mathbf{w}_i), \mathbf{o}, \mathbf{u} \in V(\mathsf{T}(\mathbf{H}))$).

10: **end if**

11: **Output**: $\sum_{x \in [n] \setminus \mathcal{J}_M} Y(x, \mathbf{H}_{\mathbf{o}}, C, \mu)$.

Proof of Lemma 5.3. Calculating $\mathcal{L}(x, y)$ for all x, y takes time $O(n^2) \cdot O(n^{3+o(1)}) = O(n^{5+o(1)})$. The cardinality of $\mathcal{H}'$ is bounded by $10^{\aleph} \cdot \binom{\aleph}{2\iota\aleph} = n^{o(1)}$, the total number of all subsets $C \subset [\aleph]$ is bounded by $2^{\aleph} = n^{o(1)}$, and the total number of all $(C_0, \ldots, C_K)$ is bounded by $2^{K\aleph} = n^{o(1)}$. Thus, according to (5.8) and (5.9), the total time complexity of computing $Y(x, \mathbf{H}, C, \mu)$ for all $x \in [n]$ and all $C \subset [\aleph]$ is bounded by

$$
O(n^{o(1)}) \cdot O\Big(n^{4(\log\log(\iota^{-1}))^3}\Big) = O\Big(n^{4(\log\log(\iota^{-1}))^3 + o(1)}\Big) .
$$

Thus, the total time complexity of Algorithm 5.2 is bounded by

$$
O(n^{5+o(1)}) + O(n) \cdot O(2^{\aleph}) \cdot O\Big(n^{4(\log\log(\iota^{-1}))^3 + o(1)}\Big) = O\Big(n^{4(\log\log(\iota^{-1}))^3 + 1 + o(1)}\Big) .
$$

The proof for the correctness of Algorithm 5.2 is almost identical to the proof of [34, Lemma 2]; the only difference is that we need to invoke the value of $\mathcal{L}(x, y)$ when we compute $Y(x, \mathbf{H}_{\mathbf{o}}, C, \mu)$ recursively. However, this can be done efficiently as we have already stored the value of all $\mathcal{L}(x, y)$ at Step 2, and we omit further details here for simplicity. ∎

A Preliminary results on graphs

Lemma A.1. *For any $N \geq \log(\iota^{-1})$ and $\mathbf{T} \in \cup_{m \leq N} \mathcal{R}_m$, we have*

$$
\#\big\{\mathbf{T}' \in \cup_{m \leq N} \mathcal{R}_m : \mathbf{T}' \sim \mathbf{T}\big\} \leq e^{6N \log\log(\iota^{-1}) / \log(\iota^{-1})} .
$$

Proof. Note that for any fixed $\mathbf{T} \in \mathcal{R}_N$, in order to have $\mathbf{T}' \sim \mathbf{T}$, there must exist

$$
\begin{aligned}
& k \leq \log^{-1}(\iota^{-1}) \cdot |V(\mathbf{T})|, k' \leq \log^{-1}(\iota^{-1}) \cdot |V(\mathbf{T}')| , \\
& \mathbf{u}_1, \ldots, \mathbf{u}_k \in V(\mathbf{T}), \mathbf{u}'_1, \ldots, \mathbf{u}'_{k'} \in V(\mathbf{T}') , \\
& 0 \leq x_1, \ldots, x_k, x'_1, \ldots, x'_{k'} \leq \log^2(\iota^{-1})
\end{aligned}
$$

such that

$$\mathbf{T} \oplus \mathcal{L}_{\mathbf{u}_1}^{(x_1)} \oplus \ldots \oplus \mathcal{L}_{\mathbf{u}_k}^{(x_k)} \cong \mathbf{T}' \oplus \mathcal{L}_{\mathbf{u}'_1}^{(x'_1)} \oplus \ldots \oplus \mathcal{L}_{\mathbf{u}'_{k'}}^{(x'_{k'})} .$$

Observe that the total number of ways to enumerate $\{\mathbf{u}_k, \mathbf{u}'_{k'}\}$ is bounded by

$$\binom{N}{N/\log(\iota^{-1})}\binom{N+\log(\iota^{-1})N}{N/\log(\iota^{-1})} \le e^{2.1N\log\log(\iota^{-1})/\log(\iota^{-1})} .$$

Additionally, the number of possible choices for $\{x_k, x'_{k'}\}$ is bounded by $\big(\log^2(\iota^{-1})\big)^{4\iota N}$. Given $\{\mathbf{u}_k, \mathbf{u}'_{k'}\}$ and $\{x_k, x'_{k'}\}$, the graph $\mathbf{T} \oplus \mathcal{L}_{\mathbf{u}_1}^{(x_1)} \oplus \ldots \oplus \mathcal{L}_{\mathbf{u}_k}^{(x_k)}$ is also determined. Since the degree of $\mathbf{u}'_i$ (for $1 \le i \le k'$) in $\mathbf{T} \oplus \mathcal{L}_{\mathbf{u}_1}^{(x_1)} \oplus \ldots \oplus \mathcal{L}_{\mathbf{u}_k}^{(x_k)}$ is bounded by $\log^2(\iota^{-1})+1$ (by Item (1) of Definition 2.6 and the fact that the degree of $\mathbf{u}'_i$ in $\mathbf{T} \oplus \mathcal{L}_{\mathbf{u}_1}^{(x_1)} \oplus \ldots \oplus \mathcal{L}_{\mathbf{u}_k}^{(x_k)}$ and the degree of $\mathbf{u}'_i$ in $\mathbf{T}$ differs at most 1), the enumeration of $\mathcal{L}_{\mathbf{u}'_1}^{(x'_1)}, \ldots, \mathcal{L}_{\mathbf{u}'_{k'}}^{(x'_{k'})}$ is bounded by $(\log(\iota^{-1}))^{3N/\log(\iota^{-1})}$. Combining these bounds, we have

$$\begin{aligned}\#\big\{\mathbf{T}' \in \mathcal{H} : \mathbf{T} \sim \mathbf{T}'\big\} &\le e^{2.1N\log\log(\iota^{-1})/\log(\iota^{-1})} \cdot \big(\log(\iota^{-1})\big)^{8\iota N + \frac{3N}{\log(\iota^{-1})}} \\ &\le e^{6N\log\log(\iota^{-1})/\log(\iota^{-1})} ,\end{aligned}$$

as desired. ∎

The following lemmas provide several properties of the subgraphs of a given graph S.

Lemma A.2 ([9], Lemma A.3). *For $H \subset S$, we can decompose $E(S) \setminus E(H)$ into $\mathtt{m}$ cycles $C_1, \ldots, C_{\mathtt{m}}$ and $\mathtt{t}$ paths $P_1, \ldots, P_{\mathtt{t}}$ for some $\mathtt{m}, \mathtt{t} \ge 0$ such that the following hold:*

(1) $C_1, \ldots, C_{\mathtt{m}}$ are vertex-disjoint (i.e., $V(C_{\mathtt{i}}) \cap V(C_{\mathtt{j}}) = \emptyset$ for all $\mathtt{i} \ne \mathtt{j}$) and $V(C_{\mathtt{i}}) \cap V(H) = \emptyset$ for all $1 \le \mathtt{i} \le \mathtt{m}$.

(2) $\mathrm{EndP}(P_{\mathtt{j}}) \subset V(H) \cup (\cup_{\mathtt{i}=1}^{\mathtt{m}} V(C_{\mathtt{i}})) \cup (\cup_{\mathtt{k}=1}^{\mathtt{j}-1} V(P_{\mathtt{k}})) \cup \mathcal{L}(S)$ for $1 \le \mathtt{j} \le \mathtt{t}$.

(3) $\big(V(P_{\mathtt{j}}) \setminus \mathrm{EndP}(P_{\mathtt{j}})\big) \cap \big(V(H) \cup (\cup_{\mathtt{i}=1}^{\mathtt{m}} V(C_{\mathtt{i}})) \cup (\cup_{\mathtt{k}=1}^{\mathtt{j}-1} V(P_{\mathtt{k}})) \cup \mathcal{L}(S)\big) = \emptyset$ for $1 \le \mathtt{j} \le \mathtt{t}$.

(4) $\mathtt{t} = |\mathcal{L}(S) \setminus V(H)| + \tau(S) - \tau(H)$.

Lemma A.3 ([9], Corollary A.4). *For $H \subset S$, we can decompose $E(S) \setminus E(H)$ into $\mathtt{m}$ cycles $C_1, \ldots, C_{\mathtt{m}}$ and $\mathtt{t}$ paths $P_1, \ldots, P_{\mathtt{t}}$ for some $\mathtt{m}, \mathtt{t} \ge 0$ such that the following hold:*

(1) $C_1, \ldots, C_{\mathtt{m}}$ are independent cycles in S.

(2) $V(P_{\mathtt{j}}) \cap \big(V(H) \cup (\cup_{\mathtt{i}=1}^{\mathtt{m}} V(C_{\mathtt{i}})) \cup (\cup_{\mathtt{k} \ne \mathtt{j}} V(P_{\mathtt{k}})) \cup \mathcal{L}(S)\big) = \mathrm{EndP}(P_{\mathtt{j}})$ for $1 \le \mathtt{j} \le \mathtt{t}$.

(3) $\mathtt{t} \le 5(|\mathcal{L}(S) \setminus V(H)| + \tau(S) - \tau(H))$.

Lemma A.4. *Suppose that $H \subset \mathcal{K}_n$ and $L_1, \ldots, L_k$ are self-avoiding paths with $\mathrm{EndP}(L_i) \subset V(H)$ and $|E(L_i)| \le M$. Denoting $S = H \cup L_1 \cup \ldots \cup L_k$, we can decompose $E(S) \setminus E(H)$ into $\mathtt{t}$ self-avoiding paths $P_1, \ldots, P_{\mathtt{t}}$ for some $\mathtt{t} \ge 0$ such that the following hold:*

(1) $V(P_{\mathtt{j}}) \cap \big(V(H) \cup (\cup_{\mathtt{k}=1}^{\mathtt{j}-1} V(P_{\mathtt{k}})) \cup \mathcal{L}(S)\big) = \mathrm{EndP}(P_{\mathtt{j}})$ for $1 \le \mathtt{j} \le \mathtt{t}$.

(2) $|E(P_{\mathtt{i}})| \le M$.

(3) $\mathtt{t} = \tau(S) - \tau(H)$.

Proof. We begin with the case $k = 1$, i.e., when $S = H \cup L_1$. Let $L_1 = (v_0, \ldots, v_{M_1})$ with $M_1 \le M$. It is clear that $V(H) \cap V(L_1)$ splits L_1 into a collection of new self-avoiding paths that we will denote by $\{P_{\mathtt{j}} : 1 \le \mathtt{j} \le \mathtt{t}\}$. Thus we have $\cup_{1 \le \mathtt{j} \le \mathtt{t}} E(P_{\mathtt{j}}) = E(S) \setminus E(H)$, and also Items (1) and (2) are satisfied. Furthermore, each $\mathrm{EndP}(P_{\mathtt{j}})$

is a subset of $V(H)$ by our construction, which yields $V(S) \setminus V(H) = \bigsqcup_{1 \le \mathtt{j} \le \mathtt{t}} \big(V(P_{\mathtt{j}}) \setminus \mathrm{EndP}(P_{\mathtt{j}})\big)$ combined with Item (1). Therefore we have

$$|E(S)| = |E(H)| + \sum_{\mathtt{j}=1}^{\mathtt{t}} |E(P_{\mathtt{j}})|\,,$$
$$|V(S)| = |V(H)| + \sum_{\mathtt{j}=1}^{\mathtt{t}} |V(P_{\mathtt{j}}) \setminus \mathrm{EndP}(P_{\mathtt{j}})| = |V(H)| - \mathtt{t} + \sum_{\mathtt{j}=1}^{\mathtt{t}} |E(P_{\mathtt{j}})|\,,$$

which yields Item (3). This completes our proof for $k = 1$.

For the general case, it suffices to denote $H_i = H \cup L_1 \cup \ldots \cup L_i$ (let $H_0 = H$ for convenience) and apply the case $k = 1$ to $H = H_i$ and $S = H_{i+1}$, and then a simple induction concludes the proof. □

The following lemma provides an upper bound on the number of subgraphs contained in a given connected graph J.

Lemma A.5. *Given $J \subset \mathcal{K}_n$, we have*

$$\#\big\{S \subset J : S \Vdash \mathbf{H} \text{ for some } \mathbf{H} \in \mathcal{H}\big\} \le \binom{|E(J)|}{\aleph} \cdot \aleph^{2\iota\aleph} 2^{\iota\aleph(\aleph + 5(\tau(J)+1))}\,. \tag{A.1}$$

Proof. Note that the number of possible choices of $\mathsf{T}(S)$ is bounded by $\binom{|E(J)|}{\aleph}$, and given $\mathsf{T}(S)$ the number of possible choices of $\mathsf{P}(S)$ is bounded by $\aleph^{2\iota\aleph}$. Given $\mathsf{T}(S)$ and $\mathsf{P}(S)$, each $\mathsf{L}_i(S)$ can be determined as follows: using Lemma A.3, $J \setminus \mathsf{T}(S)$ can be decomposed into self-avoiding paths $P_{\mathtt{1}}, \ldots, P_{\mathtt{t}}$ with $\mathtt{t} \le 5(\tau(J)+1)$ such that Items (1)–(4) of Lemma A.3 hold. Since $\mathsf{L}_i(S)$ is also a self-avoiding path, there exist $E_0 \subset E(\mathsf{T}(S))$ and $I \subset [\mathtt{t}]$ such that

$$E(\mathsf{L}_i(S)) = E_0 \cup \Big(\cup_{\mathtt{i} \in I} E(P_{\mathtt{i}}) \Big)\,.$$

Thus, for each $1 \le i \le \iota\aleph$ the number of possible choices of $\mathsf{L}_i(S)$ is bounded by $2^{\aleph + 5(\tau(J)+1)}$, and the desired result follows from the multiplication principle. □

The final lemma provides an upper bound on the number of graphs $\mathbf{G}$ that have a specific value of $\tau(\mathbf{G})$ and are isomorphic to a union of decorated trees.

Lemma A.6. *Recall Definition 5.1. For $r \le 4$, we have*

$$\#\begin{Bmatrix} \text{unlabeled graph } \mathbf{G} : \exists S_i \subset \mathcal{K}_n, S_i \vdash \mathbf{H}_i \text{ for some } \mathbf{H}_i \in \mathcal{H}, \\ 1 \le i \le r \text{ such that } \cup_{1 \le i \le r} \widetilde{S}_i \cong \mathbf{G}, \tau(\mathbf{G}) = j \end{Bmatrix} \le n^{o(1)} (\ell\aleph)^{5j}\,. \tag{A.2}$$

Proof. First, the enumeration of isomorphic classes for $\cup_{1 \le i \le r} \mathsf{T}(S_i)$ is bounded by $\binom{4\aleph}{2}^{4\aleph - 4} = n^{o(1)}$. Then, given $\cup_{1 \le i \le r} \mathsf{T}(S_i)$, we can decompose $\big(\cup_{1 \le i \le r} \widetilde{S}_i\big) \setminus \big(\cup_{1 \le i \le r} \mathsf{T}(S_i)\big)$ into $\mathtt{t} = 5\tau\big(\cup_{1 \le i \le r} \widetilde{S}_i\big) - 5\tau\big(\cup_{1 \le i \le r} \mathsf{T}(S_i)\big) \le 5(j+4)$ self-avoiding paths (denoted by $\{P_{\mathtt{i}}\}_{1 \le \mathtt{i} \le \mathtt{t}}$) by Lemma A.3 such that for $1 \le \mathtt{i} \le \mathtt{t}$ we have $|E(P_{\mathtt{i}})| \le \ell$ and

$$V(P_{\mathtt{i}}) \cap \Big(V\big(\cup_{1 \le i \le r} \mathsf{T}(S_i)\big) \cup \big(\cup_{\mathtt{k}=1}^{\mathtt{i}-1} V(P_{\mathtt{k}})\big) \cup \mathcal{L}\big(\cup_{1 \le i \le r} S_i\big) \Big) = \mathrm{EndP}(P_{\mathtt{i}})\,.$$

Therefore, we only need to bound the enumeration of $\{P_{\mathtt{i}}\}_{1 \le \mathtt{i} \le \mathtt{t}}$ up to isomorphism. Since it suffices to enumerate the endpoints and length of $P_{\mathtt{i}}$ for $\mathtt{i} = 1, \ldots, \mathtt{t}$ in order to enumerate $\{P_{\mathtt{i}}\}_{1 \le \mathtt{i} \le \mathtt{t}}$ up to isomorphism, the enumeration of $\{P_{\mathtt{i}}\}_{1 \le \mathtt{i} \le \mathtt{t}}$ is bounded by $((\aleph + \iota\aleph\ell)^2 \cdot \ell)^{5(j+4)} = n^{o(1)}(\ell\aleph)^{5j}$ up to isomorphism. Combining the enumerations above yields our desired result. □

B Proofs of Theorems 2.7 and 2.8 In this section, we complete the postponed proofs of Theorems 2.7 and 2.8.

B.1 Proof of Theorem 2.7 This subsection is dedicated to the proof of Theorem 2.7. Recall that for $N \geq 1$ we denote $\mathcal{R}_N$ as the set of all rooted unlabeled trees with size N. Let $\mathcal{R}_N^{(1,2,3,4)}$ be the collection of trees in $\mathcal{R}_N$ that satisfy Items (1)–(4) in Definition 2.6. We first show that it suffices to prove

$$\big|\mathcal{R}^{(1,2,3,4)}_{\lfloor(\aleph-1)/2\rfloor}\big| \geq (\alpha+o(1))^{-\aleph/2}\cdot\exp\big\{-5e^{-\frac{1}{10}\log^2(\iota^{-1})}\aleph/2\big\}\,. \tag{B.1}$$

Indeed, if (B.1) holds, consider all rooted unlabeled trees $\mathbf{T}$ whose root $\mathfrak{R}(\mathbf{T})$ has exactly two children subtrees $\mathbf{T} \hookrightarrow \mathbf{T}_1, \mathbf{T}_2$ with $\mathbf{T}_1 \in \mathcal{R}^{(1,2,3,4)}_{\lfloor(\aleph-1)/2\rfloor}$ and $\mathbf{T}_2 \in \mathcal{R}^{(1,2,3,4)}_{\lceil(\aleph-1)/2\rceil}$. To ensure $\mathbf{T} \in \widetilde{\mathcal{T}}_\aleph$, it suffices to select $\mathbf{T}_1$ and $\mathbf{T}_2$ that satisfy Item (5) in Definition 2.6. Thus, using Lemma A.1 we see that

$$\begin{aligned}\big|\widetilde{\mathcal{T}}_\aleph\big| &\geq \big|\mathcal{R}^{(1,2,3,4)}_{\lfloor(\aleph-1)/2\rfloor}\big|^2 - 2\big|\mathcal{R}_{\lceil(\aleph-1)/2\rceil}\big|\cdot\aleph e^{\frac{6\log\log(\iota^{-1})}{\log(\iota^{-1})}\aleph}\\ &= (\alpha+o(1))^{-\aleph}\cdot\exp\big\{-10e^{-\frac{1}{10}\log^2(\iota^{-1})}\aleph\big\}\,.\end{aligned}$$

The remainder of this subsection is therefore devoted to proving (B.1).

LEMMA B.1. *Denote $\mathcal{R}_N^{(1,2)}$ the set of $\mathbf{T} \subset \mathcal{R}_N$ satisfying Items (1) and (2) in Definition 2.6. We have*

$$\big|\mathcal{R}_N^{(1,2)}\big| \geq \big|\mathcal{R}_N\big|\cdot\exp\big(-e^{-\frac{1}{8}\log^2(\iota^{-1})}N\big)\,.$$

Proof. Let $\mathtt{L} = \log^2(\iota^{-1})$. Denote $\mathcal{R}_N^{(1)}$ to be the set of $\mathbf{T} \subset \mathcal{R}_N$ such that each vertex in $V(\mathbf{T})$ has at most $\mathtt{L}$ children (i.e, satisfying Item (1) in Definition 2.6). It has been shown in [21, Theorem 1] that

$$\big|\mathcal{R}_N^{(1)}\big| \geq \big|\mathcal{R}_N\big|\cdot\exp\big(-e^{-\frac{1}{4}\mathtt{L}}N\big)\,.$$

Thus, it suffices to show that

$$\big|\mathcal{R}_N^{(1,2)}\big| \geq \big|\mathcal{R}_N^{(1)}\big|\cdot\exp\big(-e^{-\frac{1}{5}\mathtt{L}}N\big)\,. \tag{B.2}$$

Denote $\Gamma_N = \big|\mathcal{R}_N^{(1)}\big|$ and $\widetilde{\Gamma}_N = \big|\mathcal{R}_N^{(1,2)}\big|$. It was known in [44, Page 585] that

$$\Gamma_{N+1} = \sum_{\substack{\mu_1+\ldots+\mu_N\leq\mathtt{L}\\ \mu_1+2\mu_2+\ldots+N\mu_N=N}}\prod_{i=1}^N\binom{\Gamma_i+\mu_i-1}{\mu_i}\,, \tag{B.3}$$

where for each i, μ_i counts the number of children trees with i vertices and the combinatorial number counts the ways to choose μ_i trees with i vertices (with repetition). In addition, it is easy to check that $\widetilde{\Gamma}_k = \Gamma_k$ for $k \leq \mathtt{L}$. We next derive an induction formula for $\widetilde{\Gamma}_{N+1}$, assuming that

$$\widetilde{\Gamma}_k \geq \Gamma_k\cdot\exp\big(-e^{-\frac{1}{5}\mathtt{L}}k\big) \text{ for } k \leq N\,. \tag{B.4}$$

Clearly, for any $\mathbf{T} \in \mathcal{R}_{N+1}^{(2)}$ with children trees $\mathbf{T}_1,\ldots,\mathbf{T}_m \in \mathcal{R}_N^{(2)}$ such that $\#\{j : |V(\mathbf{T}_j)| = i\} = \mu_i$, we see that $\mathbf{T} \in \mathcal{R}_{N+1}^{(1,2)}$ is equivalent to the requirement that each $\mathbf{T}_i$ is not an arm-path with length $\mathtt{L}-1$. Thus, we see that

$$\begin{aligned}\widetilde{\Gamma}_{N+1} &= \sum_{\substack{\mu_1+\ldots+\mu_N\leq\mathtt{L}\\ \mu_1+2\mu_2+\ldots+N\mu_N=N}}\left(\binom{\widetilde{\Gamma}_{\mathtt{L}-1}+\mu_{\mathtt{L}-1}-2}{\mu_{\mathtt{L}-1}}\cdot\prod_{i\neq\mathtt{L}-1}\binom{\widetilde{\Gamma}_i+\mu_i-1}{\mu_i}\right)\\ &= \sum_{\substack{\mu_1+\ldots+\mu_N\leq\mathtt{L}\\ \mu_1+2\mu_2+\ldots+N\mu_N=N}}\Big(1-\frac{\mu_{\mathtt{L}-1}}{\widetilde{\Gamma}_{\mathtt{L}-1}+\mu_{\mathtt{L}-1}-1}\Big)\cdot\prod_{i=1}^N\binom{\widetilde{\Gamma}_i+\mu_i-1}{\mu_i}\\ &\geq \Big(1-\mathtt{L}e^{-\mathtt{L}}\Big)\cdot\sum_{\substack{\mu_1+\ldots+\mu_N\leq\mathtt{L}\\ \mu_1+2\mu_2+\ldots+N\mu_N=N}}\prod_{i=1}^N\binom{\widetilde{\Gamma}_i+\mu_i-1}{\mu_i}\,,\end{aligned}$$

where the inequality follows from $\mu_{\mathtt{L}-1} \le \mathtt{L}$ (since the degrees are bounded by $\mathtt{L}$) and (2.3) (which implies that $\mathtt{L} \ge M$ and thus $\widetilde{\Gamma}_{\mathtt{L}-1} = \Gamma_{\mathtt{L}-1} \ge \alpha^{-\mathtt{L}}/100\mathtt{L}^2$ from (2.2)). Recalling (B.4), we have

$$\binom{\widetilde{\Gamma}_k + \mu_k - 1}{\mu_k} \ge \binom{\Gamma_k + \mu_k - 1}{\mu_k} \cdot \exp\big(- e^{-\frac{1}{5}\mathtt{L}} k \mu_k \big) .$$

Thus, we have

$$\begin{aligned}\widetilde{\Gamma}_{N+1} &\ge \Big(1 - \mathtt{L}e^{-\mathtt{L}}\Big) \cdot \sum_{\substack{\mu_1+\ldots+\mu_N \le \mathtt{L} \\ \mu_1+2\mu_2+\ldots+N\mu_N = N}} \prod_{i=1}^{N} \exp\big(- e^{-\frac{1}{5}\mathtt{L}} i \mu_i \big) \binom{\Gamma_i + \mu_i - 1}{\mu_i} \\ &= \Big(1 - \mathtt{L}e^{-\mathtt{L}}\Big) \cdot \exp\big(- N e^{-\frac{1}{5}\mathtt{L}} \big) \cdot \sum_{\substack{\mu_1+\ldots+\mu_N \le \mathtt{L} \\ \mu_1+2\mu_2+\ldots+N\mu_N = N}} \prod_{i=1}^{N} \binom{\Gamma_i + \mu_i - 1}{\mu_i} .\end{aligned}$$

Combined with (B.3), this completes the induction proof for (B.4) and thus leads to (B.2). ∎

Lemma B.2. *For all sufficiently large N and $\mathbf{T} \in \mathcal{R}_N^{(1,2)}$, we have $|V(\mathbf{T}_\iota)| \ge \frac{N}{\log^4(\iota^{-1})+1}$.*

Proof. Note that each $\mathbf{T} \in \mathcal{R}_N^{(1,2)}$ can be written as

$$\mathbf{T} = \mathbf{T}_\iota \oplus \big(\oplus_{\mathbf{u} \in V(\mathbf{T}_\iota)} \mathbf{T}_{\mathbf{u}} \big) ,$$

where $\mathbf{T}_{\mathbf{u}}$ is the unique descendant-tree of $\mathbf{T}$ induced by $\mathbf{u}$ and its descendants. From the definition of $\mathbf{T}_\iota$ and the fact that $\mathbf{T} \in \mathcal{R}_N^{(1,2)}$, we see that each $\mathbf{T}_{\mathbf{u}} \setminus \mathbf{T}_\iota$ is the union of at most $\log^2(\iota^{-1})$ trees, each of size bounded by $\log^2(\iota^{-1})$; this yields that $|V(\mathbf{T}_{\mathbf{u}} \setminus \mathbf{T}_\iota)| \le \log^4(\iota^{-1})$, concluding the desired result easily. ∎

Lemma B.3. *Denote $\mathcal{R}_N^{(1,2,4)}$ the set of $\mathbf{T} \subset \mathcal{R}_N$ satisfying Items (1), (2), (4) in Definition 2.6. Then*

$$\big|\mathcal{R}_N^{(1,2,4)}\big| \ge \big|\mathcal{R}_N^{(1,2)}\big| \cdot \exp\big(- e^{-\frac{1}{10}\log^2(\iota^{-1})} N \big) .$$

Proof. Define μ_N^* and μ_N to be the uniform measure on $\mathcal{R}_N^{(1,2,4)}$ and $\mathcal{R}_N^{(1,2)}$, respectively. Note that the lemma is equivalent to

$$\mu_N^*(\mathbf{T})/\mu_N(\mathbf{T}) \le \exp\big(e^{-\frac{1}{10}\log^2(\iota^{-1})} N \big) \text{ for all } \mathbf{T} \in \mathcal{R}_N^{(1,2,4)} . \tag{B.5}$$

We prove (B.5) by induction. Clearly (B.5) holds for $N \le \log^2(\iota^{-1})$. Now suppose that (B.5) holds for all $m \le N$. For each

$$a + b \le \log^2(\iota^{-1}) , \gamma_1 + \ldots + \gamma_{a+b} = N , \gamma_i \le \log^2(\iota^{-1}) \text{ if and only if } i \le a ,$$

denote $\mathcal{W}_{a,b;\gamma_1,\ldots,\gamma_{a+b}}$ the set of $\mathbf{T} \in \mathcal{R}_{N+1}^{(1,2)}$ such that $\mathbf{T}$ has $a+b$ children trees $\mathbf{T}_1, \ldots, \mathbf{T}_{a+b}$ with $|V(\mathbf{T}_i)| = \gamma_i$ for each $1 \le i \le a+b$; we will ignore the orders between $\mathbf{T}_i$'s with the same cardinality. In addition, denote $\widetilde{\mathcal{W}}_{\gamma_1,\ldots,\gamma_a} = \mathcal{W}_{a,0;\gamma_1,\ldots,\gamma_a}$. Denote $\mu_{a,b;\gamma_1,\ldots,\gamma_{a+b}}$ the uniform measure on $\mathcal{W}_{a,b;\gamma_1,\ldots,\gamma_{a+b}}$ and $\mu^*_{a,b;\gamma_1,\ldots,\gamma_{a+b}}$ the uniform measure on $\mathcal{R}_{N+1}^{(1,2,4)} \cap \mathcal{W}_{a,b;\gamma_1,\ldots,\gamma_{a+b}}$; denote $\widetilde{\mu}_{\gamma_1,\ldots,\gamma_a} = \mu_{a,0;\gamma_1,\ldots,\gamma_a}$. For any fixed $\{\gamma_1, \ldots, \gamma_{a+b}\}$, define

$$\varkappa = \varkappa(\gamma_1, \ldots, \gamma_{a+b}) = \prod_{m > \log^2(\iota^{-1})} \varkappa_m! , \text{ where } \varkappa_m = \#\{a+1 \le i \le a+b : \gamma_i = m\} .$$

Denote $\mathbf{T}_1 \oplus \ldots \oplus \mathbf{T}_a$ to be the unlabeled rooted tree with children trees given by $\mathbf{T}_1, \ldots, \mathbf{T}_a$. From the definition of μ^* and $\mathcal{R}_{N+1}^{(1,2,4)}$, we then have (for two measures μ and ν, we define $\mu \otimes \nu$ to be their product measure)

$$\begin{aligned}&\mu^*_{a,b;\gamma_1,\ldots,\gamma_{a+b}}(\mathbf{T}) \\ &= \mu_{a,b;\gamma_1,\ldots,\gamma_{a+b}}\big(\mathbf{T} \mid \mathbf{T}_{a+i} \in \mathcal{R}_{\gamma_{a+i}}^{(1,2,4)} \text{ for } 1 \le i \le b, \mathbf{T}_{a+i} \not\sim \mathbf{T}_{a+j} \text{ for all } 1 \le i \ne j \le b\big) \\ &= \varkappa \cdot \widetilde{\mu}_{\gamma_1,\ldots,\gamma_a} \otimes \mu^*_{\gamma_{a+1}} \otimes \ldots \otimes \mu^*_{\gamma_{a+b}}(\mathbf{T}_1 \oplus \ldots \oplus \mathbf{T}_a ; \mathbf{T}_{a+1}, \ldots, \mathbf{T}_{a+b} \mid \mathcal{G}) ,\end{aligned} \tag{B.6}$$

where

$$\mathcal{G}=\left\{\mathbf{T}_{a+i}\not\sim\mathbf{T}_{a+j}\text{ for all }1\le i\ne j\le b\right\}.$$

(Note that $\mathbf{T}_{a+i}\not\sim\mathbf{T}_{a+j}$ implies $\mathbf{T}_{a+i}\ne\mathbf{T}_{a+j}$, and the term $\varkappa$ comes from the enumeration that we can permute $\mathbf{T}_{a+i}$'s with the same cardinality.) Note that we have

$$\begin{aligned}\left|\mathcal{W}_{a,b;\gamma_1,\dots,\gamma_{a+b}}\right|&=\left|\widetilde{\mathcal{W}}_{\gamma_1,\dots,\gamma_a}\right|\cdot\prod_{m>\log^2(\iota^{-1})}\binom{|\mathcal{R}^{(1,2)}_m|+\varkappa_m-1}{\varkappa_m}\ge\left|\widetilde{\mathcal{W}}_{\gamma_1,\dots,\gamma_a}\right|\prod_{m>\log^2(\iota^{-1})}\frac{|\mathcal{R}^{(1,2)}_m|^{\varkappa_m}}{\varkappa_m!}\\&=\prod_{m>\log^2(\iota^{-1})}\frac{1}{\varkappa_m!}\cdot\left|\widetilde{\mathcal{W}}_{\gamma_1,\dots,\gamma_a}\right|\prod_{a+1\le i\le a+b}\Big(\left|\mathcal{R}^{(1,2)}_{\gamma_i}\right|+b\Big).\end{aligned}\tag{B.7}$$

By Lemma A.1 and a simple union bound, we have

$$\begin{aligned}&\widetilde{\mu}_{\gamma_1,\dots,\gamma_a}\otimes\mu^*_{\gamma_{a+1}}\otimes\dots\otimes\mu^*_{\gamma_{a+b}}(\exists i\ne j\,,\mathbf{T}_{a+i}\sim\mathbf{T}_{a+j})\\\le&\sum_{1\le i<j\le b}\mu^*_{\gamma_{a+i}}\otimes\mu^*_{\gamma_{a+j}}(\mathbf{T}_{a+i}\sim\mathbf{T}_{a+j})\\\le&\sum_{1\le i<j\le b}\exp\big(e^{-\frac{1}{10}\log^2(\iota^{-1})}(\gamma_{a+i}+\gamma_{a+j})\big)\cdot\mu_{\gamma_{a+i}}\otimes\mu_{\gamma_{a+j}}(\mathbf{T}_{a+i}\sim\mathbf{T}_{a+j})\\\le&\sum_{1\le i<j\le b}\exp\big(e^{-\frac{1}{10}\log^2(\iota^{-1})}(\gamma_{a+i}+\gamma_{a+j})\big)\cdot\alpha^{\gamma_{a+j}}\gamma_{a+i}^{1.5}\gamma_{a+j}^{1.5}\#\{\mathbf{T}\in\mathcal{R}_{\gamma_{a+j}}:\mathbf{T}\sim\mathbf{T}_{a+i}\}\\\le&\, b^2\cdot\exp\big((e^{-\frac{1}{10}\log^2(\iota^{-1})}+\tfrac{7\log\log(\iota^{-1})}{\log(\iota^{-1})}+\log\alpha)\cdot2\log^2(\iota^{-1})\big)\le\exp\big(-\tfrac{1}{8}\log^2(\iota^{-1})\big)\,,\end{aligned}$$

where the second inequality follows from (B.5), the third inequality follows from Lemma A.1, the fourth inequality follows from $\gamma_{a+i}\ge\log^2(\iota^{-1})$ and the last inequality follows from $b\le\log^2(\iota^{-1})$. Thus, we have

$$\begin{aligned}&\widetilde{\mu}_{\gamma_1,\dots,\gamma_a}\otimes\mu^*_{\gamma_{a+1}}\otimes\dots\otimes\mu^*_{\gamma_{a+b}}(\mathbf{T}_1\oplus\dots\oplus\mathbf{T}_a;\mathbf{T}_{a+1},\dots,\mathbf{T}_{a+b}\mid\mathcal{G})\\\le&\Big(1-e^{-\frac{1}{8}\log^2(\iota^{-1})}\Big)^{-1}\cdot\widetilde{\mu}_{\gamma_1,\dots,\gamma_a}\otimes\mu^*_{\gamma_{a+1}}\otimes\dots\otimes\mu^*_{\gamma_{a+b}}(\mathbf{T}_1\oplus\dots\oplus\mathbf{T}_a;\mathbf{T}_{a+1},\dots,\mathbf{T}_{a+b})\\\le&\big(1-e^{-\frac{1}{8}\log^2(\iota^{-1})}\big)^{-1}\cdot\exp\big(e^{-\frac{1}{10}\log^2(\iota^{-1})}(\gamma_{a+1}+\dots+\gamma_{a+b})\big)\times\\&\widetilde{\mu}_{\gamma_1,\dots,\gamma_a}\otimes\mu_{\gamma_{a+1}}\otimes\dots\otimes\mu_{\gamma_{a+b}}(\mathbf{T}_1\oplus\dots\oplus\mathbf{T}_a;\mathbf{T}_{a+1},\dots,\mathbf{T}_{a+b})\,,\end{aligned}$$

where the second inequality follows from (B.5). Plugging this estimation into (B.6), we have that for any $\mathbf{T}\in\mathcal{R}^{(1,2,4)}_{N+1}\cap\mathcal{W}_{a,b;\gamma_1,\dots,\gamma_{a+b}}$,

$$\begin{aligned}\frac{\mu^*_{a,b;\gamma_1,\dots,\gamma_{a+b}}(\mathbf{T})}{\mu_{a,b;\gamma_1,\dots,\gamma_{a+b}}(\mathbf{T})}&\le\big(1-e^{-\frac{1}{8}\log^2(\iota^{-1})}\big)^{-1}\cdot\exp\big(e^{-\frac{1}{10}\log^2(\iota^{-1})}(\gamma_{a+1}+\dots+\gamma_{a+b})\big)\times\\&\qquad\frac{\varkappa\cdot\widetilde{\mu}_{\gamma_1,\dots,\gamma_a}\otimes\mu_{\gamma_{a+1}}\otimes\dots\otimes\mu_{\gamma_{a+b}}(\mathbf{T}_1\oplus\dots\oplus\mathbf{T}_a;\mathbf{T}_{a+1},\dots,\mathbf{T}_{a+b})}{\mu_{a,b;\gamma_1,\dots,\gamma_{a+b}}(\mathbf{T})}\\&\le\exp\big(e^{-\frac{1}{10}\log^2(\iota^{-1})}(\gamma_{a+1}+\dots+\gamma_{a+b})+e^{-\frac{1}{8}\log^2(\iota^{-1})}+e^{-\frac{1}{2}\log^2(\iota^{-1})}\big)\\&\le\exp\big(e^{-\frac{1}{10}\log^2(\iota^{-1})}(N+1)\big)\,,\end{aligned}\tag{B.8}$$

where in the second inequality we used the fact that

$$\begin{aligned}
&\frac{\widetilde{\mu}_{\gamma_1,\ldots,\gamma_a}\otimes\mu_{\gamma_{a+1}}\otimes\ldots\otimes\mu_{\gamma_{a+b}}(\mathbf{T}_1\oplus\ldots\oplus\mathbf{T}_a;\mathbf{T}_{a+1},\ldots,\mathbf{T}_{a+b})}{\mu_{a,b;\gamma_1,\ldots,\gamma_{a+b}}(\mathbf{T})}\\
&=\frac{|\mathcal{W}_{a,b;\gamma_1,\ldots,\gamma_{a+b}}|}{|\widetilde{\mathcal{W}}_{\gamma_1,\ldots,\gamma_a}||\mathcal{R}^{(1,2)}_{\gamma_{a+1}}|\ldots|\mathcal{R}^{(1,2)}_{\gamma_{a+b}}|}\\
&\leq\frac{|\mathcal{W}_{a,b;\gamma_1,\ldots,\gamma_{a+b}}|}{|\widetilde{\mathcal{W}}_{\gamma_1,\ldots,\gamma_a}|(|\mathcal{R}^{(1,2)}_{\gamma_{a+1}}|+b)\ldots(|\mathcal{R}^{(1,2)}_{\gamma_{a+b}}|+b)}\cdot\Big(\frac{|\mathcal{R}^{(1,2)}_{\log^2(\iota^{-1})}|+\log^2(\iota^{-1})}{|\mathcal{R}^{(1,2)}_{\log^2(\iota^{-1})}|}\Big)^{\log^2(\iota^{-1})}\\
&\overset{(B.7)}{\leq}\varkappa^{-1}\cdot\Big(\frac{|\mathcal{R}^{(1,2)}_{\log^2(\iota^{-1})}|+\log^2(\iota^{-1})}{|\mathcal{R}^{(1,2)}_{\log^2(\iota^{-1})}|}\Big)^{\log^2(\iota^{-1})}\overset{\text{Lemma B.1}}{\leq}\varkappa^{-1}\cdot\exp(-e^{-\frac{1}{2}\log^2(\iota^{-1})})\,.
\end{aligned}$$

Summing (B.8) over $\{(a,b);\gamma_1,\ldots,\gamma_{a+b}\}$ yields (B.5). ∎

We can now complete the proof of Theorem 2.7.

Proof of Theorem 2.7. Combining Lemmas B.1, B.2 and B.3 yields Theorem 2.7. ∎

B.2 Proof of Theorem 2.8 This subsection is dedicated to the proof of Theorem 2.8. To this end, we first establish the following preliminary lemmas.

LEMMA B.4. *For all $\mathbf{T}\in\widetilde{\mathcal{T}}_{\aleph}$ and $K=\frac{\aleph}{\exp(25(\log\log(\iota^{-1}))^3)}$, we can select $W_0=\{(u_1,v_1),\ldots,(u_K,v_K)\}$ satisfying $u_i,v_i\in V(\mathbf{T})$ for $1\leq i\leq K$ such that:*

(1) $\mathsf{Vert}(W_0)\subset V(\mathbf{T}_\iota)$.

(2) For all $u\in\mathsf{Vert}(W_0)$, we have $\mathsf{deg}_{\mathbf{T}}(u)\leq\log^2(\iota^{-1})$.

(3) We have that $10(\log\log(\iota^{-1}))^{10}\leq\mathsf{Dist}_{\mathbf{T}}(u,v)$ for all $u,v\in\mathsf{Vert}(W_0),(u,v)\notin W_0$, and $2(\log\log(\iota^{-1}))^{10}\geq\mathsf{Dist}_{\mathbf{T}}(u,v)\geq(\log\log(\iota^{-1}))^{10}$ for all $(u,v)\in W_0$.

Proof. By Definition 2.6, we see that for $\mathbf{T}\in\widetilde{\mathcal{T}}_{\aleph}$

$$|V(\mathbf{T}_\iota)|\geq\tfrac{\aleph}{\log^4(\iota^{-1})}\,.$$

In addition, for all $u\in V(\mathbf{T})$ we have $\mathsf{deg}_{\mathbf{T}}(u)\leq\log^2(\iota^{-1})$. This implies that for all $u\in V(\mathbf{T})$

$$\#\big\{v\in V(\mathbf{T}):\mathsf{dist}_{\mathbf{T}}(u,v)\leq12(\log\log(\iota^{-1}))^2\big\}\leq\exp\big(25(\log\log(\iota^{-1}))^3\big)\,.$$

Now, suppose that we have already chosen $(u_1,v_1),\ldots,(u_k,v_k)$ for $k<K$. Clearly there exists a vertex $u_{k+1}\in V(\mathbf{T}_\iota)$ such that

$$\mathsf{Dist}_{\mathbf{T}_\iota}(u_{k+1},u_i),\mathsf{Dist}_{\mathbf{T}_\iota}(u_{k+1},v_i)\geq12(\log\log(\iota^{-1}))^2\text{ for all }1\leq i\leq k\,,$$

which yields a valid choice for (u_{k+1},v_{k+1}) and thus completes the proof by induction. ∎

Now we argue that we can choose $W_1,\ldots,W_{\mathtt{M}}$ from W_0. Specifically, our choice relies on randomly sampling a $W\subset W_0$ such that each pair of vertices $(u,v)\in W_0$ is included in W independently with probability $(1+\kappa)\iota\exp(25(\log\log(\iota^{-1}))^3)$. By standard binomial approximation, it is straightforward to check that we can choose $\kappa\in(0,1)$ such that

$$\mathbb{P}(|W|=\iota\aleph)\geq\exp\big(-4e^{-\log^2(\iota^{-1})}\aleph\big)\text{ and }\mathbb{P}(|W|\leq\iota\aleph)\leq\exp\big(-3e^{-\log^{1.5}(\iota^{-1})}\aleph\big)\,.\tag{B.9}$$

LEMMA B.5. *For each $\mathbf{T}\in\widetilde{\mathcal{T}}_{\aleph}$, we choose W_0 as in Lemma B.4. Sample a random subset $W\subset W_0$ by independently including each pair $(u,v)\in W_0$ with probability $(1+\kappa)\iota\exp(12(\log\log(\iota^{-1}))^3)$, where κ is chosen as in Lemma B.4. Then*

$$\begin{aligned}
&\mathbb{P}\Big(|W|=\iota\aleph;|W\cap V(\mathbf{T}')|\leq\big[\log^{-1}(\iota^{-1})\cdot|V(\mathbf{T}')|\big],\forall|V(\mathbf{T}')|\geq\log^2(\iota^{-1}),\mathbf{T}\hookrightarrow\mathbf{T}'\Big)\\
&\geq\ \exp\big(-10e^{-\log^2(\iota^{-1})}\aleph\big)\,.
\end{aligned}$$

Proof. We first show that for each event $\mathcal{A}$ measurable and decreasing in W,

$$\exp\left(4e^{-\log^2(\iota^{-1})}\aleph\right)\cdot\mathbb{P}(|W|=\iota\aleph,\mathcal{A})\geq\mathbb{P}(|W|\geq\iota\aleph,\mathcal{A})\,.$$

In fact, we have

$$\begin{aligned}\mathbb{P}(|W|\geq\iota\aleph,\mathcal{A})&=\sum_{k\geq\iota\aleph}\mathbb{P}(|W|=k,\mathcal{A})=\sum_{k\geq\iota\aleph}\mathbb{P}(|W|=k)\mathbb{P}(\mathcal{A}\mid|W|=k)\\&\leq\sum_{k\geq\iota\aleph}\mathbb{P}(|W|=k)\mathbb{P}(\mathcal{A}\mid|W|=\iota\aleph)=\mathbb{P}(|W|\geq\iota\aleph)\mathbb{P}(\mathcal{A}\mid|W|=\iota\aleph)\\&\overset{\text{(B.9)}}{\leq}\exp\left(4e^{-\log^2(\iota^{-1})}\aleph\right)\mathbb{P}(|W|=\iota\aleph)\mathbb{P}(\mathcal{A}\mid|W|=\iota\aleph)\\&=\exp\left(4e^{-\log^2(\iota^{-1})}\aleph\right)\mathbb{P}(|W|=\iota\aleph,\mathcal{A})\,.\end{aligned}$$

Here the first inequality follows from the fact that conditioned on $|W|=k$, W is uniformly distributed over subsets of W_0 with cardinality k and there is a natural coupling over the uniform k-subset of W_0 (denoted as W_k) and the uniform $(k+1)$-subset of W_0 (denoted as W_{k+1}) such that $W_k\subset W_{k+1}$. For each $u\in V(\mathbf{T}_\iota)$, recall that we use $\mathsf{Des}_{\mathbf{T}_\iota}(u)$ to denote the descendant tree of $\mathbf{T}_\iota$ rooted at u. Thus, recalling Definition 2.5, it suffices to show that

$$\begin{aligned}&\mathbb{P}\Big(|W\cap V(\mathbf{T}')|\leq\big[\log^{-1}(\iota^{-1})\cdot|V(\mathbf{T}')|\big]\vee\log(\iota^{-1})\,,\forall\,|V(\mathbf{T}')|\geq\log(\iota^{-1}),\mathbf{T}_\iota\hookrightarrow\mathbf{T}'\Big)\\&\quad\geq\exp\left(-2e^{-\log^2(\iota^{-1})}\aleph\right).\end{aligned}\tag{B.10}$$

For each $u\in V(\mathbf{T}_\iota)$ such that $|V(\mathsf{Des}_{\mathbf{T}_\iota}(u))|\geq\log(\iota^{-1})$, define E_u to be the event

$$\Big\{|W\cap V(\mathbf{T}')|\leq\big[\log^{-1}(\iota^{-1})\cdot|V(\mathbf{T}')|\big]\vee\log(\iota^{-1})\Big\}\,.$$

Since the events $\{E_u:u\in V(\mathbf{T}_\iota)\}$ are decreasing, by the FKG-inequality we have that the left hand side of (B.10) is bounded by

$$\begin{aligned}\mathbb{P}\Big(\cap_{u\in V(\mathbf{T}_\iota)}E_u\Big)&\geq\prod_{\substack{u\in V(\mathbf{T}_\iota)\\|V(\mathsf{Des}_{\mathbf{T}_\iota}(u))|\geq\log^2(\iota^{-1})}}\mathbb{P}\Big(|W\cap V(\mathsf{Des}_{\mathbf{T}_\iota}(u))|\leq\tfrac{|V(\mathsf{Des}_{\mathbf{T}_\iota}(u))|}{\log(\iota^{-1})}\vee\log(\iota^{-1})\Big)\\&\geq\prod_{\substack{u\in V(\mathbf{T}_\iota)\\|V(\mathsf{Des}_{\mathbf{T}_\iota}(u))|\geq\log(\iota^{-1})}}\mathbb{P}\Big(\mathrm{Binom}\big(|V(\mathsf{Des}_{\mathbf{T}_\iota}(u))|,(1+\kappa)\iota\exp(25(\log\log(\iota^{-1}))^3)\big)\\&\qquad\qquad\leq\tfrac{|V(\mathsf{Des}_{\mathbf{T}_\iota}(u))|}{\log(\iota^{-1})}\vee\log(\iota^{-1})\Big)\\&\geq\Big(1-\exp\big(-\log^{1.5}(\iota^{-1})\big)\Big)^{\aleph}\geq\exp\left(-1.5e^{-\log^{1.5}(\iota^{-1})}\aleph\right).\end{aligned}$$

∎

With this, we can complete the proof of Theorem 2.8.

Proof of Theorem 2.8. Suppose that $W_1,\dots W_{\mathtt{j}}\subset W_0$ have already been constructed. We can randomly choose $W_{\mathtt{j}+1}$ provided $\mathtt{j}\leq\exp(\iota(\log\log(\iota^{-1}))^4\aleph)$. Because each $W_{\mathtt{i}}$ is sampled from W_0, Items (1) and (4) in Theorem 2.8 hold automatically. Furthermore, by Lemma B.5, Item (2) holds with probability at least $\exp(-10e^{-\log^{1.5}(\iota^{-1})}\aleph)$. And by applying a Bernstein's inequality for Bernoulli variables and a union bound, Item (3) holds with probability at least $1-\exp(-100\iota\aleph)$. These guarantee the existence of the required $W_{\mathtt{j}+1}$ and completes the proof. (We use the probability argument to show the existence of $W_{\mathtt{j}+1}$; in our algorithm we just use brutal-force search which takes at most $O(2^{\aleph^2})=O(n^{o(1)})$ time.) ∎

C Supplementary Proofs

C.1 Proof of Lemma 3.2 The case that $r+t=1$ and $r+t=2$ can be verified via direct calculations, as

$$\mathbb{E}_{\mathbb{P}_{\sigma,\pi}}\left[\frac{A_{i,j}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right] \mathrel{\mathring{=}} \frac{\sigma_i\sigma_j\sqrt{\epsilon^2\lambda s}}{\sqrt{n}}\,,$$

$$\mathbb{E}_{\mathbb{P}_{\sigma,\pi}}\left[\left(\frac{A_{i,j}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right)\left(\frac{B_{\pi(i),\pi(j)}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right)\right] \mathrel{\mathring{=}} s(1+\epsilon\sigma_i\sigma_j)\,,$$

$$\mathbb{E}_{\mathbb{P}_{\sigma,\pi}}\left[\left(\frac{A_{i,j}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right)^2\right] = \mathbb{E}_{\mathbb{P}_{\sigma,\pi}}\left[\left(\frac{B_{\pi(i),\pi(j)}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right)^2\right] \mathrel{\mathring{=}} (1+\epsilon\sigma_i\sigma_j)\,.$$

For the case where $r\geq 3$ or $r+t\geq 3$, it suffices to note that in this case

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}_{\sigma,\pi}}\left[\left(\frac{A_{i,j}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right)^r\right] = \mathbb{E}_{\mathbb{P}_{\sigma,\pi}}\left[\left(\frac{B_{i,j}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right)^r\right]\\ \mathrel{\mathring{=}}\ & \left(\tfrac{n}{\lambda s}\right)^{\frac{r}{2}}\cdot\mathbb{P}_{\sigma,\pi}(A_{i,j}=1) \leq (1+\epsilon\sigma_i\sigma_j)\cdot(n/\epsilon^2\lambda s)^{(r-2)/2}\,,\\ &\mathbb{E}_{\mathbb{P}_{\sigma,\pi}}\left[\left(\frac{A_{i,j}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right)^r\left(\frac{B_{\pi(i),\pi(j)}-\frac{\lambda s}{n}}{\left(\frac{\lambda s}{n}(1-\frac{\lambda s}{n})\right)^{1/2}}\right)^t\right]\\ \mathrel{\mathring{=}}\ & \left(\tfrac{n}{\lambda s}\right)^{\frac{r+t}{2}}\mathbb{P}_{\sigma,\pi}(A_{i,j}=B_{\pi(i),\pi(j)}=1) \leq (1+\epsilon\sigma_i\sigma_j)\cdot(n/\epsilon^2\lambda s)^{(r+t-2)/2}\,,\end{aligned}$$

thereby completing the proof.

C.2 Proof of Lemma 3.3 Recall that ν is the uniform distribution on $\{-1,+1\}^n$. By independence, we see that

$$\mathbb{E}_{\sigma\sim\nu}\Big[\prod_{i\in I}(\sigma_{i-1}\sigma_i)\mid\sigma_0,\sigma_l\Big]=0 \text{ if } I\subsetneq[l]\,.$$

Thus,

$$\mathbb{E}_{\sigma\sim\nu}\Big[\prod_{i=1}^{l}\left(a_i+b_i\sigma_{i-1}\sigma_i\right)\mid\sigma_0,\sigma_l\Big]=\prod_{i=1}^{l}a_i+\mathbb{E}_{\sigma\sim\nu}\Big[\prod_{i=1}^{l}(\sigma_{i-1}\sigma_i)\mid\sigma_0,\sigma_l\Big]\cdot\prod_{i=1}^{l}b_i=\prod_{i=1}^{l}a_i+\sigma_0\sigma_l\cdot\prod_{i=1}^{l}b_i\,,$$

completing the proof of Lemma 3.3.

C.3 Proof of Lemma 3.4 Note that

$$\mathbb{E}_{\mathbb{P}_\pi}[\phi_{S_1,S_2}]=\mathbb{E}_{\mathbb{P}_\pi}\left[\prod_{(i,j)\in E(S_1)}\left(\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\right)^{E(S_1)_{i,j}}\prod_{(i,j)\in E(S_2)}\left(\frac{B_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\right)^{E(S_2)_{i,j}}\right].$$

For all $e=(i,j)\in E(S_1)\cup E(S_2)$, recall that $\chi(e)=(\chi_1(e),\chi_2(e))=(E(S_1)_{i,j},E(S_2)_{i,j})$. We have that $\mathbb{E}_{\mathbb{P}_\pi}[\phi_{S_1,S_2}]$ equals to

$$\begin{aligned}&\mathbb{E}_{\sigma\sim\nu}\left[\prod_{(i,j)\in E(S_1)\cup E(S_2)}\left(u_{\chi_1(i,j),\chi_2(i,j)}+v_{\chi_1(i,j),\chi_2(i,j)}\sigma_i\sigma_j\right)\right]\\ =&\sum_{E\subset E(S_1)\cup E(S_2)}\mathbb{E}_{\sigma\sim\nu}\left[\prod_{(i,j)\in E}\left(v_{\chi_1(i,j),\chi_2(i,j)}\sigma_i\sigma_j\right)\prod_{(i,j)\in E(S_1)\cup E(S_2)\setminus E}\left(u_{\chi_1(i,j),\chi_2(i,j)}\right)\right],\end{aligned}$$

which is positive by Lemma 3.2 and the fact that $\mathbb{E}\big[\prod_{(i,j)\in E}\sigma_i\sigma_j\big]\geq 0$.

C.4 Proof of Lemma 3.5 The first result (3.6) is obvious. For (3.8), we may assume that $\pi = \mathsf{id}$ without loss of generality. Denote $G = \widetilde{S}_1 \cup \widetilde{S}_2$. Using Property (3) of Lemma 2.10, we see that

$$\mathcal{L}(G) \subset \mathbb{L} := \mathcal{L}(\mathsf{T}(S_1)) \cup \mathcal{L}(\mathsf{T}(S_2)) .$$

Denote G_0 as the graph with $V(G_0) = V \cup \mathbb{L}$ and $E(G_0) = \emptyset$. We then have $\mathcal{L}(G) \subset V(G_0)$. Applying Lemma A.3 with $G_0 \subset G$, we see that $E(G)$ can be written as $E(G) = P_{\mathtt{1}} \cup \ldots \cup P_{\mathtt{t}} \cup C_{\mathtt{1}} \cup \ldots \cup C_{\mathtt{m}}$ satisfying Items (1)–(4) in Lemma A.3. In particular, we have

$$\mathtt{t} \leq 5\Big(\tau(G) - \tau(G_0)\Big) \leq 5\tau(G) + 5|V| + 5|\mathbb{L}| \leq 5\tau(G) + 5|V| + 10\aleph .$$

Also, conditioned on $\{\sigma_u : u \in V(G_0)\}$, we have that

$$\Bigg\{ \prod_{(i,j) \in P_{\mathtt{k}}} \big(u_{\chi_1(i,j),\chi_2(i,j)} + v_{\chi_1(i,j),\chi_2(i,j)} \sigma_i \sigma_j \big) : \mathtt{1} \leq \mathtt{k} \leq \mathtt{t}, \\ \prod_{(i,j) \in C_{\mathtt{k}}} \big(u_{\chi_1(i,j),\chi_2(i,j)} + v_{\chi_1(i,j),\chi_2(i,j)} \sigma_i \sigma_j \big) : \mathtt{1} \leq \mathtt{k} \leq \mathtt{m} \Bigg\}$$

is a collection of conditionally independent variables. In addition, from Lemma 3.3 we have

$$\begin{aligned}
&\mathbb{E}_{\sigma \sim \nu} \Big[\prod_{(i,j) \in P_{\mathtt{j}}} \big(u_{\chi_1(i,j),\chi_2(i,j)} + v_{\chi_1(i,j),\chi_2(i,j)} \sigma_i \sigma_j \big) \mid \{\sigma_u : u \in U\} \Big] \\
\leq& \prod_{(i,j) \in P_{\mathtt{j}}} u_{\chi_1(i,j),\chi_2(i,j)} + \prod_{w \in \mathrm{EndP}(P_{\mathtt{j}})} \sigma_w \cdot \prod_{(i,j) \in P_{\mathtt{j}}} v_{\chi_1(i,j),\chi_2(i,j)} \\
\leq& \, 2 \prod_{(i,j) \in P_{\mathtt{j}}} \max \big\{ u_{\chi_1(i,j),\chi_2(i,j)}, v_{\chi_1(i,j),\chi_2(i,j)} \big\} .
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathbb{E}_{\sigma \sim \nu} \Big[\prod_{(i,j) \in C_{\mathtt{j}}} \big(u_{\chi_1(i,j),\chi_2(i,j)} + v_{\chi_1(i,j),\chi_2(i,j)} \sigma_i \sigma_j \big) \mid \{\sigma_u : u \in U\} \Big] \\
\leq& \, 2 \prod_{(i,j) \in P_{\mathtt{j}}} \max \big\{ u_{\chi_1(i,j),\chi_2(i,j)}, v_{\chi_1(i,j),\chi_2(i,j)} \big\} .
\end{aligned}$$

Combining the above analysis with Lemma 3.2 implies that

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}} [\phi_{S_1,S_2} \mid \{\sigma_u : u \in U\}] \leq 2^{\mathtt{t}} \prod_{(i,j) \in G} \max \big\{ u_{\chi_1(i,j),\chi_2(i,j)}, v_{\chi_1(i,j),\chi_2(i,j)} \big\} ,$$

which yields the desired result as $\mathtt{t} \leq 5\tau(G) + 5|V| + 10\aleph$.

C.5 Proof of Lemma 3.6 Note that given $\mathtt{U} \subset V(T)$, T can be written as

$$T = T_{\mathtt{U}} \oplus \big(\oplus_{\mathtt{u} \in \mathtt{U}} T_{\mathtt{u}} \big) ,$$

where $T_{\mathtt{U}} \subset T$ is the minimum sub-tree of T containing $\mathtt{U}$, and $T_{\mathtt{u}}$ is the descendant tree of T rooted at $\mathtt{u}$. Thus, we have

$$\begin{aligned}
\mathbb{E}\Big[\phi_T(A)^2 \cdot \prod_{\mathtt{u} \in \mathtt{U}} \sigma_{\mathtt{u}} \Big] &= \mathbb{E}\Big[\phi_{T_{\mathtt{U}}}(A)^2 \cdot \prod_{\mathtt{u} \in \mathtt{U}} \phi_{T_{\mathtt{u}}}(A)^2 \cdot \prod_{\mathtt{u} \in \mathtt{U}} \sigma_{\mathtt{u}} \Big] \\
&= \mathbb{E}\Bigg\{ \prod_{\mathtt{u} \in \mathtt{U}} \sigma_{\mathtt{u}} \cdot \mathbb{E}\Big[\phi_{T_{\mathtt{U}}}(A)^2 \cdot \prod_{\mathtt{u} \in \mathtt{U}} \phi_{T_{\mathtt{u}}}(A)^2 \mid \{\sigma_{\mathtt{u}} : \mathtt{u} \in \mathtt{U}\} \Big] \Bigg\} . \qquad \text{(C.1)}
\end{aligned}$$

Note that conditioned on $\{\sigma_{\mathtt{u}} : \mathtt{u} \in \mathtt{U}\}$, we have that $\{\phi_{T_{\mathtt{U}}}(A), \phi_{T_{\mathtt{u}}}(A) : \mathtt{u} \in \mathtt{U}\}$ is a collection of conditionally independent variables, with

$$\mathbb{E}\Big[\phi_{T_{\mathtt{u}}}(A)^2 \mid \{\sigma_{\mathtt{u}} : \mathtt{u} \in \mathtt{U}\}\Big] = 1\,.$$

Thus, we have

$$\text{(C.1)} = \mathbb{E}\Big[\phi_{T_{\mathtt{U}}}(A)^2 \cdot \prod_{\mathtt{u} \in \mathtt{U}} \sigma_{\mathtt{u}}\Big]\,. \tag{C.2}$$

Observe that $\mathcal{L}(T_{\mathtt{U}}) \subset \mathtt{U}$. By applying Lemma A.3 to $T_{\mathtt{U}}$ and $\mathcal{L}(T_{\mathtt{U}})$ we can decompose $T_{\mathtt{U}}$ into $\mathtt{t}$ self-avoiding paths $P_{\mathtt{1}}, \ldots, P_{\mathtt{t}}$ satisfying Items (1)–(4) in Lemma A.3. In particular, we have $\mathtt{t} \leq 5|\mathcal{L}(T_{\mathtt{U}})| \leq 5|\mathtt{U}|$. Thus, we can decompose $T_{\mathtt{U}}$ into $\mathtt{t}' \leq \mathtt{t} + |\mathtt{U}|$ self-avoiding paths $P'_{\mathtt{1}}, \ldots, P'_{\mathtt{t}'}$ such that:

(I) $V(P'_{\mathtt{i}}) \cap (\cup_{\mathtt{j} \neq \mathtt{i}} V(P'_{\mathtt{j}})) = \mathrm{EndP}(V(P'_{\mathtt{i}}))$.

(II) $\mathtt{U} \subset \mathtt{W} := \cup_{\mathtt{i} \leq \mathtt{t}'} \mathrm{EndP}(V(P'_{\mathtt{i}}))$.

By conditioning on $\sigma_{\mathtt{W}} = \{\sigma_{\mathtt{u}} : \mathtt{u} \in \mathtt{W}\}$, we obtain

$$\begin{aligned}
\text{(C.2)} &= \mathbb{E}_{\sigma_{\mathtt{W}} \sim \nu_{\mathtt{W}}}\Bigg\{\prod_{\mathtt{u} \in \mathtt{U}} \sigma_{\mathtt{u}} \mathbb{E}\Big[\prod_{\mathtt{i} \leq \mathtt{t}'} \phi_{P'_{\mathtt{i}}}(A)^2 \mid \sigma_{\mathtt{W}}\Big]\Bigg\} \\
&\mathrel{\mathring{=}} \mathbb{E}_{\sigma_{\mathtt{W}} \sim \nu_{\mathtt{W}}}\Bigg\{\prod_{\mathtt{u} \in \mathtt{U}} \sigma_{\mathtt{u}} \prod_{\mathtt{i} \leq \mathtt{t}'} \Big(1 + \epsilon^{|E(P'_{\mathtt{i}})|} \prod_{u,v = \mathrm{EndP}(P'_{\mathtt{i}})} \sigma_u \sigma_v\Big)\Bigg\} \\
&= \sum_{\Lambda \subset [t']} \prod_{\mathtt{i} \in \Lambda} \epsilon^{|E(P'_{\mathtt{i}})|} \mathbb{E}_{\sigma_{\mathtt{W}} \sim \nu_{\mathtt{W}}}\Bigg\{\prod_{\mathtt{u} \in \mathtt{U} \cup \cup_{\mathtt{i} \in \Lambda} \mathrm{EndP}(P'_{\mathtt{i}})} \sigma_u\Bigg\} = \sum_{\Lambda \in \mathcal{M}} \prod_{\mathtt{i} \in \Lambda} \epsilon^{|E(P'_{\mathtt{i}})|}\,,
\end{aligned} \tag{C.3}$$

where $\mathcal{M}$ is the collection of $\Lambda \subset [t']$ such that each vertex appears an even number of times in $\mathtt{U} \cup \cup_{\mathtt{i} \in \Lambda} \mathrm{EndP}(P'_{\mathtt{i}})$. Now we fix $\Lambda \in \mathcal{M}$ and denote $\mathtt{H} = \cup_{\mathtt{i} \in \Lambda} P'_{\mathtt{i}} \subset T_{\mathtt{U}}$. Then the degree of any $\mathtt{u} \in V(T_{\mathtt{U}}) \setminus \mathtt{U}$ in $\mathtt{H}$ is even and the degree of any $\mathtt{u} \in \mathtt{U}$ in $\mathtt{H}$ is odd. Thus, for all $\mathtt{u} \in \mathtt{U}$ there exists $\mathtt{v} \in \mathtt{U}$ such that $\mathtt{u}$ is connected to $\mathtt{v}$ in $\mathtt{H}$. Denote $\mathcal{L}_{\mathtt{u},\mathtt{v}}$ be the path connecting $\mathtt{u}, \mathtt{v}$ in $\mathtt{H}$ and let $\mathcal{L}_{\mathtt{u}}$ to be the subgraph of $\mathcal{L}_{\mathtt{u},\mathtt{v}}$ induced by $\{\mathtt{w} \in V(\mathcal{L}_{\mathtt{u},\mathtt{v}}) : \mathsf{Dist}_{T_{\mathtt{U}}}(\mathtt{u}, \mathtt{w}) \leq \mathtt{d}/2\}$. We then have (note that $|E(\mathcal{L}_{\mathtt{u},\mathtt{v}})| = \mathsf{Dist}_{T_{\mathtt{U}}}(\mathtt{u}, \mathtt{v})$)

$$\mathcal{L}_{\mathtt{u}}\text{'s are disjoint and } |E(\mathcal{L}_{\mathtt{u}})| \geq \mathtt{d}/2\,.$$

Thus, for all $\Lambda \in \mathcal{M}$ we have (using $\mathsf{Dist}_{T_{\mathtt{U}}}(\mathtt{u}, \mathtt{v}) \geq \mathtt{d}$ for $\mathtt{u}, \mathtt{v} \in \mathtt{U}$)

$$\sum_{\mathtt{i} \in \Lambda} |E(P'_{\mathtt{i}})| \geq |\mathtt{U}|\mathtt{d}/2\,.$$

We then have

$$\text{(C.3)} \leq \sum_{\Lambda \in \mathcal{M}} \epsilon^{\mathtt{d}|\mathtt{U}|/2} \leq 2^{6|\mathtt{U}|} \epsilon^{\mathtt{d}|\mathtt{U}|/2}\,,$$

as desired.

C.6 Proof of Lemma 3.7 In this section, we prove Lemma 3.7. We will only prove (3.11) and the proof of (3.12) follows similarly. Denote $\Lambda = \{1 \leq p \leq m : \mathcal{L}_p \subset S\}$. In addition, define

$$\mathtt{U} = \cup_{1 \leq p \leq m} \mathrm{EndP}(\mathcal{L}_p)\,. \tag{C.4}$$

Conditioned on $\{\sigma_v : v \in \mathtt{U}\}$, we have that

$$\begin{aligned}
\Bigg\{ &\prod_{(i,j) \in E(T)} \frac{A_{i,j} - \frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1 - \frac{\lambda s}{n})}} \prod_{(i,j) \in E(T)} \frac{B_{i,j} - \frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1 - \frac{\lambda s}{n})}}\,, \\
&\prod_{(i,j) \in E(\mathcal{L}_p)} \frac{A_{i,j} - \frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1 - \frac{\lambda s}{n})}} : p \in \Lambda\,, \quad \prod_{(i,j) \in E(\mathcal{L}_q)} \frac{B_{i,j} - \frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1 - \frac{\lambda s}{n})}} : q \in [m] \setminus \Lambda \Bigg\}
\end{aligned}$$

is a collection of conditionally independent variables. In addition, using Lemma 3.3 we have for each $p \in \Lambda$

$$\mathbb{E}\Bigg[\prod_{(i,j)\in E(\mathcal{L}_p)} \frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}} \mid \{\sigma_u : u \in \mathtt{U}\}\Bigg]$$
$$= \mathbb{E}\Bigg[\prod_{(i,j)\in E(\mathcal{L}_p)} \frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}} \mid \{\sigma_u : u \in \mathrm{EndP}(\mathcal{L}_p)\}\Bigg] \mathrel{\mathring{=}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(\mathcal{L}_p)|/2} \prod_{u\in\mathrm{EndP}(\mathcal{L}_p)} \sigma_u\,,$$

where the last asymptotic equality is from the fact that $(1-\frac{\lambda s}{n})^{O(\log n)} = 1+o(1)$. Similar result holds for $\mathcal{L}_q$ for $q \in [m]\setminus\Lambda$ with A replaced by B. Thus, we have (for a vertex subset $\mathtt{V}$, we define $\nu_{\mathtt{V}}$ to be the uniform distribution of the labelings on $\mathtt{V}$)

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_S(A)\phi_K(B)] = \mathbb{E}_{\sigma_{\mathtt{U}}\sim\nu_{\mathtt{U}}}\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\phi_S(A)\phi_K(B) \mid \{\sigma_u : u\in\mathtt{U}\}\big]$$
$$= \mathbb{E}_{\sigma_{\mathtt{U}}\sim\nu_{\mathtt{U}}}\Bigg\{\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\Big[\prod_{(i,j)\in E(T)}\Big(\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\frac{B_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\Big)$$
$$\cdot\prod_{p\in\Lambda}\prod_{(i,j)\in E(\mathcal{L}_p)}\frac{A_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}}\prod_{q\in[m]\setminus\Lambda}\prod_{(i,j)\in E(\mathcal{L}_q)}\frac{B_{i,j}-\frac{\lambda s}{n}}{\sqrt{\frac{\lambda s}{n}(1-\frac{\lambda s}{n})}} \mid \sigma_{\mathtt{U}}\Big]\Bigg\}$$
$$\mathrel{\mathring{=}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\sum_{p\le m}|E(\mathcal{L}_p)|/2}\cdot\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\Big[\phi_T(A)\phi_T(B)\prod_{1\le j\le m}\prod_{u\in\mathrm{EndP}(\mathcal{L}_j)}\sigma_u\Big]\,,$$

completing the proof.

C.7 Proof of Lemma 5.2 Define $\mathcal{B}(\mathbf{H})$ to be the set of (S_1,S_2) such that $S_1,S_2\vdash\mathbf{H}$ but $S_1,S_2\not\Vdash\mathbf{H}$. Define

$$f_{\mathrm{bad}} = \sum_{\mathbf{H}\in\mathcal{H}}\frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))s^{\aleph-1}(\epsilon^2\lambda s)^{\ell\iota\aleph}}{n^{\aleph+\ell\iota\aleph}}\sum_{S_1,S_2\in\mathcal{B}(\mathbf{H})}\phi_{S_1,S_2}\,. \tag{C.5}$$

By Proposition 3.1 and Lemma 2.9, it suffices to show

$$\frac{f_{\mathrm{bad}}}{\mathbb{E}_{\mathbb{P}}[f]} = \sum_{\mathbf{H}\in\mathcal{H}}\frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))s^{\aleph-1}}{n^{\aleph+(\ell-2)\iota\aleph+o(1)}(\epsilon^2\lambda s)^{\ell\iota\aleph}}\sum_{S_1,S_2\in\mathcal{B}(\mathbf{H})}\phi_{S_1,S_2}\to 0 \tag{C.6}$$

in probability under both $\mathbb{P}$ and $\mathbb{Q}$. We will only prove (C.6) under $\mathbb{P}$, and the argument for $\mathbb{Q}$ is similar. Since f_{bad} is symmetric under π_*, it suffices to show (C.6) under $\mathbb{P}_{\mathsf{id}}$. To this end, we require additional definitions from [39].

Definition C.1. *We say a connected graph H is a tangle, if H contains at least two cycles and the diameter of H is at most $\sqrt{\log n}$. Given a multigraph $S\vdash\mathbf{H}\in\mathcal{H}$, we say that S contains t tangles if we need to delete at least t edges (counting multiplicity) so that the remaining subgraph of S contains no tangle.*

Definition C.2. *For a path $\gamma=(\gamma_1,\dots,\gamma_l)$ and a graph G, we say an edge (γ_i,γ_{i+1}) is*

- new *with respect to G, if $\gamma_{i+1}\notin V(G)\cup\{\gamma_1,\dots,\gamma_i\}$.*
- old *with respect to G, if $(\gamma_i,\gamma_{i+1})\in E(G)\cup\{(\gamma_1,\gamma_2),\dots,(\gamma_{i-1},\gamma_i)\}$.*
- returning *with respect to G, if (γ_i,γ_{i+1}) is neither new nor old with respect to G.*

We now divide f_{bad} into the following cases. For $S_1,S_2\vdash\mathbf{H}$ and $S_1,S_2\in\mathcal{B}(\mathbf{H})$, denote $G_S=\widetilde{S}_1\cup\widetilde{S}_2$. Let

$$\mathcal{B}_1(\mathbf{H}) := \Big\{(S_1,S_2)\in\mathcal{B}(\mathbf{H}) : \tau(G_S) > 20\aleph^2\Big\}\,; \tag{C.7}$$

$$\mathcal{B}_2(\mathbf{H}) := \Big\{(S_1,S_2)\in\mathcal{B}(\mathbf{H}) : \tau(G_S)\le 20\aleph^2, G_S \text{ contains at least } 40\aleph^3 \text{ tangles}\Big\}\,; \tag{C.8}$$

$$\mathcal{B}_3(\mathbf{H}) := \Big\{(S_1,S_2)\in\mathcal{B}(\mathbf{H}) : \tau(G_S)\le 20\aleph^2, G_S \text{ contains at most } 40\aleph^3 \text{ tangles}\Big\}\,. \tag{C.9}$$

Then let

$$f_{\mathcal{B}_i} = \sum_{\mathbf{H}\in\mathcal{H}} \frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))s^{\aleph-1}}{n^{\aleph+(\ell-2)\iota\aleph}(\epsilon^2\lambda s)^{\ell\iota\aleph}} \sum_{(S_1,S_2)\in\mathcal{B}_i(\mathbf{H})} \phi_{S_1,S_2} \text{ for } i=1,2,3\,. \tag{C.10}$$

Lemma C.3. *We have* $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[|f_{\mathcal{B}_1}|\big] = o(1)$.

The proof of Lemma C.3 is incorporated in Section C.8. Having established a bound for $f_{\mathcal{B}_1}$, we deal with $f_{\mathcal{B}_2}$. Define Ξ to be the event that the parent graph G does not contain any tangle. It was shown in [39, Lemma 6.1] that $\mathbb{P}(\Xi) = 1-o(1)$. We show the following estimation.

Lemma C.4. *We have* $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\mathbf{1}_\Xi|f_{\mathcal{B}_2}|\big] = o(1)$.

The proof of Lemma C.4 is postponed to Section C.9. Finally we bound $f_{\mathcal{B}_3}$.

Lemma C.5. *We have* $\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[f^2_{\mathcal{B}_3}\big] = o(1)$.

The proof of Lemma C.5 is postponed to Section C.10. We can now complete the proof of Lemma 5.2.

Proof of Lemma 5.2. Using Lemmas C.3, C.4 and C.5 respectively, we see that

$$f_{\mathcal{B}_1}, f_{\mathcal{B}_2}, f_{\mathcal{B}_3} \to 0 \text{ in probability under } \mathbb{P}_{\mathsf{id}}, \text{ respectively}\,.$$

Plugging this into (C.10) and (C.6), we see that

$$\frac{f_{\mathrm{bad}}}{\mathbb{E}_{\mathbb{P}}[f]} = f_{\mathcal{B}_1} + f_{\mathcal{B}_2} + f_{\mathcal{B}_3} \to 0$$

in probability under $\mathbb{P}_{\mathsf{id}}$, as desired. ∎

C.8 Proof of Lemma C.3 Note that we have the estimation

$$\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[|A_{i,j}-\tfrac{\lambda s}{n}|\big], \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[|B_{i,j}-\tfrac{\lambda s}{n}|\big], \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[|(A_{i,j}-\tfrac{\lambda s}{n})(B_{i,j}-\tfrac{\lambda s}{n})|\big] \le \tfrac{2\lambda s}{n}\,.$$

Combined with (2.7), it yields that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[|f_{\mathcal{B}_1}|\big] &\le \frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))s^{\aleph-1}}{n^{\aleph+(\ell-2)\iota\aleph+o(1)}(\epsilon^2\lambda s)^{\ell\iota\aleph}} \sum_{(S_1,S_2)\in\mathcal{B}_1(\mathbf{H})} \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[|\phi_{S_1,S_2}|\big] \\ &\le \sum_{(S_1,S_2)\in\mathcal{B}_1(\mathbf{H})} \frac{\mathrm{Aut}(\mathsf{T}(\mathbf{H}))s^{\aleph-1}}{n^{\aleph+(\ell-2)\iota\aleph+o(1)}(\epsilon^2\lambda s)^{\ell\iota\aleph}} (2\lambda s)^{2\ell\iota\aleph+2\aleph-2} n^{(\aleph-1+\ell\iota\aleph)-|E(G_S)|} \\ &\le n^{o(1)}\cdot \sum_{(S_1,S_2)\in\mathcal{B}_1(\mathbf{H})} (2\lambda s)^{2\ell\iota\aleph+2\aleph} n^{\aleph-|E(G_S)|}\,.\end{aligned}$$

In order to prove Lemma C.3, we first show the following lemma.

Lemma C.6. *Recall that* $G_S = \widetilde{S}_1 \cup \widetilde{S}_2$. *We have*

$$\#\Big\{(S_1,S_2)\in\mathcal{B}_1(\mathbf{H}) : |V(G_S)| = v, \tau(G_S) = m\Big\} \le n^{v+2\aleph+\max\{10\aleph^2, m/2\}+o(1)}\,.$$

Proof. Clearly, the enumeration of $(\mathsf{T}(S_1),\mathsf{T}(S_2))$ is bounded by $n^{2\aleph}$ and given $(\mathsf{T}(S_1),\mathsf{T}(S_2))$, the enumeration of $(\mathsf{P}(S_1),\mathsf{P}(S_2))$ is bounded by $(2\aleph)^{4\iota\aleph} = n^{o(1)}$. Now we give an upper bound on the enumeration of

$$(\mathsf{L}_1(S_1),\dots,\mathsf{L}_{\iota\aleph}(S_1);\mathsf{L}_1(S_2),\dots,\mathsf{L}_{\iota\aleph}(S_2))\,.$$

Denote $\mathcal{L}_{i+(j-1)\iota\aleph} = \mathsf{L}_i(S_j)\,(1\le i\le \iota\aleph, 1\le j\le 2)$ and suppose that $\mathcal{L}_i$ has n_i new edges and r_i returning edges with respect to $\mathsf{T}(S_1)\cup\mathsf{T}(S_2)\cup(\cup_{j\le i-1}\mathcal{L}_j)$. Clearly, by $-2\le\tau(\mathsf{T}(S_1)\cup\mathsf{T}(S_2))\le\aleph$ we have

$$\sum_{i=1}^{2\iota\aleph} n_i \le v \text{ and } m+2 \ge \sum_{i=1}^{2\iota\aleph} r_i \ge m-\aleph\,.$$

We now control the enumeration of $\{\mathcal{L}_i\}$. Recall that $\iota < \frac{1}{100}$. Therefore, we have $|V(G_S)| \leq |E(G_S)| \leq 2(\ell\iota+1)\aleph \leq \ell\aleph$. For each $\mathcal{L}_i$ with n_i new edges and r_i returning edges, there are $\frac{\ell!}{n_i! r_i! (\ell-n_i-r_i)!}$ choices to decide the order of old, new and returning edges. Also, we may consider the choices for vertices along the path sequentially and observe the following: for each returning edge (v_j, v_{j+1}) there are at most $|V(S_1 \cup S_2)| \leq 2(\ell-1)\iota\aleph + 2\aleph \leq 3\ell\aleph$ choices for v_{j+1}; for each new edge (v_j, v_{j+1}) there are at most n choices for v_{j+1}; and for each old edge (v_j, v_{j+1}) there are at most $\deg_{G_S}(v_j) \leq \deg_{\mathsf{T}(S_1)\cup\mathsf{T}(S_2)}(v_j) + \sum_{i=1}^{2\iota\aleph} r_i \leq 2\aleph + m$ choices for v_{j+1}. Thus, the enumeration of $\mathcal{L}_i$ is bounded by

$$
\begin{aligned}
&\frac{\ell!}{n_i! r_i! (\ell-n_i-r_i)!} \cdot n^{n_i} (3\ell\aleph)^{r_i} (2\aleph+m)^{\ell-n_i-r_i} \leq \frac{\ell^{\ell-n_i} n^{n_i} (3\ell\aleph)^{r_i} (2\aleph+m)^{\ell-n_i-r_i}}{(\ell-n_i-r_i)!} \\
&= n^{n_i} (3\ell^2\aleph)^{r_i} \frac{(\ell(\aleph+m))^{\ell-n_i-r_i}}{(\ell-n_i-r_i)!} \leq n^{n_i} (3\ell^2\aleph)^{r_i} (20(\aleph+m))^{\ell} ,
\end{aligned}
$$

where the first inequality follows from $\frac{\ell!}{n_i!} \leq \ell^{\ell-n_i}$, and the last inequality follows from the fact that the function $\frac{y^x}{x!}$ is increasing in x when $x \leq y$. Thus, the enumeration of $\{\mathcal{L}_i\}$ is bounded by

$$
\begin{aligned}
&\sum_{\substack{n_1+\cdots+n_{2\iota\aleph} \leq v \\ m-\aleph \leq r_1+\cdots+r_{2\iota\aleph} \leq m+2}} \prod_{i=1}^{2\iota\aleph} \Big(n^{n_i} (3\ell^2\aleph)^{r_i} (20(\aleph+m))^{\ell} \Big) \\
&= \sum_{\substack{n_1+\cdots+n_{2\iota\aleph} \leq v \\ m-\aleph \leq r_1+\cdots+r_{2\iota\aleph} \leq m+2}} n^{v} (3\ell^2\aleph)^{m+2} (20(\aleph+m))^{2\iota\aleph\ell} \\
&\leq n^{v+o(1)} (3\ell^2\aleph)^m \aleph^{2\iota\aleph\ell} m^{2\iota\aleph\ell} \leq e^{2\iota\ell\aleph\log(\aleph)} n^{v+\frac{\log(3\ell^2\aleph)}{\log n} m + \aleph\log m} \leq n^{v+\max\{10\aleph^2, m/2\}+o(1)} ,
\end{aligned}
$$

where in the first inequality we use the fact that

$$
\begin{aligned}
&\#\Big\{(n_1, \ldots, n_{2\iota\aleph}) : n_1 + \cdots + n_{2\iota\aleph} \leq v\Big\} \leq v^{2\iota\aleph} \overset{v=|V(G_S)|}{\leq} (\ell\aleph)^{2\iota\aleph} = n^{o(1)} ; \\
&\#\Big\{(r_1, \ldots, r_{2\iota\aleph}) : r_1 + \cdots + r_{2\iota\aleph} \leq m+2\Big\} \leq (m+2)^{2\iota\aleph} \overset{m=\tau(G_S)}{\leq} (\ell\aleph)^{2\iota\aleph} = n^{o(1)} ,
\end{aligned}
$$

in the second inequality we use the fact that $\ell \leq \log n$ (see (2.6)), and the last inequality follows from $\aleph\log m \leq 10\aleph^2 + 0.1(\log m)^2 \leq 10\aleph^2 + 0.1m$ for m larger than a sufficiently large constant. Thus, the enumeration of (S_1, S_2) is bounded by

$$
n^{2\aleph+o(1)} \cdot n^{v+\max\{10\aleph^2, m/2\}+o(1)} = n^{v+2\aleph+\max\{10\aleph^2, m/2\}+o(1)} ,
$$

as desired. ∎

Now we complete the proof of Lemma C.3.

Proof of Lemma C.3. Note that we have

$$
\begin{aligned}
&\sum_{(S_1,S_2)\in\mathcal{B}_1(\mathbf{H})} (2\lambda s)^{2\ell\iota\aleph+2\aleph} n^{\aleph-|E(G_S)|} \\
&\leq \sum_{0\leq v\leq 2\ell\aleph, m\geq 20\aleph^2} (2\lambda s)^{2\ell\iota\aleph} n^{\aleph-(v+m)+o(1)} \#\Big\{(S_1,S_2) : |V(G_S)| = v, \tau(G_S) = m\Big\} \\
&\overset{\text{Lemma C.6}}{\leq} \sum_{0\leq v\leq 2\ell\aleph, m\geq 20\aleph^2} (2\lambda s)^{2\ell\iota\aleph} n^{\aleph-(v+m)} n^{v+m/2+2\aleph+o(1)} \\
&\leq \sum_{0\leq v\leq 2\ell\aleph, m\geq 20\aleph^2} n^{-m/2+\aleph^2+o(1)} = o(1) ,
\end{aligned}
$$

leading to the desired result. ∎

C.9 Proof of Lemma C.4 Recall that for each $(S_1, S_2) \in \mathcal{B}_2(\mathbf{H})$, we denote $G_S = \widetilde{S}_1 \cup \widetilde{S}_2$. We say a subset of unordered pairs $F \subset \mathrm{U}_n$ is *pivotal* to G_S, if $G_S \setminus F$ is tangle-free (a graph is tangle-free if it does not contain any tangle) but $G_S \setminus F'$ contains at least one tangle for any $F' \subsetneq F$. In case that F is pivotal to G_S, at most two connected components of $G_S \setminus F$ can be trees (otherwise since $G_S = \widetilde{S}_1 \cup \widetilde{S}_2$ has at most two connected components, an edge could be added in $G_S \setminus F$ between two of the tree components without introducing any tangle, which contradicts to our definition). Therefore, supposing that $\mathsf{Comp}_1, \ldots, \mathsf{Comp}_i$ are all connected components of $G_S \setminus F$, we have $|E(G_S)| - |E(F)| = \sum_{j=1}^{i} |E(\mathsf{Comp}_j)| \geq \sum_{j=1}^{i} (|V(\mathsf{Comp}_j)| - \mathbf{1}_{\{\mathsf{Comp}_j \text{ is a tree}\}}) \geq |V(G_S)| - 2$, which implies $|F| \leq \tau(G_S) + 2$. Denote $\mathcal{F}(S)$ to be the collection of pivotal sets of G_S. Defining

$$\Omega_{\mathcal{F}(S)} := \cup_{F \in \mathcal{F}(S)} \Omega_F \text{ where } \Omega_F = \Big\{ G_{i,j} = 0 : (i,j) \in F \Big\}, \tag{C.11}$$

it is clear that $\Xi \subset \Omega_{\mathcal{F}(S)}$: since on the event $\cap_{F \in \mathcal{F}(S)} \Omega_F^c$ there exists a tangle $H \subset S$ with $G_{i,j} = 1$ for all $(i,j) \in E(H)$, contradicting to the definition of Ξ (which means that the edge set $\{(i,j) : G_{i,j} = 1\}$ is tangle-free). Thus, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\mathbf{1}_{\Xi} |f_{\mathcal{B}_2}|\big] &\leq \frac{1}{n^{\aleph + (\ell-2)\iota\aleph}} \sum_{(S_1,S_2) \in \mathcal{B}_2(\mathbf{H})} \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\mathbf{1}_{\Omega_{\mathcal{F}(S)}} |\phi_{S_1,S_2}|\big] \\
&\leq \frac{1}{n^{\aleph + (\ell-2)\iota\aleph}} \sum_{(S_1,S_2) \in \mathcal{B}_2(\mathbf{H})} \sum_{F \in \mathcal{F}(S)} \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\mathbf{1}_{\Omega_F} |\phi_{S_1,S_2}|\big] .
\end{aligned} \tag{C.12}$$

Since $\tau(G_S) \leq 20\aleph^2$, for all $F \in \mathcal{F}(S)$ we have $|F| \leq 20\aleph^2 + 2 \leq 100\aleph^2$. In addition, since G_S has at least $40\aleph^3$ tangles we see that

$$\sum_{(i,j) \in F} \Big(E(S_1)_{i,j} + E(S_2)_{i,j} - 2 \Big) \geq 40\aleph^3 - 200\aleph^2 \geq 35\aleph^3 \text{ for all } F \in \mathcal{F}(S) .$$

Thus, we see by definition of ϕ_{S_1,S_2} that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[\mathbf{1}_{\Omega_F} |\phi_{S_1,S_2}|\big] \\
&\leq \big(\tfrac{2\lambda s}{n}\big)^{\frac{1}{2}(\sum_{(i,j)\in F}(E(S_1)_{i,j}+E(S_2)_{i,j}))} \cdot \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{\frac{1}{2}(\sum_{(i,j)\notin F}(2-E(S_1)_{i,j}-E(S_2)_{i,j}))} \cdot 2^{|E(G_S)|} \\
&\leq \big(\tfrac{2\lambda s}{n}\big)^{|F|+\sum_{(i,j)\in F}(E(S_1)_{i,j}+E(S_2)_{i,j}-2)} \cdot \big(\tfrac{2\lambda s}{n}\big)^{\frac{1}{2}(\sum_{(i,j)}(2-E(S_1)_{i,j}-E(S_2)_{i,j}))} \cdot \big(\tfrac{2}{\epsilon^2}\big)^{4\ell\aleph} \\
&\leq \big(\tfrac{2\lambda s}{n}\big)^{|E(G_S)|+35\aleph^3-\aleph+1-\ell\iota\aleph} \cdot n^{\aleph^2} \leq n^{-30\aleph^3-|E(G_S)|+\ell\iota\aleph} ,
\end{aligned}$$

for all $F \in \mathcal{F}(S)$, where the first inequality holds by Lemma 3.2, and the third inequality follows from a crude bound $|F| \geq 0$ and

$$\sum_{(i,j) \in E(G_S)} (2 - E(S_1)_{i,j} - E(S_2)_{i,j}) = 2|E(G_S)| - 2(\ell\iota\aleph + \aleph - 1) .$$

Plugging this estimation into (C.12), we get that (note that $0 \leq \tau(G_S) \leq 20\aleph^2$)

$$\begin{aligned}
\text{(C.12)} &\leq n^{-25\aleph^3} \sum_{(S_1,S_2)\in\mathcal{B}_2} \sum_{F \in \mathcal{F}(S)} n^{-|E(G_S)|} \\
&\leq n^{-25\aleph^3} \sum_{k \leq 3\ell\iota\aleph} \sum_{m \leq 20\aleph^2} n^{-k} \cdot \#\{(S_1,S_2) : |E(\widetilde{S}_1 \cup \widetilde{S}_2)| = k, \tau(\widetilde{S}_1 \cup \widetilde{S}_2) = m\} \\
&\overset{\text{Lemma C.6}}{\leq} n^{-25\aleph^3} \sum_{k \leq 3\ell\iota\aleph} \sum_{m \leq 20\aleph^2} n^{-k} \cdot n^{(k-m)+12\aleph^2+o(1)} = o(1) .
\end{aligned} \tag{C.13}$$

This leads to the desired result.

C.10 Proof of Lemma C.5 Denote $G_\cup = \widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{K}_1 \cup \widetilde{K}_2$. By Lemma 4.4 and the fact that

$$\sum_{(i,j)\in E(G_\cup)} (E(S_1)_{i,j} + E(S_2)_{i,j} + E(K_1)_{i,j} + E(K_2)_{i,j}) = 4(\aleph - 1 + \ell\iota\aleph)\,,$$

we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{\mathsf{id}}}\big[f^2_{\mathcal{B}_3}\big] &= \sum_{\mathbf{H}\in\mathcal{H}} \frac{(\mathrm{Aut}(\mathsf{T}(\mathbf{H})))^2 s^{2\aleph-2}}{n^{2\aleph+2(\ell-2)\iota\aleph}(\epsilon^2\lambda s)^{2\ell\iota\aleph}} \sum_{(S_1,S_2),(K_1,K_2)\in\mathcal{B}_3(\mathbf{H})} \mathbb{E}_{\mathbb{P}_{\mathsf{id}}}[\phi_{S_1,S_2}\phi_{K_1,K_2}] \\
&\le \sum_{\mathbf{H}\in\mathcal{H}} \frac{n^{o(1)} s^{2\aleph-2}}{n^{2\aleph+2(\ell-2)\iota\aleph}(\epsilon^2\lambda s)^{2\ell\iota\aleph}} \sum_{(S_1,S_2),(K_1,K_2)\in\mathcal{B}_3(\mathbf{H})} \Big(\frac{n}{\epsilon^2\lambda s}\Big)^{2(\aleph-1+\ell\iota\aleph)-|E(G_\cup)|} \\
&\le \frac{n^{4\iota\aleph-2+o(1)} s^{2\aleph-2}|\mathcal{H}|}{(\epsilon^2\lambda s)^{4\ell\iota\aleph+2(\aleph-1)}} \sum_{(S_1,S_2),(K_1,K_2)\in\mathcal{B}_3(\mathbf{H})} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(G_\cup)|} \\
&= \frac{n^{4\iota\aleph-2+o(1)}}{(\epsilon^2\lambda s)^{4\ell\iota\aleph}} \sum_{(S_1,S_2),(K_1,K_2)\in\mathcal{B}_3(\mathbf{H})} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(G_\cup)|}\,. \qquad \text{(C.14)}
\end{aligned}$$

In addition, the following holds by Remark 4.6: for $G_\cup \subset \mathcal{K}_n$, if

(C.15) $$\exists (S_1,S_2),(K_1,K_2)\in\mathcal{B}_3(\mathbf{H}) \text{ such that } G_\cup = \widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{K}_1 \cup \widetilde{K}_2 \text{ and } \mathcal{L}(G_\cup)\subset V(G_{\ge 2})\,,$$

then we have

(C.16) $$4\iota\aleph - 1 - \tau(G_\cup) \le \tfrac{4\ell\iota\aleph+4\aleph-|E(G_\cup)|}{\ell/2}\,.$$

We say a simple graph $\mathbf{G}$ is with (C.15) and (C.16) if $\mathbf{G}\cong G_\cup$ for some $G_\cup$ satisfying (C.15) and (C.16). Using this terminology, we get that

$$\begin{aligned}
\text{(C.14)} \le \frac{n^{4\iota\aleph-2+o(1)}}{(\epsilon^2\lambda s)^{4\ell\iota\aleph}} &\sum_{\mathbf{G}\text{ with (C.15),(C.16)}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(\mathbf{G})|} \\
&\cdot \sum_{G_\cup\cong\mathbf{G}} \#\{(S_1,S_2),(K_1,K_2)\in\mathcal{B}_3 : \widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{K}_1 \cup \widetilde{K}_2 = G_\cup\}\,. \qquad \text{(C.17)}
\end{aligned}$$

We first need the following lemma.

Lemma C.7. *Given $J\subset\mathcal{K}_n$ with at most $\mathsf{z}=O(1)$ connected components, we have*

(C.18) $$\#\big\{S\subset J : \exists S' \text{ such that } (S,S')\in\mathcal{B}_3\big\} \le \binom{|E(J)|}{\aleph} n^{o(1)}\aleph^{2\iota\aleph} e^{\sqrt{\log n}(\log\log n)^4(\tau(J)+\mathsf{z})}\,.$$

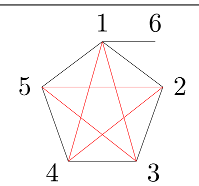
Proof. Note that the enumeration of $\mathsf{T}(S)$ is bounded by $\binom{|E(J)|}{\aleph}$, and given $\mathsf{T}(S)$ the enumeration of $\mathsf{P}(S)$ is bounded by $\aleph^{2\iota\aleph}$. Given $\mathsf{T}(S)$ and $\mathsf{P}(S)$, we now explain how to choose $\mathsf{L}_i(S)$. By Lemma A.3, $J\setminus\mathsf{T}(S)$ can be decomposed into at most $5(\tau(J)+1)$ self-avoiding paths such that they only overlap at endpoints. Now define the trace of $\mathsf{L}_i(S)$ by $\mathsf{Tr}_i = \widetilde{\mathsf{L}_i(S)}$. Since Tr_i is the union of a subset of these $5(\tau(J)+1)$ self-avoiding paths, the enumeration for Tr_i is bounded by $2^{5(\tau(J)+1)}$.

Given Tr_i, we will bound the enumeration of $\mathsf{L}_i(S)$ by constructing a surjection from certain tuples of variables (see (C.19) below) to realizations of $\mathsf{L}_i(S)$, and then bound the enumeration of such tuples. The following proof is similar to the arguments in [39]. We will introduce some definitions first to make our motivations in the following proof clearer. Recall Definition C.1. Define $\mathcal{T}=\mathcal{T}(\mathsf{L}_i(S))$ to be the family of minimal multigraphs T such that $\mathsf{L}_i(S)\setminus T$ is tangle-free. By definition of $\mathcal{B}_3$, $\mathsf{L}_i(S)$ has at most $40\aleph^3$ tangles and therefore $|E(T)|\le 40\aleph^3$ for $T\in\mathcal{T}(\mathsf{L}_i(S))$. Given any $T\in\mathcal{T}$, any vertex v in Tr_i and any neighbor u of v in Tr_i, we say u is a *short* neighbor of v if it is in a cycle in $\mathsf{L}_i(S)\setminus T$ with length bounded by $\sqrt{\log n}$; by definition of $\mathcal{T}$, every vertex has at most two short neighbors. We say u is a *tangled* neighbor of v if it is not a short neighbor of v, but it is in a cycle in $\mathsf{L}_i(S)$ with length bounded by $\sqrt{\log n}$. We say u is a *long* neighbor of v if it is neither a short nor a tangled neighbor of v. Define $V_{\geq 3}^{(i)} = \{v \in V(\mathsf{Tr}_i) : d_{\mathsf{Tr}_i}(v) \geq 3\}$, which is determined by Tr_i (here $d_{\mathsf{Tr}_i}(v)$ is the degree of v in Tr_i). Also, for each $\mathsf{L}_i(S)$ such that $\mathsf{L}_i(S) = (v_0, v_1, \ldots, v_\ell)$ where $(v_0, v_\ell) \in \mathsf{P}(S)$, and given any $T \in \mathcal{T}$ we define a list $\mathsf{List}_T^{(i)}(v)$ with ℓ lines $\mathsf{List}_T^{(i)}(v)_1, \ldots, \mathsf{List}_T^{(i)}(v)_\ell$ on each $v \in V_{\geq 3}^{(i)}$ such that the following hold: (i) each line has three possible states: blank, non-short-coming and non-short-going; (ii) the state of $\mathsf{List}_T^{(i)}(v)_j$ is non-short-coming (respectively, non-short-going) if and only if v_{j-1} (respectively, v_{j+1}) exists, $v = v_j$ and v_{j-1} (respectively, v_{j+1}) is not a short neighbor of v_j. Next we prove the following claim: given $\mathsf{L}_i(S) = (v_0, v_1, \ldots, v_\ell)$, $T \in \mathcal{T}(\mathsf{L}_i(S))$ and $v \in V_{\geq 3}^{(i)}$ (which is determined by $\mathsf{L}_i(S)$), there are at most $\frac{\ell}{\sqrt{\log n}} + 40\aleph^3$ lines in $\mathsf{List}_T^{(i)}(v)$ with a non-short-coming or non-short-going state. To this end, (by symmetry) it suffices to deal with the non-short-going case. By definition of long and tangled neighbors, every time a step of $\mathsf{L}_i(S)$ goes to a long neighbor (of some vertex v) it takes at least $\sqrt{\log n}$ steps to go back to v, and every time a step of $\mathsf{L}_i(S)$ goes to a tangled neighbor (of some vertex v) it either takes at least $\sqrt{\log n}$ steps to go back to v, or traverses an edge in T at least once. Therefore, our claim follows from $|E(T)| \leq 40\aleph^3$.

Now we construct a mapping

$$\begin{aligned} \mathcal{X} : &\Big(\mathsf{Tr}_i, (v_0, v_1, v_\ell), \big(s(v), \mathsf{H}(v)\big)_{v \in V_{\geq 3}^{(i)}}, \big(w_{v,j} : 1 \leq j \leq \tfrac{2\ell}{\sqrt{\log n}} + 80\aleph^3, v \in V_{\geq 3}^{(i)}\big)\Big) \\ &\longrightarrow (v_0, v_1, \ldots, v_\ell)\,, \end{aligned} \tag{C.19}$$

where the notations in C.19 are specified as follows: for any $v \in V_{\geq 3}^{(i)}$, $s(v) = \{v', v''\}$ where $v', v'' \in \mathsf{Nei}_{\mathsf{Tr}_i}(v)$; $\mathsf{H}(v)$ is a list with ℓ lines $\mathsf{H}(v)_1, \ldots, \mathsf{H}(v)_\ell$ such that each line has three possible states: blank, non-short coming and non-short-going, and at most $\frac{\ell}{\sqrt{\log n}} + 40\aleph^3$ lines of $\mathsf{H}(v)$ are in the state of non-short-coming (going); for $v \in V_{\geq 3}^{(i)}$ each $(w_{v,\cdot})$ is a vertex tuple where $w_{v,j} \in V(\mathsf{Tr}_i)$ for $1 \leq j \leq \frac{2\ell}{\sqrt{\log n}} + 80\aleph^3$. Also, the image $(v_0, v_1, \ldots, v_\ell)$ serves as the non-backtracking path $\mathsf{L}_i(S)$ we hope for. The mapping is constructed as follows: First, we start from v_0 and choose the steps of $\mathsf{L}_i(S)$ one by one. For the xth step starting at v_{x-1}, if $x = 1$ then v_1 is already in the input; else if $d_{\mathsf{Tr}_i}(v_{x-1}) \leq 2$, the choice of v_x is unique by the fact that $(v_0, v_1, \ldots, v_\ell)$ is non-backtracking; else if the state of the xth line of $\mathsf{H}(v_{x-1})$ is blank, we set v_x the element in $s(v_{x-1})$ such that $v_x \neq v_{x-2}$; else we set $v_x = w_{v_{x-1},y}$ if $x = \mathsf{ind}(y, \mathsf{H}(v))$, where $\mathsf{ind}(y, \mathsf{H}(v))$ is the yth smallest index z such that $\mathsf{H}(v)_z$ is not in the blank state. Then we have obtained $(v_0, v_1, \ldots, v_\ell) = \mathsf{L}_i(S)$.



v	$S(v)$	$H(v)$	w_v
1	$\{5, 2\}$	NSG at 5th line; NSC at 10th, 13th line	[3,2,6,arbitrary]
2	$\{1, 3\}$	NSC at 8th line; NSG at 11th line	[4,4,arbitrary]
3	$\{2, 4\}$	NSC at 6th line	[5,arbitrary]
4	$\{3, 5\}$	NSC at 9th, 12th line	[1,1,arbitrary]
5	$\{4, 1\}$	NSC at 7th line	[1,2,arbitrary]

Table C.1: An example of a non-backtracking path and its preimage. Left: an example of Tr_i; Right: a preimage of the non-backtracking path $\mathsf{L}_i(S) = (2, 3, 4, 5, 1, 3, 5, 2, 4, 1, 2, 4, 1, 6)$, where $V_{\geq 3}^{(i)} = \{1, 2, 3, 4, 5\}$. We omitted $(v_0, v_\ell) = (2, 6)$ and $v_1 = 3$ in the preimage table, and NSC (NSG) in the table refers to non-short-coming (non-short-going). The edges in Tr_i which are in a possible T are colored red. The lines of $H(v)$'s which are not mentioned are in the blank state. The notation w_v stands for the sequence $\{w_{v,j}\}_{1 \leq j \leq \frac{2\ell}{\sqrt{\log n}} + 80\aleph^3}$, and for any $1 \leq v \leq 5$, an "arbitrary" in the end of w_v means an arbitrary sequence of vertices with appropriate length.

Then we claim that given any Tr_i and (v_0, v_ℓ), $\mathcal{X}$ is a surjection to all non-backtracking paths $\mathsf{L}_i(S)$ with at most $40\aleph^3$ tangles and with $\widetilde{\mathsf{L}_i(S)} = \mathsf{Tr}_i$. It suffices to show that for each such $\mathsf{L}_i(S)$ we can construct a preimage of $\mathcal{X}$ with image $\mathsf{L}_i(S)$. To this end, consider an instance $\mathsf{L}_i(S) = (v_0, v_1, \ldots, v_\ell)$ with trace Tr_i, and an arbitrary $T \in \mathcal{T}(\mathsf{L}_i(S))$ with $|T| \leq 40\aleph^3$. Then for any $v \in V_{\geq 3}^{(i)}$, we determine the corresponding variables in our preimage by

$$\begin{aligned} &s(v) = \{u \in \mathsf{Nei}_{\mathsf{Tr}_i}(v) : u \text{ is a short neighbor of } v \text{ with respect to } T\}\,, \\ &\mathsf{H}(v) = \mathsf{List}_T^{(i)}(v)\,,\ w_{v,y} = v_{\mathsf{ind}(y, \mathsf{H}(v))}\,. \end{aligned}$$

Now we argue that $\mathcal{X}$ indeed maps

$$\Big(\mathsf{Tr}_i, (v_0, v_\ell), v_1, \big(s(v), \mathsf{H}(v)\big)_{v \in V_{\geq 3}^{(i)}}, \big(w_{v,j} : 1 \leq j \leq \tfrac{2\ell}{\sqrt{\log n}} + 80\aleph^3, v \in V_{\geq 3}^{(i)}\big)\Big)$$

to $(v_0, v_1, \ldots, v_\ell) = \mathsf{L}_i(S)$. Otherwise, suppose that the image is $(v_0', v_1', \ldots, v_\ell') \neq (v_0, v_1, \ldots, v_\ell)$ and there exists a minimal x such that $v_x' \neq v_x$. Then by definition of our map we have $x \geq 2$; we also have $d_{\mathsf{Tr}_i}(v_{x-1}) \geq 3$ since both $(v_0', v_1', \ldots, v_\ell')$ and $(v_0, v_1, \ldots, v_\ell)$ are non-backtracking, making it impossible for $v_{x-2} = v_{x-2}'$, $v_{x-1} = v_{x-1}'$ and $v_x \neq v_x'$ when $d_{\mathsf{Tr}_i}(v_{x-1}) \leq 2$. If the state of the xth line of $\mathsf{H}(v_{x-1}) = \mathsf{List}_T^{(i)}(v_{x-1})$ is blank, then by definition of $\mathsf{List}_T^{(i)}$ we have that both v_{x-2} and v_x are short neighbors of v_{x-1}, and since $(v_0, v_1, \ldots, v_\ell)$ is non-backtracking we have $s(v_{x-1}) = \{v_{x-2}, v_x\}$. By the definition of $\mathcal{X}$, we have $v_x' \in s(v_{x-1}') = s(v_{x-1})$ and $v_x' \neq v_{x-2}' = v_{x-2}$, which gives $v_x' = v_x$, contradicting $v_x' \neq v_x$. Therefore, the state of the xth line of $\mathsf{H}(v_{x-1}) = \mathsf{List}_T^{(i)}(v_{x-1})$ is non-short-coming or non-short-going. However, in both cases by definition of $\mathcal{X}$ we have $v_x' = w_{v,y} = v_{\mathsf{ind}(y, \mathsf{H}(v))}$ where $\mathsf{ind}(y, \mathsf{H}(v)) = x$, which is a contradiction. Therefore, we have shown that $\mathcal{X}$ is a surjection, which gives that given Tr_i and (v_0, v_ℓ), the enumeration of non-backtracking paths $\mathsf{L}_i(S)$ with at most $40\aleph^3$ tangles and with $\widetilde{\mathsf{L}_i(S)} = \mathsf{Tr}_i$ is bounded by the enumeration of (C.19).

Finally, we bound the enumeration of (C.19) given Tr_i and (v_0, v_ℓ). Since $|V(\mathsf{Tr}_i)| \leq |E(\mathsf{Tr}_i)| + 1 \leq \ell + 1$, the enumeration of v_1 and $\big(w_{v,j} : 1 \leq j \leq \frac{2\ell}{\sqrt{\log n}} + 80\aleph^3, v \in V_{\geq 3}^{(i)}\big)$ is bounded by $(\ell+1)^{1 + (2\ell/\sqrt{\log n} + 80\aleph^3)|V_{\geq 3}^{(i)}|}$. Then, since for each $v \in V_{\geq 3}^{(i)}$ there are at most $\frac{2\ell}{\sqrt{\log n}} + 80\aleph^3$ lines with states that are not blank, the enumeration of $\big(\mathsf{H}(v)\big)_{v \in V_{\geq 3}^{(i)}}$ is bounded by $(\ell+1)^{(2\ell/\sqrt{\log n} + 80\aleph^3)|V_{\geq 3}^{(i)}|}$. Also, by a direct upper bound the enumeration of $\big(s(v)\big)_{v \in V_{\geq 3}^{(i)}}$ is bounded by $(2\ell+2)^{2|V_{\geq 3}^{(i)}|}$. Therefore, by the fact that $|V_{\geq 3}^{(i)}| \leq \tau(\mathsf{L}_i(S)) \leq \tau(S) + 1 \leq \tau(J) + \mathsf{z}$ we obtain that given Tr_i and (v_0, v_ℓ) the enumeration of (C.19) is bounded by

$$(2\ell+2)^{1 + (4\ell/\sqrt{\log n} + 2 + 160\aleph^3)(\tau(J) + \mathsf{z})} \leq e^{\sqrt{\log n}(\log\log n)^3(\tau(J)+\mathsf{z})}\,.$$

Thus, combining all the enumerations above, the enumeration of S is bounded by

$$\binom{|E(J)|}{\aleph} \cdot \aleph^{2\iota\aleph} \cdot 2^{\iota\aleph \cdot 5(\tau(J)+1)} \cdot e^{\aleph\sqrt{\log n}(\log\log n)^3(\tau(J)+\mathsf{z})}\,,$$

and the desired result follows from $\aleph = o(\log\log n)$ and $e^{\sqrt{\log n}(\log\log n)^4 \mathsf{z}} = n^{o(1)}$. ∎

Now we see that

$$\begin{aligned}
\text{(C.17)} &\leq \frac{n^{4\iota\aleph - 2 + o(1)}}{(\epsilon^2\lambda s)^{4\ell\iota\aleph}} \sum_{\mathbf{G} \text{ with (C.15),(C.16)}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(\mathbf{G})|} \sum_{G \cong \mathbf{G}} \binom{|E(G)|}{\aleph}^4 \aleph^{8\iota\aleph} e^{4\sqrt{\log n}(\log\log n)^4 \tau(G)} \\
&= \frac{n^{4\iota\aleph - 2 + o(1)}}{(\epsilon^2\lambda s)^{4\ell\iota\aleph}} \sum_{\mathbf{G} \text{ with (C.15),(C.16)}} \big(\tfrac{\epsilon^2\lambda s}{n}\big)^{|E(\mathbf{G})|} e^{4\sqrt{\log n}(\log\log n)^4 \tau(\mathbf{G})} \#\{G \subset \mathcal{K}_n : G \cong \mathbf{G}\} \\
&\leq n^{4\iota\aleph - 2 + o(1)} \sum_{\substack{0 \leq x \leq 4\iota\ell\aleph + 4\aleph \\ \frac{2x - 8\aleph - 4\ell\iota\aleph - \ell}{\ell} \leq y \leq 2\aleph^2}} (\epsilon^2\lambda s)^{-(4\ell\iota\aleph + 4\aleph - x)} n^{-x} e^{4\sqrt{\log n}(\log\log n)^4 y} n^{-y+x} \\
&\qquad \cdot \#\{\mathbf{G} : \mathbf{G} \text{ with (C.15), (C.16)}, |E(\mathbf{G})| = x, \tau(\mathbf{G}) = y\} \\
&\leq n^{4\iota\aleph - 2 + o(1)} \sum_{0 \leq x \leq 4\iota\ell\aleph + 4\aleph} n^{-2\frac{4\ell\iota\aleph + 4\aleph - x}{l}} \big(\tfrac{n}{(\ell\aleph)^2}\big)^{-\frac{2x - 8\aleph - 4\ell\iota\aleph - \ell}{\ell}} \\
&= n^{-1+o(1)}\,, \qquad \text{(C.20)}
\end{aligned}$$

where in the first inequality we apply Lemma C.7, the third inequality holds by (C.16) and $(\epsilon^2\lambda s)^{4\aleph} = n^{o(1)}$, and the last inequality holds by $e^{4\sqrt{\log n}(\log\log n)^4 y} \leq e^{8\sqrt{\log n}(\log\log n)^4 \aleph^2} = n^{o(1)}$ and Lemma A.6. This leads to the desired result.

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